

FACTORIZATION OF ASPLUND OPERATORS

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ABSTRACT. We give necessary and sufficient conditions for an operator $A : X \rightarrow Y$ on a Banach space having a shrinking FDD to factor through a Banach space Z such that the Szlenk index of Z is equal to the Szlenk index of A . We also prove that for every ordinal $\xi \in (0, \omega_1) \setminus \{\omega^\eta : \eta < \omega_1 \text{ a limit ordinal}\}$, there exists a Banach space \mathfrak{G}_ξ having a shrinking basis and Szlenk index ω^ξ such that for any separable Banach space X and any operator $A : X \rightarrow Y$ having Szlenk index less than ω^ξ , A factors through a subspace and through a quotient of \mathfrak{G}_ξ , and if X has a shrinking FDD, A factors through \mathfrak{G}_ξ .

1. INTRODUCTION

A celebrated result in Banach space theory is the factorization theorem of Davis, Figiel, Johnson, and Pełczyński [14], which states that any weakly compact operator factors through a reflexive Banach space. Since then, a number of classes of operators have been shown to be characterized by such factorization property. Beauzamy [4] showed that any Rosenthal operator (that is, any operator not preserving an isomorphic copy of ℓ_1) factors through a Banach space which contains no isomorphic copy of ℓ_1 , and Reĭnov [24], Heinrich [19], and Stegall [27] independently showed that any Asplund operator factors through an Asplund Banach space. In contrast, Beauzamy [3] showed that there exist super weakly compact operators which do not factor through any superreflexive Banach space. This turns out to be a particular case of a quantitative factorization problem. More precisely, there exists an ordinal index, called the *James index*, denoted by \mathcal{J} , which takes an operator and returns an ordinal if that operator is weakly compact, and (by convention) returns the symbol ∞ if that operator is not weakly compact. It was shown in [2] that if an operator A satisfies $\mathcal{J}(A) \leq \omega^{\omega^\xi}$, then A factors through a Banach space Z with $\mathcal{J}(I_Z) \leq \omega^{\omega^{\xi+1}}$, which is a quantified version of the David, Figiel, Johnson, Pełczyński factorization result. Given that the class of super weakly compact operators is precisely the class of operators whose James index does not exceed ω , Beauzamy's negative factorization result from [3] witnesses that the passage from the upper estimate $\mathcal{J}(A) \leq \omega^{\omega^\xi}$ to a strictly larger upper estimate on the James index of I_Z for a space through which A factors is necessary. Similarly, Brooker [5] showed that an operator with Szlenk index ω^ξ always factors through a Banach space with Szlenk index not more than $\omega^{\xi+1}$, and that there exist certain ordinals ξ and operators for which the passage from ω^ξ to $\omega^{\xi+1}$ is optimal. Such quantified factorization theorems yield useful information regarding universal factorization spaces (see [2] for further information), generalizing Figiel's example [16] of a separable, reflexive Banach space Z such that every compact operator factors through a subspace of Z .

The main goal of this work is to extend Brooker's result regarding factorization of Asplund operators. That is, if $A : X \rightarrow Y$ is an operator and $Sz(A) = \omega^\xi$, one would like to know when A factors through a Banach space Z such that $Sz(Z) = \omega^\xi$, or when the passage to $\omega^{\xi+1}$ is optimal. We completely solve this problem in the case that the domain of the operator has a shrinking basis. All relevant notions regarding the ε -Szlenk indices will be defined later. This result extends the factorization result of Kutzarova and Prus, which is the $\xi = 1$ case of the following theorem.

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Theorem 1.1. *Fix an ordinal $0 < \xi < \omega_1$. Suppose that X is a Banach space with a shrinking basis and $A : X \rightarrow Y$ is an operator with $Sz(A) = \omega^\xi$.*

- (i) *If $\xi = \omega^\zeta$, where ζ is a limit ordinal, then A does not factor through any Banach space Z with $Sz(Z) = Sz(A)$.*
- (ii) *$\xi = 1$ or $\xi = \omega^{\zeta+1}$ for some ordinal ζ , then A factors through a Banach space Z with $Sz(Z) = Sz(A)$ if and only if there exists an ordinal $\gamma < \omega^\xi$ such that for all $n \in \mathbb{N}$, $Sz(A, 1/2^n) \leq \gamma^n$.*
- (iii) *If $\xi = \beta + \gamma$ for some $\beta, \gamma < \xi$, then A factors through a Banach space Z with $Sz(Z) = Sz(A)$.*

It follows from standard facts about ordinals that the cases listed in Theorem 1.1 are exhaustive.

The second major result of the paper is to establish an optimal result regarding universal Asplund spaces.

Theorem 1.2. *Fix an ordinal $\xi \in (0, \omega_1) \setminus \{\omega^\eta : \eta \text{ a limit ordinal}\}$. Then there exists a Banach space \mathfrak{G}_ξ with shrinking basis and $Sz(\mathfrak{G}_\xi) = \omega^\xi$ such that if $A : X \rightarrow Y$ is a separable range operator with $Sz(A) < \omega^\xi$, then A factors through both a subspace and a quotient of \mathfrak{G}_ξ .*

2. COORDINATE SYSTEMS

Throughout, \mathbb{K} will denote the scalar field, which is either \mathbb{R} or \mathbb{C} . Given a subset S of a Banach space, $[S]$ will denote the closed span of S . Given a Banach space X and $K \subset X^*$ weak*-compact, we let $r_K \equiv 0$ if $K = \emptyset$, and otherwise we let $r_K(x) = \max_{x^* \in K} \operatorname{Re} x^*(x)$.

We recall that a *Markushevich basis* (or *M-basis*) for a Banach space is a biorthogonal system $(x_i, x_i^*)_{i \in I} \subset X \times X^*$ such that $[x_i : i \in I] = X$ and $\bigcap_{i \in I} \ker(x_i^*) = \{0\}$. For us, an *FMD* for the Banach space X will be a sequence $\mathbf{F} = (F_n)_{n=1}^\infty$ of subspaces of X such that there exist an *M-basis* $(x_i, x_i^*)_{i=1}^\infty$ and a sequence $0 = k_0 < k_1 < \dots$ of natural numbers such that for each $n \in \mathbb{N}$,

$$F_n = [x_i : k_{n-1} < i \leq k_n].$$

If $K \subset X^*$, we say that an FMD \mathbf{F} for X is *K-shrinking* provided that there exist an *M-basis* $(x_i, x_i^*)_{i=1}^\infty$ and $0 = k_0 < k_1 < \dots$ such that

$$F_n = [x_i : k_{n-1} < i \leq k_n]$$

and such that $K \subset [x_i^* : i \in \mathbb{N}]$. In the case that $A : X \rightarrow Y$ is an operator and $K = A^*B_{Y^*}$, we will say \mathbf{F} is *A-shrinking* rather than *K-shrinking*. If $K = B_{X^*}$, we will simply say \mathbf{F} is *shrinking*.

We say a sequence $(u_i)_{i=1}^\infty$ in X is a *block sequence with respect to \mathbf{F}* provided that there exist natural numbers $0 = k_0 < k_1 < \dots$ such that for each $i \in \mathbb{N}$, $u_i \in [F_j : k_{i-1} < j \leq k_i]$.

We will be primarily concerned with separable Banach spaces, and exclusively concerned with weak*-fragmentable sets. We recall that if X is a Banach space and $K \subset X^*$ is weak*-compact, we say K is *weak*-fragmentable* if for any $\varepsilon > 0$ and any non-empty subset L of K , there exists a weak*-open set $v \subset X^*$ such that $v \cap L \neq \emptyset$ and $\operatorname{diam}(v \cap L) \leq \varepsilon$. It is a consequence of the Baire category theorem and topological considerations that if X is a separable Banach space and $K \subset X^*$ is weak*-compact, then K is weak*-fragmentable if and only if it is norm separable.

One benefit of the notion of a *K-shrinking FMD* is that if X is separable and $K \subset X^*$ is norm separable, then X admits a *K-shrinking FMD*. Indeed, assume K is norm separable and does not lie in the span of finitely many vectors (otherwise the result is trivial). We may fix $(v_n)_{n=1}^\infty \subset X$ norm dense in X , $(v_n^*)_{n=1}^\infty$ weak*-dense in X^* , and $(u_n^*)_{n=1}^\infty$ norm dense in K . By the usual method of constructing an *M-basis* for a separable Banach space, one recursively selects an *M-basis* $(x_n, x_n^*)_{n=1}^\infty$ having the property that for each $n \in \mathbb{N}$, $v_n \in [x_i : i \leq 3n]$, $v_n^*, u_n^* \in [x_i^* : i \leq 3n]$. The weakening of the notion of shrinking FMD to the notion of a *K-shrinking FMD* allows us to study norm separable subsets of the duals of Banach spaces with non-separable duals, for example $K = A^*B_{Y^*}$, where $A : \ell_1 \rightarrow Y$ is an Asplund operator.

The primary property of a K -shrinking FMD, say F , with which we will be concerned is that a bounded block sequence $(y_n)_{n=1}^\infty$ with respect to F must be $\sigma(X, K)$ -null. Indeed, suppose $(x_i, x_i^*)_{i=1}^\infty$ is an M -basis such that $K \subset [x_n^* : n \in \mathbb{N}]$ and $0 = k_0 < k_1 < \dots$ is such that for each $n \in \mathbb{N}$, $F_n = [x_i : k_{n-1} < i \leq k_n]$. Then if $(y_n)_{n=1}^\infty$ is a bounded block sequence with respect to F , to see that $(y_n)_{n=1}^\infty$ is $\sigma(X, K)$ -null, it is sufficient to know that $(y_n)_{n=1}^\infty$ is pointwise null on a subset of X^* the closed span of which contains K . We then note that $\{x_n^* : n \in \mathbb{N}\}$ is such a set.

Given an FMD F for the Banach space X , a weak*-compact subset $K \subset X^*$, and an infinite subset M of \mathbb{N} , we define a seminorm $\langle \cdot \rangle_{X, F, K, M}$ on c_{00} by

$$\left\langle \sum_{i=1}^{\infty} a_i e_i \right\rangle_{X, F, K, M} = \max \left\{ r_K \left(\sum_{i=1}^{\infty} a_i x_i \right) : (x_i)_{i=1}^{\infty} \in B_X^{\mathbb{N}} \cap \prod_{i=1}^{\infty} [F_j : m_{i-1} < j \leq m_i] \right\},$$

where $m_0 = 0$ and $M = \{m_1, m_2, \dots\}$, $m_1 < m_2 < \dots$. It is evident that for any $(a_i)_{i=1}^{\infty} \in c_{00}$ and any sequence $(\varepsilon_i)_{i=1}^{\infty}$ of unimodular scalars,

$$\left\langle \sum_{i=1}^{\infty} a_i e_i \right\rangle_{X, F, K, M} = \left\langle \sum_{i=1}^{\infty} a_i \varepsilon_i e_i \right\rangle_{X, F, K, M}$$

for any infinite subset M of \mathbb{N} .

We recall that a *finite dimensional decomposition* (or *FDD*) for a Banach space X is a sequence $F = (F_n)_{n=1}^\infty$ of finite dimensional, non-zero subspaces of X such that for any $x \in X$, there exists a unique sequence $(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty F_n$ such that $x = \sum_{n=1}^\infty x_n$. From this it follows that for each $n \in \mathbb{N}$, the projection $P_n^F : X \rightarrow F_n$ given by $P_n^F x = x_n$, where $x = \sum_{m=1}^\infty x_m$ and $(x_m)_{m=1}^\infty \in \prod_{m=1}^\infty F_m$, is well-defined and bounded. Furthermore, for a (finite or infinite) interval $I \subset \mathbb{N}$, we let $I^F = \sum_{n \in I} P_n^F$. It follows from the principle of uniform boundedness that

$$\sup \{ \|I^F\| : I \subset \mathbb{N} \text{ is an interval} \} < \infty.$$

We refer to this quantity as the *projection constant* of F in X . If the projection constant of F in X is 1, we say F is *bimonotone*. It is well-known that if F is an FDD for X , then there exists an equivalent norm $|\cdot|$ on X such that F is a bimonotone FDD for $(X, |\cdot|)$. We also remark that any FDD is also an FMD.

If F is a bimonotone FDD for X , then $F_n^* = (P_n^F)^*(X^*) \subset X^*$ isometrically and canonically. Then $F^* := (F_n^*)_{n=1}^\infty$ is a bimonotone FDD for its closed span in X^* . We let $X^{(*)}$ denote this closed span. We say F is *shrinking* provided that $X^{(*)} = X^*$, which occurs if and only if any bounded block sequence with respect to F is weakly null. Let us note that $X^{(*)^{(*)}} = X$.

Let F be an FDD for X . For $x \in X$, we let $\text{supp}_F(x) = \{n \in \mathbb{N} : P_n^F x \neq 0\}$. We let $c_{00}(F)$ denote the set of those $x \in X$ such that $\text{supp}_F(x)$ is finite. We write $n < x$ (resp. $n \leq x$) to mean that $n < \min \text{supp}_F(x)$ (resp. $n \leq \min \text{supp}_F(x)$). We write $x < y$ to mean that $\max \text{supp}_F(x) < \min \text{supp}_F(y)$.

Of course, any Schauder basis $(x_i)_{i=1}^\infty$ gives rise to the FDD $(\text{span}(x_i))_{i=1}^\infty$, and each of the definitions above for an FDD can be adapted to a Schauder basis. In particular, if $(x_i)_{i=1}^\infty$ is a Schauder basis, we let $E^{(*)} = [x_i^* : i \in \mathbb{N}] \subset E^*$ denote the closed span of the coordinate functionals. Throughout, we let E denote the FDD arising from the canonical c_{00} basis.

We say a Banach space E is a *sequence space* provided that the canonical c_{00} basis is a normalized basis for E having the property that for any scalar sequence $(a_i)_{i=1}^n$ and any unimodular scalars $(\varepsilon_i)_{i=1}^n$,

$$\left\| \sum_{i=1}^n a_i e_i \right\| = \left\| \sum_{i=1}^n a_i \varepsilon_i e_i \right\|.$$

We say the sequence space E has property

- (i) R provided that for any strictly increasing sequences $(k_i)_{i=1}^\infty, (l_i)_{i=1}^\infty$ of natural numbers such that $k_i \leq l_i$ for each $i \in \mathbb{N}$, any $n \in \mathbb{N}$, and any scalars $(a_i)_{i=1}^n$,

$$\left\| \sum_{i=1}^n a_i e_{k_i} \right\| \leq \left\| \sum_{i=1}^n a_i e_{l_i} \right\|,$$

- (ii) S provided that there exists a constant C such that for any strictly increasing sequences $(k_i)_{i=1}^\infty, (l_i)_{i=1}^\infty$ of natural numbers such that $l_i < k_{i+1}$ for all $i \in \mathbb{N}$, any $n \in \mathbb{N}$, and any scalars $(a_i)_{i=1}^n$,

$$\left\| \sum_{i=1}^n a_i e_{l_i} \right\| \leq C \left\| \sum_{i=1}^n a_i e_{k_i} \right\|,$$

- (iii) T provided that there exists a constant C such that for any strictly increasing sequence $(k_i)_{i=1}^\infty$ of natural numbers, any $n \in \mathbb{N}$, and any sequence $(x_i)_{i=1}^n \subset X$ such that $x_i \in [e_j : k_{i-1} < j \leq k_i]$ (where $k_0 = 0$),

$$\left\| \sum_{i=1}^n x_i \right\| \leq C \left\| \sum_{i=1}^n \|x_i\| e_{k_i} \right\|.$$

Given a Banach space X with FDD \mathbf{F} and a sequence space E , we define three quantities on $c_{00}(\mathbf{F})$. We let

$$\|x\|_{X_{\check{\vee}}^E(\mathbf{F})} = \sup \left\{ \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}} x\|_{X e_{\max I_i}} \right\|_E : I_1 < I_2 < \dots, I_i \text{ an interval} \right\},$$

$$[x]_{X_{\check{\wedge}}^E(\mathbf{F})} = \inf \left\{ \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}} x\|_{X e_{\max I_i}} \right\|_E : I_1 < I_2 < \dots, I_i \text{ an interval, } \mathbb{N} = \cup_{i=1}^{\infty} I_i \right\},$$

and

$$\|x\|_{X_{\check{\wedge}}^E(\mathbf{F})} = \inf \left\{ \sum_{i=1}^n [x_i]_{X_{\check{\wedge}}^E(\mathbf{F})} : n \in \mathbb{N}, x_i \in c_{00}(\mathbf{F}), x = \sum_{i=1}^n x_i \right\}.$$

Proposition 2.1. *Let E be a sequence space and let X be a Banach space with bimonotone FDD \mathbf{F} .*

- (i) \mathbf{F} is a bimonotone FDD for both $X_{\check{\vee}}^E(\mathbf{F})$ and $X_{\check{\wedge}}^E(\mathbf{F})$, and $[I^{\mathbf{F}} \cdot]_{X_{\check{\vee}}^E(\mathbf{F})} \leq [\cdot]_{X_{\check{\vee}}^E(\mathbf{F})}$ on $c_{00}(\mathbf{F})$.
(ii) $(X_{\check{\vee}}^E(\mathbf{F}))^{(*)} = (X^{(*)})_{\check{\wedge}}^{E^{(*)}}(\mathbf{F}^*)$ and $(X_{\check{\wedge}}^E(\mathbf{F}))^{(*)} = (X^{(*)})_{\check{\vee}}^{E^{(*)}}(\mathbf{F}^*)$

Proof. Throughout the proof, for ease of notation, we write $X_{\check{\vee}}$ and $X_{\check{\wedge}}$ in place of $X_{\check{\vee}}^E(\mathbf{F})$ and $X_{\check{\wedge}}^E(\mathbf{F})$, respectively.

