ON TWO LETTER IDENTITIES IN LIE RINGS

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Abstract. Let $L = L(a,b)$ be a free Lie ring on two letters a, b. We investigate the kernel I of the map $L \oplus L \to L$ given by $(A,B) \mapsto [A,a]+[B,b]$. Any homogeneous element of L of degree ≥ 2 can be presented as $[A,a]+[B,b]$. Then I measures how far such a presentation from being unique. Elements of I can be interpreted as identities $[A(a,b),a] = [B(a,b),b]$ in Lie rings. The kernel I can be decomposed into a direct sum $I = \bigoplus_{n,m} I_{n,m}$, where elements of $I_{n,m}$ correspond to identities on commutators of weight $n+m$, where the letter a occurs n times and the letter b occurs m times. We give a full description of $I_{2,m}$; describe the rank of $I_{3,m}$; and present a concrete non-trivial element in $I_{3,3n}$ for $n \geq 1$.

Introduction

It is easy to check that the following identity is satisfied in any Lie ring (=Lie algebra over $\mathbb{Z}$)

$$[a,b,b,a] = [a,b,a,b], \tag{1}$$

where $[x_1,\dots,x_n]$ is the left-normed bracket of elements $x_1,\dots,x_n$ defined by recursion $[x_1,\dots,x_n] := [[x_1,\dots,x_{n-1}],x_n]$. We denote by $[a,_i b]$ the Engel brackets of a, b :

$$[a,_0 b] = a, \qquad [a,_{i+1} b] = [[a,_i b], b].$$

For example, $[a,_3 b] = [a,b,b,b]$. In [1] the second author together with Roman Mikhailov generalized the identity (1) as follows

$$[[a,_{2n} b],a] = \left[\sum_{i=0}^{n-1} (-1)^i [[a,_{2n-1-i} b],[a,_i b]] \ , \ b \right], \tag{2}$$

where $n \geq 1$. This identity is crucial in their proof that the wedge of two circles $S^1 \vee S^1$ is a $\mathbb{Q}$-bad space in sense of Bousfield-Kan. Note that the letter a occurs twice in each commutator of this identity and the letter b occurs $2n$ times. Moreover, the identity has the form $[A,a] = [B,b]$.

We are interested in identities of the form:

$$[A(a,b),a] = [B(a,b),b], \tag{3}$$

where A and B are some expressions on letters a and b. These identities can be interpreted as an equalities in the free Lie ring $L = L(a,b)$. Note that a description of all identities of such kind would give a full description of the intersection $[L,a] \cap [L,b]$. Consider a $\mathbb{Z}$-linear map

$$\Theta : L \oplus L \to L,$$
$$\Theta(A,B) = [A,a]+[B,b].$$

Then the problem of describing identities of type (3) can be formalised as the problem of describing

$$I := \mathrm{Ker}(\Theta).$$

Any homogeneous element of L of degree ≥ 2 can be presented as $[A,a]+[B,b]$. So I measures how far this presentation from being unique. The problem of describing of I is different from the problem formulated on the formal language of identities, because A, B here are not just formal expressions but they are elements of the free Lie ring. For example, the identity $[[b,b],a] = [[a,a],b]$ is not interesting for us because $([b,b],-[a,a]) = (0,0)$ in I. This work is devoted to the study of I.

The Lie ring L has a natural grading by the weight of a commutator: $L = \bigoplus_{n\geq 1} L_n$. Moreover, $L_n = \bigoplus_{k+m=n} L_{k,m}$, where $L_{k,m} \subseteq L_{k+m}$ is an abelian group generated by multiple commutators with k letters a and m letters b. We can consider the following restrictions of the map Θ

$$\Theta_n : L_{n-1} \oplus L_{n-1} \to L_n, \qquad \Theta_{k,l} : L_{k-1,l} \oplus L_{k,l-1} \to L_{k,l}$$

and set $I_n = \mathrm{Ker}(\Theta_n)$ and $I_{k,l} = \mathrm{Ker}(\Theta_{k,l})$. It is easy to check that

$$I = \bigoplus_{n\geq 1} I_n, \qquad I_n = \bigoplus_{k+l=n} I_{k,l}.$$

The main results of the paper are the full description of $I_{2,n}$; the description of the rank of the free abelian group $I_{3,n}$; and the description of a concrete series of elements from $I_{3,3n}$ for any $n \geq 1$.

