# DUALIZING, PROJECTING, AND RESTRICTING GKZ SYSTEMS

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ABSTRACT. Let $A$ be an integer matrix, and assume that its semigroup ring $\mathbb{C}[\mathbb{N}A]$ is normal. Fix a face $F$ of the cone of $A$. We show that the projection and restriction of an $A$-hypergeometric system to the coordinate subspace corresponding to $F$ are isomorphic; moreover, they are essentially $F$-hypergeometric.

We also show that, if $A$ is in addition homogeneous, the holonomic dual of an $A$-hypergeometric system is itself $A$-hypergeometric. This extends a result from [Wal07], proving a conjecture of Nobuki Takayama in the normal homogeneous case.

## 1. Introduction

Let $A \in \mathbb{Z}^{d\times n}$ be an integer matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ such that $\mathbb{Z}A = \mathbb{Z}^d$; we abuse notation and also use $A$ to denote the set of its columns. Assume that $\mathbb{N}A$ is pointed, i.e. that $\mathbb{N}A \cap -\mathbb{N}A = 0$. Associated to this data, Gel′fand, Graev, Kapranov, and Zelevinskiĭ defined in [GGZ87, GZK89] a family of modules over the sheaf $\mathcal{D}_{\mathbb{C}^n}$ of algebraic linear partial differential operators on $\mathbb{C}^n$ today referred to either as *GKZ-* or *A-hypergeometric* systems. These systems are defined as follows:

The *Euler operators* of $A$ are the operators $E_i := a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n$ $(i = 1, \ldots, d)$, and the *toric ideal* of $A$ is the $\mathbb{C}[\partial_1, \ldots, \partial_n]$-ideal $I_A := \langle \partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-} \mid A\mathbf{u} = 0, \mathbf{u} \in \mathbb{Z}^n\rangle$. The *A-hypergeometric system* corresponding to the parameter $\beta \in \mathbb{C}^d$ is then defined to be

$$\mathcal{M}_A(\beta) := \mathcal{D}_{\mathbb{C}^n} / \left(\mathcal{D}_{\mathbb{C}^n} I_A + \mathcal{D}_{\mathbb{C}^n}\{E_1 - \beta_1, \ldots, E_d - \beta_d\}\right). \tag{1.0.1}$$

If the condition that $\mathbb{Z}A = \mathbb{Z}^d$ is relaxed, $\mathcal{M}_A(\beta)$ may still be defined as above by first choosing a $\mathbb{Z}$-basis of $\mathbb{Z}A$; the resulting $\mathcal{D}_{\mathbb{C}^n}$-module is independent of this choice.

1.1. **Projection and restriction.** Explicit formulas for restriction (i.e. pullback via the $D$-module inverse image) to a coordinate subspace were computed in [CJT03, Th. 4.4] and [FFCJ11, Th. 4.2] for

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certain classes of GKZ systems. These formulas were generalized in [FFW11, Th. 2.2] under certain hypotheses about the genericity of the parameter $\beta$ and the size of the coordinate subspace. We focus on a different situation, and explicitly compute, when the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is normal, the restriction of $\mathcal{M}_A(\beta)$ to the coordinate subspace $\mathbb{C}^F$ corresponding to a face $F \preceq A$ (see (2.3.4) for the notation $\mathbb{C}^F$). Moreover, we show (Theorem 5.8) that this restriction is in fact equal to the projection (i.e. the pushforward via the $D$-module direct image) of $\mathcal{M}_A(\beta)$ to $\mathbb{C}^F$. Note that, unless $F = A$, the subspace $\mathbb{C}^F$ does not satisfy the size requirements of [FFW11, Th. 2.2], hence there is no nontrivial overlap between this paper and [FFW11].

Our approach is to use the notion of mixed and dual mixed Gauss–Manin systems (see §2.4) introduced in [Ste17]. We first study these in slightly more generality in §3. In §4, we generalize the notion of quasi-equivariant $D$-module (introduced by T. Reichelt and U. Walther in [RW17]) to what we are calling twistedly quasi-equivariant $D$-modules (Definition 4.2). We then follow a similar process to that in [RW17] to relate the restriction and projection of such modules (Lemma 4.4) and to show that mixed and dual mixed Gauss–Manin systems are twistedly quasi-equivariant (Proposition 4.5). These results are combined in §5 first to compute the restriction and projection to $\mathbb{C}^F$ of dual mixed Gauss–Manin and mixed Gauss–Manin systems, respectively (Theorem 5.4), and then to do the same for normal $A$-hypergeometric systems (Theorem 5.8).

1.2. **Duality.** N. Takayama conjectured that the holonomic dual of an $A$-hypergeometric system is itself a GKZ system (after applying the coordinate transformation $x \mapsto -x$ if $A$ is non-homogeneous, i.e. if the columns of $A$ do not all lie in a hyperplane). U. Walther, in [Wal07], provided a class of counterexamples to this conjecture. However, each of these counterexamples is rank-jumping (i.e. the holonomic rank is higher than expected), and in the same paper, Walther shows that for generic parameters, Takayama's conjecture does indeed hold. In particular, when the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is normal, he proves ([Wal07, Prop. 4.4]) that the set of all parameters $\beta$ for which the holonomic dual of $\mathcal{M}_A(\beta)$ is not a GKZ system has codimension at least three. We show in Theorem 6.3 using the notion of mixed and dual mixed Gauss–Manin systems that if $A$ is homogeneous, this set is in fact empty.

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## 2. Notation and conventions

In §2.1, we define various notations and conventions related to varieties, derived categories, $D$-modules, and mixed Hodge modules. §2.2 recalls the notions of fiber and cofiber support. §2.3 defines various notations related to the semigroup $\mathbb{N}A$, and in §2.4 we recall and discuss the notions of mixed and dual mixed Gauss–Manin parameters and systems.

### 2.1. **General geometric conventions/notation.**

Varieties, smooth or otherwise, are not required to be irreducible, are defined over $\mathbb{C}$, and are always considered with the Zariski topology. The closure of a subset $Z$ of a topological space $X$ is written $\overline{Z}$. If $X$ is a smooth variety, denote by $\mathcal{D}_X$ its sheaf of algebraic linear partial differential operators.

2.1.1. *Derived categories.* The category of mixed Hodge modules on a variety $X$ is denoted $\mathrm{MHM}(X)$. The bounded derived category of $\mathrm{MHM}(X)$ is denoted $\mathrm{D^b\,MHM}(X)$. If $X$ is smooth, the bounded derived category of $\mathcal{D}_X$-modules with coherent and holonomic cohomology are denoted by $\mathrm{D^b_c}(\mathcal{D}_X)$ and $\mathrm{D^b_h}(\mathcal{D}_X)$, respectively. If $Z$ is a closed subvariety, a superscript $Z$ in the notation for any of these categories denotes the full subcategory of objects whose cohomology is supported in $Z$.

2.1.2. *$D$-module functors.* (cf. [HTT08]) The holonomic duality functor ([HTT08, Def. 2.6.1]) is denoted $\mathbb{D}$. Let $f\colon X \to Y$ be a morphism of smooth varieties. We write $f_+$ for the $D$-module direct image, $f_\dagger := \mathbb{D} f_+ \mathbb{D}$ for the $D$-module exceptional direct image, $f^+ := \mathrm{L}f^*[\dim X - \dim Y]$ for the $D$-module inverse image, and $f^\dagger := \mathbb{D} f^+ \mathbb{D}$ for the $D$-module exceptional direct image. If $X_1$ and $X_2$ are smooth varieties and $\mathcal{M}_i^\bullet \in \mathrm{D}^b(\mathcal{D}_{X_i})$ $(i = 1, 2)$, the exterior tensor product (see [HTT08, p38]) of $\mathcal{M}_1^\bullet$ and $\mathcal{M}_2^\bullet$ is

$$\mathcal{M}_1^\bullet \boxtimes \mathcal{M}_2^\bullet := \mathcal{D}_{X_1 \times X_2} \otimes_{p_1^{-1}\mathcal{D}_{X_1} \otimes_{\mathbb{C}} p_1^{-1}\mathcal{D}_{X_2}} (p_1^{-1}\mathcal{M}_1^\bullet \otimes_{\mathbb{C}} p_1^{-1}\mathcal{M}_2^\bullet).$$

Note that [HTT08] denotes the functors $f_+$, $f^+$, $f_\dagger$, and $f^\dagger$ by $\int_f$, $f^\dagger$, $\int_{f!}$, and $f^\bigstar$, respectively. They define the first two on pages 33 and 40, respectively, while they define the second two in Def. 3.2.13 on page 91.