- (i) Let I be an interval in \mathbb{N} . Then for any $x \in c_{00}(\mathbf{F})$,

$$\begin{aligned} \|I^{\mathbf{F}} x\|_{X_{\check{\vee}}} &= \sup \left\{ \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}} I^{\mathbf{F}} x\|_{X e_{\max I_i}} \right\|_E \right\} = \sup \left\{ \left\| \sum_{i=1}^{\infty} \|I^{\mathbf{F}} I_i^{\mathbf{F}} x\|_{X e_{\max I_i}} \right\|_E \right\} \\ &\leq \sup \left\{ \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}} x\|_{X e_{\max I_i}} \right\|_E \right\} = \|x\|_{X_{\check{\vee}}}. \end{aligned}$$

Here, each supremum is taken over the set of all sequences of intervals $I_1 < I_2 < \dots$ with $\cup_{i=1}^{\infty} I_i = \mathbb{N}$. Replacing the suprema above with infima gives that $[I^{\mathbf{F}} x]_{X_{\check{\wedge}}} \leq [x]_{X_{\check{\wedge}}}$, and

$$\begin{aligned} \|I^{\mathbf{F}} x\|_{X_{\check{\wedge}}} &= \inf \left\{ \sum_{i=1}^n [x_i]_{X_{\check{\wedge}}} : I^{\mathbf{F}} x = \sum_{i=1}^n x_i \right\} = \inf \left\{ \sum_{i=1}^n [I^{\mathbf{F}} x_i]_{X_{\check{\wedge}}} : x = \sum_{i=1}^n x_i \right\} \\ &\leq \left\{ \sum_{i=1}^n [x_i]_{X_{\check{\wedge}}} : x = \sum_{i=1}^n x_i \right\} = \|x\|_{X_{\check{\wedge}}}. \end{aligned}$$

(ii) In the proof, we let $X_\lambda^* = (X^{(*)})_\lambda^{E^{(*)}}(\mathbf{F}^*)$ and $(X_\vee^E(\mathbf{F}))^{(*)} = (X_\vee)^{(*)}$. Fix $x \in c_{00}(\mathbf{F})$, $x^* \in c_{00}(\mathbf{F}^*)$, and a sequence of intervals $I_1 < I_2 < \dots$ with $\cup_i I_i = \mathbb{N}$. Then

$$\begin{aligned} |x^*(x)| &\leq \sum_{i=1}^{\infty} |I_i^{\mathbf{F}^*} x^*(I_i^{\mathbf{F}} x)| \leq \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}^*} x^*\|_{X^*} \|I_i^{\mathbf{F}} x\|_X \\ &= \left(\sum_{i=1}^{\infty} \|I_i^{\mathbf{F}^*} x^*\|_{X^*} e_{\max I_i}^* \right) \left(\sum_{i=1}^{\infty} \|I_i^{\mathbf{F}} x\|_X e_{\max I_i} \right) \\ &\leq \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}^*} x^*\|_{X^*} e_{\max I_i}^* \right\|_{E^{(*)}} \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}} x\|_X e_{\max I_i} \right\|_E \\ &\leq \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}^*} x^*\|_{X^*} e_{\max I_i}^* \right\|_{E^{(*)}} \|x\|_{X_\vee}. \end{aligned}$$

Taking the infimum over such sequences $(I_i)_{i=1}^{\infty}$ yields that $|x^*(x)| \leq [x^*]_{X_\lambda^*} \|x\|_{X_\vee}$ for any $x^* \in c_{00}(\mathbf{F}^*)$ and $x \in c_{00}(\mathbf{F})$. Now for any $x^* \in c_{00}(\mathbf{F}^*)$ and $x \in c_{00}(\mathbf{F})$,

$$|x^*(x)| \leq \inf \left\{ \sum_{i=1}^n |x_i^*(x)| : x^* = \sum_{i=1}^n x_i^* \right\} \leq \|x\|_{X_\vee} \inf \left\{ \sum_{i=1}^n [x_i^*]_{X_\lambda^*} : x^* = \sum_{i=1}^n x_i^* \right\} = \|x^*\|_{X_\lambda^*} \|x\|_{X_\vee}.$$

This yields that the formal identity from X_λ^* to $(X_\vee)^{(*)}$ is well-defined with norm 1. Restricting the adjoint of the formal identity to $c_{00}(\mathbf{F})$ yields that the formal identity from $X_\vee = (X_\vee)^{(*)^{(*)}}$ to $(X_\lambda^*)^{(*)}$ has norm 1.

Now fix $x \in c_{00}$ with $\|x\|_{X_\vee} > 1$. Fix $I_1 < I_2 < \dots$ such that $\|\sum_{i=1}^{\infty} \|I_i^{\mathbf{F}} x\|_X e_{\max I_i}\|_E > 1$. We may fix $(a_i)_{i=1}^{\infty} \in c_{00}$ such that $\|\sum_{i=1}^{\infty} a_i e_{\max I_i}^*\|_{E^{(*)}} = 1$ and $\sum_{i=1}^{\infty} a_i \|I_i^{\mathbf{F}} x\|_X > 1$. We may also fix $(x_i^*)_{i=1}^{\infty} \in S_{X^*}^{\mathbb{N}}$ such that $x_i^* = I_i^{\mathbf{F}^*} x_i^*$ and $x_i^*(x_i) = \|x_i\|_X$ for all $i \in \mathbb{N}$. Now let $x^* = \sum_{i=1}^{\infty} a_i x_i^* \in c_{00}(\mathbf{F}^*)$. Note that

$$\|x^*\|_{X_\lambda^*} \leq [x^*]_{X_\lambda^*} \leq \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{F}^*} x_i^*\|_{X^*} e_{\max I_i}^* \right\|_{E^{(*)}} = 1$$

and

$$x^*(x) = \sum_{i=1}^{\infty} a_i \|I_i^{\mathbf{F}} x\|_X > 1,$$

whence $\|x\|_{(X_\lambda^*)^{(*)}} > 1$. This yields that the formal identity from $(X_\lambda^*)^{(*)}$ to X_\vee has norm 1, and is therefore an isometric isomorphism by the last fact from the previous paragraph. Restricting the adjoint of the formal identity to $c_{00}(\mathbf{F}^*)$ yields that the formal identity from $X_\lambda^* = (X_\lambda^*)^{(*)^{(*)}}$ to $(X_\vee)^{(*)}$ has norm 1, and is therefore also an isometric isomorphism. \square

Remark 2.2. It follows from standard arguments that the closed unit ball of $X_\lambda^E(\mathbf{F})$ is the closed, convex hull of those $x \in c_{00}(\mathbf{F})$ such that $[x]_{X_\lambda^E(\mathbf{F})} \leq 1$. Furthermore, it follows from the fact that for any interval $I \subset \mathbb{N}$, $[I^{\mathbf{F}} \cdot]_{X_\lambda^E(\mathbf{F})} \leq [\cdot]_{X_\lambda^E(\mathbf{F})}$ that any $x \in [F_j : j \in I]$ lies in the closed convex hull of vectors $y \in [F_j : j \in I]$ such that $[y]_{X_\lambda^E(\mathbf{F})} \leq \|x\|_{X_\lambda^E(\mathbf{F})}$.

Lemma 2.3. *Let \mathbf{F} be a bimonotone FDD for the Banach space X and let E be a sequence space. With $K = B_{(X_\lambda^E(\mathbf{F}))^*}$, for any infinite subset M of \mathbb{N} ,*

$$\left\langle \cdot \right\rangle_{X_\lambda^E(\mathbf{F}), \mathbf{F}, K, M} \leq 2 \left\langle \cdot \right\rangle_{E, E, B_{E^*}, M}.$$

Furthermore, if E has property R , the inequality holds without the factor of 2.

Proof. Using Remark 2.2, it is sufficient to show that for any $(a_i)_{i=1}^\infty \in c_{00}$, any infinite subset M of \mathbb{N} , and any $(y_i)_{i=1}^\infty \in \prod_{i=1}^\infty [F_j : m_{i-1} < j \leq m_i]$ with $[y_i]_{X_\wedge} \leq 1$ for all $i \in \mathbb{N}$,

$$\left\| \sum_{i=1}^\infty a_i y_i \right\|_{X_\wedge} \leq 2 \left\langle \sum_{i=1}^\infty a_i e_i \right\rangle_{E, E, B_{E^*}, M},$$

and that if E has property R , the same estimate holds without the factor of 2.

First suppose that $0 \leq m < n$ and $0 \neq y \in [F_j : m < j \leq n]$ is such that $[y]_{X_\wedge} \leq 1$. Then there exists a sequence $(I_i)_{i=1}^\infty$ of intervals with $I_1 < I_2 < \dots$ and $\cup_{i=1}^\infty I_i = \mathbb{N}$ such that

$$\left\| \sum_{i=1}^\infty \|I_i y\|_{X e_{\max I_i}} \right\|_E \leq 1.$$

Let $k = \min\{i : I_i y \neq 0\}$, $l = \max\{i : I_i y \neq 0\}$, and let $J = (m, n]$. Note that for each $k \leq i < l$, $\max(J \cap I_i) = \max I_i$ and $\max(J \cap I_l) \leq \max I_l$. Also, for each $i \in \mathbb{N}$, $(J \cap I_i)y = I_i y$. Furthermore, by 1-unconditionality,

$$\left\| \sum_{i=1}^\infty \|I_i y\|_{X e_{\max I_i}} \right\|_E = \left\| \sum_{i=k}^l \|I_i y\|_{X e_{\max I_i}} \right\|_E \leq \left\| \sum_{i=k}^{l-1} \|(J \cap I_i)y\|_{X e_{\max(J \cap I_i)}} \right\| + \|(J \cap I_l)y\|_X \leq 2.$$

Now if E has property R , then

$$\begin{aligned} 1 &\geq \left\| \sum_{i=1}^\infty \|I_i y\|_{X e_{\max I_i}} \right\|_E = \left\| \sum_{i=k}^l \|I_i y\|_{X e_{\max I_i}} \right\|_E \\ &\geq \left\| \sum_{i=k}^{l-1} \|(J \cap I_i)y\|_{X e_{\max(J \cap I_i)}} \right\| + \|(J \cap I_l)y\|_{X e_{\max(J \cap I_l)}}. \end{aligned}$$

To summarize, if $y \in [F_j : m < j \leq n]$ has $[y]_{X_\wedge} \leq 1$, then there exist $p \in \mathbb{N}$, intervals $J_1 < \dots < J_p$ such that $\cup_{i=1}^p J_i = (m, n]$, and $v \in 2B_E \cap [e_j : m < j \leq n]$ such that $v = \sum_{i=1}^p \|J_i y\|_{X e_{\max J_i}}$, and if E has property R , the factor of 2 can be omitted.

Now suppose that $0 = m_0 < m_1 < \dots$, $y_i \in [F_j : m_{i-1} < j \leq m_i]$, and $[y_i]_{X_\wedge} \leq 1$ for all $i \in \mathbb{N}$. Applying the previous paragraph to each y_i and concatenating the resulting sequences of intervals yields the existence of some sequence $I_1 < I_2 < \dots$ with $\cup_{i=1}^\infty I_i = \mathbb{N}$, $(v_i)_{i=1}^\infty \subset 2B_E$, and $0 = k_0 < k_1 < \dots$ such that for each $n \in \mathbb{N}$, $(m_{n-1}, m_n] = \cup_{i=k_{n-1}+1}^{k_n} I_i$ and

$$v_n = \sum_{i=k_{n-1}+1}^{k_n} \|I_i y_n\|_{X e_{\max I_i}}.$$

Now for any $(a_i)_{i=1}^\infty \in c_{00}$,

$$\left\| \sum_{i=1}^\infty a_i y_i \right\|_{X_\wedge(\mathbb{F})} \leq \left\| \sum_{i=1}^\infty \|I_i^{\mathbb{F}} \sum_{j=1}^\infty a_j y_j\|_{X e_{\max I_i}} \right\|_E = \left\| \sum_{i=1}^\infty a_i v_i \right\|_E \leq 2 \left\langle \sum_{i=1}^\infty a_i e_i \right\rangle_{E, E, B_{E^*}, M}.$$

If E has property R , we can omit the factor of 2. □

3. COMBINATORICS

Throughout, we let $2^{\mathbb{N}}$ denote the power set of \mathbb{N} and topologize this set with the Cantor topology. Given a subset M of \mathbb{N} , we let $[M]$ (resp. $[M]^{<\mathbb{N}}$) denote set of infinite (resp. finite) subsets of M . For convenience, we often write subsets of \mathbb{N} as sequences, where a set E is identified with the (possibly empty) sequence obtained by listing the members of E in strictly increasing order. Henceforth, if we write $(m_i)_{i=1}^r \in [\mathbb{N}]^{<\mathbb{N}}$ (resp.

$(m_i)_{i=1}^\infty \in [\mathbb{N}]$, it will be assumed that $m_1 < \dots < m_r$ (resp. $m_1 < m_2 < \dots$). Given $M = (m_n)_{n=1}^\infty \in [\mathbb{N}]$ and $\mathcal{F} \subset [\mathbb{N}]^{<\mathbb{N}}$, we define

$$\mathcal{F}(M) = \{(m_n)_{n \in E} : E \in \mathcal{F}\}$$

and

$$\mathcal{F}(M^{-1}) = \{E : (m_n)_{n \in E} \in \mathcal{F}\}.$$

Given $(m_i)_{i=1}^r, (n_i)_{i=1}^r \in [\mathbb{N}]^{<\mathbb{N}}$, we say $(n_i)_{i=1}^r$ is a *spread* of $(m_i)_{i=1}^r$ if $m_i \leq n_i$ for each $1 \leq i \leq r$. We agree that \emptyset is a spread of \emptyset . We write $E \preceq F$ if either $E = \emptyset$ or $E = (m_i)_{i=1}^r$ and $F = (m_i)_{i=1}^s$ for some $r \leq s$. In this case, we say E is an *initial segment* of F . For $E, F \subset \mathbb{N}$, we write $E < F$ to mean that either $E = \emptyset, F = \emptyset$, or $\max E < \min F$. Given $n \in \mathbb{N}$ and $E \subset \mathbb{N}$, we write $n \leq E$ (resp. $n < E$) to mean that $n \leq \min E$ (resp. $n < \min E$).

We say $\mathcal{G} \subset [\mathbb{N}]^{<\mathbb{N}}$ is

- (i) *compact* if it is compact in the Cantor topology,
- (ii) *hereditary* if $E \subset F \in \mathcal{G}$ implies $E \in \mathcal{G}$,
- (iii) *spreading* if whenever $E \in \mathcal{G}$ and F is a spread of E , $F \in \mathcal{G}$,
- (iv) *regular* if it is compact, hereditary, and spreading.

Given a topological space K and a subset L of K , L' denotes the *Cantor Bendixson derivative* of L , which consists of those members of L which are not relatively isolated in L . We define by transfinite induction the higher order transfinite derivatives of L by

$$\begin{aligned} L^0 &= L, \\ L^{\xi+1} &= (L^\xi)', \end{aligned}$$

and if ξ is a limit ordinal,

$$L^\xi = \bigcap_{\zeta < \xi} L^\zeta.$$

We recall that K is said to be *scattered* if there exists an ordinal ξ such that $K^\xi = \emptyset$. In this case, we define the *Cantor Bendixson index* of K by $CB(K) = \min\{\xi : K^\xi = \emptyset\}$. If $K^\xi \neq \emptyset$ for all ordinals ξ , we write $CB(K) = \infty$. We agree to the convention that $\xi < \infty$ for all ordinals ξ , and therefore $CB(K) < \infty$ simply means that $CB(K)$ is an ordinal, and K is scattered.

Of course, if ξ is a limit ordinal, K is a compact topological space, and $K^\zeta \neq \emptyset$ for all $\zeta < \xi$, then $(K^\zeta)_{\zeta < \xi}$ is a collection of compact subsets of K with the finite intersection property, so $K^\xi = \bigcap_{\zeta < \xi} K^\zeta \neq \emptyset$. From this it follows that for a compact topological space, $CB(K)$ cannot be a limit ordinal.

We recall the following, which is well known. The proof is standard, so we omit it.

Fact 3.1. *Let $\mathcal{G} \subset [\mathbb{N}]^{<\mathbb{N}}$ be hereditary. The following are equivalent.*

- (i) *There does not exist $M \in [\mathbb{N}]$ such that $[M]^{<\mathbb{N}} \subset \mathcal{G}$.*
- (ii) *\mathcal{G} is compact.*
- (iii) *$CB(\mathcal{G}) < \infty$.*
- (iv) *$CB(\mathcal{G}) < \omega_1$.*

For each $n \in \mathbb{N}$, we let $\mathcal{A}_n = \{E \in [\mathbb{N}]^{<\mathbb{N}} : |E| \leq n\}$. It is clear that \mathcal{A}_n is regular. Also of importance are the Schreier families, $(\mathcal{S}_\xi)_{\xi < \omega_1}$. We recall these families. We let

$$\mathcal{S}_0 = \mathcal{A}_1,$$

$$\mathcal{S}_{\xi+1} = \{\emptyset\} \cup \left\{ \bigcup_{i=1}^n E_i : \emptyset \neq E_i \in \mathcal{S}_\xi, n \leq E_1 < \dots < E_n \right\},$$

and if $\xi < \omega_1$ is a limit ordinal, there exists a sequence $\xi_n \uparrow \xi$ such that

$$\mathcal{S}_\xi = \{E \in [\mathbb{N}]^{<\mathbb{N}} : \exists n \leq E \in \mathcal{S}_{\xi_n+1}\}.$$

We note that the sequence $(\xi_n)_{n=1}^\infty$ has the property that for any $n \in \mathbb{N}$, $\mathcal{S}_{\xi_{n+1}} \subset \mathcal{S}_{\xi_n}$. The existence of such families with the last indicated property is discussed, for example, in [6].

Given two non-empty regular families \mathcal{F}, \mathcal{G} , we let

$$\mathcal{F}[\mathcal{G}] = \{\emptyset\} \cup \left\{ \bigcup_{i=1}^n E_i : \emptyset \neq E_i \in \mathcal{G}, E_1 < \dots < E_n, (\min E_i)_{i=1}^n \in \mathcal{F} \right\}.$$

We let $\mathcal{F}[\mathcal{G}] = \emptyset$ if either $\mathcal{F} = \emptyset$ or $\mathcal{G} = \emptyset$.