The rank of a free abelian group X is called "dimension of X" in this paper and it is denoted by $\dim X$. It well known that the dimension of L_n can be computed by the Necklace polynomial

$$\dim L_n = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d,$$

where μ is the Mobius function.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$\dim L_n$	2	1	2	3	6	9	18	30	56	99	186	335	630

Since the map Θ_n is an epimorphsm, we obtain

$$\dim I_n = 2 \cdot \dim\, L_{n-1} - \dim\, L_n.$$

We prove the following

$$\dim\, I_{2,m} = \begin{cases} 0, \text{ if } m \text{ is odd} \\ 1, \text{ if } m \text{ is even} \end{cases}, \qquad \dim\, I_{3,m} = \left\lfloor \frac{m+1}{2} \right\rfloor - \left\lfloor \frac{m-1}{3} \right\rfloor - 1.$$

(Proposition 1.4, Proposition 2.5). For $n \leq 13$ we obtain the following table for dimensions.

n	2	3	4	5	6	7	8	9	10	11	12	13
$\dim I_{2,n-2}$	0	0	1	0	1	0	1	0	1	0	1	0
$\dim I_{3,n-3}$	0	0	0	0	1	0	1	1	1	1	2	1
$\dim I_n$	3	0	1	0	3	0	6	4	13	12	37	40

The computation of $\dim I_{2,m}$ shows that there are no any other elements in $I_{2,m}$ except those that come from the identities (2). In particular, all non-trivial identities corresponding to elements of $I_{2,n}$ have even weight.

Note that $I_n = 0$ for odd $n < 9$. This can be interpreted as the fact that there is no a non-trivial identity of the type $[A, a] = [B, b]$ on two letters of odd weight lesser than 9. However, we have found a non-trivial identity of this type of weight 9 (Theorem 2.8). If we set

$$C_n = [a,_n b],$$

then the following identity of weight 9 holds in any Lie ring

$$(4) \qquad \big[2[C_5, C_1] + 5[C_4, C_2],\ a\ \big] = \big[2[C_4, C_1, C_0] + 3[C_3, C_2, C_0] - 2[C_3, C_1, C_1] + [C_2, C_1, C_2],\ b\ \big].$$

The main result of this paper is a concrete series of identities that correspond to non-trivial elements in $I_{3,3n}$ that generalise the identity (4) (Theorem 2.3). Namely, for any $n \geq 1$, the following identity is satisfied in $L_{3,3n}$.

$$\left[\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{n+1} \alpha_{n+1-k,k} [C_{2n+1-k}, C_{n+k-1}]\ ,\ a\right] = \left[\sum_{i=0}^{n} \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{i+1} \alpha_{i-j,j} [C_{n+i-j}, C_{n+j-1}, C_{n-i}]\ ,\ b\right],$$

where $\alpha_{0,0} = 1$ and $\alpha_{i,j} = 2\binom{i+j-1}{j} + \binom{i+j-2}{j-1} - \binom{i+j-2}{j-2} - 2\binom{i+j-1}{j-2}$ for $i, j \geq 0$ and $(i, j) \neq (0, 0)$. For $n = 2$ we obtain the identity (4). For $n = 1$ we obtain the identity

$$[\ 3[a, b, b, [a, b]] + 2[a, b, b, a]\ ,\ a] \ = \ [\ -[a, b, a, [a, b]] + 2[a, b, b, a, a]\ ,\ b]$$

that holds in any Lie ring.

1. Identities corresponding to elements in $I_{2,m}$

Definition 1.1. If w is a Lyndon word, we denote by $[w]$ the corresponding element of the Lyndon-Shirshov basis of the free Lie algebra L (see [2]). If w is a letter, then $[w] = w$. If w is not a letter then w has a standard factorisation $w = uv$ and $[w]$ is defined by recursion $[w] = [[u], [v]]$. For example, $[a] = a$ and $[ab^n] = [[ab^{n-1}], b] = C_n$.

Lemma 1.2. *The following set is a basis of $L_{2,n}$ with $n \in \mathbb{N}$.*

$$\{\, [C_k, C_l] \mid k > l,\ k + l = n\ \ k, l, m \in \mathbb{N} \,\}.$$

Proof. The intersection of the Lyndon-Shirshov basis with $L_{k,m}$ is a basis of $L_{k,m}$. The basis of $L_{2,m}$ consists of commutators of Lyndon words with 2 letters "a" and m letters "b".

$$[ab^lab^k] = [[ab^l],[ab^k]] = -[[ab^l],[ab^k]] = -[C_k, C_l].$$

Word ab^lab^k is a Lyndon word only when $k > l$. The assertion follows. □

Lemma 1.3. *For any $n \in \mathbb{N}$ the following is satisfied:*

$$\dim L_{2,n} = \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Proof. Consider the basis from lemma 1.2. Hence $L_{2,n} = \langle \{[C_{n_1}, C_{n_2}] \mid n_1, n_2 \in \mathbb{N}_0, n_1 > n_2 \text{ and } n_1 + n_2 = n\} \rangle$. Total number of words with 2 letters a and n letters b starting with a is $n+1$. However, in our case $n_1 > n_2$. Hence for odd n number of such commutators is $\frac{n+1}{2}$ and for even n it is $\frac{n}{2}$. □

Proposition 1.4. *For any $m \in \mathbb{N}$ we have*

$$\dim\, I_{2,m} = \begin{cases} 0, \ \textit{if } m \textit{ is odd} \\ 1, \ \textit{if } m \textit{ is even.} \end{cases}$$

