A Hopf lemma and regularity for fractional p-Laplacians *

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Abstract

In this paper, we study qualitative properties of the fractional p-Laplacian. Specifically, we establish a Hopf type lemma for positive weak super-solutions of the fractional $p-$Laplacian equation with Dirichlet condition. Moreover, an optimal condition is obtained to ensure $(-\triangle)_p^s u \in C^1(\mathbb{R}^n)$ for smooth functions u.

Keywords: Fractional $p-$Laplacian; Hopf type lemma; regularity.

1 Introduction and main results

The fractional $p-$Laplacian is defined by the singular integral

$$\begin{aligned}(-\triangle)_p^s u(x) :=& \quad C_{n,s,p} P.V. \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy \\ \equiv& \quad C_{n,s,p} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy,\end{aligned} \tag{1.1}$$

where $C_{n,s,p}$ is a positive constant depending only on n, s, and p, $s \in (0,1)$, and $p > 1$. Denote

$$L_{sp}(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \Big| \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{(1+|x|)^{n+sp}} dx < \infty \right\}.$$

If $u \in C^{1,1}_{\text{loc}} \cap L_{sp}(\mathbb{R}^n)$, then (1.1) is well defined. Clearly, when $p=2$, (1.1) becomes the fractional Laplacian which arises in many fields such as phase transitions, flame propagation, stratified materials and others (see [1, 6, 27]). In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Levy process (see [28]). The fractional

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$p-$Laplacian also has many applications, for instance, it is used to study the non-local "Tug-of-War" game (see [2, 3, 22]). The interest on these nonlocal operators continues to grow in recent years. We refer to [24] for the recent progress on these nonlocal operators.

Due to the non-locality of these kinds of operators, many traditional methods in studying the local differential operators no longer work. To overcome this difficulty, Cafarelli and Silvestre [16] introduced the *extension method* which turns nonlocal problems involving the fractional Laplacian ($p = 2$) into local ones in higher dimensions, then the classical theories for local elliptic partial differential equations can be applied. We refer to [5, 15] and references therein for broad applications of this method.

Another useful method to study the fractional Laplacian is the *integral equations method*, which turns a given fractional Laplacian equation into its equivalent integral equation, and then various properties of the original equation can be obtained by investigating the integral equation, see [7, 14, 29] and references therein.

However, so far as we know, there has neither been any extension method nor the integral equations method that work for the fractional $p-$Laplacian equation when $p \neq 2$. The nonlinearity, the singularity ($1 < p < 2$) and degeneracy ($p > 2$) of the operator $(-\triangle)^s_p$ render many powerful methods to study the fractional Laplacian ($p = 2$) no longer effective.

Recently, Chen et al. have developed a *direct method of moving planes* to investigate the nonlocal problems, which can be used to study not only the fractional Laplacian but also the fully nonlinear nonlocal operator

$$F_\alpha(u(x)) := C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy,$$

where $\alpha > 0$, $G : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function. The fractional p-Laplacian is a special case in which $G(t) = |t|^{p-2}t$ and $\alpha = sp$. This direct method has been successfully applied to obtain symmetry, monotonicity, nonexistence and other qualitative properties of solutions for various nonlocal problems, see e.g., [8, 10, 11, 12, 13].

In the present paper, we will continue to study qualitative properties for fractional p-Laplacian. We will establish a Hopf type lemma in general domains for super solutions to fractional p-Laplacian equations with a Dirichlet condition; and for any given smooth function u, we will obtain an optimal condition for $(-\Delta)^s_p u$ to be continuously differentiable.

It is well-known that the Hopf lemma is a very powerful tool in the study of various differential equations. For example, it has been successfully used in the "second" step of the moving planes method.

In the case of fractional Laplacian ($p = 2$), Fall and Jarohs [19, Proposition 3.3] proved a Hopf lemma for the entire antisymmetric supersolution of the problem

$$(-\triangle)^s u(x) = c(x)u(x) \quad \text{in } \Omega. \tag{1.2}$$

Greco and Servadei [20] obtained a Hopf type lemma to (1.2) under the assumptions that $c(x) \leq 0$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. Chen and Li [9] established a Hopf lemma for anti-symmetric function on a half space through a rather delicate analysis. More recently, Jin and Li [23] extended the results in [9] to the fractional $p-$Laplacian with $p > 3$ for positive anti-symmetric functions on the boundary of a half space. In this paper, we shall establish a Hopf type lemma for the positive weak supersolution of (1.3) on the boundary of more general domains.

Before stating our main results, we first introduce some definitions on fractional Sobolev spaces, and one can see [18, 21] for more details. For any domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, define

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \Big| \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy < \infty \right\}$$

equipped with the norm

$$||u||_{W^{s,p}(\Omega)} := ||u||_{L^p(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}},$$

and

$$W_0^{s,p}(\Omega) := \{ u \in W^{s,p}(\mathbb{R}^n) \mid u = 0 \text{ in } \mathbb{R}^n \backslash \Omega \}.$$

If $\Omega \subset \mathbb{R}^n$ is bounded, set

$$\widetilde{W}^{s,p}(\Omega) := \left\{ u \in L^p_{loc}(\mathbb{R}^n) \Big| \, \exists U \supset\supset \Omega \text{ such that } ||u||_{W^{s,p}(U)} + \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{(1+|x|)^{n+sp}} dx < \infty \right\}.$$