Given a regular family \mathcal{G} , we let $MAX(\mathcal{G})$ denote the set of maximal members of \mathcal{G} with respect to inclusion (noting that this is also the set of maximal members of \mathcal{G} with respect to the initial segment ordering). We note that for each $\xi < \omega_1$ and any $\emptyset \neq E \in \mathcal{S}_\xi$, either $E \in MAX(\mathcal{S}_\xi)$ or $E \cup (1 + \max E) \in \mathcal{S}_\xi$. From this it follows that for any $M = (m_i)_{i=1}^\infty \in [\mathbb{N}]$, there exist unique $0 = k_0 < k_1 < \dots$ such that for each $i \in \mathbb{N}$, $(m_j)_{j=k_{i-1}+1}^{k_i} \in MAX(\mathcal{S}_\xi)$. We define $M_{\mathcal{S}_\xi} = (m_j)_{j=1}^{k_1}$, and $M_{\mathcal{S}_\xi, i} = (m_j)_{j=k_{i-1}+1}^{k_i}$.

The following facts are collected in [6].

- Proposition 3.2.** (i) For any non-empty regular families \mathcal{F}, \mathcal{G} , $\mathcal{F}[\mathcal{G}]$ is regular. Furthermore, if $CB(\mathcal{F}) = \beta + 1$ and $CB(\mathcal{G}) = \alpha + 1$, then $CB(\mathcal{F}[\mathcal{G}]) = \alpha\beta + 1$.
- (ii) For any $n \in \mathbb{N}$, $CB(\mathcal{A}_n) = n + 1$.
- (iii) For any $\xi < \omega_1$, $CB(\mathcal{S}_\xi) = \omega^\xi + 1$.
- (iv) If \mathcal{F} is regular and $M \in [\mathbb{N}]$, then $\mathcal{F}(M^{-1})$ is regular and $CB(\mathcal{F}) = CB(\mathcal{F}(M^{-1}))$.
- (v) For regular families \mathcal{F}, \mathcal{G} , there exists $M \in [\mathbb{N}]$ such that $\mathcal{F}(M) \subset \mathcal{G}$ if and only if there exists $M \in [\mathbb{N}]$ such that $\mathcal{F} \subset \mathcal{G}(M^{-1})$ if and only if $CB(\mathcal{F}) \leq CB(\mathcal{G})$.

For a probability measure \mathbb{P} on \mathbb{N} , we write $\mathbb{P}(n)$ to mean $\mathbb{P}(\{n\})$. Furthermore, we let $\text{supp}(\mathbb{P}) = \{n \in \mathbb{N} : \mathbb{P}(n) > 0\}$. We will recall the repeated averages hierarchy, introduced in [1]. For each countable ordinal ξ , we will define a collection $\mathfrak{S}_\xi = \{\mathbb{S}_{M,n}^\xi : M \in [\mathbb{N}], n \in \mathbb{N}\}$ of probability measures on \mathbb{N} . If $M = (m_i)_{i=1}^\infty$, we let $\mathbb{S}_{M,n}^0 = \delta_{m_n}$, the Dirac measure at m_n . If \mathfrak{S}_ξ has been defined and $M \in [\mathbb{N}]$, we let $M_1 = M$, $p_0 = s_0 = 0$, $p_1 = \min M_1$. Now assume that M_1, \dots, M_{n-1} , s_0, \dots, s_{n-1} , \dots, p_{n-1} , and $\mathbb{S}_{M,1}^{\xi+1}, \dots, \mathbb{S}_{M,n-1}^{\xi+1}$ have been defined such that $\mathbb{S}_{M,i}^\xi = p_i^{-1} \sum_{j=s_{i-1}+1}^{s_i} \mathbb{S}_{M,j}^\xi$ and $s_i = s_{i-1} + p_i$. Let $M_n = M \setminus \cup_{j=1}^{s_{n-1}} \text{supp}(\mathbb{S}_{M,j}^\xi)$, $p_n = \min M_n$, $s_n = s_{n-1} + p_n$, and $\mathbb{S}_{M,n}^{\xi+1} = \sum_{j=s_{n-1}+1}^{s_n} \mathbb{S}_{M,j}^\xi$. Now assume that ξ is a countable limit ordinal and \mathfrak{S}_ζ has been defined for each $\zeta < \xi$. Let $(\xi_n)_{n=1}^\infty$ be the sequence such that

$$\mathcal{S}_\xi = \{\emptyset\} \cup \{E \in [\mathbb{N}]^{<\mathbb{N}} : \emptyset \neq E \in \mathcal{S}_{\xi_{\min E+1}}\}.$$

Then let $M_1 = M$, $p_1 = \min M_1$, and $\mathbb{S}_{M,1}^\xi = \mathbb{S}_{M,1}^{\xi_{p_1+1}}$. Now assuming that M_1, \dots, M_{n-1} , p_1, \dots, p_{n-1} , and $\mathbb{S}_{M,1}^\xi, \dots, \mathbb{S}_{M,n-1}^\xi$ have been defined, let $M_n = M \setminus \cup_{i=1}^{p_{n-1}} \text{supp}(\mathbb{S}_{M,i}^\xi)$, $p_n = \min M_n$, and $\mathbb{S}_{M,n}^\xi = \mathbb{S}_{M,n}^{\xi_{p_n+1}}$.

We isolate the following properties of the collections \mathfrak{S}_ξ , shown in [1].

- Proposition 3.3.** (i) For each ordinal ξ , each $M \in [\mathbb{N}]$, and each $n \in \mathbb{N}$, $\text{supp}(\mathbb{S}_{M,n}^\xi) = M_{\mathcal{S}_\xi}$.
- (ii) If $M, N \in [\mathbb{N}]$ and $r_1 < \dots < r_k$ are such that $\cup_{i=1}^k \text{supp}(\mathbb{S}_{M,r_i}^\xi)$ is an initial segment of N , then $\mathbb{S}_{N,i}^\xi = \mathbb{S}_{M,r_i}^\xi$ for each $1 \leq i \leq k$.

The second property above is called the *permanence property*.

Let us recall the following result of Gasparis.

Theorem 3.4. [18] If $\mathcal{F}, \mathcal{G} \subset [\mathbb{N}]^{<\mathbb{N}}$ are hereditary, then for any $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that either

$$\mathcal{F} \cap [N]^{<\mathbb{N}} \subset \mathcal{G} \quad \text{or} \quad \mathcal{G} \cap [N]^{<\mathbb{N}} \subset \mathcal{F}.$$

In particular, if \mathcal{G} is regular and $CB(\mathcal{F}) < CB(\mathcal{G})$, then for any $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that $\mathcal{F} \cap [N]^{<\mathbb{N}} \subset \mathcal{G}$.

The first statement was proved directly in [18], while the second follows from the fact that for any regular \mathcal{G} and $N \in [\mathbb{N}]$, $CB(\mathcal{G} \cap [N]^{<\mathbb{N}}) = CB(\mathcal{G})$.

We also will need the following, shown in [13].

Proposition 3.5. (i) For any countable ordinal ξ , if \mathcal{H} is regular with $CB(\mathcal{H}) \leq \omega^\xi + 1$, and $q \in \mathbb{N}$, then for any $\varepsilon > 0$ and $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that

$$\sup\{\mathbb{S}_{P,1}^\xi(E) : E \in \mathcal{H}, \min E \leq q, P \in [N]\} \leq \varepsilon.$$

(ii) If $\xi < \omega_1$ and if \mathcal{H} is a regular family with $CB(\mathcal{H}) \leq \omega^\xi$, then for any $\varepsilon > 0$ and $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that

$$\sup\{\mathbb{S}_{P,1}^\xi(E) : E \in \mathcal{H}, P \in [N]\} \leq \varepsilon.$$

Given a regular family \mathcal{G} and $M \in [\mathbb{N}]$, let $\mathcal{G} \bowtie M = \{(i, F) : i \in F \in [M]^{<\mathbb{N}} \cap \text{MAX}(\mathcal{G})\}$. Given a function $f : \mathcal{G} \bowtie M \rightarrow \mathbb{R}$ and $N \in [M]$, we let

$$\|f\|_N = \sup\{|f(i, F)| : (i, F) \in \mathcal{G} \bowtie N\}.$$

The next result combines an argument of Schlumprecht ([25, Corollary 4.10] with [13, Lemma 3.10]).

Lemma 3.6. Fix a countable ordinal ξ and $\varepsilon \in \mathbb{R}$. Let $f : \mathcal{S}_\xi \times \mathcal{Q} \rightarrow \mathbb{R}$ be a bounded function. If there exists $L \in [\mathcal{Q}]$ such that

$$[L] \subset \left\{ M \in [\mathbb{N}] : \sum_{j \in M_{\mathcal{S}_\xi}} f(j, M_{\mathcal{S}_\xi}) \geq \varepsilon \right\},$$

then for any $M \in [L]$ and $\delta < \varepsilon$, there exists $P \in [M]$ such that for any $E \in \mathcal{S}_\xi$, there exists $F \in \text{MAX}(\mathcal{S}_\xi) \cap [M]^{<\mathbb{N}}$ such that $P(E) \subset F$ and for each $j \in P(E)$, $f(j, F) \geq \delta$.

We next recall a special case of the infinite Ramsey theorem, the proof of which was achieved in steps by Nash-Williams [22], Galvin and Prikry [17], Silver [26], and Ellentuck [15].

Theorem 3.7. If $\mathcal{V} \subset [\mathbb{N}]$ is closed, then for any $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that either

$$[N] \subset \mathcal{V} \quad \text{or} \quad [N] \cap \mathcal{V} = \emptyset.$$

4. SCHREIER, MIXED SCHREIER, AND BAERNSTEIN SPACES

Given $F \subset \mathbb{N}$, we let F denote the projection from c_{00} to itself given by $Fx = (1_F(i)e_i^*(x))_{i=1}^\infty$. Given a regular family \mathcal{G} containing all singletons, we let $X_{\mathcal{G}}$ be the completion of c_{00} with respect to the norm

$$\|x\|_{\mathcal{G}} = \max\{\|Fx\|_{\ell_1} : F \in \mathcal{G}\}.$$

These are the *Schreier spaces*. Given $1 < p \leq \infty$, we let $X_{\mathcal{G},p}$ denote the completion of c_{00} with respect to the norm

$$\|x\|_{\mathcal{G},p} = \sup\left\{ \left\| \sum_{i=1}^{\infty} \|F_i x\|_{\ell_1} e_i \right\|_{\ell_p} : F_1 < F_2 < \dots, F_i \in \mathcal{G} \right\}.$$

These are the *Baernstein spaces*. For convenience, if $\mathcal{G} = \mathcal{S}_\xi$, we write $\|\cdot\|_\xi$ in place of $\|\cdot\|_{\mathcal{S}_\xi}$ and we write $\|\cdot\|_{\xi,p}$ in place of $\|\cdot\|_{\mathcal{S}_\xi,p}$.

Given a sequence $\mathcal{G}_0, \mathcal{G}_1, \dots$ of regular families such that \mathcal{G}_0 contains all singletons and a sequence $1 = \vartheta_0 > \vartheta_1 > \dots$ with $\lim_n \vartheta_n = 0$, we let $X(\mathcal{G}_n, \vartheta_n)$ denote the completion of c_{00} with respect to the norm

$$\|x\|_{\mathcal{G}_n, \vartheta_n} = \sup\{\vartheta_n \|x\|_{\mathcal{G}_n} : n \in \mathbb{N} \cup \{0\}\}.$$

We will refer to these spaces as the *mixed Schreier spaces*. Note that the Schreier, Baernstein, and mixed Schreier spaces have properties *R* and *S*. Note also that the Schreier and Baernstein spaces satisfy property *T*.

Lemma 4.1. Fix $\xi < \omega_1$, $1 < p \leq \infty$, regular families $\mathcal{G}_0, \mathcal{G}_1, \dots$, and a null sequence $(\vartheta_n)_{n=0}^\infty$ such that $1 = \vartheta_0 > \vartheta_1 > \dots$. Let $1/p + 1/q = 1$.

(i) If $0 < \varepsilon \leq 1$ and $m \in \mathbb{N}$ are such that $1/m^{1/q} < \varepsilon$, then

$$CB(\{F \in [\mathbb{N}]^{<\mathbb{N}} : (\forall x \in \text{co}(e_i : i \in F))(\|x\|_{\xi,p} \geq \varepsilon)\}) \leq \omega^\xi m.$$

(ii) $CB(\{F \in [\mathbb{N}]^{<\mathbb{N}} : (\forall x \in \text{co}(e_i : i \in F))(\|x\|_{\xi,p} \geq 1/m^{1/q})\}) = \omega^\xi m + 1$.

(iii) If $0 < \xi$ and $CB(\mathcal{G}_n) \leq \omega^\xi$ for all $n \in \mathbb{N}$, then for any $0 < \varepsilon \leq 1$,

$$CB(\{F \in [\mathbb{N}]^{<\mathbb{N}} : (\forall x \in \text{co}(e_i : i \in F))(\|x\|_{\mathcal{G}_n, \vartheta_n} \geq \varepsilon)\}) < \omega^\xi.$$

Proof. (i) Fix $1 < p \leq \infty$ and for $0 < \varepsilon \leq 1$, let

$$\mathcal{B}_\varepsilon = \{E \in [\mathbb{N}]^{<\mathbb{N}} : (\forall x \in \text{co}(e_i : i \in E))(\|x\|_{\xi,p} \geq \varepsilon)\}.$$

It is clear that \mathcal{B}_ε is hereditary, and since $X_{\xi,p}$ has property R , \mathcal{B}_ε is spreading. Fix $0 < 1/m^{1/q} < \varepsilon \leq 1$ and suppose that $CB(\mathcal{B}_\varepsilon) \geq \omega^\xi m$. If $\xi = 0$, then $X_{\xi,p} = \ell_p$ (resp. c_0 if $p = \infty$). Then if $E \in \mathcal{B}_\varepsilon$ and $r = |E| > 0$, then

$$\varepsilon \leq \|r^{-1} \sum_{i \in E} e_i\|_{\ell_p} = 1/r^{1/q},$$

and $r < m$. This means $CB(\mathcal{B}_\varepsilon) < m + 1$, and $CB(\mathcal{B}_\varepsilon) \leq m$.

Now suppose $0 < \xi$. Then if $CB(\mathcal{B}_\varepsilon) \geq \omega^\xi m$, since $\omega^\xi m$ is a limit ordinal, $CB(\mathcal{B}_\varepsilon) \geq \omega^\xi m + 1$. This means there exists $M = (m_i)_{i=1}^\infty \in [\mathbb{N}]$ such that $\mathcal{A}_m[\mathcal{S}_\xi](M) \subset \mathcal{B}_\varepsilon$. Let $\mathcal{G} = \mathcal{S}_\xi(M^{-1})$ and note that $CB(\mathcal{G}) = \omega^\xi + 1$. Fix $\delta > 0$ such that $\varepsilon > 1/m^{1/q} + m\delta$. Recursively select $N_1 = \mathbb{N}$, $n_1 \in N_1$, $N_2 \in [N_1]$, $n_1 < n_2 \in N_2$, \dots , such that for each $k \in \mathbb{N}$,

$$\sup\{\mathbb{S}_{N,1}^\xi(E) : E \in \mathcal{G}, \min E \leq n_{k-1}, N \in [N_k]\} \leq \delta.$$

Now let $N = (n_i)_{i=1}^\infty$.

For each $i \in \mathbb{N}$, let $x_i = \sum_{j=1}^\infty \mathbb{S}_{N,i}^\xi(j)e_{m_j}$ and let $P_i = M \setminus \cup_{j=1}^{i-1} \text{supp}(\mathbb{S}_{N,j}^\xi)$. Suppose that for some $F \in \mathcal{S}_\xi$ and $i \in \mathbb{N}$, $Fx_i \neq 0$. Fix $i < j$. Let $\max \text{supp}(x_i) = m_{n_{s-1}}$ and let $G = (n_{s-1}) \cup \{i : m_i \in F \cap \text{supp}(x_j)\}$. Then $M(G)$ is a spread of a subset of F , so $G \in \mathcal{G}$ and $\min G \leq n_{s-1}$. Furthermore, since $P_s \in [N_s]$,

$$\|Fx_j\|_{\ell_1} \leq \mathbb{S}_{N,s}^\xi(G) = \mathbb{S}_{P_s,1}^\xi(G) \leq \delta.$$

Note that $\cup_{i=1}^m \text{supp}(x_i) \in \mathcal{A}_m[\mathcal{S}_\xi](M) \subset \mathcal{B}_\varepsilon$. From this it follows that with $x = m^{-1} \sum_{i=1}^m x_i$, $\|x\|_{\xi,p} \geq \varepsilon$, whence there exist $F_1 < \dots < F_r$, $F_j \in \mathcal{S}_\xi$ such that

$$\varepsilon \leq \left\| \sum_{j=1}^r \|F_j x\|_{\ell_1} e_j \right\|_{\ell_p}.$$

By omitting extraneous sets, we may assume that $F_j x \neq \emptyset$ for each $1 \leq j \leq r$. Let T_1, \dots, T_m be such that $j \in T_i$ if and only if $F_j x_l = \emptyset$ for each $l < i$ and $F_j x_i \neq \emptyset$. Note that for each $1 \leq i < m$ and $j \in T_i \setminus \{\max T_i\}$, $F_j x_l = 0$ for each $i < l \leq m$, and if $j = \max T_i$, $\|F_j x_l\|_{\ell_1} \leq \delta$ for each $i < l \leq m$. From this it follows that

$$\left\| \sum_{j=1}^r \|F_j x\|_{\ell_1} e_j \right\|_{\ell_p} \leq \left\| \sum_{i=1}^m \frac{1}{m} \left(\sum_{j \in T_i} \|F_j x_i\|_{\ell_1} + \delta m \right) e_j \right\|_{\ell_p} \leq \frac{1}{m} \left\| \sum_{i=1}^m \|x_i\|_{\ell_1} e_j \right\|_{\ell_p} + \delta m = 1/m^{1/q} + \delta m < \varepsilon.$$

(ii) If $\xi = 0$, then $\mathcal{B}_{1/m^{1/q}} = \mathcal{A}_m$. Now assume $0 < \xi$. It is easy to verify that $\mathcal{A}_m[\mathcal{S}_\xi] \subset \mathcal{B}_{1/m^{1/q}}$, whence $CB(\mathcal{B}_{1/m^{1/q}}) \geq CB(\mathcal{A}_m[\mathcal{S}_\xi]) = \omega^\xi m + 1$. Seeking a contradiction, assume $CB(\mathcal{B}_{1/m^{1/q}}) > \omega^\xi m + 1$. This means there exists n_0 such that $\mathcal{H} = \{E \in [\mathbb{N}]^{<\mathbb{N}} : (n_0) \cup E \in \mathcal{B}_{1/m^{1/q}}\}$ has $CB(\mathcal{H}) \geq \omega^\xi m + 1$. From this it follows that there exists $M \in [\mathbb{N}]$ such that $\mathcal{A}_m[\mathcal{S}_\xi](M) \subset \mathcal{H}$. Let $\mathcal{G} = \mathcal{H}(M^{-1})$. Arguing as above, we fix

$\delta > 0$ such that $1/(m+1)^{1/q} + \delta(m+1) < 1/m^{1/q}$. We then recursively select $N_1, n_0 < n_1 \in N_1, N_2 \in [N_1], n_1 < n_2 \in N_2, \dots$ such that for each $k \in \mathbb{N}$,

$$\sup\{\mathbb{S}_{P,1}^\xi(E) : E \in \mathcal{H}, \min E \leq n_{k-1}, E \in \mathcal{G}, P \in [N_k]\} \leq \delta.$$

Let $N = (n_i)_{i=1}^\infty$. We argue as in (i) to deduce that

$$\frac{1}{m^{1/q}} \leq \left\| \frac{1}{m+1} (e_{n_0} + \sum_{i=1}^m \sum_{j=1}^\infty \mathbb{S}_{N,i}^\xi(j) e_{m_j}) \right\|_{\xi,p} \leq \frac{1}{(m+1)^{1/q}} + \delta(m+1),$$

a contradiction.