Proof. By definition, $I_{2,m} = \operatorname{Ker}\Theta_{2,m}$. Then $\dim I_{2,m} = \dim(\ker\Theta_{2,m}) = \dim(L_{1,m} \oplus L_{2,m-1}) - \dim(\operatorname{Im}\Theta_{2,m}) = \dim(L_{1,m} \oplus L_{2,m-1}) - \dim L_{2,m} = \dim L_{1,m} + \dim L_{2,m-1} - \dim L_{2,m} = 1 + \dim L_{2,m-1} - \dim L_{2,m} = 1 + \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m+1}{2} \right\rfloor$ (see lemma 1.3). Let m be even, then $\dim I_{2,m} = 1 + \frac{m}{2} - \left\lfloor \frac{m}{2} - \frac{1}{2} \right\rfloor = 1 + \frac{m}{2} - \frac{m}{2} = 1$. Consider the case of odd m. Then $\dim I_{2,m} = 1 + \left\lceil \frac{m-1}{2} \right\rceil - \frac{m+1}{2} = 1 + \frac{m}{2} - \frac{1}{2} - \frac{m}{2} - \frac{1}{2} = 0$. □

Theorem 1.5. *For any $m \in \mathbb{N}$ the following is satisfied*

$$I_{2,m} = \begin{cases} 0, \ \textit{if } m \textit{ is odd} \\ \langle\, (\, C_m,\, \sum_{i=1}^{\frac{m}{2}} (-1)^i [C_{m-i}, C_{i-1}]\,)\,\rangle, \ \textit{if } m \textit{ is even.} \end{cases}$$

Proof. Triviality of the kernel for odd m can be easily proven using lemma 1.3. Consider the case when m is even. Then basis of $L_{1,m}$ consists of one element $[ab^m]$, i.e. $L_{1,m} = \{\alpha[ab^m] \mid \alpha \in \mathbb{Z}\}$. Basis of $L_{2,m-1}$ consists of $\left\lfloor \frac{m}{2} \right\rfloor$ elements (according to lemma 1.3). Because m is even $\left\lfloor \frac{m}{2} \right\rfloor = \frac{m}{2}$. Hence the following equality is true

$$L_{2,m-1} = \left\{ \alpha_1[aab^{m-1}] + \alpha_2[abab^{m-2}] + \cdots + \alpha_{\frac{m}{2}}[ab^{\frac{m}{2}-1}ab^{m-\frac{m}{2}}] \mid \alpha_1, \alpha_2, \ldots, \alpha_{\frac{m}{2}} \in \mathbb{Z} \right\}.$$

By definition, $I_{2,m} = \ker\Theta_{2,m}$. We can apply map $\Theta_{2,m}$ to arbitrary element of $L_{1,m} \oplus L_{2,m-1}$ that is expressed as basis elements and equate the obtained to zero. Jacobi identity implies the following

$$[[ab^nab^m], b] = [[ab^{n+1}], [ab^m]] + [ab^nab^{m+1}].$$

We can use this equality to transform an image of an element from $L_{2,m-1}$.

$$\Theta_{2,m}\left(\alpha[ab^m], \sum_{i=1}^{\frac{m}{2}} \alpha_i [ab^{i-1}ab^{m-i}]\right) = \alpha[[ab^m], a] + \sum_{i=1}^{\frac{m}{2}} \alpha_i [[ab^{i-1}ab^{m-i}], b] =$$

$$= -\alpha[a, [ab^m]] + \sum_{i=1}^{\frac{m}{2}} \alpha_i \left([[ab^i], [ab^{m-i}]] + [ab^{i-1}ab^{m-i+1}]\right) =$$

For all $i \neq \frac{m}{2}$ commutator $[[ab^i], [ab^{m-i}]] = [ab^iab^{m-i}]$. Then the sum can be rewritten as follows

$$= -\alpha[aab^m] + \alpha_1[abab^{m-1}] + \alpha_1[aab^m] + \alpha_2[ab^2ab^{m-2}] + \alpha_2[abab^{m-1}] + \cdots + \alpha_{i-1}[ab^{i-1}ab^{m-i+1}] +$$
$$+\alpha_{i-1}[[ab^{i-2}ab^{m-i+2}]] + \alpha_i[ab^iab^{m-i}] + \alpha_i[ab^{i-1}ab^{m-i+1}] + \alpha_{i+1}[ab^{i+1}ab^{m-i+1}] + \alpha_{i+1}[ab^iab^{m-i}] + \cdots +$$
$$+\alpha_{\frac{m}{2}}[[ab^{\frac{m}{2}}], [ab^{m-\frac{m}{2}}]] + \alpha_{\frac{m}{2}}[ab^{\frac{m}{2}-1}ab^{m-\frac{m}{2}+1}] = 0.$$