If $\Omega \subset \mathbb{R}^n$ is unbounded, set

$$\widetilde{W}^{s,p}_{\text{loc}}(\Omega) := \{ u \in L^p_{\text{loc}}(\mathbb{R}^n) \mid u \in \widetilde{W}^{s,p}(\Omega') \text{ for any } \Omega' \subset\subset \Omega \}.$$

Next, we present two definitions of solutions to fractional $p-$Laplacian equation with Dirichlet condition

$$\begin{cases} (-\triangle)_p^s u(x) = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.3}$$

Definition 1.1. *We say that u is a classical supersolution (subsolution) of the Dirichlet problem* (1.3)*, if* (1.1) *is well-defined for any $x \in \Omega$, moreover, there hold*

$$\begin{cases} (-\triangle)_p^s u(x) \geq (\leq) f(x) & \textit{in } \Omega, \\ u \geq (\leq) 0 & \textit{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.4}$$

Furthermore, if u is both a supersolution and a subsolution of (1.3)*, then we say it is a solution to* (1.3)*.*

Definition 1.2. *Let $f \in W^{-s,p'}(\Omega)$, we say that $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak supersolution of* (1.3)*, if there hold*

$$(u+\epsilon)^- \in W_0^{s,p}(\Omega) \textit{ for any } \epsilon > 0,$$

and

$$C_{n,s,p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+sp}} dxdy \geq \langle f, \phi \rangle$$

for any $\phi \in W_0^{s,p}(\Omega)$ with $\phi \geq 0$ in Ω. The weak subsolution can be defined similarly. Moreover, if u is both a weak supersolution and a weak subsolution of (1.3)*, then we say it is a weak solution to* (1.3)*.*

One of our main results is

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $C^{1,1}$ boundary. If it is bounded, we assume $u \in \widetilde{W}^{s,p}(\Omega) \cap C(\Omega)$; if it is unbounded, we assume $u \in \widetilde{W}^{s,p}_{loc}(\Omega) \cap C(\Omega)$.*

Suppose

$$\begin{cases} (-\triangle)_p^s u \geq 0 & in\ \Omega, \\ u > 0 & in\ \Omega, \\ u = 0 & in\ \mathbb{R}^n \backslash \Omega \end{cases} \tag{1.5}$$

in the weak sense, then

$$\liminf_{d(x)\to 0} \frac{u(x)}{d^s(x)} > 0,$$

where $d(x) := dist(x, \Omega^c)$.

The other main result is concerning the regularity of $(-\Delta)_p^s u$.

The regularity of solutions of the fractional $p-$Laplacian equations has attracted considerable attention in recent years, and it has been well understood for the fractional Laplacian equations ($p = 2$). Specifically, the Schauder interior estimate of the solution is similar to that of the Poisson equation (associated with the regular Laplacian), which states roughly that if $f \in C^\gamma(\Omega)$ and $u \in C^{1,1}_{\text{loc}}(\Omega) \cap L_{2s}(\mathbb{R}^n)$ is a solution of

$$\begin{cases} (-\triangle)^s u(x) = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.6}$$

then the regularity of the solution u can be raised by the order of $2s$ in any proper subset of Ω, the same order as the operator $(-\Delta)^s$. By introducing the proper weighted Hölder norms as in the case of Poisson equations, one shall be able to control a weighted $C^{2s+\gamma}$ norm of u in Ω in terms of another weighted C^γ norm of f in Ω. However, when considering the regularity of the solution up to the boundary, the situation in the fractional order equation is quite different from that in the integer order equation (when $s = 1$, the Poisson equation). In fact, Ros-Oton and Serra [25] proved that if $u \in C^{1,1}_{\text{loc}}(\Omega) \cap L_{2s}(\mathbb{R}^n)$ is a solution of (1.6) with $f \in L^\infty(\Omega)$, then u is C^s up to the boundary; and this is optimal in general. Later, Chen et al. [14] proved the similar results by a simpler method.

For the fractional $p-$Laplacian, the study of the regularity becomes quite complicated. So far there are very few results. Di Castro and Kuusi [17] showed that if $u \in \widetilde{W}^{s,p}(\Omega)$ satisfies $(-\triangle)_p^s u = 0$ in Ω, then u is locally γ-Hölder continuous for small γ. Brasco et al. [4] established a higher Hölder regularity for the fractional $p-$Laplacian equation in the superquadratic case ($p > 2$). Indeed, the authors have verified that if $u \in W^{s,p}_{\text{loc}}(\Omega) \cap L_{sp}(\mathbb{R}^n)$ is a local weak solution of

$$(-\triangle)_p^s u = f \quad \text{in } \Omega, \tag{1.7}$$

where $f \in L^q_{\text{loc}}(\Omega)$ with

$$\begin{cases} q > \frac{n}{sp} & \text{if } sp \leq n, \\ q \geq 1 & \text{if } sp > n, \end{cases} \tag{1.8}$$

then $u \in C^{\delta}_{\text{loc}}(\Omega)$ for every $0 < \delta < \Theta(n, s, p, q)$ with

$$\Theta(n, s, p, q) = \min\left\{\frac{1}{p-1}\left(sp - \frac{n}{q}\right), 1\right\}.$$

Iannizzotto et al. [21] proved that the solutions of (1.7) with $f \in L^{\infty}(\Omega)$ belong to $C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s]$.