(iii) Let $X = X(\mathcal{G}_n, \vartheta_n)$ and for $0 < \varepsilon \leq 1$, let $\mathcal{B}_\varepsilon = \{E \in [\mathbb{N}]^{<\mathbb{N}} : (\forall x \in \text{co}(e_i : i \in E)(\|x\|_X \geq \varepsilon))\}$. Note that, since X has property R , \mathcal{B} is spreading and hereditary. If $CB(\mathcal{B}_\varepsilon) \geq \omega^\xi$, then since ω^ξ is a limit ordinal, $CB(\mathcal{B}_\varepsilon) > \omega^\xi$. Then there exists $M \in [\mathbb{N}]$ such that $\mathcal{S}_\xi(M) \subset \mathcal{B}_\varepsilon$. Fix $n_1 \in \mathbb{N}$ such that $\vartheta_{n_1} < \varepsilon$ and $N \in [\mathbb{N}]$ such that

$$\sup\{\mathbb{S}_{N,1}^\xi(E) : E \in \cup_{i=1}^{n_1} \mathcal{G}_i(M^{-1})\} < \varepsilon/2.$$

We may do this, since

$$CB(\cup_{i=1}^{n_1} \mathcal{G}_i(M^{-1})) = \max_{1 \leq i \leq n_1} CB(\mathcal{G}_i) < \omega^\xi.$$

Then let $x = \sum_{j=1}^\infty \mathbb{S}_{N,1}^\xi(j) e_{m_j}$ and note that, since $\text{supp}(x) \in \mathcal{B}_\varepsilon$, $\|x\|_X \geq \varepsilon$. However, if $F \in \cup_{i=1}^{n_1} \mathcal{G}_i$ and $G = \{i : m_i \in F \cap \text{supp}(x)\}$,

$$\|Fx\|_{\ell_1} \leq \mathbb{S}_{N,1}^\xi(G) \leq \varepsilon/2.$$

Thus

$$\varepsilon \leq \|x\|_X \leq \max\{\varepsilon/2, \sup_{n \geq n_1} \vartheta_n \|x\|_{\ell_1}\} \leq \max\{\varepsilon/2, \vartheta_{n_1}\} < \varepsilon,$$

a contradiction. □

Fix $\xi \in (0, \omega_1) \setminus \{\omega^\eta : \eta \text{ a limit ordinal}\}$. If $\xi = \omega^{\zeta+1}$, let us say that the mixed Schreier space $X(\mathcal{G}_n, \vartheta_n)$ is ξ -well-constructed provided that there exist $0 < \vartheta < 1$ and a regular family \mathcal{G} with $\omega^{\omega^\zeta} < CB(\mathcal{G}) < \omega^{\omega^{\zeta+1}}$ such that

$$\begin{aligned} \mathcal{G}_0 &= \mathcal{S}_0, \\ \mathcal{G}_n &= \mathcal{G}[\mathcal{G}_{n-1}] \end{aligned}$$

for $n \in \mathbb{N}$, and $\vartheta_n = \vartheta^n$ for all $n \in \mathbb{N} \cup \{0\}$. Note that such a sequence exists. Indeed, we may take $\mathcal{G}_0 = \mathcal{S}_\beta$ for some $\omega^\zeta < \beta < \omega^{\zeta+1}$ and then $\mathcal{G}_n = \mathcal{S}_\beta[\mathcal{G}_{n-1}]$.

If $\xi = 1$, let us say that the mixed Schreier space $X(\mathcal{G}_n, \vartheta_n)$ is ξ -well-constructed provided that there exist $0 < \vartheta < 1$ and a regular family \mathcal{G} with $1 < CB(\mathcal{G}) < \omega$ such that

$$\begin{aligned} \mathcal{G}_0 &= \mathcal{S}_0, \\ \mathcal{G}_n &= \mathcal{G}[\mathcal{G}_{n-1}] \end{aligned}$$

for $n \in \mathbb{N}$, and $\vartheta_n = \vartheta^n$ for all $n \in \mathbb{N} \cup \{0\}$. Note that such a sequence $\mathcal{G}_0, \mathcal{G}_1, \dots$ exists. Indeed, we may fix $l \in \mathbb{N}$ and take $\mathcal{G}_n = \mathcal{A}_{l^n}$ for all $n \in \mathbb{N} \cup \{0\}$.

Now assume that $\xi \in (0, \omega_1) \setminus \{\omega^\eta : \eta < \omega_1\}$. Let us say $X(\mathcal{G}_n, \vartheta_n)$ is ξ -well-constructed provided that there exist some ordinals $\beta, \gamma < \xi$ such that $\beta + \gamma = \xi$, $CB(\mathcal{G}_0) = \omega^\beta + 1$ and there exist regular families $\mathcal{F}_1, \mathcal{F}_2, \dots$ such that $\mathcal{G}_n = \mathcal{G}_n[\mathcal{G}_0]$ and $CB(\mathcal{F}_n) \uparrow \omega^\gamma$. Note that there is no requirement that $(\vartheta_n)_{n=0}^\infty$ be a geometric sequence in this case. Note that such β, γ and such a sequence of $\mathcal{G}_0, \mathcal{G}_1, \dots$ exists. Indeed, by basic facts about ordinals, if $\xi \in (0, \omega_1) \setminus \{\omega^\eta : \eta < \omega_1\}$, there exist $\beta, \gamma < \xi$ with $\beta + \gamma = \xi$. If $\gamma = \zeta + 1$, let $\mathcal{G}_0 = \mathcal{S}_\beta, m_1 < m_2 < \dots$ be natural numbers, and $\mathcal{F}_n = \mathcal{A}_{m_n}[\mathcal{S}_\zeta]$. If γ is a limit ordinal, let $\mathcal{G}_0 = \mathcal{S}_\beta, \gamma_n \uparrow \gamma$, and $\mathcal{F}_n = \mathcal{S}_{\gamma_n}$.

5. SZLENK INDEX

Given a Banach space X , a weak*-compact subset K of X^* , and $\varepsilon > 0$, we let $s_\varepsilon(K)$ denote the set of those $x^* \in K$ such that for any weak*-neighborhood v of x^* , $\text{diam}(v \cap K) > \varepsilon$. We let $s_\varepsilon(K) = K$ for any $\varepsilon \leq 0$. We then define the transfinite derivations by

$$\begin{aligned} s_\varepsilon^0(K) &= K, \\ s_\varepsilon^{\xi+1}(K) &= s_\varepsilon(s_\varepsilon^\xi(K)), \end{aligned}$$

and if ξ is a limit ordinal, let

$$s_\varepsilon^\xi(K) = \bigcap_{\zeta < \xi} s_\varepsilon^\zeta(K).$$

If there exists an ordinal ξ such that $s_\varepsilon^\xi(K) = \emptyset$, we let $Sz(K, \varepsilon)$ be the minimum such ordinal, and otherwise we write $Sz(K, \varepsilon) = \infty$. We agree to the convention that $Sz(K, \varepsilon) < \infty$ means there exists an ordinal ξ such that $s_\varepsilon^\xi(K) = \emptyset$. If $Sz(K, \varepsilon) < \infty$ for all $\varepsilon > 0$, then we let $Sz(K) = \sup_{\varepsilon > 0} Sz(K, \varepsilon)$, and otherwise we write $Sz(K) = \infty$. If $A : X \rightarrow Y$ is an operator, we write $Sz(A, \varepsilon)$ and $Sz(A)$ in place of $Sz(A^*B_{Y^*}, \varepsilon)$ and $Sz(A^*B_{Y^*})$, respectively. If X is a Banach space, we write $Sz(X, \varepsilon)$ and $Sz(X)$ in place of $Sz(I_X, \varepsilon)$ and $Sz(I_X)$.

If $K \subset X^*$ is weak*-compact and $Sz(K) < \infty$, then for any $\varepsilon > 0$, there exists a minimum ordinal ζ such that $s_\varepsilon^{\omega^\zeta}(K) = \emptyset$. We let $Sz_\zeta(K, \varepsilon)$ be this minimum ordinal. If $Sz(K, \varepsilon) = \infty$, then $Sz_\zeta(K, \varepsilon) = \infty$. If $Sz(K) \leq \omega^{\xi+1}$, then $Sz_\zeta(K, \varepsilon) < \omega$ for each $\varepsilon > 0$. We then define

$$p_\zeta(K) = \limsup_{\varepsilon \rightarrow 0^+} \frac{\log Sz_\zeta(K, \varepsilon)}{|\log(\varepsilon)|},$$

noting that this value need not be finite. If $Sz(K) = \infty$ or $Sz(K) > \omega^{\xi+1}$, we let $p_\zeta(K) = \infty$.

The following is a generalization of a result from [12].

Lemma 5.1. *Fix $\xi \in (0, \omega_1) \setminus \{\omega^\eta : \eta \text{ a limit ordinal}\}$ and let $X = X(\mathcal{G}_n, \vartheta_n)$ be a ξ -well-constructed mixed Schreier space.*

- (i) *If $\vartheta_n < \varepsilon$, $Sz(X, \varepsilon) \geq CB(\mathcal{G}_n)$.*
- (ii) *$Sz(X) = \omega^\xi$.*

Proof. (i) It is straightforward to see that $\vartheta_n \sum_{i \in F} e_i^* \in B_{X^*}$ for any $F \in \mathcal{G}_n$, and if $F, G \in \mathcal{G}_n$ are distinct, $\|\vartheta_n \sum_{i \in F} e_i^* - \vartheta_n \sum_{i \in G} e_i^*\| \geq \vartheta_n$. Furthermore, if $(F_j)_{j=1}^\infty \subset \mathcal{G}_n$ and $F_j \rightarrow F$ in the Cantor topology, then $\vartheta_n \sum_{i \in F_j} e_i^* \xrightarrow{\text{weak}^*} \vartheta_n \sum_{i \in F} e_i^*$. From this and an easy induction argument it follows that for every ordinal η , $\{\vartheta_n \sum_{i \in F} e_i^* : F \in \mathcal{G}_n^\eta\} \subset s_\varepsilon^\eta(B_{X^*})$. In particular, if $CB(\mathcal{G}_n) = \zeta_n + 1$, then $0 = \vartheta_n \sum_{i \in \emptyset} e_i^* \in s_\varepsilon^{\zeta_n}(B_{X^*})$ and $Sz(X, \varepsilon) \geq \zeta_n + 1 = CB(\mathcal{G}_n)$.

(ii) Part (i) yields that $Sz(B_{X^*}) \geq \omega^\xi$. We focus on the reverse estimate. Let $K_n = \{\vartheta_n \sum_{i \in F} e_i^* : F \in \mathcal{G}_n\}$ and let $K = \bigcup_{n=0}^\infty K_n$. Note that there exists $r > 0$ such that $rB_{X^*} \subset \overline{\text{abs co}}^{\text{weak}^*}(K)$ (we may take $r = 1/2$ if $\mathbb{K} = \mathbb{R}$ and $r = 1/2\sqrt{2}$ if $\mathbb{K} = \mathbb{C}$). From this it follows that $Sz(X) = Sz(B_{X^*}) = Sz(rB_{X^*}) \leq Sz(\overline{\text{abs co}}^{\text{weak}^*}(K))$. By the main theorem of [9], $Sz(\overline{\text{abs co}}^{\text{weak}^*}(K)) \leq \omega^\xi$ if $Sz(K) \leq \omega^\xi$, whence it is sufficient to prove that $Sz(K) \leq \omega^\xi$. Note also that K and K_n are weak*-compact. For any ordinal η and any $\varepsilon > 0$,

$$s_\varepsilon^\eta(K) \subset \{0\} \cup \bigcup_{n=0}^\infty s_\varepsilon^\eta(K_n).$$

Thus it suffices to show that for any $\varepsilon > 0$, $\sup_n Sz(K_n, \varepsilon) < \omega^\xi$.

We first note that for any $n \in \mathbb{N} \cup \{0\}$, any $\varepsilon > 0$, and any ordinal η ,

$$s_\varepsilon^\eta(K_n) \subset \{\vartheta_n \sum_{i \in F} e_i^* : F \in \mathcal{G}_n^\eta\},$$

whence we obtain the estimate $Sz(K_n, \varepsilon) \leq CB(\mathcal{G}_n)$. We now argue that if $n, m \in \mathbb{N}$ are such that $\vartheta_m < 2\varepsilon$ and $m < n$,

- (i) if $\xi = \omega^{\zeta+1}$ or $\xi = 1$, $Sz(K_n, \varepsilon) \leq CB(\mathcal{G}_m) = (CB(\mathcal{G}) - 1)^m + 1$,
- (ii) if $\xi = \beta + \gamma$ for β, γ as in the definition of ξ -well-constructed, $Sz(K_n, \varepsilon) \leq \omega^\gamma + 1$.

Then for any $\varepsilon > 0$, if $m \in \mathbb{N}$ is such that $\vartheta_m < 2\varepsilon$, we obtain the estimate

$$Sz(K_n, \varepsilon) \leq CB(\mathcal{G}_m) < \omega^\xi$$

in the case $\xi = \omega^{\zeta+1}$ or $\xi = 1$, and

$$Sz(K_n, \varepsilon) \leq \max\left\{\max_{0 \leq n \leq m} CB(\mathcal{G}_m), \omega^\gamma + 1\right\} < \omega^\xi$$

in the remaining case. These estimates will finish the proof.

We will use the following fact: If $\mathcal{A}[\mathcal{B}]$ are regular families and $E \prec F \in \mathcal{A}[\mathcal{B}]$, then either $E \in \mathcal{A}'[\mathcal{B}]$ or $F \setminus E \in \mathcal{B}$. Write $F = \cup_{i=1}^k F_i$, $F_1 < \dots < F_k$, $\emptyset \neq F_i \in \mathcal{B}$, $(\min F_i)_{i=1}^l \in \mathcal{A}$. Then either $F \setminus E \subset F_k$ and therefore $F \setminus E$ lies in \mathcal{B} by heredity, or there exists $0 \leq l < k$ such that $E = \cup_{i=1}^l (E \cap F_i)$ and $E \cap F_i \neq \emptyset$ for each $1 \leq i \leq l$. In the second case, since $(\min(E \cap F_i))_{i=1}^l = (\min F_i)_{i=1}^l \prec (\min F_i)_{i=1}^k$, $E = \cup_{i=1}^l E \cap F_i$ witnesses the fact that $E \in \mathcal{A}'[\mathcal{B}]$.