Last but one element of sum is equals to 0 because $\alpha_{\frac{m}{2}}[[ab^{\frac{m}{2}}], [ab^{m-\frac{m}{2}}]] = \alpha_{\frac{m}{2}}[[ab^{\frac{m}{2}}], [ab^{\frac{m}{2}}]] = 0$. It is easy to see that for equality we need such coefficients α and α_i that terms of the sum will be reduced. Let $\alpha = 1$, hence $\alpha_1 = 1$ because we need $-\alpha[aab^m]$ and $\alpha_1[aab^m]$ to be reduced. Other coefficients can be obtained similarly. Commutators with coefficients α_i and α_{i+1} will be reduced. Hence $\ker\Theta_{2,m}$ is generated by element $[ab^m] = C_m$ and sum $[aab^{m-1}] - [abab^{m-2}] + \cdots \mp [ab^{\frac{m}{2}-1}ab^{m-\frac{m}{2}+1}] \pm [ab^{\frac{m}{2}}ab^{m-\frac{m}{2}}] = \sum_{i=1}^{\frac{m}{2}} (-1)^{i+1}[ab^{i-1}ab^{m-i}] = = \sum_{i=1}^{\frac{m}{2}} (-1)^{i+1}[[ab^{i-1}], [ab^{m-i}]] = \sum_{i=1}^{\frac{m}{2}} (-1)^i [[ab^{m-i}], [ab^{i-1}]] = \sum_{i=1}^{\frac{m}{2}} (-1)^i [C_{m-i}, C_{i-1}]$ □

2. Identities corresponding to elements in $I_{3,m}$

Lemma 2.1. *For any $n \in \mathbb{N}$ the following set is a basis of $L_{3,n}$.*

$$\{\, [C_k, C_l, C_m] \mid k > l,\ k \geq m,\ k + l + m = n\ \ k, l, m \in \mathbb{N}_0 \,\}.$$

Proof. Lyndon words commutators of length $n+3$ with 3 letters "a" and n letters "b" construct the basis of $L_{3,n}$. It is easy to prove that $ab^iab^jab^t$ is a Lyndon word if and only if $i \leq j$ and $i < t$, where $i, j, t \in \mathbb{N}_0$. Consider two cases:

(1) $j < t$ then $[ab^iab^jab^t] = [[ab^i], [ab^jab^t]] = [[ab^i], [[ab^j], [ab^t]]] = [[ab^t], [ab^j], [ab^i]] = [C_t, C_j, C_i]$. Take $t = k$, $j = l$, $i = m$ then $k > l$, $k > m$ and $l \geq m$.

(2) $j \geq t$ then $[ab^iab^jab^t] = [[ab^iab^j], [ab^t]] = [[ab^i], [ab^j], [ab^t]] = [C_i, C_j, C_t] = -[C_j, C_i, C_t]$. Take $j = k$, $u = l$, $t = m$ then $k \geq m$, $l \leq k$ and $m > l$. Hence $k \geq m > l$, so $k > l$.

If we unite conditions of both cases, we get $k > l$ and $k \geq m$ for arbitrary $k, l, m \in \mathbb{N}_0$. □

2.1. Generalized identity with three letters "a".

Lemma 2.2. *For expression $\alpha_{i,j} = 2\binom{i+j-1}{j} + \binom{i+j-2}{j-1} - \binom{i+j-2}{j-2} - 2\binom{i+j-1}{j-2}$, where $i, j \in \mathbb{N}_0$ the following conditions are satisfied:*

1) $\alpha_{i-1,j} + \alpha_{i,j-1} = \alpha_{i,j}$, *when $i \neq j$ and $j \neq 0$*
2) $\alpha_{i,i-1} = \alpha_{i,i}$, *when $i \geq 2$*
3) $\alpha_{i,o} = 2$, *when $i \geq 1$*

Proof. We can use the recurrence relation for binomial coefficient to prove the first condition:

$$\alpha_{i-1,j} + \alpha_{i,j-1} = 2\binom{i+j-2}{j} + \binom{i+j-3}{j-1} - \binom{i+j-3}{j-2} - 2\binom{i+j-2}{j-2} +$$
$$+ 2\binom{i+j-2}{j-1} + \binom{i+j-3}{j-2} - \binom{i+j-3}{j-3} - 2\binom{i+j-2}{j-3} = 2\left(\binom{i+j-2}{j} + \binom{i+j-2}{j-1}\right) +$$
$$+ \left(\binom{i+j-3}{j-1} + \binom{i+j-3}{j-2}\right) - \left(\binom{i+j-3}{j-2} + \binom{i+j-3}{j-3}\right) - 2\left(\binom{i+j-2}{j-2} + \binom{i+j-2}{j-3}\right) =$$
$$= 2\binom{i+j-1}{j} + \binom{i+j-2}{j-1} - \binom{i+j-2}{j-2} - 2\binom{i+j-1}{j-2} = \alpha_{i,j}.$$

To prove the second condition we can substitute $j = i$ into $\alpha_{i,j}$ and express each term using recurrence relation for binomial coefficient:

$$\alpha_{i,i} = 2\binom{2i-1}{i} + \binom{2i-2}{i-1} - \binom{2i-2}{i-2} - 2\binom{2i-1}{i-2} =$$
$$= \underline{2\binom{2i-2}{i-1}} + 2\binom{2i-2}{i} + \underline{\binom{2i-3}{i-2}} + \binom{2i-3}{i-1} - \underline{\binom{2i-3}{i-3}} - \binom{2i-3}{i-2} - \underline{2\binom{2i-2}{i-3}} - 2\binom{2i-2}{i-2} =$$
$$= \alpha_{i,i-1} + 2\binom{2i-2}{i} + \binom{2i-3}{i-1} - \binom{2i-3}{i-2} - 2\binom{2i-2}{i-2}.$$