Concerning the regularities of $(-\triangle)^s_p u$ for a given smooth function u, there are more substantial technical difficulties than the local case.

For the fractional Laplacian $(-\triangle)^s$, Silvestre [26] has made a comprehensive investigation. More specifically, he has verified that if $u \in L_{2s}(\mathbb{R}^n) \cap C^{2s+\epsilon}$ (or $C^{1,2s+\epsilon-1}$ if $s > 1/2$) for some $\epsilon > 0$ in an open set Ω, then $(-\triangle)^s u$ is a continuous function in Ω for $s \in (0, 1)$. Furthermore, if $u \in C^{k,\alpha}$ and $k + \alpha - 2s$ is not an integer, then $(-\triangle)^s u \in C^{l,\beta}$, where l is the integer part of $k + \alpha - 2s$ and $\beta = k + \alpha - 2s - l$.

While for the fractional $p-$Laplacian, the singularity ($0 < p < 2$) and degeneracy ($p > 2$) of operator $(-\triangle)^s_p$ make it more complex.

For example, even for the local operator $\triangle_p$ and the sufficient smooth function $u(x) = x^2$ in $\mathbb{R}$, $-\triangle_p u(0) = \infty$ if $1 < p < 2$, and $(-\triangle_p u)'(0) = \infty$ if $1 < p < 3$ and $p \neq 2$.

In this paper, we shall consider the differentiability of $(-\triangle)^s_p u$ for $p > 2$ and establish an optimal condition such that $(-\triangle)^s_p u \in C^1(\mathbb{R}^n)$. Specifically, we prove that

Theorem 1.2. *Let $p > 2$, $u \in C^3_{loc}(\mathbb{R}^n) \cap L_{sp}(\mathbb{R}^n)$ and $|\nabla u| \in L_{sp}(\mathbb{R}^n)$. If $p > \frac{3}{2-s}$, then $(-\triangle)^s_p u \in C^1(\mathbb{R}^n)$.*

The condition $p > \frac{3}{2-s}$ is optimal as shown in the following

Theorem 1.3. *Let $p > 2$, $u(x) = \eta(x)x^2$ in $\mathbb{R}$, where $\eta \in C^{\infty}_0(-2, 2)$ is even and satisfies $0 \leq \eta(x) \leq 1$ in $\mathbb{R}$, $\eta(x) = 1$ in $(-1, 1)$ and $|\eta'(x)| \leq 1$ in $\mathbb{R}$. If $p < \frac{3}{2-s}$, then*

$$\lim_{x \to 0+} \left| \left((-\triangle)^s_p u\right)'(x) \right| = \infty. \tag{1.9}$$

And if $p = \frac{3}{2-s}$, then

$$\lim_{x \to 0+} \left((-\triangle)^s_p u\right)'(x) \neq \left((-\triangle)^s_p u\right)'(0). \tag{1.10}$$

The rest of the paper is organized as follows. Section 2 is devoted to establishing the Hopf type lemma for the positive solution of (1.5). In section 3, we first prove the differentiability of $(-\triangle)^s_p u$ under the condition $p > \frac{3}{2-s}$. Then we show that this condition is optimal by giving a counterexample when $p \leq \frac{3}{2-s}$. In the Appendix, we state some results in [21] used in the present paper for convenience.

2 Hopf type lemma

In this section, we prove the Hopf type lemma for the positive weak solution of (1.5) by constructing a suitable subsolution.

Proof of Theorem 1.1. For any given $x_0 \in \partial\Omega$, it follows from the $C^{1,1}$ property of the boundary of Ω that there exist $x_1 \in \Omega$ on the normal line to $\partial\Omega$ at x_0 and a positive constant α such that $B_\alpha(x_1) \subset \Omega$, $\overline{B}_\alpha(x_1) \cap \partial\Omega = x_0$ and $\text{dist}(x_1, \Omega^c) = |x_0 - x_1|$. Without loss of the generality, we suppose that x_0 is the origin, $\alpha = 1$ and $x_1 = e_n$ with $e_n = (0, \cdots, 1)$ the last vector of the canonical basis of $\mathbb{R}^n$. Let $r \in \left(0, \frac{1}{3\sqrt{5}}\right)$ be a constant, O denote the origin and $\eta \in C^2(\mathbb{R}^n)$ satisfy that

$$\eta(X) = 1 \text{ in } B_{2r}(O), \quad \eta(X) = 0 \text{ in } B^c_{3r}(O), \quad \text{and } |\nabla\eta| \leq \frac{1}{r} \text{ in } \mathbb{R}^n. \tag{2.1}$$

Now, define $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Psi(X) = X + \left(1 - X_n - \sqrt{((1 - X_n)^2 - |X'|^2)_+}\right)\eta(X)e_n \quad \text{for any } X \in \mathbb{R}^n, \tag{2.2}$$

where $X = (X', X_n)$. Clearly, it follows from (2.1) and (2.2) that $\Psi(X) = X$ for any $X \in B^c_{3r}(O)$. Since $r < \frac{1}{3\sqrt{5}}$, we have

$$1 - X_n \geq 2|X'| \quad \text{for any } X \in B_{3r}(O), \tag{2.3}$$

which implies

$$\Psi(X) = \begin{cases} X + \left(1 - X_n - \sqrt{(1 - X_n)^2 - |X'|^2}\right)\eta(X)e_n & \text{for any } X \in B_{3r}(O), \\ X & \text{for any } X \in B^c_{3r}(O). \end{cases} \tag{2.4}$$

Next we show the following two claims.