2.1.3. *Fourier–Laplace transform.* (cf. [Bry86, pp85-102]) The Fourier–Laplace transform is denoted by FL. By definition, $\mathrm{FL}(\mathcal{M}^\bullet)$ is the pullback of $\mathcal{M}^\bullet \in \mathrm{D}^b(\mathcal{D}_{\mathbb{C}^n})$ by the $\mathbb{C}$-algebra automorphism of $\mathcal{D}_{\mathbb{C}^n}$ taking $x_i \mapsto \partial/\partial x_i$ and $\partial/\partial x_i$ to $-x_i$. The inverse Fourier transform is denoted by $\mathrm{FL}^{-1}$ and is defined similarly.

For a description of FL in terms of $D$-module direct and inverse image functors, see [DE03].

2.1.4. *Mixed Hodge modules.* Let $\mathcal{M}^\bullet$ be a complex of mixed Hodge modules, and let $F$ be a functor of $D$-modules. If the mixed Hodge module structure on $\mathcal{M}^\bullet$ induces a mixed Hodge module structure on $F(\mathcal{M}^\bullet)$, we will always take $F(\mathcal{M}^\bullet)$ to be this induced mixed Hodge module unless otherwise specified.

2.2. **Fiber and cofiber support.** We recall from [Ste17, Def. 3.1] the notions of fiber and cofiber support. The *fiber support* of a (bounded) complex $\mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules is

$$\operatorname{fSupp} \mathcal{M}^\bullet \coloneqq \left\{ x \in X \;\middle|\; k(x) \otimes^{\mathrm{L}}_{\mathcal{O}_{X,x}} \mathcal{M}^\bullet_x \neq 0 \right\}. \tag{2.2.1}$$

If $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$, its *cofiber support* is

$$\operatorname{cofSupp} \mathcal{M}^\bullet \coloneqq \operatorname{fSupp} \mathbb{D}\mathcal{M}^\bullet. \tag{2.2.2}$$

2.3. **Toric and GKZ conventions/notation.** The semigroup ring of $A$ is $S_A \coloneqq \mathbb{C}[\mathbb{N}A] = \mathbb{C}[\partial_1, \dots, \partial_n]/I_A$. The toric variey of $A$ is $X_A \coloneqq \operatorname{Var}(I_A)$, and the torus of $A$ is $T_A \coloneqq \operatorname{Spec} \mathbb{C}[\mathbb{Z}A]$. Given $\beta \in \mathbb{C}^d$, define the $\mathcal{D}_{T_A}$-module

$$\mathcal{O}^\beta_{T_A} \coloneqq \mathcal{D}_{T_A}/\mathcal{D}_{T_A}\{\, t_i\partial_{t_i} + \beta_i \mid i = 1, \dots, d \,\} = \mathcal{O}_{T_A} t^{-\beta}. \tag{2.3.1}$$

Note that $\mathcal{O}^\beta_{T_A}$ can be defined in a coordinate-free manner (see [Ste17, eq. (2.1.9)]). Set

$$\hat{\mathcal{M}}_A(\beta) \coloneqq \mathrm{FL}^{-1}(\mathcal{M}_A(\beta)). \tag{2.3.2}$$

*Definition* 2.1. A submatrix $F$ of $A$ is called a *face* of $A$, written $F \preceq A$, if $F$ has $d$ rows and $\mathbb{R}_{\geq 0}F$ is a face of $\mathbb{R}_{\geq 0}A$. A *facet* of $A$ is a face of rank $d-1$.

The torus embedding $t \mapsto (t^{\mathbf{a}_1}, \dots, t^{\mathbf{a}_n})$ of $T_A$ into $\mathbb{C}^n$ defined by $A$ induces an action of $T_A$ on $\mathbb{C}^n$ which makes $T_A$-equivariant the inclusion $X_A \subseteq \mathbb{C}^n$. If $F \preceq A$ is a face, the $T_A$-orbit of $X_A$ corresponding to $F$ is

$$\mathrm{O}_A(F) \coloneqq T_A \cdot \mathbb{1}_F, \tag{2.3.3}$$

where the $i$th coordinate of $\mathbb{1}_F$ is 1 if $\mathbf{a}_i \in F$ and 0 otherwise. Set

$$\mathbb{C}^F \coloneqq \{\, x \in \mathbb{C}^n \mid x_i = 0 \text{ for all } \mathbf{a}_i \notin F \,\}. \tag{2.3.4}$$

*Definition* 2.2. For a facet $G \preceq \mathbb{N}A$, there is a unique linear form $h_G = h_{G,A} \colon \mathbb{Z}^d \to \mathbb{Z}$, called the *primitive integral support function* of $G$, satisfying the following conditions:

(1) $h_G(\mathbb{Z}^d) = \mathbb{Z}$.
(2) $h_G(\mathbf{a}_i) \geq 0$ for all $i$.
(3) $h_G(\mathbf{a}_i) = 0$ for all $\mathbf{a}_i \in G$.

2.3.1. *Euler–Koszul complex.* We recall the definition of Euler–Koszul complex $\mathcal{K}_\bullet^A(S_A; E_A - \beta)$ from [MMW05]:

$$\mathcal{K}_\bullet^A(S_A; E_A - \beta) \coloneqq K_\bullet\big(\cdot(E_A - \beta); \mathcal{D}_{\mathbb{C}^n}/\mathcal{D}_{\mathbb{C}^n} I_A\big);$$

i.e. it is the Koszul complex of left $\mathcal{D}_{\mathbb{C}^n}$-modules defined by the (right) action of the sequence $E_A - \beta = E_1 - \beta_1, \ldots, E_d - \beta_d$ on the left $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{D}_{\mathbb{C}^n}/\mathcal{D}_{\mathbb{C}^n} I_A$. The more general Euler–Koszul complexes defined in [MMW05] will not be needed. The inverse Fourier–Laplace transform of $\mathcal{K}_\bullet^A(S_A; E_A - \beta)$ is denoted by $\hat{\mathcal{K}}_\bullet^A(S_A; E_A - \beta)$.

2.4. **Mixed and dual mixed Gauss–Manin systems.** Given a $T_A$-stable open neighborhood $U \subseteq \mathbb{C}^n$ of $T_A$ and a $\beta \in \mathbb{C}^d$, set

$$\text{(2.4.1)} \quad \mathrm{MGM}(U, \beta) \coloneqq \varpi_\dagger \iota_+ \mathcal{O}_{T_A}^\beta \quad \text{and} \quad \mathrm{MGM}^*(U, \beta) \coloneqq \varpi_+ \iota_\dagger \mathcal{O}_{T_A}^\beta,$$

where $\iota \colon T_A \hookrightarrow U$ is the torus embedding and $\varpi \colon U \hookrightarrow \mathbb{C}^n$ is inclusion.

*Definition* 2.3. A complex $\mathcal{M}^\bullet \in \mathrm{D}_\mathrm{h}^\mathrm{b}(\mathcal{D}_{\mathbb{C}^n})$ is *mixed Gauss–Manin* (resp. *dual mixed Gauss–Manin*) if it is isomorphic to $\mathrm{MGM}(U, \beta)$ (resp. $\mathrm{MGM}^*(U, \beta)$ for some $U$ and $\beta$.