Now in either of the cases $\xi = \omega^{\zeta+1}$ or $\xi = 1$, we claim that for any ordinal η ,

$$s_\varepsilon^\eta(K_n) \subset \{\vartheta_n \sum_{i \in F} e_i^* : F \in \mathcal{G}_m^\eta[\mathcal{G}_{n-m}]\},$$

which will give the result by taking $\eta = CB(\mathcal{G}_m)$. In case (ii), we claim that

$$s_\varepsilon^\eta(K_n) \subset \{\vartheta_n \sum_{i \in F} e_i^* : F \in \mathcal{F}_n^\eta[\mathcal{G}_0]\},$$

which will give the desired conclusion taking $\eta = \omega^\gamma + 1 > CB(\mathcal{F}_n)$. We prove these results by induction on η , with the $\eta = 0$ case being equality (noting that $\mathcal{G}_n = \mathcal{G}_m[\mathcal{G}_{n-m}]$) and the limit ordinal case being obvious. Assume the result holds for some η . In case (i), let $\mathcal{A} = \mathcal{G}_m^\eta$, $\mathcal{B} = \mathcal{G}_{n-m}$. In case (ii), let $\mathcal{A} = \mathcal{F}_n^\eta$ and $\mathcal{B} = \mathcal{G}_0$. We must show that

$$s_\varepsilon(\{\vartheta_n \sum_{i \in F} e_i^* : F \in \mathcal{A}[\mathcal{B}]\}) \subset \{\vartheta_n \sum_{i \in F} e_i^* : F \in \mathcal{A}'[\mathcal{B}]\},$$

which will complete the induction and the proof. Now if $\vartheta_n \sum_{i \in E} e_i^* \in s_\varepsilon(\{\vartheta_n \sum_{i \in F} e_i^* : F \in \mathcal{A}[\mathcal{B}]\})$, there exists $F \in \mathcal{A}[\mathcal{B}]$ with $E \prec F$ such that

$$\varepsilon/2 < \|\vartheta_n \sum_{i \in F \setminus E} e_i^*\|.$$

It suffices to show that $E \in \mathcal{A}'[\mathcal{B}]$. If $E \notin \mathcal{A}'[\mathcal{B}]$, then $F \setminus E \in \mathcal{B}$. In case (i), $\vartheta^{n-m} \sum_{i \in F \setminus E} e_i^* \in K_{n-m} \subset B_{X^*}$, whence

$$\varepsilon/2 < \vartheta^m \|\vartheta^{n-m} \sum_{i \in F \setminus E} e_i^*\| \leq \vartheta^m,$$

a contradiction. In case (ii), $\sum_{i \in F \setminus E} e_i^* \in K_0 \subset B_{X^*}$, whence

$$\varepsilon/2 < \vartheta_n \|\sum_{i \in F \setminus E} e_i^*\| \leq \vartheta_n < \vartheta_m,$$

a contradiction. □

Proposition 5.2. *Suppose X is a Banach space, Z is a subspace of X with $\dim X/Z < \infty$, $K \subset X^*$ is weak*-compact, and $x^* \in s_\varepsilon(K)$. Then for any $0 < \delta < \varepsilon/4$ and any weak*-neighborhood v of x^* , there exist $y^* \in K$ and $z \in B_Z$ such that $\operatorname{Re} y^*(z) > \delta$.*

Proof. Fix $R > 1$ such that $K \subset RB_{X^*}$. We may fix a net $(x_\lambda^*) \subset K$ converging weak* to x^* and such that $\|x_\lambda^* - x^*\| > \varepsilon/2$ for all λ . For each λ , we may fix $x_\lambda \in B_X$ such that $\operatorname{Re}(x_\lambda^* - x^*)(x_\lambda)$. Fix $\eta > 0$ such that $\delta + 3R\eta < \varepsilon/4$. After passing to a subnet, we may assume that $|x^*(x_{\lambda_2} - x_{\lambda_1})| < \eta$ and $\|x_{\lambda_2} - x_{\lambda_1}\|_{X/Z} < \eta$ for all λ_1, λ_2 . Now fix any λ_1 and then choose λ_2 such that $|(x_{\lambda_2}^* - x^*)(x_{\lambda_1})| < \eta$ and such that $x_{\lambda_2}^* \in v$. We may now fix $z \in B_Z$ such that $\|\frac{x_{\lambda_2} - x_{\lambda_1}}{2} - z\| < 2\eta$ and let $y^* = x_{\lambda_2}^* \in K$. Now note that

$$\begin{aligned} \operatorname{Re} y^*(z) &> \operatorname{Re} x_{\lambda_2}^* \left(\frac{x_{\lambda_2} - x_{\lambda_1}}{2} \right) - R\eta \\ &> \operatorname{Re} (x_{\lambda_2}^* - x^*) \left(\frac{x_{\lambda_2} - x_{\lambda_1}}{2} \right) - 2R\eta \\ &> \operatorname{Re} (x_{\lambda_2}^* - x^*)(x_{\lambda_2}/2) - 3R\eta > \varepsilon/4 - 3R\eta > \delta. \end{aligned}$$

□

The following can be compared to Proposition 5 of [23].

Corollary 5.3. *Suppose \mathcal{G} is a regular family with $CB(\mathcal{G}) = \xi + 1$. If F is any FMD for X , $K \subset X^*$ is weak*-compact, and $x^* \in s_\varepsilon^\xi(K)$, then for any $0 < \delta < \varepsilon/4$, there exist a collection $(x_t)_{t \in \mathcal{G} \setminus \max(\mathcal{G})} \subset B_X$ and a collection $(x_t^*)_{t \in \mathcal{G}} \subset K$ such that $x_\emptyset^* = x^*$, $x_E \in \operatorname{span}\{F_j : j > \max E\}$, and if $\emptyset \prec E \preceq F \in \mathcal{G}$, then $\operatorname{Re} x_F^*(x_E) > \delta$.*

Proof. Define $\mu : \mathcal{G} \rightarrow [0, \xi]$ by letting $\mu(E) = \max\{\zeta : E \in \mathcal{G}^\zeta\}$. We will define $(x_E)_{E \in \mathcal{G} \setminus \max(\mathcal{G})}$ and $(x_E^*)_{E \in \mathcal{G}}$ recursively to have each of the properties mentioned in the corollary, and to have the property that for each $E \in \mathcal{G}$, $x_E^* \in s_\varepsilon^{\mu(E)}(K)$. We let $x_\emptyset^* = x^*$.

Now suppose that $\emptyset \prec E \in \mathcal{G}$, $x_{E^-}^* \in s_\varepsilon^{\mu(E^-)}(K)$, and $x_G \in \operatorname{span}(F_j : j \in \mathbb{N})$ have been defined for each $\emptyset \prec G \prec E$. If $E^- = \emptyset$, let $v = X^*$, and otherwise let $v = \{y^* \in Y^* : (\forall \emptyset \prec G \prec E)(\operatorname{Re} y^*(x_G) > \delta)\}$, which is a weak*-neighborhood of $x_{E^-}^*$. Let $p = \max E$ and $Z = [F_j : j > p]$. Note that $\mu(E) < \mu(E^-)$. Since $x_{E^-}^* \in s_\varepsilon^{\mu(E^-)}(K) \subset s_\varepsilon^{\mu(E)+1}(K) = s_\varepsilon(s_\varepsilon^{\mu(E)}(K))$, Proposition 5.2 yields the existence of $z \in B_Z$ and $x_E^* \in s_\varepsilon^{\mu(E)}(K) \cap v$ such that $\operatorname{Re} x_E^*(z) > \delta$. By density of $\operatorname{span}\{F_j : j > p\}$ in $[F_j : j > p]$, we may fix $x_E \in B_X \cap \operatorname{span}\{F_j : j > p\}$ such that $\operatorname{Re} x_E^*(x_E) > \delta$. This completes the recursive construction, and the collections $(x_E^*)_{E \in \mathcal{G}}$, $(x_E)_{E \in \mathcal{G} \setminus \max(\mathcal{G})}$ are easily seen to satisfy the conclusions. □

Now for a Banach space X , an FMD F of X , $K \subset X^*$ weak*-compact, and $\varepsilon > 0$, let $\mathcal{H}(X, F, K, \varepsilon) = \emptyset$ if $K = \emptyset$, and otherwise let $\mathcal{H}(X, F, K, \varepsilon)$ denote the collection consisting of \emptyset together with all $(k_i)_{i=1}^n \in [\mathbb{N}]^{< \mathbb{N}}$ such that (with $k_0 = 0$), there exist $x^* \in K$ and $(u_i)_{i=1}^n \in B_X \cap \prod_{i=1}^n [F_j : k_{i-1} < j \leq k_i]$ such that $|x^*(u_i)| \geq \varepsilon$ for all $1 \leq i \leq n$ (equivalently, such that $\operatorname{Re} x^*(u_i) \geq \varepsilon$ for all $1 \leq i \leq n$).

Lemma 5.4. *For any Banach space X , any weak*-compact subset K of X^* , any K -shrinking FMD F of X , $0 < \delta < \varepsilon$ and any ordinal ξ ,*

$$\mathcal{H}(X, F, K, \varepsilon)^{2^\xi} \subset \mathcal{H}(X, F, s_\delta^\xi(K), \varepsilon).$$

In particular, if $\mathcal{H}(X, F, K, \varepsilon)^{2^\xi} \neq \emptyset$, then $s_\delta^\xi(K) \neq \emptyset$.

Proof. In the proof, we will repeatedly use the fact that for a weak*-compact subset L of X^* , $\mathcal{H}(X, F, L, \varepsilon) \neq \emptyset$ if and only if $L \neq \emptyset$ if and only if $\emptyset \in \mathcal{H}(X, F, L, \varepsilon)$.

We induct on ξ . The $\xi = 0$ case is trivial.

Assume ξ is a limit ordinal and the result holds for all $\zeta < \xi$. Note that by the properties of ordinals, $2^\xi = \xi$ and $2^\zeta < \xi$ for every $\zeta < \xi$. Suppose that for some $n \in \mathbb{N} \cup \{0\}$,

$$(k_i)_{i=1}^n \in \mathcal{H}(X, F, K, \varepsilon)^{2^\xi} = \mathcal{H}(X, F, K, \varepsilon)^\xi = \bigcap_{\zeta < \xi} \mathcal{H}(X, F, K, \varepsilon)^{2^\zeta}.$$

Here, if $n = 0$, $(k_i)_{i=1}^0$ denotes the empty sequence by convention.

If $n > 0$, then for every $\zeta < \xi$, we may fix $x_\zeta^* \in s_\delta^\zeta(K)$ and $(u_i^\zeta)_{i=1}^n \in B_X \cap \prod_{i=1}^n [F_j : k_{i-1} < j \leq k_i] =: C$ such that for each $1 \leq i \leq n$, $\operatorname{Re} x_\zeta^*(u_i^\zeta) \geq \varepsilon$. If C is endowed with the product of the norm topology and K is endowed with its weak*-topology, by compactness of $C \times K$, we may fix

$$(u_1, \dots, u_n, x^*) \in \bigcap_{\eta < \xi} \overline{\{(u_1^\eta, \dots, u_n^\eta, x_\eta^*) : \eta < \zeta < \xi\}}.$$

Obviously $x^* \in s_\delta^\xi(K)$ and $\operatorname{Re} x^*(u_i) \geq \varepsilon$ for each $1 \leq i \leq n$, witnessing that $(k_i)_{i=1}^n \in \mathcal{H}(X, F, s_\delta^\xi(K), \varepsilon)$. If $n = 0$, we omit reference to u_i^ζ , u_i , and C in the previous argument and use the fact at the beginning of the proof to deduce that $(k_i)_{i=1}^0 = \emptyset \in \mathcal{H}(X, F, s_\delta^\xi(K), \varepsilon)$.

Assume the result holds for ξ and $(k_i)_{i=1}^n \in \mathcal{H}(X, F, K, \varepsilon)^{2(\xi+1)} = \mathcal{H}(X, F, K, \varepsilon)^{2\xi+2}$. First suppose $n > 0$. Then there exists a sequence $l_1 < m_1 < l_2 < m_2 < \dots$ such that for all $t \in \mathbb{N}$, $(k_i)_{i=1}^n \wedge (l_t, m_t) \in \mathcal{H}(X, F, K, \varepsilon)^{2\xi}$. By the inductive hypothesis, for each $t \in \mathbb{N}$, there exist $x_t^* \in s_\delta^\xi(K) \subset K$, $(u_i^t)_{i=1}^n \in B_X \cap \prod_{i=1}^n [F_j : k_{i-1} < j \leq k_i] =: C$, and $v_t \in B_X \cap [E_j : l_t < j \leq m_t]$ such that $\operatorname{Re} x_t^*(v_t) \geq \varepsilon$ and for each $1 \leq i \leq n$, $\operatorname{Re} x_t^*(u_i^t) \geq \varepsilon$. We may pass to a subsequence and use the sequential compactness of C with the product of its norm topology and K with its weak*-topology to assume $u_i^t \rightarrow u_i$ and $x_t^* \xrightarrow{\text{weak}^*} x^*$. Obviously $\operatorname{Re} x^*(u_i) \geq \varepsilon$ for all $1 \leq i \leq n$. Since F is K -shrinking and $v_t \in B_X \cap [E_j : l_t < j \leq m_t]$,

$$\liminf_t \|x_t^* - x^*\| \geq \liminf_t \operatorname{Re} (x_t^* - x^*)(v_t) \geq \varepsilon > \delta.$$

Since $(x_t^*)_{t=1}^\infty \subset s_\delta^\xi(K)$, $x^* \in s_\delta^{\xi+1}(K)$. This yields that $(k_i)_{i=1}^n \in \mathcal{H}(X, F, s_\delta^{\xi+1}(K), \varepsilon)$. If $n = 0$, we omit reference to u_i^t and u_i in the previous argument and use the remark at the beginning of the proof to deduce that $(k_i)_{i=1}^0 = \emptyset \in \mathcal{H}(K, F, s_\delta^{\xi+1}, \varepsilon)$. □

Corollary 5.5. *If X is a Banach space, $K \subset X^*$ is weak*-compact, F is a K -shrinking FDD for X , then for any $\varepsilon > 0$,*

$$Sz(K, 5\varepsilon) \leq CB(\mathcal{H}(X, F, K, \varepsilon)) \leq 2Sz(K, \varepsilon/2).$$

In particular, if K is convex and not norm compact,

$$Sz(K) = \sup_{\varepsilon > 0} CB(\mathcal{H}(X, F, K, \varepsilon)).$$

Proof. The proof of the first part follows from Corollary 5.3 and Lemma 5.4. The second part follows from the fact that if K is convex and not norm compact, either $Sz(K) = \infty = \sup_{\varepsilon > 0} CB(\mathcal{H}(X, F, K, \varepsilon))$, and otherwise $Sz(K) = \omega^\xi$ for some $0 < \xi < \omega_1$. In this case, for each $\varepsilon > 0$, $2Sz(K, \varepsilon/2) < \omega^\xi$, so

$$\omega^\xi = \sup_{\varepsilon > 0} Sz(K, 5\varepsilon) \leq \sup_{\varepsilon > 0} CB(\mathcal{H}(X, F, K, \varepsilon)) \leq \sup_{\varepsilon > 0} 2Sz(K, \varepsilon/2) = \omega^\xi.$$

□

We next prove a generalization of a result of Schlumprecht, which was shown in the case $K = B_{X^*}$.

Lemma 5.6. *Suppose X is a Banach space X , $K \subset X^*$ is weak*-compact, F is a K -shrinking FMD for X , and $0 < \xi < \omega_1$. Then $Sz(K) \leq \omega^\xi$ if and only if for any $\varepsilon > 0$ and any $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that*

$$\sup \left\{ \left\langle \sum_{j=1}^{\infty} \mathbb{S}_{N,1}^\xi(n_j) e_j \right\rangle_{X, F, K, N} : N \in [M] \right\} \leq \varepsilon.$$

Proof. Throughout the proof, for ease of notation, let $\langle \cdot \rangle_M = \langle \cdot \rangle_{X, F, K, M}$. First suppose that $Sz(K) > \omega^\xi$. Fix ε such that $Sz(K, 15\varepsilon) > \omega^\xi$. Then by Corollary 5.3, there exist collections $(x_E)_{E \in \mathcal{S}_\xi \setminus \{\emptyset\}} \subset B_X$ and $(x_E^*)_{E \in MAX(\mathcal{S}_\xi)} \subset K$ such that for every $\emptyset \prec E \preceq F \in MAX(\mathcal{S}_\xi)$, $\text{Re } x_F^*(x_E) \geq 3\varepsilon$, and such that $u_E \in \text{span}\{F_j : j > \max E\}$. Seeking a contradiction, suppose that $M = (m_i)_{i=1}^\infty \in [\mathbb{N}]$ is such that

$$\left\langle \sum_{j=1}^\infty \mathbb{S}_{N,1}^\xi(n_j)e_j \right\rangle_N \leq \varepsilon$$

for all $N \in [M]$. Fix $3 \leq n_1$. Assuming that $n_1 < \dots < n_k$ have been chosen, if $(n_1, \dots, n_k) \in \mathcal{S}_\xi$, fix $n_k < n_{k+1} \in M$ such that $x_{(n_1, \dots, n_k)} \in \text{span}\{F_j : j < n_{k+1}\}$. If $(n_1, \dots, n_k) \notin \mathcal{S}_\xi$, fix $n_k < n_{k+1} \in M$ arbitrary. By compactness of \mathcal{S}_ξ together with the fact that for each $\emptyset \neq E \in \mathcal{S}_\xi \setminus MAX(\mathcal{S}_\xi)$, $E \cup (1 + \max E) \in \mathcal{S}_\xi$, there exists $k \in \mathbb{N}$ such that $G = (n_1, \dots, n_k) \in MAX(\mathcal{S}_\xi)$. Let $n_0 = 0$ and $x_1 = 0 \in [F_j : n_0 < j \leq n_1]$. For $1 < i \leq k$, let $x_i = x_{(n_1, \dots, n_{i-1})} \in [F_j : n_{i-1} < j \leq n_i]$. Then

$$\varepsilon \geq \left\langle \sum_{i=1}^\infty \mathbb{S}_{N,1}^\xi(n_j)e_j \right\rangle_N \geq \text{Re } x_G^* \left(\sum_{j=1}^k \mathbb{S}_{N,1}^\xi(n_j)x_j \right) \geq 3\varepsilon(1 - \mathbb{S}_{N,1}^\xi(n_1)) \geq 3\varepsilon(2/3) = 2\varepsilon,$$

a contradiction.