Claim 1. Ψ is a $C^{1,1}$ diffeomorphism of $\mathbb{R}^n$. We firstly show that Ψ is a bijection in $B_{3r}(O)$. Noting that if there exist $X = (X', X_n)$, $Y = (Y', Y_n) \in \mathbb{R}^n$ such that $\Psi(X) = \Psi(Y)$, then $X' = Y'$. For any given $X' \in \mathbb{R}^{n-1}$ with $|X'| \leq 3r$, define

$$h : \{X_n \in \mathbb{R} | (X', X_n) \in B_{3r}(O)\} \to \mathbb{R}$$

as

$$h(X_n) = X_n + \left(1 - X_n - \sqrt{(1 - X_n)^2 - |X'|^2}\right)\eta(X). \tag{2.5}$$

Now we show that h is strictly monotone. Direct calculation implies that

$$h'(X_n) = 1 + \frac{|X'|^2}{1 - X_n + \sqrt{(1 - X_n)^2 - |X'|^2}}\frac{\partial\eta}{\partial X_n}(X) + \left(-1 + \frac{1 - X_n}{\sqrt{(1 - X_n)^2 - |X'|^2}}\right)\eta(X). \tag{2.6}$$

Noting that if $|X'| = 0$ then $h' = 1$. Furthermore, if $|X'| \neq 0$, the last term in (2.6) is positive and the second term can be rewritten as

$$\frac{|X'|}{1 - X_n + \sqrt{(1 - X_n)^2 - |X'|^2}} \cdot \frac{\partial\eta}{\partial X_n}(X)|X'| := J_1 \cdot J_2.$$

It then follows from $|X'| \leq 3r$ and (2.1) that $|J_2| \leq 3$. Moreover, thanks to (2.3), we can verify that $J_1 \leq \frac{1}{2+\sqrt{3}}$. Consequently, there holds

$$h'(X_n) \geq 1 - \frac{3}{2 + \sqrt{3}} > 0,$$

that is, h is strictly increasing in $\{X_n \in \mathbb{R} | (X', X_n) \in B_{3r}(O)\}$. This together with (2.4) shows that $\Psi \in C^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ is a diffeomorphism of $\mathbb{R}^n$.

Claim 2. There holds

$$B_1(e_n) \cap B_r(0) \subset \Psi(B_{\sqrt{2}r}(O) \cap \mathbb{R}^n_+). \tag{2.7}$$

Indeed, it suffices to show that for any $x \in B_1(e_n) \cap B_r(0)$, there is $X \in B_{\sqrt{2}r}(O) \cap \mathbb{R}^n_+$ such that $\Psi(X) = x$. To this end, choose

$$X = \left(x', 1 - \sqrt{(1-x_n)^2 + |x'|^2}\right).$$

Then

$$\begin{aligned} |X|^2 &= |x'|^2 + 1 - 2\sqrt{(1-x_n)^2 + |x'|^2} + (1-x_n)^2 + |x'|^2 \\ &\le 2|x'|^2 + 1 - 2(1-x_n) + (1-x_n)^2 \\ &\le 2|x|^2 \\ &\le 2r^2, \end{aligned}$$

which shows that $X \in B_{\sqrt{2}r}(O) \cap \mathbb{R}^n_+$. It then follows from (2.4) that

$$\begin{aligned} \Psi(X) &= X + \left(1 - X_n - \sqrt{(1-X_n)^2 - |X'|^2}\right) e_n \\ &= \left(x', 1 - \sqrt{(1-x_n)^2 + |x'|^2}\right) + \left(\sqrt{(1-x_n)^2 + |x'|^2} - (1-x_n)\right) e_n \\ &= x. \end{aligned}$$

The claim is then verified.

Now, we define $\rho : \mathbb{R}^n \to \mathbb{R}$ as

$$\rho(x) = \operatorname{dist}(x, B_1^c(e_n)) \quad \text{in } \mathbb{R}^n. \tag{2.8}$$

Then there holds

$$\rho(\Psi(X)) = (X_n)_+ \quad \text{for any } X \in B_{2r}(O). \tag{2.9}$$

Indeed, It follows from (2.1) and (2.4) that

$$\Psi(X) = \left(X', 1 - \sqrt{(1-X_n)^2 - |X'|^2}\right) \quad \text{for any } X \in B_{2r}(O).$$

By a direct calculation, we see that $\Psi(X) \in \partial B_{1-X_n}(e_n)$ for any $X \in B_{2r}(O)$, which together with (2.8) leads to (2.9). Next we show by a similar calculation as in [21] that there exists a positive constant C_1 such that

$$(-\triangle)^s_p \rho^s(x) \le C_1 \quad \text{in } B_1(e_n) \cap B_r(0). \tag{2.10}$$