*Definition* 2.4. A parameter $\beta \in \mathbb{C}^d$ is *mixed Gauss–Manin* (resp. *dual mixed Gauss–Manin*) if $\hat{\mathcal{K}}_A(S_A; E_A - \beta)$ is mixed Gauss–Manin (resp. dual mixed Gauss–Manin).

Note that the definitions of mixed and dual mixed Gauss–Manin parameters in [Ste17, Def. 8.15] is different than that in Definition 2.4. However, the two definitions are equivalent by [Ste17, Th. 8.17 and 8.19].

## 3. Alternating direct images

In this section we discuss a generalization of mixed and dual mixed Gauss–Manin systems which we will refer to by the name "alternating direct images".

In §3.1, we characterize in terms of fiber and cofiber support when a $D$-module or mixed Hodge module is isomorphic to a given alternating direct image.

In §3.2, we use the results of §3.1 to characterize, under a certain openness condition, when a $D$-module or mixed Hodge module is isomorphic to *some* alternating direct image.

In §3.3, we specialize Corollaries 3.5 and 3.6 to the GKZ case (Theorem 3.7). As a consequence, we obtain Corollary 3.8, which states that for GKZ systems, being dual mixed Gauss–Manin is the same as being mixed Gauss–Manin and not rank-jumping.

### 3.1. **Characterizing alternating direct images passing through a fixed $U$.**

Let
$$Z \xrightarrow{\iota} U \xrightarrow{\varpi} X$$
be inclusions of smooth (locally closed) subvarieties, where $U$ is open in $X$, and set $\varphi \coloneqq \varpi \circ \iota$. We associate to this situation the *alternating direct image* functors $\varpi_+\iota_\dagger$ and $\varpi_\dagger\iota_+$.

*Remark* 3.1. Note that if $\mathcal{N}^\bullet$ is in $\mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ or $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$, then $\varphi^+\mathcal{N}^\bullet$ is canonically isomorphic to $\varphi^\dagger\mathcal{N}^\bullet$. To see this, notice that because $\varpi$ is an open embedding, $\varpi^\dagger = \varpi^+$; now shrink $U$ so that $\iota$ is a closed immersion, then apply Kashiwara's equivalence.

**Lemma 3.2.** *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$). Then $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ is the unique object in $\mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ (resp. in $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$) such that*

*(1) the restriction to $Z$ is isomorphic to $\mathcal{M}^\bullet$;*
*(2) the fiber support is contained in $U$; and*
*(3) the cofiber support intersected with $U$ is contained in $Z$.*

*Proof.* We first show that $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ satisfies the required properties. Because both $\iota$ and $\varpi$ are inclusions of (locally closed) subvarieties, $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ is supported on $\overline{Z}$. Applying $\varphi^+$ to $\varpi_+\iota_\dagger\mathcal{M}^\bullet$, we get
$$\varphi^+\varpi_+\iota_\dagger\mathcal{M}^\bullet = \iota^+\iota_\dagger\mathcal{M}^\bullet = \iota^\dagger\iota_\dagger\mathcal{M}^\bullet = \mathcal{M}^\bullet,$$
where the second equality follows for the same reason as in Remark 3.1. So the restriction to $Z$ is $\mathcal{M}^\bullet$. Let $i_x$ denote inclusion of a point $x \in X$. If $x \notin U$, then $i_x^+\varpi_+\iota_\dagger\mathcal{M}^\bullet$ vanishes by [Ste17, Lem. 3.3], so the fiber support is contained in $U$. If $x \in U \setminus Z$, then also by [Ste17, Lem. 3.3],
$$i_x^\dagger\varpi_+\iota_\dagger\mathcal{M}^\bullet = i_x^\dagger\iota_\dagger\mathcal{M}^\bullet = 0.$$
So, the cofiber support intersected with $U$ is contained in $Z$.

We now prove uniqueness. Suppose $\mathcal{N}^\bullet$ also satisfies the properties. Then the equality of $\varphi^+\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ induces a morphism $f\colon \iota_\dagger\mathcal{M}^\bullet \to \varpi^+\mathcal{N}^\bullet$. By property 3, $i_x^\dagger f = 0$ for all $x \in U \setminus Z$, while by property 1, the restriction $\iota^+ f$ is an equality. Hence, cone$(f)$ has empty fiber support, and therefore it vanishes by [Ste17, Cor. 3.6]. Thus, $f$ is an isomorphism. By duality, the same argument applied to the case $Z = U$ and $\mathcal{M}^\bullet = \varpi^+\mathcal{N}^\bullet$ gives an isomorphism $\mathcal{N}^\bullet \to \varpi_+\iota_\dagger\mathcal{M}^\bullet$. ☐

**Lemma 3.3.** *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$). Then $\varpi_\dagger\iota_+\mathcal{M}^\bullet$ is the unique object in $\mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ (resp. in $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$) such that*

1. *the restriction to $Z$ equals $\mathcal{M}^\bullet$;*
2. *the cofiber support is contained in $U$; and*
3. *the fiber support intersected with $U$ is contained in $Z$.*

*Proof.* This follows from Lemma 3.2 by duality. ☐

*Remark* 3.4. Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$. Lemmas 3.2 and 3.3 imply that if there are open neighborhoods $U$ and $U'$ of $Z$ such that $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ and $\varpi'_\dagger\iota'_+\mathcal{M}^\bullet$ are isomorphic as $\mathcal{D}_X$-modules, then they are also isomorphic as mixed Hodge modules.

3.2. **The relatively open (co)fiber support case.** If the fiber support of $\varphi_\dagger\mathcal{M}^\bullet$ is relatively open, then we may shrink $U$ so that $U\cap\overline{Z} = \mathrm{fSupp}\,\varpi_+\iota_\dagger\mathcal{M}^\bullet$ without changing $\varpi_+\iota_\dagger\mathcal{M}^\bullet$. Similarly, if the cofiber support of $\varphi_+\mathcal{M}^\bullet$ is relatively open, then we may shrink $U$ so that $U\cap\overline{Z} = \mathrm{cofSupp}\,\varpi_\dagger\iota_+\mathcal{M}^\bullet$ without changing $\varpi_\dagger\iota_+\mathcal{M}^\bullet$. As an immediate consequence, we get the following corollaries of Lemmas 3.2 and 3.3:

**Corollary 3.5.** *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$), and assume that the fiber support of $\varphi_\dagger\mathcal{M}^\bullet$ is relatively open. Let $\mathcal{N}^\bullet \in \mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ (resp. in $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$). Then there exists an open neighborhood $U \subseteq X$ of $Z$ such that (in the notation of §3.1) $\varpi_+\iota_\dagger\mathcal{M}^\bullet \cong \mathcal{N}^\bullet$ if and only if all of the following conditions hold:*

1. *$\varphi^+\mathcal{N}^\bullet \cong \mathcal{M}^\bullet$;*
2. *$\mathrm{fSupp}\,\mathcal{N}^\bullet \cap \mathrm{cofSupp}\,\mathcal{N}^\bullet \subseteq Z$; and*
3. *$\mathrm{fSupp}\,\mathcal{N}^\bullet$ is relatively open.*

**Corollary 3.6.** *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$), and assume that the cofiber support of $\varphi_+\mathcal{M}^\bullet$ is relatively open. Let $\mathcal{N}^\bullet \in \mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ (resp. in $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$). Then there exists an open neighborhood $U \subseteq X$ of $Z$ such that (in the notation of §3.1) $\varpi_\dagger\iota_+\mathcal{M}^\bullet \cong \mathcal{N}^\bullet$ if and only if all of the following conditions hold:*

1. *$\varphi^+\mathcal{N}^\bullet \cong \mathcal{M}^\bullet$;*
2. *$\mathrm{fSupp}\,\mathcal{N}^\bullet \cap \mathrm{cofSupp}\,\mathcal{N}^\bullet \subseteq Z$; and*
3. *$\mathrm{cofSupp}\,\mathcal{N}^\bullet$ is relatively open.*

3.3. **A different characterization of mixed and dual mixed Gauss–Manin parameters.** Specializing Corollaries 3.5 and 3.6 to the GKZ case, we get Theorem 3.7 below. Before stating it, we recall the definition of the set of *A-exceptional parameters*. This is the set $\mathcal{E}_A$ of parameters $\beta$ for which the holonomic rank of $\mathcal{M}_A(\beta)$ is larger than for a generic parameter. Note that $\mathcal{E}_A$ also has a description in terms of local cohomology (see [MMW05]).