Now suppose that $Sz(K) \leq \omega^\xi$. Fix $\varepsilon > 0$ and let \mathcal{V} denote the set of $M \in [\mathbb{N}]$ such that

$$\left\langle \sum_{j=1}^\infty \mathbb{S}_{M,1}^\xi(n_j)e_j \right\rangle_M \leq \varepsilon.$$

By the permanence properties of the measures $\mathbb{S}_{M,i}^\xi$, it follows that \mathcal{V} is closed. Then there exists $M \in [\mathbb{N}]$ such that either $[M] \cap \mathcal{V} = \emptyset$ or $[M] \subset \mathcal{V}$. We will show that $[M] \subset \mathcal{V}$, which will finish the proof. Seeking a contradiction, assume $[M] \cap \mathcal{V} = \emptyset$. For each $F = (n_i)_{i=1}^k \in MAX(\mathcal{S}_\xi) \cap [M]^{<\mathbb{N}}$, we may fix $N_F \in [M]$ such that F is an initial segment of N_F . Then since

$$\varepsilon < \left\langle \sum_{j=1}^\infty \mathbb{S}_{N_F,1}^\xi(n_j)e_j \right\rangle_{N_F},$$

there exist $x_F^* \in K$ and $(x_i^F)_{i=1}^k \in B_X^k \cap \prod_{i=1}^k [F_j : n_{i-1} < j \leq n_i]$ such that $\text{Re } x_F^* \left(\sum_{j=1}^k \mathbb{S}_{N_F,1}^\xi(n_j)x_i^F \right) \geq \varepsilon$. Define $f : \mathcal{S}_\xi \times M \rightarrow \mathbb{R}$ as follows: If $F = (n_i)_{i=1}^k \in MAX(\mathcal{S}_\xi) \cap [M]^{<\mathbb{N}}$, let $f(n_i, F) = \text{Re } x_F^*(x_i^F)$. Then by Proposition 3.5, there exists $P \in [\mathbb{N}]$ such that for any $E \in \mathcal{S}_\xi$, there exists $P(E) \subset F \in MAX(\mathcal{S}_\xi) \cap [M]^{<\mathbb{N}}$ such that for each $j \in P(E)$, $f(j, F) \geq \varepsilon/2$. From this it easily follows that $\mathcal{S}_\xi(P) \subset \mathcal{H}(X, F, K, \varepsilon/2)$, and $CB(\mathcal{H}(X, F, K, \varepsilon/2)) \geq CB(\mathcal{S}_\xi) = \omega^\xi + 1$. But since $Sz(K, \varepsilon/3) < \omega^\xi$,

$$CB(\mathcal{H}(X, F, K, \varepsilon/2)) \leq 2Sz(K, \varepsilon/2) < \omega^\xi,$$

a contradiction. □

Corollary 5.7. *Suppose E is a sequence space the canonical basis of which is shrinking. Suppose that X is a Banach space with bimonotone FDD F . Then F is shrinking in $X_\wedge^E(F)$ and*

$$Sz(X_\wedge^E(F)) \leq Sz(E).$$

Proof. First, we recall the following easy fact. For any Banach space Z with FDD G , then G is a shrinking FDD for Z if and only if for every $\varepsilon > 0$, $CB(\mathcal{H}(Z, G, B_{Z^*}, \varepsilon)) < \omega_1$. Since

$$\langle \cdot \rangle_{X_\wedge^E(F), F, B_{X_\wedge^E(F)^*}, M} \leq 2 \langle \cdot \rangle_{E, E, B_{E^*}, M}$$

for all $M \in [\mathbb{N}]$,

$$CB(\mathcal{H}(X_\wedge^E(F), F, B_{X_\wedge^E(F)^*}, \varepsilon)) \leq CB(\mathcal{H}(E, E, B_{E^*}, \varepsilon/2))$$

for all $\varepsilon > 0$. Since \mathbf{E} is shrinking in E , the latter value is countable for each $\varepsilon > 0$, as is the former. From this it follows that \mathbf{F} is shrinking in $X_\wedge^E(\mathbf{F})$.

Since $\dim E = \infty$ and E^* is separable, $Sz(E) = \omega^\xi$ for some $0 < \xi < \omega_1$. Since

$$\langle \cdot \rangle_{X_\wedge^E(\mathbf{F}), \mathbf{F}, B_{(X_\wedge^E(\mathbf{F}))^*}, N} \leq 2 \langle \cdot \rangle_{E, E, B_{E^*}, N}$$

for any $N \in [\mathbb{N}]$, an appeal to Lemma 5.6 gives the result. \square

Corollary 5.8. *Fix $\xi < \omega_1$ and p, q with $1 \leq q < \infty$ and $1/p + 1/q = 1$. Suppose X is a Banach space, $K \subset X^*$, and \mathbf{F} is a K -shrinking FMD for X . Then $\mathfrak{p}_\xi(K) \leq q$ if and only if for each $1 < r < p$, there exist a blocking \mathbf{G} of \mathbf{F} , a sequence $(m_n)_{n=1}^\infty \in [\mathbb{N}]$, and a constant $C \geq 0$ such that for each $0 = r_0 < r_1 < \dots$, each $(u_i)_{i=1}^\infty \in B_X^{\leq \mathbb{N}} \cap \prod_{i=1}^\infty [G_j : r_{i-1} < j \leq r_i]$, and each $(a_i)_{i=1}^\infty \in c_{00}$,*

$$r_K \left(\sum_{i=1}^\infty a_i u_i \right) \leq C \left\| \sum_{i=1}^\infty a_i e_{m_{r_i}} \right\|_{\xi, r}.$$

Proof. First suppose that $\mathfrak{p}_\xi(K) \leq q$. Fix $1 < r < \alpha < p$ and let $1 = 1/r + 1/s = 1/\alpha + 1/\beta$. Fix $m \in \mathbb{N}$ such that $2^m > \sup_{x^* \in K} \|x^*\|$. Since $\mathfrak{p}_\xi(K) \leq q$, there exists $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$2Sz_\xi(K, 2^m/2^{1+n/\beta}) < l2^n.$$

Recursively select $M_1 \supset M_2 \supset \dots$ such that for each $n \in \mathbb{N}$, either

$$\mathcal{H}(X, \mathbf{F}, K, 2^m/2^{n/\beta}) \cap [M_n]^{< \mathbb{N}} \subset \mathcal{A}_{l2^n}[\mathcal{S}_\xi]$$

or

$$\mathcal{A}_{l2^n}[\mathcal{S}_\xi] \cap [M_n]^{< \mathbb{N}} \subset \mathcal{H}(X, \mathbf{F}, K, 2^m/2^{n/\beta}).$$

But since

$$CB(\mathcal{H}(X, \mathbf{F}, K, 2^m/2^{n/\beta})) < \omega^\xi l 2^n + 1 = CB(\mathcal{A}_{l2^n}[\mathcal{S}_\xi]),$$

the first inclusion must hold. Now fix $m_1 < m_2 < \dots$, $m_n \in M_n$. For each $n \in \mathbb{N}$, let $G_n = [F_j : m_{n-1} < j \leq m_n]$ and let $\mathbf{G} = (G_n)_{n=1}^\infty$. Let

$$C = \sum_{n=1}^\infty \frac{n2^m}{2^{(n-1)/\beta}} + \frac{2^m (l2^n)^{1/s}}{2^{(n-1)/\beta}} < \infty.$$

Now fix $0 = r_0 < r_1 < \dots$, $(u_i)_{i=1}^\infty \in B_X^{\leq \mathbb{N}} \cap \prod_{i=1}^\infty [G_j : r_{i-1} < j \leq r_i]$, $(a_i)_{i=1}^\infty \in c_{00}$, and $x^* \in K$ such that

$$r_K \left(\sum_{i=1}^\infty a_i u_i \right) = \operatorname{Re} x^* \left(\sum_{i=1}^\infty a_i u_i \right).$$

For each $n \in \mathbb{N}$, let

$$B_n = \{i < n : |x^*(u_i)| \in (2^m/2^{n/\beta}, 2^m/2^{(n-1)/\beta})\}$$

and

$$C_n = \{i \geq n : |x^*(u_i)| \in (2^m/2^{n/\beta}, 2^m/2^{(n-1)/\beta})\}.$$

Then for any $n \in \mathbb{N}$,

$$\operatorname{Re} x^* \left(\sum_{i \in B_n} a_i u_i \right) \leq \frac{2^m}{2^{(n-1)/\beta}} \sum_{i \in B_n} |a_i| \leq \frac{n2^m}{2^{(n-1)/\beta}} \left\| \sum_{i=1}^\infty a_i e_{m_{r_i}} \right\|_{\xi, r}.$$

For any $n \in \mathbb{N}$,

$$(m_{r_i})_{i \in C_n} \in \mathcal{H}(X, \mathbf{F}, K, 2^m/2^{n/\beta}) \cap [M_n]^{< \mathbb{N}} \subset \mathcal{A}_{l2^n}[\mathcal{S}_\xi].$$

Now write $(m_{r_i})_{i \in C_n} = \cup_{i=1}^k E_i$, $k \leq l2^n$, $\emptyset \neq E_i \in \mathcal{S}_\xi$. Then

$$\begin{aligned} \operatorname{Re} x^* \left(\sum_{i \in C_n} a_i u_i \right) &\leq \frac{2^m}{2^{(n-1)/\beta}} \sum_{i \in C_n} |a_i| = \frac{2^m}{2^{(n-1)/\beta}} \sum_{j=1}^k \sum_{i \in E_j} |a_i| \\ &\leq \frac{2^m (l2^n)^{1/s}}{2^{(n-1)/\beta}} \left(\sum_{j=1}^k \left(\sum_{i \in E_j} |a_i| \right)^r \right)^{1/r} \\ &\leq \frac{2^m (l2^n)^{1/s}}{2^{(n-1)/\beta}} \left\| \sum_{i=1}^{\infty} a_i e_{m_{r_i}} \right\|_{\xi, r}. \end{aligned}$$

Then

$$r_K \left(\sum_{i=1}^{\infty} a_i u_i \right) \leq \sum_{n=1}^{\infty} \operatorname{Re} x^* \left(\sum_{i \in B_n \cup C_n} a_i u_i \right) \leq C \left\| \sum_{i=1}^{\infty} a_i e_{m_{r_i}} \right\|_{\xi, r}.$$

Now suppose that for every $1 < r < p$, the blocking \mathbf{G} , the sequence $(m_n)_{n=1}^{\infty}$, and the constant C exist. Now fix $1 < r < p$, let $1/r + 1/s = 1$, and let \mathbf{G} , $(m_n)_{n=1}^{\infty}$, and C be as in the statement. By replacing C with a larger value if necessary, we may assume $C \geq 1$. For each $0 < \varepsilon \leq 1$, let

$$\mathcal{B}_\varepsilon = \{E \in [\mathbb{N}]^{<\mathbb{N}} : (\forall x \in \operatorname{co}(e_i : i \in E)) (\|x\|_{\xi, r} \geq \varepsilon)\}$$

and note that $CB(\mathcal{B}_\varepsilon) \leq \omega^\xi[\varepsilon^{-s}] + 1$ for all $\varepsilon \in (0, 1]$ by Lemma 4.1. Let $\mathcal{C}_\varepsilon = \{E : M(E) \in \mathcal{B}_\varepsilon\}$ and note that \mathcal{C}_ε is homeomorphic to \mathcal{B}_ε , whence $CB(\mathcal{C}_\varepsilon) \leq \omega^\xi[\varepsilon^{-s}] + 1$ for all $\varepsilon \in (0, 1]$. Now suppose that $(r_i)_{i=1}^l \in \mathcal{H}(X, \mathbf{G}, K, \varepsilon)$. Fix any $r_l < r_{l+1} < r_{l+2} < \dots$. By definition, there exist $(u_i)_{i=1}^l \in B_X^l \cap \prod_{i=1}^l [G_j : r_{i-1} < j \leq r_i]$ and $x^* \in K$ such that $\operatorname{Re} x^*(u_i) \geq \varepsilon$ for each $1 \leq i \leq l$. Let $u_i = 0$ for all $i > l$. Fix non-negative scalars $(a_i)_{i=1}^l$ summing to 1 and note that

$$C \left\| \sum_{i=1}^l a_i e_{m_{r_i}} \right\| \geq r_K \left(\sum_{i=1}^l a_i u_i \right) \geq \operatorname{Re} x^* \left(\sum_{i=1}^l a_i u_i \right) \geq \varepsilon,$$

whence $(m_{r_i})_{i=1}^l \in \mathcal{B}_{\varepsilon/C}$ and $(r_i)_{i=1}^l \in \mathcal{C}_{\varepsilon/C}$. We have shown that

$$\mathcal{H}(X, \mathbf{G}, K, \varepsilon) \subset \mathcal{C}_{\varepsilon/C},$$

whence

$$Sz(K, 5\varepsilon) \leq CB(\mathcal{H}(X, \mathbf{G}, K, \varepsilon)) \leq CB(\mathcal{C}_{\varepsilon/C}) \leq \omega^\xi[(\varepsilon/C)^{-s}] + 1.$$

From this it easily follows that there exists a constant D such that for any $0 < \varepsilon < 1$, $Sz_\xi(K, \varepsilon) \leq D/\varepsilon^s$, and $\mathfrak{p}_\xi(K) \leq s$. Since $1 < r < p$ was arbitrary, $\mathfrak{p}_\xi(K) \leq q$. \square

We next collect an embedding theorem which combines results from [7] and [8].

Theorem 5.9. *Fix $\xi < \omega_1$.*

- (i) *If X is a Banach space with separable dual and $Sz(X) \leq \omega^\xi$, then there exists a Banach space W with bimonotone FDD \mathbf{F} such that X is isomorphic to both a subspace and a quotient of $W_\wedge^{X_\xi}(\mathbf{F})$.*
- (ii) *If X is a Banach space, $1/p + 1/q = 1$, and $\mathfrak{p}_\xi(X) < q$, then there exists a Banach space W with bimonotone FDD \mathbf{F} such that X is isomorphic to both a subspace and a quotient of $W_\wedge^{X_{\xi, p}}(\mathbf{F})$.*

Proof. (i) Let $\mathcal{E} = \{(n_1, \dots, n_{2k}) : k \in \mathbb{N}, n_1 < \dots < n_{2k}\}$. We first remark that it was shown in [7] that if $Sz(X) \leq \omega^\xi$, then there exists a constant C such that for any collection $(x_E)_{E \in \mathcal{E}} \subset B_X$ such that for each $n_1 < \dots < n_{2k-1}$, $(x_{(n_1, \dots, n_{2k-1}, n_k)})_{n_k > n_{2k-1}}$ is weakly null, there exist $n_1 < n_2 < \dots$ such that for any $(a_i)_{i=1}^{\infty} \in c_{00}$,

$$\left\| \sum_{i=1}^{\infty} a_i x_{(n_1, \dots, n_{2i})} \right\| \leq C \left\| \sum_{i=1}^{\infty} a_i e_{n_{2i-1}} \right\|_{X_\xi}.$$

From the main embedding theorem of [7], since the canonical basis of X_ξ is shrinking and has properties R , S , and T , there exist Banach spaces U, V with bimonotone FDDs \mathbf{G} and \mathbf{H} such that X is isomorphic to a subspace of \widehat{U} and to a quotient of \widehat{V} , where the norm on \widehat{U} is given by

$$\|u\|_{\widehat{U}} = \sup \left\{ \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{G}} u\|_{U e_{\min I_i}} \|X_\xi : I_1 < I_2 < \dots, I_i \text{ an interval} \right\| \right\}$$

and the norm of \widehat{V} is given by

$$\|v\|_{\widehat{V}} = \sup \left\{ \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{H}} v\|_{V e_{\min I_i}} \|X_\xi : I_1 < I_2 < \dots, I_i \text{ an interval} \right\| \right\}.$$

Since X_ξ has properties S and T , the norms of \widehat{U} and \widehat{V} are equivalent to $\|\cdot\|_{U \widehat{\wedge}^{\mathbf{G}}(X_\xi)}$ and $\|\cdot\|_{V \widehat{\wedge}^{\mathbf{H}}(X_\xi)}$, respectively. Let $F_n = G_n \oplus_\infty H_n$ and $W = U \oplus_\infty V$. Then X is isomorphic to a subspace and a quotient of $W \widehat{\wedge}^{\mathbf{F}}(X_\xi)$.

(ii) This is similar to (i). We only need to show that if X is a Banach space with separable dual and $\mathfrak{p}_\xi(X) < q$, then there exists constant C' such that for any $(x_E)_{E \in \mathcal{E}} \subset B_X$ such that for each $n_1 < \dots < n_{2k-1}$, $(x_{(n_1, \dots, n_{2k-1}, n_k)})_{n_k > n_{2k-1}}$ is weakly null, there exist $n_1 < n_2 < \dots$ such that for any $(a_i)_{i=1}^\infty \in c_{00}$,

$$\left\| \sum_{i=1}^{\infty} a_i x_{(n_1, \dots, n_{2i})} \right\| \leq C' \left\| \sum_{i=1}^{\infty} a_i e_{n_{2i-1}} \right\|_{X_{\xi, p}}.$$

We note that, as shown in [8], the canonical basis of $X_{\xi, p}$ is shrinking, and $X_{\xi, p}$ has properties R , S , and T , so the main embedding theorem from [7] applies. In order to find the indicated constant C' , we note that by Corollary 5.8, there exist an FMD $\mathbf{l} = (I_n)_{n=1}^\infty$ for X and $m_1 < m_2 < \dots$ such that for any $0 = r_0 < r_1 < \dots$, any $(u_i)_{i=1}^\infty \subset B_X \cap \prod_{i=1}^\infty [I_j : r_{i-1} < j \leq r_i]$, and any $(a_i)_{i=1}^\infty \in c_{00}$,

$$\left\| \sum_{i=1}^{\infty} a_i u_i \right\| \leq C \left\| \sum_{i=1}^{\infty} a_i e_{m_{r_i}} \right\|_{X_{\xi, p}}.$$

Now note that, since $X_{\xi, p}$ has property S , there exists a constant D such that for any $s_1 < t_1 < s_2 < t_2 < \dots$ and any $(a_i)_{i=1}^\infty \in c_{00}$,

$$\left\| \sum_{i=1}^{\infty} a_i e_{t_i} \right\|_{\xi, p} \leq D \left\| \sum_{i=1}^{\infty} a_i e_{s_i} \right\|_{\xi, p}.$$

Let $C' = CD + 1$. Fix a sequence of positive numbers $(\varepsilon_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty \varepsilon_n = 1$ and suppose $(x_E)_{E \in \mathcal{E}}$ is as above. Let us recursively select $n_1 < n_2 < \dots$, $t_1 < t_2 < \dots$, and $u_i \in B_X$ such that

- (i) $\|x_{(n_1, \dots, n_{2i})} - u_i\|_X < \varepsilon_i$,
- (ii) $u_i \in [I_j : t_{i-1} < j \leq t_i]$,
- (iii) $n_{2i-1} < m_{t_i} < n_{2i+1}$.