Indeed, thanks to lemma A.3, we only need to show that there exists $f \in L^\infty(B_1(e_n) \cap B_r(0))$ such that

$$\lim_{\epsilon \to 0} \int_{\{|\Psi^{-1}(x) - \Psi^{-1}(y)| > \epsilon\}} \frac{G(\rho^s(x) - \rho^s(y))}{|x-y|^{n+ps}} dy = f \quad \textit{in } L^1(B_1(e_n) \cap B_r(0)),$$

where $G(t) = |t|^{p-2}t$ for any $t \in \mathbb{R}$. Making a change of variables $X = \Psi^{-1}(x)$, then for any $x \in B_1(e_n) \cap B_r(0)$, there exists $X \in B_{\sqrt{2}r}(O) \cap \mathbb{R}^n_+$ such that $\Psi(X) = x$ and

$$
\begin{aligned}
&\int_{\{|\Psi^{-1}(x)-\Psi^{-1}(y)|>\epsilon\}} \frac{G(\rho^s(x) - \rho^s(y))}{|x-y|^{n+ps}} dy \\
=&\int_{B^c_\epsilon(X)} \frac{G\left(\rho^s(\Psi(X)) - \rho^s(\Psi(Y))\right)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY \\
=&\int_{B^c_\epsilon(X)\cap B_{2r}(O)} \frac{G\left((X_n)^s_+ - (Y_n)^s_+\right)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY + \int_{B^c_{2r}(O)} \frac{G\left(\rho^s(\Psi(X)) - \rho^s(\Psi(Y))\right)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY \\
=&\int_{B^c_\epsilon(X)} \frac{G\left((X_n)^s_+ - (Y_n)^s_+\right)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY \\
&+ \int_{B^c_{2r}(O)} \frac{G\left(\rho^s(\Psi(X)) - \rho^s(\Psi(Y))\right) - G\left((X_n)^s_+ - (Y_n)^s_+\right)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY \\
:=&J_1(X) + J_2(X),
\end{aligned} \tag{2.11}
$$

where the second equality follows from (2.7) and (2.9). Noting that Ψ is a $C^{1,1}$ diffeomorphism of $\mathbb{R}^n$ and $\Psi = I$ in $B^c_{3r}(O)$, Lemma A.2 then yields that there exists $f_1 \in L^\infty(B_1(e_n) \cap B_r(0))$ such that

$$
\begin{aligned}
\lim_{\epsilon\to 0} J_1(X) &= \lim_{\epsilon\to 0} \int_{B^c_\epsilon(X)} \frac{G\left((X_n)^s_+ - (Y_n)^s_+\right)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY \\
&= \lim_{\epsilon\to 0} \int_{\{|\Psi^{-1}(x)-\Psi^{-1}(y)|>\epsilon\}} \frac{G\left((\Psi^{-1}(x)\cdot e_n)^s_+ - (\Psi^{-1}(y)\cdot e_n)^s_+\right)}{|x-y|^{n+sp}} dy \\
&= f_1(\Psi(X)) \quad \text{in } L^1\left(\Psi^{-1}(B_1(e_n) \cap B_r(0))\right).
\end{aligned} \tag{2.12}
$$

Thanks to (2.7), there exists a positive constant C such that for any $x \in B_1(e_n) \cap B_r(0)$ and $Y \in B^c_{2r}(O)$,

$$|X - Y| \geq C(1 + |Y|).$$

Hence, we have

$$
\begin{aligned}
|J_2(X)| &\leq \int_{B^c_{2r}(O)} \frac{|\rho^s(\Psi(X)) - \rho^s(\Psi(Y))|^{p-1} + \left|(X_n)^s_+ - (Y_n)^s_+\right|^{p-1}}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY \\
&\leq C_\Psi \int_{B^c_{2r}(O)} \frac{1}{|X-Y|^{n+s}} dY \\
&\leq C_\Psi \int_{B^c_{2r}(O)} \frac{1}{(1+|Y|)^{n+s}} dY \\
&\leq C_\Psi,
\end{aligned}
$$

where the notation C_Ψ above may denote different positive constants. This together with (2.11) and (2.12) shows that

$$\lim_{\epsilon\to 0} \int_{B^c_\epsilon(X)} \frac{G\left(\rho^s(\Psi(X)) - \rho^s(\Psi(Y))\right)}{|\Psi(X) - \Psi(Y)|^{n+ps}} J_\Psi(Y) dY = f_1(\Psi(X)) + J_2(X)$$

in $L^1\left(\Psi^{-1}(B_1(e_n)\cap B_r(0))\right)$, with $f_1\circ\Psi$ and J_2 belong to $L^\infty\left(\Psi^{-1}(B_1(e_n)\cap B_r(0))\right)$. It then follows that

$$\lim_{\epsilon\to 0}\int_{\{|\Psi^{-1}(x)-\Psi^{-1}(y)|>\epsilon\}}\frac{G(\rho^s(x)-\rho^s(y))}{|x-y|^{n+ps}}dy = f_1(x)+J_2\circ\Psi^{-1}(x)\quad\text{in } L^1(B_1(e_n)\cap B_r(0)).$$

Consequently, (2.10) follows.