**Theorem 3.7.** *Let $\beta \in \mathbb{C}^d$.*

(1) *$\beta$ is dual mixed Gauss–Manin for $A$ if and only if*

$$\beta \notin \mathcal{E}_A \quad \textit{and} \quad \operatorname{fSupp} \hat{\mathcal{M}}_A(\beta) \cap \operatorname{cofSupp} \hat{\mathcal{M}}_A(\beta) = T_A.$$

(2) *$\beta$ is mixed Gauss–Manin for $A$ if and only if*

$$\operatorname{fSupp} \hat{\mathcal{K}}^A_\bullet(S_A; E_A - \beta) \cap \operatorname{cofSupp} \hat{\mathcal{K}}^A_\bullet(S_A; E_A - \beta) = T_A.$$

*Proof.* (1) By [Ste17, Th. 8.17], a dual mixed Gauss–Manin parameter is not $A$-exceptional. By [Ste17, Lemma 8.8], if $\beta \notin \mathcal{E}_A$, then the fiber support of $\hat{\mathcal{M}}_A(\beta)$ is relatively open; in particular, as $\varphi_\dagger \mathcal{O}^\beta_{T_A}$ is isomorphic to $\hat{\mathcal{M}}_A(\beta')$ for some $\beta' \notin \mathcal{E}_A$ ([Ste17, Rmk. 8.16]), the fiber support of $\varphi_\dagger \mathcal{O}^\beta_{T_A}$ is relatively open. Now use Corollary 3.5.

(2) By [Sai01, Prop. 2.2 (4)], the orbit-cone correspondence, and [Ste17, Th. 7.4], the cofiber support of $\hat{\mathcal{K}}^A_\bullet(S_A; E_A - \beta)$ is relatively open for all $\beta$. In particular, as $\varphi_+ \mathcal{O}^\beta_{T_A}$ is isomorphic to $\hat{\mathcal{K}}^A_\bullet(S_A; E_A - \beta)$ for some $\beta'$ ([SW09, Cor. 3.7]), the cofiber support of $\varphi_+ \mathcal{O}^\beta_{T_A}$ is relatively open. Now use Corollary 3.6. ☐

**Corollary 3.8.** *A parameter is dual mixed Gauss–Manin for $A$ if and only if it is mixed Gauss–Manin for $A$ and not $A$-exceptional.*

## 4. Twisted quasi-equivariance

Reichelt and Walther introduced in [RW17, Def. 3.2] the notion of a quasi-equivariant $\mathcal{D}_E$ module. For the purposes of this paper, we need to generalize this notion slightly (Definition 4.2) to incorporate a "twist" by a rank one integrable connection on $\mathbb{C}^*$ à la [Hot98]. In Lemma 4.4, this generalization is used to to relate certain projections and restrictions of twistedly equivariant $D$-modules. Proposition 4.5 shows that, when properly interpreted, every mixed and dual mixed Gauss–Manin module is twistedly equivariant. Note that Lemma 4.4 and proposition 4.5 are generalizations of [RW17, Lem. 3.3 and 3.4].

We begin by recalling the notion of a fibered $\mathbb{C}^*$-action on a trivial vector bundle. Let $\pi\colon E \to X$ be a trivial vector bundle on a smooth affine variety $X$, and denote by

$$i\colon X \hookrightarrow E$$

the zero section. Set

$$E^* \coloneqq E \setminus i(X).$$

*Definition* 4.1 ([RW17, Def. 3.1]). A $\mathbb{C}^*$ action $\mu\colon \mathbb{C}^* \times E \to E$ is *fibered* if

(1) $\mu$ preserves fibers;
(2) $\mu$ extends under the inclusion $\mathbb{C}^* \hookrightarrow \mathbb{C}$ to a morphism (also denoted $\mu$) $\mathbb{C} \times E \to E$;
(3) $0 \in \mathbb{C}$ multiplies into the zero section, i.e. $\mu\colon \{0\} \times E \to i(X)$; and
(4) $\mathbb{C}$ fixes the zero section.

*Definition* 4.2. Let $\mu\colon \mathbb{C}^* \times E \to E$ be a fibered action on $E$, let $\mu'$ be the restriction of this action to $E^*$, and let $\lambda \in \mathbb{C}$. A complex $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_E)$ is $\lambda$*-twistedly* $\mathbb{C}^*$*-quasi-equivariant* if

$$\mu'^* \mathcal{M}^\bullet_{|E^*} \cong \mathcal{O}^\lambda_{\mathbb{C}^*} \boxtimes \mathcal{M}^\bullet_{|E^*}. \tag{4.0.1}$$

A complex $\mathcal{M}^\bullet$ is *twistedly* $\mathbb{C}^*$*-quasi-equivariant* if it is $\lambda$-twistedly $\mathbb{C}^*$-quasi-equivariant for some $\lambda$.

*Remark* 4.3. Note that because $\mu'$ is smooth of relative dimension 1, (4.0.1) is equivalent to

$$\mu'^+ \mathcal{M}^\bullet_{|E^*} \cong \mathcal{O}^\lambda_{\mathbb{C}^*}[1] \boxtimes \mathcal{M}^\bullet_{|E^*} \tag{4.0.2}$$

and also to

$$\mu'^\dagger \mathcal{M}^\bullet_{|E^*} \cong \mathcal{O}^\lambda_{\mathbb{C}^*}[-1] \boxtimes \mathcal{M}^\bullet_{|E^*}. \tag{4.0.3}$$

The following lemma is proved in exactly the same way as is [RW17, Lem. 3.3]. The only change to the proof is that "$\mathcal{O}_{\mathbb{G}_{\mathrm{m}}}$" must be replaced throughout with "$\mathcal{O}^\lambda_{\mathbb{C}^*}$". No issues occur with doing so, and no issues occur with the passage to the derived category as opposed to modules.

**Lemma 4.4.** *If $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_E)$ is $\lambda$-twistedly $\mathbb{C}^*$-quasi-equivariant, then $\pi_+\mathcal{M}^\bullet \cong i^\dagger\mathcal{M}^\bullet$ and $\pi_\dagger\mathcal{M}^\bullet \cong i^+\mathcal{M}^\bullet$.*

We now generalize [RW17, Lem. 3.4]. The basic idea of the proof is the same. However, sufficiently many technical details need to be modified that we feel it necessary to provide the proof in full.

**Proposition 4.5.** *Let $F \preceq A$ be a face, and view $\mathbb{C}^n$ as a vector bundle over $\mathbb{C}^F$ via the coordinate projection $\pi\colon \mathbb{C}^n \to \mathbb{C}^F$. Let $\beta \in \mathbb{C}^d$. Then there exists a fibered $\mathbb{C}^*$-action on $\mathbb{C}^n$ such that for all $T_A$-stable open* *neighborhoods $U \subseteq \mathbb{C}^n$ of $T_A$, both* $\mathrm{MGM}(U,\beta)$ *and* $\mathrm{MGM}^*(U,\beta)$ *are twistedly quasi-equivariant.*

*Proof.* Write $E$ for $\mathbb{C}^n$ viewed as vector bundle over $\mathbb{C}^F$. Since $\mathbb{N}A$ is pointed and $F$ is a face, there exists a $\mathbf{u} \in \mathbb{Z}^d$ such that $\langle \mathbf{a}_i, \mathbf{u}\rangle = 0$ for $\mathbf{a}_i \in F$ and $\langle \mathbf{a}_i, \mathbf{u}\rangle > 0$ for $\mathbf{a}_i \notin F$. We show that the monomial action $\mu\colon \mathbb{C}^* \times E \to E$ induced by $\mathbf{v} := A^\top \mathbf{u}$, i.e. $t \cdot (x_1, \ldots, x_n) = (t^{v_1}x_1, \ldots, t^{v_n}x_n)$, satisfies the requirements of the proposition.