We may fix $n_1 = 1$, $n_2 = 2$, $u_1 \in B_X \cap \text{span}\{I_j : j \in \mathbb{N}\}$ such that $\|x_{(n_1, n_2)} - u_1\| < \varepsilon_1$, and $t_1 \in \mathbb{N}$ such that $u_1 \in \text{span}\{I_j : j \leq t_1\}$. Now assume that $n_1 < \dots < n_{2i}$, $t_1 < \dots < t_i$, u_1, \dots, u_i have been chosen. Fix $n_{2i+1} > m_{t_i}$. Then choose $n_{2i+2} > n_{2i+1}$ such that

$$d(x_{(n_1, \dots, n_{2i+2})}, B_X \cap [I_j : j > t_i]) < \varepsilon_{i+1},$$

$u_{i+1} \in B_X \cap \text{span}\{I_j : j > t_i\}$ such that $\|x_{(n_1, \dots, n_{2i+2})} - u_{i+1}\| < \varepsilon_{i+1}$, and $t_{i+1} > n_{2i+2}$ such that $u_{i+1} \in \text{span}\{I_j : t_i < j \leq t_{i+1}\}$. Now for any $(a_i)_{i=1}^\infty \in c_{00}$, letting $a = \|(a_i)_{i=1}^\infty\|_{c_0}$,

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} a_i x_{(n_1, \dots, n_{2i})} \right\| &\leq a + \left\| \sum_{i=1}^{\infty} a_i u_i \right\| \leq a + C \left\| \sum_{i=1}^{\infty} a_i e_{m_{t_i}} \right\|_{\xi, p} \\ &\leq (1 + CD) \left\| \sum_{i=1}^{\infty} a_i e_{n_{2i-1}} \right\|_{\xi, p} = C' \left\| \sum_{i=1}^{\infty} a_i e_{n_{2i-1}} \right\|_{\xi, p}. \end{aligned}$$

□

Corollary 5.10. *Suppose X is a Banach space, $K \subset X^*$, \mathbf{F} is a K -shrinking FMD for X , $0 < \xi < \omega_1$, and $Sz(K) \leq \omega^\xi$. Then there exist a blocking \mathbf{G} of \mathbf{F} , a sequence $(m_n)_{n=1}^\infty \in [\mathbb{N}]$, and a constant $C \geq 0$ such that for each $0 = r_0 < r_1 < \dots$, each $(u_i)_{i=1}^\infty \in B_X^{\leq \mathbb{N}} \cap \prod_{i=1}^\infty [G_j : r_{i-1} < j \leq r_i]$, and each $(a_i)_{i=1}^\infty \in c_{00}$,*

$$r_K \left(\sum_{i=1}^\infty a_i u_i \right) \leq C \left\| \sum_{i=1}^\infty a_i e_{m_{r_i}} \right\|_\xi.$$

Proof. Fix $R > \sup_{x^* \in K} \|x^*\|$. As in Corollary 5.8, we recursively select $M_1 \supset M_2 \supset \dots$ such that for all $n \in \mathbb{N}$,

$$\mathcal{H}(X, \mathbf{F}, K, 2^m/2^n) \cap [M_n]^{< \mathbb{N}} \subset \mathcal{S}_\xi.$$

We may do this, since

$$CB(\mathcal{H}(X, \mathbf{F}, K, 2^m/2^n)) < \omega^\xi.$$

Now fix $m_1 < m_2 < \dots$, $m_n \in M_n$ and let $G_n = [F_j : m_{n-1} < j \leq m_n]$, $0 = r_0 < r_1 < \dots$, $(u_i)_{i=1}^\infty \in B_X^{\leq \mathbb{N}} \cap \prod_{i=1}^\infty [G_j : r_{i-1} < j \leq r_i]$, and $(a_i)_{i=1}^\infty \in c_{00}$,

$$r_K \left(\sum_{i=1}^\infty a_i u_i \right) \leq \left\| \sum_{i=1}^\infty a_i e_{m_{r_i}} \right\|_{X_\xi} \sum_{n=1}^\infty \frac{2^m}{2^{n-1}} n.$$

□

6. FACTORIZATION AND UNIVERSALITY

We first recall a construction of Schechtman. There exists a sequence $\mathbf{U} = (U_n)_{n=1}^\infty$ of finite dimensional spaces which form a bimonotone FDD for a Banach space \mathfrak{U} such that if X is any Banach space with bimonotone FDD $\mathbf{F} = (F_n)_{n=1}^\infty$ and if $m_1 < m_2 < \dots$ are natural numbers, then there exist a sequence $k_1 < k_2 < \dots$ of natural numbers, a sequence $I_n : F_n \rightarrow U_{k_n}$ of isomorphisms, and a projection $P : \mathfrak{U} \rightarrow [U_{k_i} : i \in \mathbb{N}]$ such that

- (i) $m_i < k_i$ for all $i \in \mathbb{N}$,
- (ii) $\|I_n\|, \|I_n^{-1}\| \leq 2$,
- (iii) the map $x = \sum_{n=1}^\infty x_n \mapsto \sum_{n=1}^\infty I_n x_n$ defines an isomorphism $I : X \rightarrow [U_{k_i} : i \in \mathbb{N}]$ such that $\|I\|, \|I^{-1}\| \leq 2$,
- (iv) $\|P\| = 1$.

In the sequel, the symbol \mathfrak{U} will be reserved for this space and the symbol \mathbf{U} will denote the FDD $(U_n)_{n=1}^\infty$ of \mathfrak{U} .

Proposition 6.1. (i) *If $P : \mathfrak{U} \rightarrow [U_{k_i} : i \in \mathbb{N}]$ is the norm 1 projection given by $P \sum_{i=1}^\infty x_i = \sum_{i=1}^\infty x_{k_i}$, then for any sequence space E , $P : \mathfrak{U}_\wedge^E(\mathbf{U}) \rightarrow \mathfrak{U}_\wedge^E(\mathbf{U})$ is also norm 1.*

(ii) *If E is a sequence space with property R , $k_1 < k_2 < \dots$, $\mathfrak{V} = [U_{k_i} : i \in \mathbb{N}] \subset \mathfrak{U}$, $V_n = U_{k_n}$, the projection $P : \mathfrak{U} \rightarrow \mathfrak{V}$ given by $P \sum_{i=1}^\infty x_i = \sum_{i=1}^\infty x_{k_i}$ is norm 1, $\mathbf{V} = (V_n)_{n=1}^\infty$, and the norms $\|\cdot\|_\mathfrak{U}$ and $\|\cdot\|_{\mathfrak{V}_\wedge^E(\mathbf{V})}$ are equivalent on \mathfrak{V} , then the norms $\|\cdot\|_\mathfrak{U}$ and $\|\cdot\|_{\mathfrak{U}_\wedge^E(\mathbf{U})}$ are equivalent on \mathfrak{V} .*

(iii) *Suppose $\mathcal{G}, \mathcal{G}_0, \mathcal{G}_1, \dots$ are regular families such that $CB(\mathcal{G}) < \sup_n CB(\mathcal{G}_n)$ and $1 = \vartheta_0 > \vartheta_1 > \dots$, $\lim_n \vartheta_n = 0$. Let $E = X(\mathcal{G}_n, \vartheta_n)$ be the mixed Schreier space. If W is a Banach space with FDD \mathbf{F} such that the norms $\|\cdot\|_W$ and $\|\cdot\|_{W_\wedge^{X\mathcal{G}}(\mathbf{F})}$ are equivalent, then W embeds complementably into $\mathfrak{U}_\wedge^E(\mathbf{U})$.*

Proof. (i) For any $x \in c_{00}(\mathbf{U})$,

$$\begin{aligned} [Px]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} &= \inf \left\{ \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{U}} Px\|_{\mathfrak{U}e_{\max I_i}} \|E : I_1 < I_2 < \dots, \cup_{i=1}^{\infty} I_i = \mathbb{N} \right\| \right\} \\ &= \inf \left\{ \left\| \sum_{i=1}^{\infty} \|PI_i^{\mathbf{U}} x\|_{\mathfrak{U}e_{\max I_i}} \|E : I_1 < I_2 < \dots, \cup_{i=1}^{\infty} I_i = \mathbb{N} \right\| \right\} \\ &\leq \inf \left\{ \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{U}} x\|_{\mathfrak{U}e_{\max I_i}} \|E : I_1 < I_2 < \dots, \cup_{i=1}^{\infty} I_i = \mathbb{N} \right\| \right\} \\ &= [x]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} \end{aligned}$$

and

$$\begin{aligned} \|Px\|_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} &= \inf \left\{ \sum_{i=1}^n [x_i]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} : n \in \mathbb{N}, Px = \sum_{i=1}^n x_i \right\} \\ &\leq \inf \left\{ \sum_{i=1}^n [Px_i]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} : n \in \mathbb{N}, x = \sum_{i=1}^n x_i \right\} \\ &\leq \inf \left\{ \sum_{i=1}^n [x_i]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} : n \in \mathbb{N}, x = \sum_{i=1}^n x_i \right\} \\ &= \|x\|_{\mathfrak{U}_{\wedge}^E(\mathbf{U})}. \end{aligned}$$

(ii) Of course, $\|\cdot\|_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} \leq \|\cdot\|_{\mathfrak{U}}$. To establish the reverse inequality, it is sufficient to prove that $\|x\|_{\mathfrak{V}_{\wedge}^E(\mathbf{V})} \leq \|x\|_{\mathfrak{U}_{\wedge}^E(\mathbf{U})}$ for all $x \in c_{00}(\mathbf{V})$. To that end, fix $x \in c_{00}(\mathbf{V})$ and intervals $I_1 < I_2 < \dots$ such that $\cup_{i=1}^{\infty} I_i = \mathbb{N}$. Let J_i be such that for all $j \in \mathbb{N}$, $J_j = \{i : k_i \in I_j\}$. Let $S = \{j : J_j \neq \emptyset\}$ and note that $(J_i)_{i \in S}$ are successive, $\mathbb{N} = \cup_{i \in S} J_i$, and $\max J_i \leq k_{\max J_i} \leq \max I_i$ for all $i \in S$. Furthermore,

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} \|I_i^{\mathbf{U}} x\|_{\mathfrak{U}e_{\max I_i}} \|E \right\| &= \left\| \sum_{i \in S} \|I_i^{\mathbf{U}} x\|_{\mathfrak{U}e_{\max I_i}} \|E \right\| = \left\| \sum_{i \in S} \|J_i^{\mathbf{V}}\|_{\mathfrak{U}e_{\max I_i}} \|E \right\| \geq \left\| \sum_{i \in S} \|J_i^{\mathbf{V}} x\|_{\mathfrak{U}e_{\max J_i}} \|E \right\| \\ &\geq [x]_{\mathfrak{V}_{\wedge}^E(\mathbf{V})}. \end{aligned}$$

From this it follows that $[x]_{\mathfrak{V}_{\wedge}^E(\mathbf{V})} \leq [x]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})}$ for any $x \in c_{00}(\mathbf{V})$. Now for any $x \in c_{00}(\mathbf{V})$,

$$\begin{aligned} \|x\|_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} &= \inf \left\{ \sum_{i=1}^n [x_i]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} : x_i \in c_{00}(\mathbf{U}), x = \sum_{i=1}^n x_i \right\} \\ &\geq \inf \left\{ \sum_{i=1}^n [Px_i]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} : x_i \in c_{00}(\mathbf{U}), x = \sum_{i=1}^n x_i \right\} \\ &= \inf \left\{ \sum_{i=1}^n [x_i]_{\mathfrak{U}_{\wedge}^E(\mathbf{U})} : x \in c_{00}(\mathbf{V}), x = \sum_{i=1}^n x_i \right\} \\ &\geq \inf \left\{ \sum_{i=1}^n [x_i]_{\mathfrak{V}_{\wedge}^E(\mathbf{V})} : x \in c_{00}(\mathbf{V}), x = \sum_{i=1}^n x_i \right\} \\ &= \|x\|_{\mathfrak{V}_{\wedge}^E(\mathbf{V})}. \end{aligned}$$

(iii) Fix $l \in \mathbb{N}$ such that $CB(\mathcal{G}) < CB(\mathcal{G}_l)$ and $M = (m_n)_{n=1}^{\infty}$ such that $\mathcal{G}(M) \subset \mathcal{G}_l$. By renorming W , we may assume F is bimonotone in W and we may assume $W = W_{\wedge}^{X_{\mathcal{G}}}(\mathbf{F})$. Select $k_1 < k_2 < \dots$, $I_n : F_n \rightarrow U_{k_n}$, and $P : \mathfrak{U} \rightarrow \mathfrak{V} = [U_{k_n} : n \in \mathbb{N}] = [V_n : n \in \mathbb{N}]$ satisfying (i)-(iv) as in the discussion of \mathfrak{U} . Let us first note that for any $(a_i)_{i=1}^{\infty} \in c_{00}$,

$$\left\| \sum_{i=1}^{\infty} a_i e_{k_i} \|E \right\| \geq \vartheta_l \left\| \sum_{i=1}^{\infty} a_i e_i \|X_{\mathcal{G}} \right\|.$$

Indeed, fix $F \in \mathcal{G}$ such that

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{X_{\mathcal{G}}} = \left\| F \sum_{i=1}^{\infty} a_i e_i \right\|_{\ell_1}.$$

Then $H := \{k_i : i \in F\}$ is a spread of $M(F)$, and therefore lies in \mathcal{G}_l . From this it follows that

$$\left\| \sum_{i=1}^{\infty} a_i e_{k_i} \right\|_E \geq \vartheta_l \left\| H \sum_{i=1}^{\infty} a_i e_{k_i} \right\|_{\ell_1} = \vartheta_l \left\| F \sum_{i=1}^{\infty} a_i e_i \right\|_{\ell_1} = \vartheta_l \left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{X_{\mathcal{G}}}.$$

We note that, since $I : W \rightarrow \mathfrak{V}$ is an isomorphism which takes F_n to V_n , the norms $\|\cdot\|_{\mathfrak{U}}$ and $\|\cdot\|_{\mathfrak{V}^{\wedge, X_{\mathcal{G}}}(V)}$ are equivalent on \mathfrak{V} . Since $\|\cdot\|_{\mathfrak{U}^E(\mathfrak{U})} \leq \|\cdot\|_{\mathfrak{U}}$, we know $\sum_{n=1}^{\infty} w_n \mapsto \sum_{n=1}^{\infty} I_n w_n$ extends to a bounded, linear map from W into $\mathfrak{U}^E(\mathfrak{U})$. In order to know this is an isomorphic embedding, it is sufficient to know that

$$\vartheta_l \|x\|_{\mathfrak{V}^{\wedge, X_{\mathcal{G}}}(V)} \leq \|x\|_{\mathfrak{U}^E(\mathfrak{U})}$$

for all $x \in c_{00}(V)$. To that end, fix $x \in c_{00}(V)$ and intervals $I_1 < I_2 < \dots$ with $\cup_{i=1}^{\infty} I_i = \mathbb{N}$. Let J_1, J_2, \dots and S be as in (ii) and note that $k_{\max J_i} \leq \max I_i$ for all $i \in S$. Then

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} \|I_i^U x\|_{\mathfrak{U}^E(\mathfrak{U})} e_{\max I_i} \right\|_E &= \left\| \sum_{i \in S} \|I_i^U x\|_{\mathfrak{U}^E(\mathfrak{U})} e_{\max I_i} \right\|_E \geq \left\| \sum_{i \in S} \|J_i^V x\|_{\mathfrak{U}^E(\mathfrak{U})} e_{k_{\max J_i}} \right\|_E \\ &\geq \vartheta_l \left\| \sum_{i \in S} \|J_i^V x\|_{\mathfrak{U}^E(\mathfrak{U})} e_{\max J_i} \right\|_{X_{\mathcal{G}}}. \end{aligned}$$

From this it follows that

$$\vartheta_l [x]_{\mathfrak{V}^{\wedge, X_{\mathcal{G}}}(V)} \leq [x]_{\mathfrak{U}^E(\mathfrak{U})}$$

for all $x \in c_{00}(V)$. We now reach the desired conclusion as in (ii), deducing that the image of I is complemented in $\mathfrak{U}^E(\mathfrak{U})$ by (i). □

Theorem 6.2. Fix $0 < \xi < \omega_1$. Let X be a Banach space with shrinking FDD F and let $A : X \rightarrow Y$ be an operator with $Sz(A) = \omega^\xi$.

- (i) If $\xi = \omega^{\zeta+1}$, then A factors through a Banach space Z with Szlenk index ω^ξ if and only if there exists $\gamma < \omega^\zeta$ such that for each $n \in \mathbb{N}$, $Sz(A, 2^{-n}) \leq \gamma^n$.
- (ii) If $\xi = \omega^\zeta$, ζ a limit ordinal, A does not factor through any Banach space Z with $Sz(Z) = Sz(A)$.
- (iii) If $\xi = \beta + \gamma$ for some $\beta, \gamma < \xi$, then A factors through a Banach space Z with $Sz(A) = \omega^\xi$.

Remark 6.3. If X is a Banach space with bimonotone FDD G , $A : X \rightarrow Y$ is an operator, E is a sequence space, and $C \geq 1$ are such that for any $0 = r_0 < r_1 < \dots$ and any $(x_i)_{i=1}^{\infty} \in B_X^{\mathbb{N}} \cap [G_j : r_{i-1} < j \leq r_i]$, then for any $(a_i)_{i=1}^{\infty} \in c_{00}$,

$$\left\| A \sum_{i=1}^{\infty} a_i x_i \right\| \leq C \left\| \sum_{i=1}^{\infty} a_i e_{r_i} \right\|.$$

Then A factors through $X_{\wedge}^E(G)$.