Now let $D\subset\subset B_1^c(e_n)\cap\Omega$ be a bounded smooth domain, and $\beta>0$ be a positive constant to be determined below. Set

$$\underline{u}(x)=\beta\rho^s(x)+\chi_D(x)u(x), \tag{2.13}$$

where ρ is defined by (2.8), and χ_D is the characteristic function of D, namely,

$$\chi_D(x)=\begin{cases}1, & x\in D,\\ 0, & x\notin D.\end{cases}$$

It follows from $D\subset\subset B_1^c(e_n)\cap\Omega$ that there is a positive constant C_D such that

$$|x-y|\geq C_D\quad\text{for any } x\in B_1(e_n),\ y\in D. \tag{2.14}$$

For any $x\in B_1(e_n)\cap B_r(0)$, direct calculation (we omit the term '$C_{n,s,p}\lim\limits_{\epsilon\to 0}$' in the following calculation for convenience) shows that

$$\begin{aligned}
(-\triangle)_p^s\underline{u}(x) &= \int_{B_\epsilon^c(x)}\frac{G(\underline{u}(x)-\underline{u}(y))}{|x-y|^{n+ps}}dy\\
&= \int_{B_\epsilon^c(x)}\frac{G\left(\beta\rho^s(x)-\beta\rho^s(y)-\chi_D(y)u(y)\right)}{|x-y|^{n+ps}}dy\\
&= \int_{B_\epsilon^c(x)\cap B_1(e_n)}\frac{G\left(\beta\rho^s(x)-\beta\rho^s(y)\right)}{|x-y|^{n+ps}}dy+\int_{B_1^c(e_n)}\frac{G\left(\beta\rho^s(x)-\chi_D(y)u(y)\right)}{|x-y|^{n+ps}}dy\\
&= \int_{B_\epsilon^c(x)}\frac{G\left(\beta\rho^s(x)-\beta\rho^s(y)\right)}{|x-y|^{n+ps}}dy\\
&\quad+\int_{B_1^c(e_n)}\frac{G\left(\beta\rho^s(x)-\chi_D(y)u(y)\right)-G\left(\beta\rho^s(x)-\beta\rho^s(y)\right)}{|x-y|^{n+ps}}dy\\
&= \beta^{p-1}(-\triangle)_p^s\rho^s(x)+\int_D\frac{G\left(\beta\rho^s(x)-u(y)\right)-G\left(\beta\rho^s(x)\right)}{|x-y|^{n+ps}}dy\\
&\leq \beta^{p-1}C_1+\overline{C}_DA(x),
\end{aligned}\tag{2.15}$$

where

$$A(x)=\int_D G\left(\beta\rho^s(x)-u(y)\right)-G\left(\beta\rho^s(x)\right)dy, \tag{2.16}$$

and the last inequality holds due to (2.10). Let

$$M_0=\min_{x\in D}u(x)>0,\quad \beta\leq\frac{1}{2}M_0,$$

then it follows from the monotonicity of G that

$$A(x) \leq \int_D G(\beta\rho^s(x) - u(y))dy \leq \int_D G\left(\frac{1}{2}M_0 - M_0\right)dy = -\left(\frac{1}{2}\right)^{p-1} M_0^{p-1}|D|. \quad (2.17)$$

It then follows from (2.15), (2.16) and (2.17) that

$$(-\triangle)_p^s \underline{u}(x) \leq M_1\beta^{p-1} - M_2 \quad \text{in } B_1(e_n) \cap B_r(0),$$

where M_1 and M_2 are some positive constants. In view of (2.8) and (2.13), there holds

$$\underline{u}(x) \leq u(x) \ \text{ in } \ B_1^c(e_n).$$

Let

$$M_3 = \inf_{x \in B_1(e_n) \cap B_r^c(0)} u(x) > 0, \quad \beta < \frac{1}{2} \min\left\{M_0, M_3, \left(\frac{M_2}{M_1}\right)^{\frac{1}{p-1}}\right\},$$

then we have

$$\begin{cases} (-\triangle)_p^s \underline{u} < 0 & \text{in } B_1(e_n) \cap B_r(0), \\ \underline{u}(x) \leq u(x) & \text{in } (B_1(e_n) \cap B_r(0))^c. \end{cases}$$

The comparison principle then yields that

$$u(x) \geq \underline{u}(x) \ \text{ in } \ \mathbb{R}^n.$$

By the definition of ρ, we have $\rho(te_n) = d(te_n)$ for any $t \in (0,1)$, and

$$\frac{u(te_n)}{d^s(te_n)} = \frac{u(te_n)}{\rho^s(te_n)} = \beta\frac{u(te_n)}{\underline{u}(te_n)} \geq \beta > 0.$$

The proof is complete. □

3 Regularity

This section is devoted to the study of regularity of $(-\triangle)_p^s u$. We first prove the differentiability of $(-\triangle)_p^s u$ under the assumptions of Theorem 1.2, then we show that the condition $p > \frac{3}{2-s}$ is optimal by giving a counterexample when $p \leq \frac{3}{2-s}$.

Proof of theorem 1.2. For any $x \in \mathbb{R}^n$, by making change of variables, $(-\triangle)_p^s u(x)$ can be rewritten as

$$(-\triangle)_p^s u(x) = \frac{1}{2}C_{n,s,p}\int_{\mathbb{R}^n} \frac{G(u(x) - u(x-y)) + G(u(x) - u(x+y))}{|y|^{n+sp}}dy,$$

where $G(t) = |t|^{p-2}t$ for any $t \in \mathbb{R}$. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1}{|y|^{n+sp}}\left(\frac{\partial G(u(x) - u(x+y))}{\partial x_i} + \frac{\partial G(u(x) - u(x-y))}{\partial x_i}\right)dy \\
=&\int_{|y|\leq 1} \frac{1}{|y|^{n+sp}}\left(\frac{\partial G(u(x) - u(x+y))}{\partial x_i} + \frac{\partial G(u(x) - u(x-y))}{\partial x_i}\right)dy \\
&+\int_{|y|>1} \frac{1}{|y|^{n+sp}}\left(\frac{\partial G(u(x) - u(x+y))}{\partial x_i} + \frac{\partial G(u(x) - u(x-y))}{\partial x_i}\right)dy \\
:=&I_1 + I_2.
\end{aligned} \quad (3.1)$$