Step 1: $\mu$ is a fibered action.

*Proof of Step 1.* Condition (1) of Definition 4.1 holds because $v_i = 0$ for all $\mathbf{a}_i \in F$. Because in addition $v_i > 0$ for all $\mathbf{a}_i \notin F$, the action extends to $\mathbb{C}$; so, condition (2) holds. Conditions (3) and (4) follow immediately from the definition of this extension. This finishes the proof of Step 1.

Step 2: $\tilde{\mu}^*\mathcal{O}^\beta_{T_A} \cong \mathcal{O}^{\langle \mathbf{u},\beta\rangle}_{\mathbb{C}^*} \boxtimes \mathcal{O}^\beta_{T_A}$, where $\tilde{\mu}$ denotes the monomial action on $T_A$ induced by $\mathbf{u}$.

*Proof of Step 2.* Let $f\colon \tilde{\mu}^*\mathcal{O}^\beta_{T_A} \to \mathcal{O}^{\langle \mathbf{u},\beta\rangle}_{\mathbb{C}^*} \boxtimes \mathcal{O}^\beta_{T_A}$ be the $\mathcal{O}_{\mathbb{C}^*\times T_A}$-module isomorphism taking the generator $1 \otimes t^{-\beta}$ to the generator $s^{-\langle \mathbf{u},\beta\rangle} \otimes t^{-\beta}$, where $s$ denotes the coordinate on $\mathbb{C}^*$. The action of $1 \otimes t_i\partial_{t_i}$ on both generators is multiplication by $-\beta_i$, while the action of $s\partial_s$ on both generators is multiplication by $-\langle \mathbf{u},\beta\rangle$. Therefore, $f$ is an isomorphism of $\mathcal{D}_{\mathbb{C}^*\times T_A}$-modules. This finishes the proof of Step 2.

Step 3: Both $\mathrm{MGM}(U,\beta)$ and $\mathrm{MGM}^*(U,\beta)$ are $\langle \mathbf{u},\beta\rangle$-twistedly quasi-equivariant.

*Proof of Step 3.* Since the two statements are equivalent via duality, we only prove the first. Consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{C}^* \times T_A & \xrightarrow{\mathrm{id}\times\iota'} & \mathbb{C}^* \times (U \cap E^*) & \xrightarrow{\mathrm{id}\times\varpi'} & \mathbb{C}^* \times E^* \\
\big\downarrow{\scriptstyle\tilde{\mu}} & & \big\downarrow{\scriptstyle\mu''} & & \big\downarrow{\scriptstyle\mu'} \\
T_A & \xrightarrow{\iota'} & U \cap E^* & \xrightarrow{\varpi'} & E^*
\end{array}
\tag{4.0.4}
$$

Here, $\iota'$ is the torus embedding, $\varpi'$ is inclusion, $\mu'$ is the restriction of $\mu$ to $E^*$, and $\mu''$ is the restriction of $\mu$ to $U \cap E^*$. By construction, the action $\mu$ factors through the action of $T_A$. So, because $U$ is $T_A$-stable, it is also $\mathbb{C}^*$-stable, and therefore both squares in (4.0.4) are Cartesian. Then

$$
\begin{aligned}
\mu'^\dagger \,\mathrm{MGM}(U,\beta)_{|E^*} &\cong \mu'^\dagger \varpi'_\dagger \iota'_+ \mathcal{O}^\beta_{T_A} \\
&\cong (\mathrm{id}\times\varpi')_\dagger \mu''^\dagger \iota'_+ \mathcal{O}^\beta_{T_A} \\
&\cong (\mathrm{id}\times\varpi')_\dagger (\mathrm{id}\times\iota')_+ \tilde{\mu}^\dagger \mathcal{O}^\beta_{T_A} \\
&\cong (\mathrm{id}\times\varpi')_\dagger (\mathrm{id}\times\iota')_+ (\mathcal{O}^{\langle \mathbf{u},\beta\rangle}_{\mathbb{C}^*}[-1] \boxtimes \mathcal{O}^\beta_{T_A})
\end{aligned}
$$

$$\cong \mathcal{O}_{\mathbb{C}^*}^{\langle \mathbf{u},\beta\rangle}[-1] \boxtimes \varpi'_\dagger \iota'_+ \mathcal{O}_{T_A}^{\beta}$$
$$\cong \mathcal{O}_{\mathbb{C}^*}^{\langle \mathbf{u},\beta\rangle}[-1] \boxtimes \mathrm{MGM}(U,\beta)_{|E^*},$$

where the second isomorphism is by base change, the third is by base change together with the fact that $\mu''$ and $\tilde{\mu}$ are smooth of the same relative dimension, and the fourth is by Step 2 and the smoothness of $\tilde{\mu}$. Now use Remark 4.3. This finishes the proof of Step 3 and thereby the proposition. □

## 5. Projections and restrictions

In §5.1, we use the framework of a $\mathbb{C}^*$-fibered vector bundle to show that the projection and restriction of alternating direct images are also alternating direct images. We apply this in §5.2 to mixed and dual mixed Gauss–Manin systems.

In §5.3, we specialize these results to the case of normal $S_A$, culminating in Theorem 5.8, where we compute the restriction and projection of $\mathcal{M}_A(\beta)$ to the coordinate subspace corresponding to a face of $A$.

5.1. **Restricting and projecting twistedly quasi-equivariant alternating direct images.** Let $X$ be a smooth affine variety, $\pi\colon E \to X$ a $\mathbb{C}^*$-fibered vector bundle, and as before, denote by $i\colon X \hookrightarrow E$ the zero section. Consider the following diagrams:

$$Z \xrightarrow{\ \iota\ } U \xrightarrow{\ \varpi\ } E \quad \text{and} \quad i^{-1}(U)\cap\pi(Z) \xrightarrow{\ \iota'\ } i^{-1}(U) \xrightarrow{\ \varpi'\ } X.$$

Here, $Z$ is smooth and locally closed in $X$, $U$ is an open subset of $E$ containing $Z$, and the morphisms are inclusion. Set $\varphi := \varpi\circ\iota$ and $\varphi' := \varpi'\circ\iota'$.

**Proposition 5.1.** *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_Z)$. Assume that $U \supseteq \pi^{-1}(i^{-1}(U))$ and $\pi(Z)$ is locally closed.*

*(1) If $\mathcal{N}^\bullet := \varpi_+\iota_\dagger\mathcal{M}^\bullet$ is twistedly $\mathbb{C}^*$-quasi-equivariant, then*
$$i^+\mathcal{N}^\bullet \cong \varpi'_+\iota'_\dagger(i\circ\varphi')^+\mathcal{N}^\bullet.$$

*(2) If $\mathcal{N}^\bullet := \varpi_\dagger\iota_+\mathcal{M}^\bullet$ is twistedly $\mathbb{C}^*$-quasi-equivariant, then*
$$\pi_+\mathcal{N}^\bullet \cong \varpi'_\dagger\iota'_+(i\circ\varphi')^\dagger\mathcal{N}^\bullet.$$

*Proof.* (1) By Lemma 3.2, the fiber support of $i^+\mathcal{N}^\bullet$ is contained in $i^{-1}(U)$. Suppose $x \in i^{-1}(U)\cap \operatorname{cofSupp} i^+\mathcal{N}^\bullet$. Then by Lemma 4.4 and the base change formula, $(\pi|_{E_x})_\dagger i^\dagger_{E_x}\mathcal{N}^\bullet \neq 0$, where $E_x := \pi^{-1}(x)$ is the fiber of $E$ over $x$, and $i_{E_x}\colon E_x \hookrightarrow E$ is inclusion. So, $i^\dagger_{E_x}\mathcal{N}^\bullet \neq 0$, and therefore $E_x \cap \operatorname{cofSupp}\mathcal{N}^\bullet \neq \emptyset$. On the other hand, $x \in i^{-1}(U)$, so because $U \supseteq \pi^{-1}(i^{-1}(U))$, we have that $E_x \subseteq U$. Hence, $E_x \cap$

cofSupp $\mathcal{N}^\bullet$ is a non-empty subset of $Z$ by Lemma 3.2, and therefore $\pi(x) \in \pi(Z) \cap i^{-1}(U)$. Thus,

$$i^+\mathcal{N}^\bullet \cong \varpi'_+\iota'_\dagger\varphi'^+i^+\mathcal{N}^\bullet \cong \varpi'_+\iota'_\dagger(i\circ\varphi')^+\mathcal{N}^\bullet.$$

(2) This follows from (1) by duality together with Lemma 4.4. □

It may appear at first that the assumption that $U \supseteq \pi^{-1}(i^{-1}(U))$ in Proposition 5.1 is too restrictive to apply in the situation of Proposition 4.5. However, as we will see in Lemma 5.3, $U$ can always be enlarged to satisfy this assumption without changing $\mathrm{MGM}(U,\beta)$ or $\mathrm{MGM}^*(U,\beta)$.