All we need to prove now is that $2\binom{2i-2}{i} + \binom{2i-3}{i-1} - \binom{2i-3}{i-2} - 2\binom{2i-2}{i-2} = 0$. Using symmetric property of binomial coefficient, i.e. $\binom{n}{k} = \binom{n}{n-k}$, all terms will be reduced:

$$2\binom{2i-2}{i} + \binom{2i-3}{i-1} - \binom{2i-3}{i-2} - 2\binom{2i-2}{i-2} = \binom{2i-3}{i-2} + 2\binom{2i-2}{i-2} - \binom{2i-3}{i-2} - 2\binom{2i-2}{i-2} = 0.$$

Consider the case, when $j = 0$. We need to mention that for $k < 0$ binomial coefficient $\binom{n}{k} = 0$. Then the expression will be as follows.

$$\alpha_{i,0} = 2\binom{i-1}{0} + \binom{i-2}{-1} - \binom{i-2}{-2} - 2\binom{i-1}{-2} = 2\binom{i-1}{0} = 2.$$

□

Theorem 2.3. *For any $n \in \mathbb{N}$, the following identity is satisfied in $L_{3,3n}$.*

$$\left[\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{n+1} \alpha_{n+1-k,k} [C_{2n+1-k}, C_{n+k-1}], a\right] = \left[\sum_{i=0}^{n} \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{i+1} \alpha_{i-j,j} [C_{n+i-j}, C_{n+j-1}, C_{n-i}], b\right],$$

where $\alpha_{0,0} = 1$ and $\alpha_{i,j} = 2\binom{i+j-1}{j} + \binom{i+j-2}{j-1} - \binom{i+j-2}{j-2} - 2\binom{i+j-1}{j-2}$, where $i, j \in \mathbb{N}$.

Proof. Consider $n, k \in \mathbb{N}$. Lets prove the following identity for any $k \le n$:

$$\left[\sum_{i=0}^{k}\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor}(-1)^{i+1}\alpha_{i-j,j}[C_{n+i-j}, C_{n+j-1}, C_{n-i}], b\right] = \sum_{t=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+1}\alpha_{k+1-t,t}[C_{n+k+1-t}, C_{n-1+t}, C_{n-k}].$$

Denote the left part as ω_k and the right part as θ_k. We will prove this equality using mathematical induction with variable k.

$\boxed{1}$ We can expand the sum and use lemma 2.2

$$\omega_1 = [-\alpha_{0,0}[C_n, C_{n-1}, C_n] + \alpha_{1,0}[C_{n+1}, C_{n-1}, C_{n-1}], b] = -1[C_n, C_{n-1}, C_n, b] + 2[C_{n+1}, C_{n-1}, C_{n-1}, b] =$$
$$= -1[C_{n+1}, C_{n-1}, C_n] + 1[C_{n+1}, C_n, C_{n-1}] + 2[C_{n+1}, C_{n-1}, C_n] + 2[C_{n+2}, C_{n-1}, C_{n-1}] + 2[C_{n+1}, C_{n-1}, C_{n-1}] =$$
$$= 2[C_{n+2}, C_{n-1}, C_{n-1}] + 3[C_n, C_{n-1}, C_n] = \alpha_{2,0}[C_{n+2}, C_{n-1}, C_{n-1}] + \alpha_{1,1}[C_n, C_{n-1}, C_n] = \theta_1$$

$\boxed{2}$ We need to prove that $\omega_k = \theta_k$ implies $\omega_{k+1} = \theta_{k+1}$. We can expand ω_{k+1} as a sum of ω_k and the last element of the first sum in ω_{k+1}:

$$\omega_{k+1} = \underbrace{\omega_k}_{\theta_k} + \left[\sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+2}\alpha_{k+1-j,j}[C_{n+k+1-j}, C_{n+j-1}, C_{n-k-1}], b\right]$$

By expanding the commutator as follows $[C_{n+k+1-j}, C_{n+j-1}, C_{n-k-1}, b] = [C_{n+k+2-j}, C_{n+j-1}, C_{n-k-1}, b] + [C_{n+k+1-j}, C_{n+j}, C_{n-k-1}, b] + [C_{n+k+1-j}, C_{n+j-1}, C_{n-k}, b]$, we can express the second term as three different sums. One of them will be reduced by θ_k.

$$\omega_{k+1} = \sum_{t=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+1}\alpha_{k+1-t,t}[C_{n+k+1-t}, C_{n-1+t}, C_{n-k}] + \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+2}\alpha_{k+1-j,j}[C_{n+k+2-j}, C_{n+j-1}, C_{n-k-1}] +$$
$$+ \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+2}\alpha_{k+1-j,j}[C_{n+k+1-j}, C_{n+j}, C_{n-k-1}] + \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+2}\alpha_{k+1-j,j}[C_{n+k+1-j}, C_{n+j-1}, C_{n-k}] =$$
$$= \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+2}\alpha_{k+1-j,j}[C_{n+k+2-j}, C_{n+j-1}, C_{n-k-1}] + \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+2}\alpha_{k+1-j,j}[C_{n+k+1-j}, C_{n+j}, C_{n-k-1}]$$