By a direct calculation, we obtain

$$\begin{aligned}\frac{\partial G(u(x)-u(x-y))}{\partial x_i} &= (p-1)|u(x)-u(x-y)|^{p-2}\left(\frac{\partial u(x)}{\partial x_i}-\frac{\partial u(x-y)}{\partial x_i}\right) \\ &= (p-1)|u(x)-u(x-y)|^{p-2}(v(x)-v(x-y))\end{aligned} \tag{3.2}$$

and

$$\frac{\partial G(u(x)-u(x+y))}{\partial x_i} = (p-1)|u(x)-u(x+y)|^{p-2}(v(x)-v(x+y)), \tag{3.3}$$

where $v(x) := \frac{\partial u}{\partial x_i}(x)$.

Now, we verify that $|I_2| < \infty$. Indeed,

$$\begin{aligned}|I_2| \le& \int_{|y|>1}\frac{1}{|y|^{n+sp}}\left(\left|\frac{\partial G(u(x)-u(x+y))}{\partial x_i}\right|+\left|\frac{\partial G(u(x)-u(x-y))}{\partial x_i}\right|\right)dy \\ \le& C_p\left(\int_{|y|>1}\frac{|u(x)|^{p-2}|v(x)|+|u(x)|^{p-2}|v(x-y)|}{|y|^{n+sp}}dy\right. \\ &+\int_{|y|>1}\frac{|u(x-y)|^{p-2}|v(x)|+|u(x-y)|^{p-2}|v(x-y)|}{|y|^{n+sp}}dy \\ &+\int_{|y|>1}\frac{|u(x)|^{p-2}|v(x)|+|u(x)|^{p-2}|v(x+y)|}{|y|^{n+sp}}dy \\ &\left.+\int_{|y|>1}\frac{|u(x+y)|^{p-2}|v(x)|+|u(x+y)|^{p-2}|v(x+y)|}{|y|^{n+sp}}dy\right).\end{aligned}$$

It follows from $u \in L_{sp}(\mathbb{R}^n)$, $|\nabla u| \in L_{sp}(\mathbb{R}^n)$ and the Hölder inequality that $|I_2| < \infty$.

For the term I_1, using the Taylor expansion formula, there hold

$$\begin{aligned}\frac{\partial G(u(x)-u(x+y))}{\partial x_i} =&(p-1)|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}(-\nabla v(x)\cdot y+O(|y|^2)) \\ =&(p-1)|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}(-\nabla v(x)\cdot y) \\ &+(p-1)|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}O(|y|^2) \\ :=&(p-1)J_1+(p-1)J_2,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial G(u(x)-u(x-y))}{\partial x_i} =&(p-1)|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}(\nabla v(x)\cdot y) \\ &+(p-1)|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}O(|y|^2) \\ :=&(p-1)J_3+(p-1)J_4,\end{aligned}$$

where the notation $O(|y|^2)$ denotes that there exist some positive constant C such that $|O(|y|^2)| \le C|y|^2$. Consequently, there is a positive constant C such that

$$|J_2|+|J_4| \le C|y|^p. \tag{3.4}$$

Now, we consider the terms J_1 and J_3.

Case 1. $\nabla u(x)\cdot y=0$. Then it follows from the definitions of J_1 and J_3 that there exist two positive constants C_1 and C_3 such that

$$|J_1|<C_1|y|^{2p-3} \text{ and } |J_3|<C_3|y|^{2p-3}.$$

Cases 2. $\nabla u(x)\cdot y\neq 0$. Then we rewrite J_1 and J_3 respectively as

$$\begin{aligned}J_1&=|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}(-\nabla v(x)\cdot y)\\&=\left(|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}-|\nabla u(x)\cdot y|^{p-2}\right)(-\nabla v(x)\cdot y)-|\nabla u(x)\cdot y|^{p-2}\nabla v(x)\cdot y,\end{aligned}$$

and

$$J_3=\left(|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}-|\nabla u(x)\cdot y|^{p-2}\right)(\nabla v(x)\cdot y)+|\nabla u(x)\cdot y|^{p-2}\nabla v(x)\cdot y.$$

It follows that

$$\begin{aligned}|J_1+J_3|\leq&C\left(|\nabla u(x)\cdot y+O(|y|^2)|^{p-2}-|\nabla u(x)\cdot y|^{p-2}\right)|\nabla v(x)\cdot y|\\\leq&C|\nabla u(x)\cdot y|^{p-4}|2O(|y|^2)\nabla u(x)\cdot y+O(|y|^4)||\nabla v(x)\cdot y|\\\leq&C|y|^p.\end{aligned}$$