5.2. **Restricting and projecting GKZ systems.** Before stating Theorem 5.4, we recall the below facts about mixed and dual mixed Gauss–Manin systems. Also recall from (2.3.3) that $\mathrm{O}_A(F)$ is the $T_A$-orbit of the toric variety $X_A$ which corresponds to $F$.

Here and in the rest of this article, we follow that convention that $\bigwedge\mathbb{C}^k$ lives in cohomological degrees $-k$ through 0.

**Fact 5.2.** *Let $\beta \in \mathbb{C}^d$, and let $U \subseteq \mathbb{C}^n$ be a $T_A$-stable open neighborhood of $T_A$. Write $i_{\mathrm{O}_A(F)}$ for the inclusion $\mathrm{O}_A(F) \hookrightarrow \mathbb{C}^n$.*

*(1) If $\mathrm{O}_A(F) \subseteq \mathrm{cofSupp}\,\mathrm{MGM}(U,\beta)$, then*

$$i^\dagger_{\mathrm{O}_A(F)}\,\mathrm{MGM}(U,\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F} \mathcal{O}^\lambda_{T_F} \otimes_{\mathbb{C}} \bigwedge\mathbb{C}^{d_{A/F}},$$

*where the direct sum is over those $\lambda+\mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F$ for which $\beta-\lambda \in \mathbb{Z}^d$. This follows from* [Ste17, Lem. 8.14(b), Rem. 8.16, and Eq. (8.3.3)].

*(2) If $\mathrm{O}_A(F) \subseteq \mathrm{fSupp}\,\mathrm{MGM}^*(U,\beta)$, then*

$$i^+_{\mathrm{O}_A(F)}\,\mathrm{MGM}^*(U,\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F} \mathcal{O}^\lambda_{T_F} \otimes_{\mathbb{C}} \bigwedge\mathbb{C}^{d_{A/F}},$$

*where the direct sum is over those $\lambda+\mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F$ for which $\beta-\lambda \in \mathbb{Z}^d$. This follows from Fact 5.2(1) and* [Ste17, Rmk. 8.18].

Let $F \preceq A$ be a face, and let $\pi_F\colon \mathbb{C}^n \to \mathbb{C}^F$ and $i_F\colon \mathbb{C}^F \hookrightarrow \mathbb{C}^n$ be coordinate projection and inclusion, respectively.

**Lemma 5.3.** *Let $\beta \in \mathbb{C}^n$ and $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_{\mathbb{C}^n})$. Let $U \subseteq \mathbb{C}^n$ be a $T_A$-stable open neighborhood of $T_A$, and let $U' = U \cup \pi_F^{-1}(i_F^{-1}(U))$. Then*

$$\mathrm{MGM}^*(U,\beta) \cong \mathrm{MGM}^*(U',\beta) \quad \textit{and} \quad \mathrm{MGM}(U,\beta) \cong \mathrm{MGM}(U',\beta).$$

*Proof.* It suffices to show that $U'\cap X_A = U \cap X_A$. The containment $U'\cap X_A \supseteq U\cap X_A$ is immediate. For the other containment, let $\mathrm{O}_A(G) \subseteq U'$, and suppose $\mathrm{O}_A(G) \subseteq \pi_F^{-1}(i_F^{-1}(U))$. Then $i_F(\pi_F(\mathrm{O}_A(G))) \subseteq U$.

But $i_F(\pi_F(\mathrm{O}_A(G))) = i_F(\mathrm{O}_F(G\cap F)) = \mathrm{O}_A(G\cap F)$, so $\mathrm{O}_A(G\cap F) \subseteq U$. Therefore, because $U$ is open, the orbit-cone correspondence implies that $\mathrm{O}_A(G) \subseteq U$. Thus, $U' \cap X_A = U \cap X_A$. □

**Theorem 5.4.** *Let $\beta \in \mathbb{C}^n$, and let $U \subseteq \mathbb{C}^n$ be a $T_A$-stable open neighborhood of $T_A$.*

*(1) If $\beta \notin \mathbb{C}F + \mathbb{Z}^d$, then $\pi_{F+} \operatorname{MGM}_A(U,\beta) = i_F^+ \operatorname{MGM}^*_A(U,\beta) = 0$.*
*(2) If $\beta \in \mathbb{C}F + \mathbb{Z}^d$, then*

$$i_F^+ \operatorname{MGM}^*_A(U,\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F} \operatorname{MGM}^*_F\big(i_F^{-1}(U),\lambda\big) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}$$

*and*

$$\pi_{F+} \operatorname{MGM}_A(U,\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F} \operatorname{MGM}_F\big(i_F^{-1}(U),\lambda\big) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}},$$

*where the direct sums are over those $\lambda + \mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F$ for which $\beta - \lambda \in \mathbb{Z}^d$.*

*Proof.* We only prove the dual MGM case. The MGM case follows by duality together with Lemma 4.4.

For ease of notation, set $\pi = \pi_F$ and $i = i_F$. By Lemma 5.3, we may replace $U$ with $U \cup \pi^{-1}(i^{-1}(U))$ (note that this leaves $i_F^{-1}(U)$ unchanged) to assume that $U \supseteq \pi^{-1}(i^{-1}(U))$. In addition, $\pi(T_A) = T_F$, which is locally closed in $\mathbb{C}^F$. Therefore, Proposition 5.1(1) applies to give

$$i^+ \operatorname{MGM}^*_A(U,\beta) \cong \varpi'_+ \iota'_\dagger (i \circ \varphi_F)^+ \operatorname{MGM}^*_A(U,\beta),$$

where $\iota' = \varphi_F \colon T_F \hookrightarrow i^{-1}(U) \cap T_F$ and $\varpi' \colon i^{-1}(U) \cap T_F \hookrightarrow \mathbb{C}^F$ is inclusion. If $\beta \notin \mathbb{C}F + \mathbb{Z}^d$, then $i^{-1}(U) \cap T_F = i^{-1}(U \cap \mathrm{O}_A(F)) = \emptyset$ by Fact 5.2, and therefore, $i^+ \operatorname{MGM}^*_A(U,\beta) = 0$. So, assume that $\beta \in \mathbb{C}F + \mathbb{Z}^d$. Then $i^{-1}(U) \cap T_F = i^{-1}(\mathrm{O}_A(F))$, and therefore $i \circ \varphi'$ is just the inclusion $\mathrm{O}_A(F) \hookrightarrow \mathbb{C}^n$. Now use Fact 5.2 together with the additivity of the $D$-module functors. □

5.3. **Normal case.** Throughout this section, $S_A$ is assumed to be normal. Lemma 5.5 is a technical lemma which we will use (both in this section and in §6) to move a parameter $\beta$ within the class of those parameters whose $A$-hypergeometric system is isomorphic to that of $\beta$. Lemma 5.6 will be needed in the proof of Theorem 5.8. Recall from Definition 2.2 the definition of the primitive integral support functions $h_G$.