Denote commutators in sums as a_j and b_j correspondingly. We can show that for any $j \ge 0$ it is satisfied that $a_{j+1} = b_j$.

$$a_{j+1} = [C_{n+k+2-j-1}, C_{n+j+1-1}, C_{n-k-1}] = [C_{n+k+1-j}, C_{n+j}, C_{n-k-1}] = b_j.$$

Because of that, the expression can be written as a sum of a_0, $b_{\lfloor \frac{k+1}{2} \rfloor}$ with corresponding coefficients and one sum on index j with summed coefficients.

$$\omega_{k+1} = \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor - 1}(-1)^{k+2}(\alpha_{k-j,j+1} + \alpha_{k+1-j,j})[C_{n+k+1-j}, C_{n+j}, C_{n-k-1}] + (-1)^{k+2}\alpha_{k+1,0}[C_{n+k+2}, C_{n-1}, C_{n-k-1}] -$$
$$+ (-1)^{k+2}\alpha_{k+1-\lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{k+1}{2} \rfloor}[C_{n+k+1-\lfloor \frac{k+1}{2} \rfloor}, C_{n+\lfloor \frac{k+1}{2} \rfloor}, C_{n-k-1}].$$

Coefficients of commutators, in obtained sum on index j, can be transformed using the first case of lemma 2.2. Also, we can change index of sum by subtracting 1 from it. Coefficient of a_0 can be rewritten using the third case of lemma 2.2. Then $b_{\lfloor \frac{k+1}{2} \rfloor}$ will become a zero element of sum on index j.

$$\omega_{k+1} = \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+2}\alpha_{k+2-j,j}[C_{n+k+2-j}, C_{n-1+j}, C_{n-k-1}] + (-1)^{k+2}\alpha_{k+2,0}[C_{n+k+2}, C_{n-1}, C_{n-k-1}] +$$
$$+ (-1)^{k+2}\alpha_{k+1-\lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{k+1}{2} \rfloor}[C_{n+k+1-\lfloor \frac{k+1}{2} \rfloor}, C_{n+\lfloor \frac{k+1}{2} \rfloor}, C_{n-k-1}] =$$
$$= \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor}(-1)^{k+2}\alpha_{k+2-j,j}[C_{n+k+2-j}, C_{n-1+j}, C_{n-k-1}] + (-1)^{k+2}\alpha_{k+1-\lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{k+1}{2} \rfloor} b_{\lfloor \frac{k+1}{2} \rfloor}.$$

Consider two cases:

1) k is odd. Then $\lfloor\frac{k+2}{2}\rfloor = \lfloor\frac{k+1}{2}+\frac{1}{2}\rfloor = \frac{k+1}{2}+\lfloor\frac{1}{2}\rfloor = \frac{k+1}{2} = \lfloor\frac{k+1}{2}\rfloor$, hence $b_{\lfloor\frac{k+1}{2}\rfloor} = 0$. It is true because $b_{\lfloor\frac{k+1}{2}\rfloor} = [C_{n+k+1-\frac{k+1}{2}}, C_{n+\frac{k+1}{2}}, C_{n-k-1}] = [0, C_{n-k-1}] = 0$. Consequently ω_{k+1} can be expressed as follows.

$$\omega_{k+1} = \sum_{j=0}^{\lfloor\frac{k+2}{2}\rfloor} (-1)^{k+2}\alpha_{k+2-j,j}[C_{n+k+2-j}, C_{n-1+j}, C_{n-k-1}] = \theta_{k+1}.$$

2) k is even. Then $\lfloor\frac{k+2}{2}\rfloor = \frac{k+2}{2} = \lfloor\frac{k}{2}\rfloor + 1 = \lfloor\frac{k+1}{2}\rfloor + 1$, hence $\alpha_{k+2-\lfloor\frac{k+2}{2}\rfloor,\lfloor\frac{k+2}{2}\rfloor} = \alpha_{\lfloor\frac{k+1}{2}\rfloor+1,\lfloor\frac{k+1}{2}\rfloor}$ because of the second case of lemma 2.2. It is important to mention that $\lfloor\frac{k+1}{2}\rfloor = \lfloor\frac{k}{2}+\frac{1}{2}\rfloor = \frac{k}{2}$. Hence $\alpha_{k+1-\lfloor\frac{k+1}{2}\rfloor,\lfloor\frac{k+1}{2}\rfloor} = \alpha_{\frac{k}{2}+1,\lfloor\frac{k+1}{2}\rfloor} = \alpha_{\lfloor\frac{k+1}{2}\rfloor+1,\lfloor\frac{k+1}{2}\rfloor}$. We can express θ_{k+1} as a sum up to $\lfloor\frac{k+2}{2}\rfloor - 1 = \lfloor\frac{k+1}{2}\rfloor$ and the last addendum with $j = \lfloor\frac{k+2}{2}\rfloor = \lfloor\frac{k+1}{2}\rfloor + 1$:

$$\theta_{k+1} = \sum_{j=0}^{\lfloor\frac{k+1}{2}\rfloor} (-1)^{k+2}\alpha_{k+2-j,j}[C_{n+k+2-j}, C_{n-1+j}, C_{n-k-1}] +$$
$$+(-1)^{k+2}\alpha_{k+2-\lfloor\frac{k+2}{2}\rfloor,\lfloor\frac{k+2}{2}\rfloor}[C_{n+k+2-\lfloor\frac{k+1}{2}\rfloor-1}, C_{n-1+\lfloor\frac{k+1}{2}\rfloor+1}, C_{n-k-1}] =$$
$$= \sum_{j=0}^{\lfloor\frac{k+1}{2}\rfloor} (-1)^{k+2}\alpha_{k+2-j,j}[C_{n+k+2-j}, C_{n-1+j}, C_{n-k-1}] + (-1)^{k+2}\alpha_{k+1-\lfloor\frac{k+1}{2}\rfloor,\lfloor\frac{k+1}{2}\rfloor} b_{\lfloor\frac{k+1}{2}\rfloor} = \omega_{k+1}$$

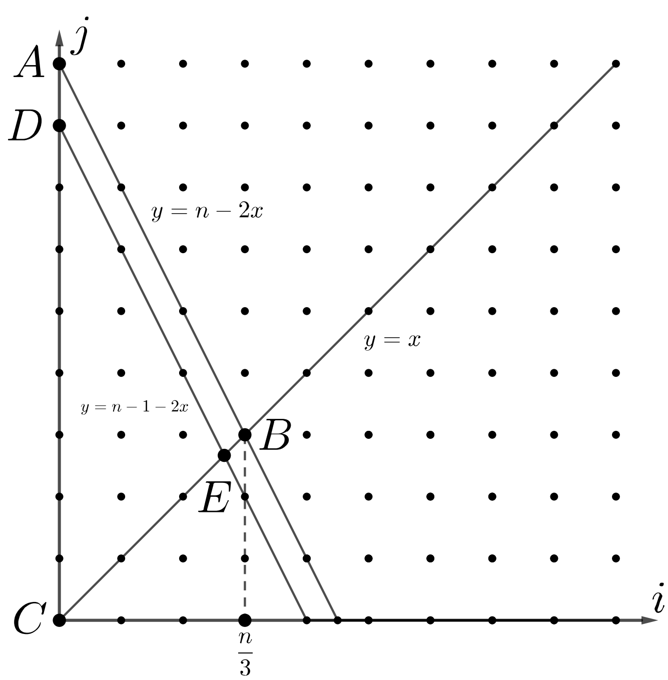
Consequently, the identity above is satisfied for any $n, k \in \mathbb{N}$, such that $k \leq n$. If we substitute $k = n$, we will get the original identity. □

2.2. Additional results.

Lemma 2.4. *For any $n \in \mathbb{N}$ the following is satisfied:*

$$\dim L_{3,n} - \dim L_{3,n-1} = \left\lfloor \frac{n-1}{3} \right\rfloor + 1.$$

Proof. To calculate this expression, we need to count all Lyndon words of form $ab^{n_1}ab^{n_2}ab^{n_3}$, where $n_1, n_2, n_3 \in \mathbb{N}_0$ and $n_1 + n_2 + n_3 = n$. Let $n_1 = i$ and $n_2 = j$, hence $n_3 = n - i - j$. As it was mentioned before, $ab^{n_1}ab^{n_2}ab^{n_3}$ is a Lyndon word if and only if $n_1 \leq n_2$ and $n_1 < n_3$, where $n_1, n_2, n_3 \in \mathbb{N}$. We can portray integer points that satisfy these conditions on coordinate plane by drawing plots of functions $y = x$ and $y = n - 2x$.



Abscissa of functions intersection point is $\frac{n}{3}$. $ab^iab^jab^{n-i-j}$ is a Lyndon word if point (i, j) belongs to $\triangle ABC$ (without point on the line $y = n - 2x$). Then $\dim L_{3,n}$ equals to number of integer points in $\triangle DEC$. Hence $\dim L_{3,n} - \dim L_{3,n-1}$ equals to number of integer points on segment DE, i.e. $\lfloor \frac{n-1}{3} \rfloor + 1$. □

Proposition 2.5. *For any $m \in \mathbb{N}$ the following is satisfied:*

$$\dim I_{3,m} = \left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m-1}{3} \right\rfloor - 1.$$

Proof. By definition, $I_{3,m} = \ker \Theta_{3,m}$. Hence, according to lemmas 1.4 and 2.4, $\dim I_{3,m} = \dim \ker \Theta_{3,m} = \dim(L_{2,m} \oplus L_{3,m-1}) - \dim L_{3,m} = \dim L_{2,m} - (\dim L_{3,m} - \dim L_{3,m-1}) = \lceil \frac{m}{2} \rceil - \lfloor \frac{m-1}{3} \rfloor - 1$. □