Indeed, since $\|\cdot\|_{X_{\wedge}^E(G)} \leq \|\cdot\|_X$, the formal inclusion $I : X \rightarrow X_{\wedge}^E(G)$ is well-defined. Fix $x \in c_{00}(G)$ and suppose

$$[x]_{X_{\wedge}^E(G)} = \left\| \sum_{i=1}^{\infty} \|I_i^G x\|_X e_{\max I_i} \right\|_E.$$

Now if $I_i^G x \neq 0$, let $a_i = \|I_i^G x\|_X$ and let $x_i = a_i^{-1} I_i^G x$. If $I_i^G x = 0$, let $a_i = 0$ and $x_i = 0$. Then $x = \sum_{i=1}^{\infty} a_i x_i$ and, by hypothesis,

$$\|Ax\| = \left\| A \sum_{i=1}^{\infty} a_i x_i \right\| \leq C \left\| \sum_{i=1}^{\infty} a_i e_{\max I_i} \right\|_E = C [x]_{X_{\wedge}^E(G)}.$$

Now for any $x \in c_{00}(\mathbb{G})$,

$$\|Ax\| \leq \inf \left\{ \sum_{i=1}^n \|Ax_i\| : n \in \mathbb{N}, x = \sum_{i=1}^n x_i \right\} \leq C \inf \left\{ \sum_{i=1}^n [x_i]_{X_{\wedge}^E(\mathbb{G})} : n \in \mathbb{N}, x = \sum_{i=1}^n x_i \right\} = C\|x\|_{X_{\wedge}^E(\mathbb{G})}.$$

From this it follows that $A|_{c_{00}(\mathbb{G})}$ extends to a norm at most C operator $J : X_{\wedge}^E(\mathbb{G}) \rightarrow Y$, and $A = JI$.

Proof. (i) Suppose $\xi = \omega^{\zeta+1}$. First suppose that Z is a Banach space such that A factors through Z and $Sz(Z) = \omega^{\zeta+1}$. Then there exists a constant $C \geq 1$ such that $Sz(A, \varepsilon) \leq Sz(Z, \varepsilon/C)$ for all $0 < \varepsilon < 1$. Since $Sz(Z) = \omega^{\zeta+1}$, there exists $\gamma < \omega^{\zeta+1}$ such that $Sz(Z, 1/2C) < \gamma$. For all $n \in \mathbb{N}$,

$$Sz(A, 1/2^n) \leq Sz(Z, 1/C2^n) \leq Sz(Z, 1/(2C)^n) \leq Sz(Z, 1/2C)^n < \gamma^n.$$

For the converse, suppose there exists $\gamma < \omega^{\zeta+1}$ such that $Sz(A, 1/2^n) < \gamma^n$ for all $n \in \mathbb{N}$. Fix $m \in \mathbb{N}$ such that $2^m > \|A\|$. Fix a regular family \mathcal{G} with $2\gamma < CB(\mathcal{G}) < \omega^{\zeta+1}$. Let $\mathcal{G}_0 = \mathcal{S}_0$ and $\mathcal{G}_n = \mathcal{G}[\mathcal{G}_{n-1}]$ for $n \in \mathbb{N}$. Let $\vartheta = 2/3$ and let $X = X(\mathcal{G}_n, \vartheta^n)$. Note that $Sz(X) = \omega^{\zeta+1}$ by Lemma 5.1. Recursively select $\mathbb{N} \supset M_1 \supset M_2 \supset \dots$ such that for each $n \in \mathbb{N}$, either

$$\mathcal{H}(X, F, A^*B_{Y^*}, 1/2^{m+n}) \cap [M_n]^{<\mathbb{N}} \subset \mathcal{G}_n$$

or

$$\mathcal{G}_n \cap [M_n]^{<\mathbb{N}} \subset \mathcal{G}(X, F, A^*B_{Y^*}, 2^m/2^n).$$

Now since

$$CB(\mathcal{H}(X, F, A^*B_{Y^*}, 2^m/2^n)) \leq 2Sz(A, 1/2^n) < 2\gamma^n \leq (2\gamma)^n < CB(\mathcal{G}_n),$$

it must be the case that

$$\mathcal{H}(X, F, A^*B_{Y^*}, 2^m/2^n) \cap [M_n]^{<\mathbb{N}} \subset \mathcal{G}_n$$

for all $n \in \mathbb{N}$. Fix $0 = m_0 < m_1 < m_2 < \dots$ such that $m_n \in M_n$. For each $n \in \mathbb{N}$, let $G_n = [F_j : m_{n-1} < j \leq m_n]$ and let $\mathbb{G} = (G_n)_{n=1}^{\infty}$. Let E be the sequence space whose norm is given by $\|\sum_{i=1}^{\infty} a_i e_i\|_E = \|\sum_{i=1}^{\infty} a_i e_{m_i}\|_X$. Let

$$C = 2^{m+1} \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} + \frac{3^n}{4^n} < \infty.$$

Suppose $0 = r_0 < r_1 < \dots$ and $(x_n)_{n=1}^{\infty} \in B_X^{\mathbb{N}} \cap \prod_{n=1}^{\infty} [G_j : r_{n-1} < j \leq r_n]$. Fix $(a_i)_{i=1}^{\infty} \in c_{00}$ and $y^* \in B_{Y^*}$ such that

$$\|A \sum_{i=1}^{\infty} a_i x_i\| = A^* y^* \left(\sum_{i=1}^{\infty} a_i x_i \right).$$

For each $n \in \mathbb{N}$, let

$$B_n = \{i < n : |A^* y^*(x_i)| \in (2^m/2^n, 2^m/2^{n-1}]\}$$

and

$$C_n = \{i \geq n : |A^* y^*(x_i)| \in (2^m/2^n, 2^m/2^{n-1}]\}.$$

Note that $|B_n| < n$, whence

$$A^* y^* \left(\sum_{i \in B_n} a_i x_i \right) \leq \frac{2^m}{2^{n-1}} \sum_{i \in B_n} |a_i| \leq \frac{n2^m}{2^{n-1}} \left\| \sum_{i \in B_n} e_{m_{r_i}} \right\| \leq \frac{n2^m}{2^{n-1}} \left\| \sum_{i \in B_n} e_{r_i} \right\|_E.$$

Note also that

$$(m_{r_i})_{i \in C_n} \in \mathcal{H}(X, F, A^*B_{Y^*}, 2^m/2^n) \cap [M_n]^{<\mathbb{N}} \subset \mathcal{G}_n.$$

From this it follows that

$$A^* y^* \left(\sum_{i \in C_n} a_i x_i \right) \leq \frac{2^m}{2^{n-1}} \sum_{i \in C_n} |a_i| \leq \frac{2^m}{2^{n-1}} \cdot \frac{3^n}{2^n} \left\| \sum_{i \in C_n} e_{m_{r_i}} \right\| = \frac{n2^{m+1}3^n}{4^n} \left\| \sum_{i \in C_n} e_{r_i} \right\|_E.$$

Then

$$\|A \sum_{i=1}^{\infty} a_i x_i\| = A^* y^* \left(\sum_{i=1}^{\infty} a_i x_i \right) \leq C \left\| \sum_{i=1}^{\infty} a_i e_{r_i} \right\|_E.$$

Then as noted in Remark 6.3, A factors through $X_{\wedge}^E(\mathbf{G})$. By Corollary 5.7, $Sz(X_{\wedge}^E(\mathbf{G})) = \omega^{\omega^{\zeta+1}}$.

(ii) By [9], there is no Banach space with Szlenk index $\omega^{\omega^{\zeta}}$, ζ a limit ordinal.

(iii) Write $\xi = \beta + \gamma$ with $\beta, \gamma < \xi$. Fix $m \in \mathbb{N}$ such that $2^m > \|A\|$. We may fix an increasing sequence γ_n of ordinals such that $\gamma_n \uparrow \omega^{\gamma}$ and $2Sz(A, 2^m/2^{n+1}) < \omega^{\beta} \gamma_n$. Fix a sequence of regular families \mathcal{F}_n with $\gamma_n < CB(\mathcal{F}_n)$ and let $\mathcal{G}_0 = \mathcal{S}_{\beta}$, $\mathcal{G}_n = \mathcal{F}_n[\mathcal{G}_0]$. Let $X = X(\mathcal{G}_n, (2/3)^n)$ be the mixed Schreier space and note that X is ξ -well-constructed. As in (i), for each $n \in \mathbb{N}$, we find M_n such that $\mathcal{H}(X, \mathbf{F}, A^* B_{Y^*}, 2^m/2^n) \cap [M_n]^{<\mathbb{N}} \subset \mathcal{G}_n$. We then select $m_1 < m_2 < \dots$ such that $m_n \in M_n$. Arguing as in (i), with \mathbf{G} , E , C defined in the same way, we deduce that A factors through $X_{\wedge}^E(\mathbf{F})$, which has Szlenk index ω^{ξ} . \square

Remark 6.4. If X has a shrinking FDD and $A : X \rightarrow Y$ is an operator with $Sz(A) = \omega$, then A factors through a Banach space Z with $Sz(Z) = \omega$ if and only if there exists $l \in \mathbb{N}$ such that $Sz(A, 1/2^n) \leq l^n$ for all $n \in \mathbb{N}$. This has already been shown in [21]. Our argument above is essentially a transfinite extension of this fact. Indeed, in this case, $\mathfrak{p}_0(A) < \infty$. Therefore if $1/r + 1/s = 1$ and $\mathfrak{p}_0(A) < s < \infty$, A factors through $X_{\wedge}^{X_0, r}(\mathbf{G}) = X_{\wedge}^{\ell_r}(\mathbf{G})$ as a consequence of Corollary 5.8 and Remark 6.3.

Theorem 6.5. *For any $\xi \in (0, \omega_1) \setminus \{\omega^n : \eta \text{ a limit ordinal}\}$, there exists a Banach space \mathfrak{G}_{ξ} with a shrinking basis and $Sz(\mathfrak{G}_{\xi}) = \omega^{\xi}$ such that if $A : X \rightarrow Y$ is a separable range operator with $Sz(A) < \omega^{\xi}$, then A factors through a subspace and through a quotient of \mathfrak{G}_{ξ} . Moreover, if X has a shrinking FDD, A may be taken to factor through \mathfrak{G}_{ξ} .*

Proof. Case 1: ξ is a successor, say $\xi = \zeta + 1$. Let $\mathfrak{G}_{\xi} = \mathfrak{U}_{\wedge}^{X_{\zeta, 2}}(\mathbf{U})$. By a technique of Pełczyński, for each $n \in \mathbb{N}$, there exists a finite dimensional space I_n having basis with basis constant not more than 2 and such that $U_n \leq I_n$ and U_n is 2-complemented in I_n . Let $P_n : I_n \rightarrow U_n$ be a projection with norm not more than 2 and let $J_n = \ker(P_n)$. Let $\mathfrak{G}_{\xi} = \mathfrak{G}_{\xi} \oplus_{\infty} (\oplus_{n=1}^{\infty} J_n)_{c_0}$. Then \mathfrak{G}_{ξ} has a shrinking basis and

$$Sz(\mathfrak{G}_{\xi}) = \max\{Sz(\mathfrak{G}_{\xi}), Sz(\oplus_{n=1}^{\infty} J_n)_{c_0}\} = \omega^{\zeta+1}.$$

Now suppose that $A : X \rightarrow Y$ is an operator with separable range and $Sz(A) \leq \omega^{\zeta}$. Then by [10], A factors through a separable Banach space Z with $\mathfrak{p}_{\zeta}(Z) < 2$. By Theorem 5.9, there exists a Banach space W with FDD \mathbf{F} such that Z is isomorphic to both a subspace and to a quotient of $W_{\wedge}^{X_{\zeta, 2}}(\mathbf{F})$. By Proposition 6.1, $W_{\wedge}^{X_{\zeta, 2}}(\mathbf{F})$ is isomorphic to a complemented subspace of \mathfrak{G}_{ξ} , and is therefore isomorphic to a complemented subspace of \mathfrak{G}_{ξ} . Then A factors through Z , which is isomorphic to both a subspace and to a quotient of \mathfrak{G}_{ξ} .

If X has a shrinking FDD, say \mathbf{F} , then by Corollary 5.8 and Remark 6.3, there exists a blocking \mathbf{G} of \mathbf{F} such that A factors through $X_{\wedge}^{X_{\zeta, 2}}(\mathbf{G})$, since $\mathfrak{p}_{\zeta}(A) = 0$. The space $X_{\wedge}^{X_{\zeta, 2}}(\mathbf{G})$ is isomorphic to a complemented subspace of \mathfrak{G}_{ξ} , whence A can be taken to factor through \mathfrak{G}_{ξ} .

Case 2: ξ is a limit ordinal. Let $E = X(\mathcal{G}_n, \vartheta_n)$ be a ξ -well-constructed mixed Schreier space, so that $Sz(E) = \omega^{\xi}$, $\mathfrak{G}_{\xi} = \mathfrak{U}_{\wedge}^E(\mathbf{U})$, and $\mathfrak{G}_{\xi} = \mathfrak{G}_{\xi} \oplus_{\infty} (\oplus_{n=1}^{\infty} J_n)_{c_0}$, where J_n is chosen as in the previous case. Then $Sz(\mathfrak{G}_{\xi}) = \omega^{\xi}$.

Now suppose that $A : X \rightarrow Y$ is a separable range operator with $Sz(A) = \omega^{\zeta} < \omega^{\xi}$. Then A factors through a separable Banach space Z with $Sz(Z) \leq \omega^{\zeta+1}$ by [5], and Z is isomorphic to both a subspace and to a quotient of a Banach space $W_{\wedge}^{X_{\zeta+1}}(\mathbf{F})$, where W is some Banach space and \mathbf{F} is an FDD for W by Theorem 5.9. Then by Proposition 6.1, $W_{\wedge}^{X_{\zeta+1}}(\mathbf{F})$ is isomorphic to a complemented subspace of \mathfrak{G}_{ξ} .

If X has a shrinking FDD, say \mathbf{F} , then by Corollary 5.10 and Remark 6.3, there exists a blocking \mathbf{G} of \mathbf{F} such that A factors through $X_{\wedge}^{X_{\zeta+1}}(\mathbf{G})$. By Proposition 6.1, $X_{\wedge}^{X_{\zeta+1}}(\mathbf{G})$ embeds complementably in \mathfrak{G}_{ξ} , and so A can be taken to factor through \mathfrak{G}_{ξ} in this case.

□

Remark 6.6. By a result of Johnson and Szankowski [20], there is no separable Banach space through which all compact operators factor. A consequence of this result is that for $0 < \xi < \omega_1$, there cannot be a separable Banach space such that every operator with Szlenk index less than ω^ξ factors through that space, since every compact operator has Szlenk index 1. Thus having the operators in Theorem 6.5 factor through a subspace or a quotient rather than through the whole space is necessary.

Remark 6.7. For any ordinal $\xi < \omega_1$ and any Banach space Z with $Sz(Z) = \omega^\xi$, there is a separable Banach space X (which can be taken to be a mixed Schreier space if $0 < \xi$) with $Sz(Z) = Sz(X)$ such that X is not isomorphic to any subspace of any quotient of Z . Indeed, if $\xi = 0$ and $Sz(Z) = \omega^\xi = 1$, $\dim Z < \infty$, and we simply take X to have $\dim Z < \dim X < \infty$.

If $Z = \omega^{\zeta+1}$, we fix $Sz(Z, 1/2) < \gamma < \omega^{\omega^{\zeta+1}}$ and let X be a ξ -well-constructed mixed Schreier space with $Sz(X) = \omega^{\omega^{\zeta+1}}$ such that $Sz(X, (2/3)^n) \geq \gamma^n$. If $\xi = \beta + \gamma$ for $\beta, \gamma < \xi$, we fix $\gamma_n < \omega^\gamma$ such that $Sz(Z, (1/2)^n) < \omega^\beta \gamma_n$ and construct a mixed Schreier space X such that $Sz(X, (2/3)^n) > \omega^\beta \gamma_n$ and $Sz(X) = \omega^\xi$. In either of these two cases, X cannot be isomorphic to a subspace of a quotient of Z , otherwise there would exist some $C \geq 1$ such that

$$Sz(X, \varepsilon) \leq Sz(Z, \varepsilon/C)$$

for all $0 < \varepsilon < 1$. But if $n \in \mathbb{N}$ is such that $(1/2)^n < (2/3)^n/C$, our choice of X yields that

$$\gamma^n \leq Sz(X, (2/3)^n) \leq Sz(Z, (2/3)^n/C) \leq Sz(Z, (1/2)^n) < \gamma^n$$

in the first case, and

$$\omega^\beta \gamma_n \leq Sz(X, (2/3)^n) \leq Sz(Z, (2/3)^n/C) \leq Sz(Z, (1/2)^n) < \omega^\beta \gamma_n$$

in the second case.

Remark 6.8. For $\xi = \omega^\zeta$, ζ a limit ordinal, there is no Banach space with Szlenk index ω^ξ , which is the reason Theorem 6.5 is limited to $\xi \in (0, \omega_1) \setminus \{\omega^\eta : \eta \text{ a limit ordinal}\}$. However, for any countable limit ordinal η , we may fix a sequence $\eta_n \uparrow \eta$ and define $\mathfrak{H}_\eta = (\oplus_{n=1}^\infty \mathfrak{G}_{\omega^{\eta_n}})_{c_0}$. We may define a diagonal operator $A_\eta : \mathfrak{H}_\eta \rightarrow \mathfrak{H}_\eta$ by $A_\eta|_{\mathfrak{G}_{\omega^{\eta_n}}} = \frac{1}{n} I_{\mathfrak{G}_{\omega^{\eta_n}}}$. Then

$$Sz(A_\eta) = \sup_n Sz(\mathfrak{G}_{\omega^{\eta_n}}) = \sup_n \omega^{\omega^{\eta_n}+1} = \omega^{\omega^\eta},$$

and if $A : X \rightarrow Y$ is any operator between separable Banach spaces with $Sz(A) < \omega^{\omega^\eta}$, there exist a subspace Z of \mathfrak{H}_η and operators $R : X \rightarrow Z$, $L : Z \rightarrow Y$ such that $A_\eta(Z) = Z$ and such that $LA_\eta R = A$.

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