**Lemma 5.5.** *Let $\beta \in \mathbb{C}^d$. Then there exists a $\gamma \in \mathbb{Z}^d$ such that for all facets $G \preceq A$,*

*(1) $h_G(\gamma) \neq 0$ if $h_G(\beta) \notin \mathbb{Z}$;*

*(2)* $h_G(\gamma) > 0$ *if* $h_G(\beta) \in \mathbb{N}$*; and*
*(3)* $h_G(\gamma) < 0$ *if* $h_G(\beta) \in \mathbb{Z}_{<0}$*.*

*Proof.* Consider the system of equations

$$\{\, h_G(x) = h_G(\beta) \mid G \preceq A \text{ is a facet with } h_G(\beta) \in \mathbb{Z}\,\}.$$

This has a solution in $\mathbb{C}^d$, namely $\beta$, and therefore has a solution in $\mathbb{R}^d$. Let $\alpha$ be one such solution. Then $\alpha$ describes a hyperplane

$$H_\alpha = \{\, f \in (\mathbb{R}^d)^* \mid f(\alpha) = 0\,\}.$$

Denote by $H_\alpha^{\geq 0}$ the set of $f \in (\mathbb{R}^d)^*$ such that $f(\alpha) \geq 0$, and similarly for $H_\alpha^{>0}$, $H_\alpha^{\leq 0}$, and $H_\alpha^{<0}$.

Let us now consider the sets $P_\alpha = \{\, h_G \mid h_G(\alpha) \geq 0\,\}$ and $N_\alpha = \{\, h_G \mid h_G(\alpha) < 0\,\}$. By construction, $\mathbb{R}_{\geq 0}P_\alpha \cap \mathbb{R}_{\geq 0}N_\alpha = \{0\}$. Let $Z$ be an affine hyperplane in $(\mathbb{R}^d)^*$ transverse to the dual cone $(\mathbb{R}_{\geq 0}A)^\vee$. Then $Z \cap \mathbb{R}_{\geq 0}P_\alpha$ and $Z \cap \mathbb{R}_{\geq 0}N_\alpha$ are convex, compact, and disjoint. Hence, there exists a hyperplane $L$ in $Z$ separating $Z \cap \mathbb{R}_{\geq 0}P_\alpha$ and $Z \cap \mathbb{R}_{\geq 0}N_\alpha$. Choose a $\gamma \in \mathbb{R}^d$ such that $H_\gamma \cap Z = L$ and $H_\gamma^{>0} \supseteq Z \cap \mathbb{R}_{\geq 0}P_\alpha$. Then $H_\gamma^{<0} \supseteq Z \cap \mathbb{R}_{\geq 0}N_\alpha$. Then by convexity, $H_\gamma^{>0} \supseteq \mathbb{R}_{\geq 0}P_\alpha$ and $H_\gamma^{<0} \supseteq \mathbb{R}_{\geq 0}N_\alpha$. In particular, $H_\gamma^{>0} \supseteq P_\alpha$ and $H_\gamma^{<0} \supseteq N_\alpha$. Because $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$, we may modify $\gamma$ so that it is in $\mathbb{Q}^d$. Clearing denominators, we may take $\gamma$ to be in $\mathbb{Z}^d$. □

Note that because we are in the normal case, we may define

$$\operatorname{sRes}(A) = \mathbb{C}^d \setminus \left\{\, \beta \in \mathbb{C}^d \;\middle|\; h_G(\beta) \geq 0 \text{ whenever } h_G(\beta) \in \mathbb{Z}\,\right\}. \tag{5.3.1}$$

We will take this as the definition of $\operatorname{sRes}(A)$ since we are only dealing with normal $A$. However, (5.3.1) follows from the general definition given in [SW09] by applying [Ste17, Th. 9.3 and Lem. 9.1] along with [SW09, Cor. 3.8]

**Lemma 5.6.** *Let* $\beta \in \mathbb{C}F + \mathbb{Z}^d$*, and let* $F \preceq A$ *be a face. Then there exists a* $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ *such that for all facets* $F'$*,*

*(1)* $h_{F'}(\lambda) \in \mathbb{N}$ *implies that* $h_G(\beta) \in \mathbb{N}$ *for all facets* $G$ *of* $A$ *with* $G \cap F = F'$*; and*
*(2)* $h_{F'}(\lambda) \in \mathbb{Z}_{<0}$ *implies that* $h_G(\beta) \in \mathbb{Z}_{<0}$ *for all facets* $G$ *of* $A$ *with* $G \cap F = F'$*.*

*Proof.* Step 1: The lemma holds for $\beta \in \mathbb{Z}^d$.

*Proof of Step 1.* By induction on the rank of $F$, we may assume that $F$ is a facet of $A$. Let $F_1, \dots, F_\ell$ be the facets of $F$. For each $i$, let $G_i$ be the facet of $A$ whose intersection with $F$ is $F_i$. For each $I \subseteq \{1, \dots, \ell\}$, consider the sets

$$X_I := \{\, x \in \mathbb{R}F \mid h_{F_i}(x) \geq 0 \text{ for all } i \in I\,\}$$

$$Y_I := \left\{ x \in \mathbb{R}^d \;\middle|\; h_{G_i}(x) \geq 0 \text{ for all } i \in I \right\}.$$

When $X_I$ is nonempty, neither is $Y_I$, and $X_I$ and $Y_I$ are chambers of the arrangments $\{\mathbb{R}F_1, \dots, \mathbb{R}F_\ell\}$ and $\{\mathbb{R}G_1, \dots, \mathbb{R}G_\ell\}$, respectively. But these two arrangements are combinatorially equivalent by construction, so they have the same number of chambers. Hence, $X_I$ is nonempty if and only if $Y_I$ is nonempty. Since both arrangements are central, $X_I \cap \mathbb{Z}F$ is nonempty if and only if $Y_I \cap \mathbb{Z}^d$ is nonempty. Therefore, if $\beta \in Y_I$, then any $\lambda \in X_I \cap \mathbb{Z}F$ has the required properties. This finishes the proof of Step 1.

Step 2: The lemma holds for general $\beta$.

*Proof of Step 2.* Apply Lemma 5.5 to $\beta$ to get a $\gamma \in \mathbb{Z}^d$. Apply Step 1 to $\gamma$ to get an $\alpha \in \mathbb{Z}F$. Let $\lambda_0 \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d) \setminus \operatorname{sRes}(A)$. By adding sufficiently many copies of $\sum_{\mathbf{a}_i \in F} \mathbf{a}_i$ to $\lambda_0$, we may assume that

$$h_{F'}(\lambda_0) \geq |h_{F'}(\alpha)| \tag{5.3.2}$$

for all facets $F'$ of $F$ with $h_{F'}(\lambda_0) \in \mathbb{Z}$. Set $\lambda = \lambda_0 + \alpha$. Let $F'$ be a facet of $F$, and let $G$ be a facet of $A$ with $G \cap F = F'$.

Suppose $h_{F'}(\lambda) \in \mathbb{N}$. Then because $h_{F'}(\alpha) \in \mathbb{Z}$, $h_{F'}(\lambda_0)$ must be an integer and therefore a non-negative integer. Then by (5.3.2), $h_{F'}(\alpha) \geq 0$. Hence, $h_G(\gamma) \geq 0$, which by construction of $\gamma$ means that $h_G(\beta) \in \mathbb{N}$.

Next, suppose $h_{F'}(\lambda) \in \mathbb{Z}_{<0}$. As before, this implies that $h_{F'}(\lambda_0)$ is a non-negative integer. But then $h_{F'}(\alpha)$ must be negative. Hence, $h_G(\gamma) \leq 0$, which by construction of $\gamma$ means that $h_G(\beta) \in \mathbb{Z}_{<0}$. This finishes the proof of Step 2 and thereby the lemma. ☐

The following example shows that even if $h_G(\beta) \in \mathbb{Z}$ for every facet $G$ of $A$ with $G \cap F = F'$, it is still possible that $h_{F'}(\lambda) \notin \mathbb{Z}$.