Lemma 2.6. *For $k > l$, $k \geq m$ the following is satisfied:*

$$[C_k, C_l, C_m, b] = \begin{cases} [C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] + [C_k, C_l, C_{m+1}], \textit{if } k > l+1,\ k \geq m+1 \\ [C_{k+1}, C_l, C_m] + [C_k, C_l, C_{m+1}], \textit{if } k = l+1,\ k \geq m+1 \\ 2[C_{k+1}, C_l, C_m] - [C_{k+1}, C_{l+1}, C_{m-1}], \textit{if } k = l+1,\ k = m \\ 2[C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] - [C_{k+1}, C_k, C_l], \textit{if } k > l+1,\ k = m. \end{cases}$$

Proof. It is easy to rewrite the expression in the first case using Jacobi identity:

$$[C_k, C_l, C_m, b] = [C_k, C_l, b, C_m] + [C_k, C_l, [C_m, b]] = [C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] + [C_k, C_l, C_{m+1}].$$

Second case:

$$[C_k, C_l, C_m, b] = [C_{l+2}, C_l, C_m] + [C_{l+1}, C_{l+1}, C_m] + [C_{l+1}, C_l, C_{m+1}] = [C_{k+1}, C_l, C_m] + [C_k, C_l, C_{m+1}].$$

Third case:

$$[C_k, C_l, C_m, b] = [C_{k+1}, C_l, C_m] + [C_k, C_l, C_{m+1}] = [C_{k+1}, C_l, C_m] + [C_k, C_{m+1}, C_l] + [C_{m+1}, C_l, C_k] =$$
$$= 2[C_{k+1}, C_l, C_m] - [C_{k+1}, C_{l+1}, C_{m-1}].$$

Fourth case:

$$[C_k, C_l, C_m, b] = [C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] + [C_k, C_l, C_{m+1}] = [C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] +$$
$$+ [C_k, C_{m+1}, C_l] - [C_l, C_{m+1}, k] = [C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] + [C_k, C_{m+1}, C_l] - [C_l, C_{m+1}, k] =$$
$$= 2[C_{k+1}, C_l, C_m] + [C_k, C_{l+1}, C_m] - [C_{k+1}, C_k, C_l].$$

□

Theorem 2.7. *The kernel of $\Theta_{3,3}$ is generated by the following element:*

$$(3[C_2, C_1] + 2[C_3, C_0], [C_1, C_0, C_1] - 2[C_2, C_0, C_0]).$$

Proof. According to lemma 2.5 $\dim I_{3,3} = \lceil \frac{3}{2} \rceil - \lfloor \frac{2}{3} \rfloor - 1 = 1$. Consequently, we have to provide only one identity to describe the whole $I_{3,3}$. Substitute $n = 1$ into identity from theorem 2.3:

$$[\alpha_{2,0}[C_3, C_0] + \alpha_{1,1}[C_2, C_1], a] = [-\alpha_{0,0}[C_1, C_0, C_1] + \alpha_{1,0}[C_2, C_0, C_0], b].$$

By definition of $\alpha_{i,j}$, $\alpha_{2,0} = 2$, $\alpha_{1,1} = 3$, $\alpha_{0,0} = 1$ and $\alpha_{1,0} = 2$. We can move right part of the equality to the left side and it will become an image of the element from $L_{2,3} \oplus L_{3,2}$:

$$[2[C_3, C_0] + 3[C_2, C_1], a] + [[C_1, C_0, C_1] - 2[C_2, C_0, C_0], b] = \Theta_{3,3}(2[C_3, C_0] + 3[C_2, C_1], [C_1, C_0, C_1] - 2[C_2, C_0, C_0]) = 0$$

As a result, we obtained the element that generates all identities in $L_{3,3}$ that is equivalent to description of $I_{3,3}$. □

Theorem 2.8. *The abelian group $I_{3,6}$ is generated by the following element*

$$(-2[C_5, C_1] - 5[C_4, C_2]\ ,\ 2[C_4, C_1, C_0] + 3[C_3, C_2, C_0] - 2[C_3, C_1, C_1] + [C_2, C_1, C_2])$$

Proof. Similarly to proof of the theorem 2.7. $\dim I_{3,6} = \lceil \frac{6}{2} \rceil - \lfloor \frac{5}{3} \rfloor - 1 = 1$. Substitute $n = 2$ into identity from theorem 2.3:

$$[-\alpha_{3,0}[C_5, C_1] - \alpha_{2,1}[C_4, C_2], a] = [-\alpha_{0,0}[C_2, C_1, C_2] + \alpha_{1,0}[C_3, C_1, C_1] - \alpha_{2,0}[C_4, C_1, C_0] - \alpha_{1,1}[C_3, C_2, C_0], b].$$

Coefficients will be $\alpha_{3,0} = 2$, $\alpha_{2,1} = 5$, $\alpha_{0,0} = 1$, $\alpha_{1,0} = 2$, $\alpha_{2,0} = 2$ and $\alpha_{1,1} = 3$. Again, we've found an element of $L_{2,6} \oplus L_{3,5}$ that generates all possible identities. Coefficients in the right part will be multiplied by -1 because of moving to the left side. □

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