*Example* 5.7. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad F = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The only facet of $F$ is $\emptyset$, and the only facet of $A$ whose intersection with $F$ is $\emptyset$ is the facet $G = [1, 0]^\top$. The primitive integral support functions of these facets are $h_{\emptyset,F}(c, 2c) = c$ and $h_{G,A}(a, b) = b$. Then $h_{G,A}(c, 2c) = 2c$, so $h_{G,A}|_{\mathbb{C}F} = 2h_{\emptyset,F}$.

Consider the parameter $\beta = (1/2, 1)$. This parameter is already in $\mathbb{C}F$. Since $h_{\emptyset,F}(\beta) = 1/2$ is not in $\mathbb{Z}$, the same is true of $h_{\emptyset,F}(\lambda)$ for every $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^2)$. However, $h_{G,A}(\beta) = 2 \in \mathbb{Z}$.

**Theorem 5.8.** *Assume $S_A$ is normal, let $F \preceq A$ be a face, and let $\beta \in \mathbb{C}^d$.*

*(1) If $\beta \notin \mathbb{C}F + \mathbb{Z}^d$, then $\pi_{F+}\mathcal{M}_A(\beta) = i_F^+\mathcal{M}_A(\beta) = 0$.*

*(2) If $\beta \in \mathbb{C}F + \mathbb{Z}^d$, then there exists a $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ such that*
$$\pi_{F+}\mathcal{M}_A(\beta) \cong \mathcal{M}_F(\lambda) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}} \cong i_F^+\mathcal{M}_A(\beta).$$

*Proof.* Recall that the Fourier–Laplace transform interchanges $\pi_{F+}$ and $i_F^+$. Therefore, the theorem is equivalent to the same statement with $\mathcal{M}_A(\beta)$ and $\mathcal{M}_F(\lambda)$ replaced with $\hat{\mathcal{M}}_A(\beta)$ and $\hat{\mathcal{M}}_F(\lambda)$, respectively. We prove this Fourier–Laplace transformed statement.

Choose open subsets $U, V$ of $\mathbb{C}^n$ such that $U \cap X_A = \operatorname{fSupp} \hat{\mathcal{M}}_A(\beta)$ and $V \cap X_A = \operatorname{cofSupp} \hat{\mathcal{M}}_A(\beta)$. [Ste17, Th. 9.3] establishes that

$$\mathrm{MGM}_A^*(U, \beta) \cong \hat{\mathcal{M}}_A(\beta) \cong \mathrm{MGM}_A(V, \beta). \tag{5.3.3}$$

If $\beta \notin \mathbb{C}F + \mathbb{Z}^d$, then Theorem 5.4(1) applies to give the Fourier–Laplace transformed version of (1).

Suppose $\beta \in \mathbb{C}F + \mathbb{Z}^d$. By normality, the direct sums in Theorem 5.4(2) collapse to a single summand. Therefore, taking into account (5.3.3), it remains to show that there exists a $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ such that

$$\mathrm{MGM}_F^*\big(i_F^{-1}(U), \beta\big) \cong \hat{\mathcal{M}}_F(\lambda) \cong \mathrm{MGM}_F\big(i_F^{-1}(V), \beta\big). \tag{5.3.4}$$

By [Ste17, Th. 9.3 together with Lem. 9.1(c) and (d)], (5.3.4) holds for any $\lambda$ satisfying the conditions in Lemma 5.6. Now use that Lemma 5.6 guarantees that such a $\lambda$ exists. □

## 6. Duality of normal GKZ systems

Throughout this section, $S_A$ is assumed to be normal. In Theorem 6.3, we assume in addition that $A$ is homogeneous (Recall that $A$ is *homogeneous* if its columns all lie in a hyperplane).

Lemma 6.1 shows that for all parameters $\beta$, there is a parameter $\beta' \in -\beta + \mathbb{Z}^d$ such that $\hat{\mathcal{M}}_A(\beta')$ has the cofiber support one would expect for the holonomic dual of $\hat{\mathcal{M}}_A(\beta)$. Proposition 6.2 uses this to prove that this $\hat{\mathcal{M}}_A(\beta')$ is indeed the holonomic dual of $\hat{\mathcal{M}}_A(\beta)$. The Fourier–Laplace transform of this result, together with a monodromicity argument, gives Theorem 6.3.

**Lemma 6.1.** *Let $\beta \in \mathbb{C}^d$. Then there exists a $\beta' \in -\beta + \mathbb{Z}^d$ such that*
$$\operatorname{cofSupp} \hat{\mathcal{M}}_A(\beta') = \operatorname{fSupp} \hat{\mathcal{M}}_A(\beta).$$
*If $\beta$ does not lie on the $\mathbb{C}$-span of any facet, then $\beta'$ may be taken to be $-\beta$.*

*Proof.* By [Ste17, Lem. 9.1(c) and (d)], it suffices to show that there exists a $\beta' \in -\beta + \mathbb{Z}^d$ for all facets $G \preceq A$,
$$h_G(\beta') \in \mathbb{N} \quad \text{if and only if} \quad h_G(\beta) \in \mathbb{Z}_{<0}.$$

Choose $\gamma \in \mathbb{Z}^d$ as in Lemma 5.5. Then $\hat{\mathcal{M}}_A(\beta)$ and $\hat{\mathcal{M}}_A(\beta+\gamma)$ have the same fiber support (by [Ste17, Lem. 9.1(c)]) and are therefore isomorphic by [Ste17, Th. 9.2]. Moreover, $\beta+\gamma$ does not lie on the $\mathbb{C}$-span of any facet. Replacing $\beta$ with $\beta+\gamma$, we may assume that $\beta$ itself does not lie on the $\mathbb{C}$-span of any facet.

Let $\beta' = -\beta$. Then $h_G(\beta)$ is never zero, so $h_G(\beta') \in \mathbb{N}$ if and only if $h_G(\beta) \in \mathbb{Z}_{<0}$, as hoped. ☐

**Proposition 6.2.** *Let $\beta \in \mathbb{C}^d$. Then there exists a $\beta' \in -\beta + \mathbb{Z}^d$ such that $\mathbb{D}\hat{\mathcal{M}}_A(\beta) \cong \hat{\mathcal{M}}_A(\beta')$. If $\beta$ does not lie on the $\mathbb{C}$-span of any facet, then $\beta'$ may be taken to be $-\beta$.*

*Proof.* By [Ste17, Th. 9.2], there exists an open $U \subseteq \mathbb{C}^n$ with $U \cap X_A = \operatorname{fSupp} \hat{\mathcal{M}}_A(\beta)$ such that $\hat{\mathcal{M}}_A(\beta) \cong \mathrm{MGM}^*(U, \beta)$. Applying the holonomic duality functor gives $\mathbb{D}\hat{\mathcal{M}}_A(\beta) \cong \mathrm{MGM}(U, -\beta)$. Now use [Ste17, Th. 9.2] again along with Lemma 6.1. ☐

**Theorem 6.3.** *Assume that $A$ is homogeneous. Let $\beta \in \mathbb{C}^d$. Then there exists a $\beta' \in -\beta + \mathbb{Z}^d$ such that $\mathbb{D}\mathcal{M}_A(\beta) \cong \mathcal{M}_A(\beta')$. If $\beta$ does not lie on the $\mathbb{C}$-span of any facet, then $\beta'$ may be taken to be $-\beta$.*

*Proof.* By [Rei14, Lem. 1.13], the homogeneity condition implies that every $A$-hypergeometric system is monodromic. By [Bry86, Prop. 6.13] (or rather the restatement of it for $D$-modules which appears in [Rei14, Th. 1.4]), if $\mathcal{M}$ is monodromic, then $\mathbb{D}\,\mathrm{FL}\,\mathcal{M} \cong \mathrm{FL}\,\mathbb{D}\mathcal{M}$. Now use Proposition 6.2. ☐

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