To summary, we conclude that there exists a positive constant C independent of y such that

$$|J_1+J_3|+|J_2|+|J_4|\leq C(|y|^{2p-3}+|y|^p) \text{ for any } y\in B_1(0).$$

The assumption $p>\frac{3}{2-s}$ further implies

$$|I_1|\leq(p-1)\int_{|y|\leq 1}\frac{1}{|y|^{n+sp}}(|J_2|+|J_4|+|J_1+J_3|)<\infty,$$

that is,

$$\int_{\mathbb{R}^n}\frac{1}{|y|^{n+sp}}\left|\frac{\partial G(u(x)-u(x+y))}{\partial x_i}+\frac{\partial G(u(x)-u(x-y))}{\partial x_i}\right|dy<\infty \text{ for any } x\in\mathbb{R}^n.$$

By exchanging the order of integration and differentiation, we derive that $(-\triangle)_p^s u$ is differentiable in $\mathbb{R}^n$, and then we conclude $(-\triangle)_p^s u\in C(\mathbb{R}^n)$ by exchanging the order of integration and limit. The proof is complete. □

Theorem 1.2 verifies that in the case $p>2$, if one assumes in addition that $p>\frac{3}{2-s}$, then $(-\triangle)_p^s u\in C^1(\mathbb{R}^n)$ for any u satisfying $u\in C^3_{\text{loc}}(\mathbb{R}^n)\cap L_{sp}(\mathbb{R}^n)$ and $|\nabla u|\in L_{sp}(\mathbb{R}^n)$. It seems from the proof of Theorem 1.2 that $p>\frac{3}{2-s}$ is a technical assumption. While, the counterexample in Theorem 1.3 shows that this condition is optimal to ensure $(-\triangle)_p^s u\in C(\mathbb{R}^n)$ for any u satisfying $u\in C^3_{\text{loc}}(\mathbb{R}^n)\cap L_{sp}(\mathbb{R}^n)$ and $|\nabla u|\in L_{sp}(\mathbb{R}^n)$.

Proof of Theorem 1.3. By virtue of the definition, we have

$$(-\triangle)_p^s u(x)=\frac{1}{2}C_{s,p}\int_{-\infty}^{+\infty}\frac{G(u(x)-u(x+y))+G(u(x)-u(x-y))}{|y|^{1+sp}}dy.$$

For the convenience of writing, we set

$$F(x,y) := \frac{G(u(x)-u(x+y))+G(u(x)-u(x-y))}{|y|^{1+sp}}. \tag{3.5}$$

It follows from a straightforward calculation that

$$\begin{aligned}\frac{\partial F(x,y)}{\partial x} =& \frac{(p-1)}{|y|^{1+sp}}\big(|u(x)-u(x+y)|^{p-2}(u'(x)-u'(x+y))\\ &+|u(x)-u(x-y)|^{p-2}(u'(x)-u'(x-y))\big)\\ =&\frac{(p-1)}{|y|^{1+sp}}|\eta(x)x^2-\eta(x+y)(x+y)^2|^{p-2}\\ &\times\big[\eta'(x)x^2+2\eta(x)x-\eta'(x+y)(x+y)^2-2\eta(x+y)(x+y)\big]\\ &+\frac{(p-1)}{|y|^{1+sp}}|\eta(x)x^2-\eta(x-y)(x-y)^2|^{p-2}\\ &\times\big[\eta'(x)x^2+2\eta(x)x-\eta'(x-y)(x-y)^2-2\eta(x-y)(x-y)\big].\end{aligned} \tag{3.6}$$

Let

$$f(x,y)=|\eta(x)x^2-\eta(x+y)(x+y)^2|^{p-2}\big[\eta'(x)x^2+2\eta(x)x-\eta'(x+y)(x+y)^2-2\eta(x+y)(x+y)\big],$$

then we can rewrite (3.6) as

$$\frac{\partial F(x,y)}{\partial x}=\frac{(p-1)}{|y|^{1+sp}}[f(x,y)+f(x,-y)].$$

Note that for any $0<x<\frac{1}{8}$, there hold

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{\partial F(x,y)}{\partial x}dy\\ &=(p-1)\int_{-\infty}^{+\infty}\frac{f(x,y)+f(x,-y)}{|y|^{1+sp}}dy\\ &=2(p-1)\int_{0}^{+\infty}\frac{f(x,y)+f(x,-y)}{|y|^{1+sp}}dy\\ &=2(p-1)\left(\int_0^{\frac{1}{2}}\frac{f(x,y)+f(x,-y)}{|y|^{1+sp}}dy+\int_{\frac{1}{2}}^{\frac{5}{2}}\frac{f(x,y)+f(x,-y)}{|y|^{1+sp}}dy\right.\\ &\qquad\left.+\int_{\frac{5}{2}}^{\infty}\frac{f(x,y)+f(x,-y)}{|y|^{1+sp}}dy\right)\\ &:=2(p-1)(I_1+I_2+I_3).\end{aligned} \tag{3.7}$$

For I_3, in view of $y>\frac{5}{2}$ and $0<x<\frac{1}{8}$, there hold $|x-y|>2$ and $|x+y|>2$, which along with the properties of η implies that

$$f(x,y)+f(x,-y)=4x^{2p-3}.$$

Hence, we have

$$I_3 = \int_{\frac{5}{2}}^{\infty} \frac{4x^{2p-3}}{y^{1+sp}} dy = C_1 x^{2p-3}, \tag{3.8}$$

where C_1 is a positive constant independent of x.

For I_2, thanks to $\frac{1}{2} < y < \frac{5}{2}$ and $0 < x < \frac{1}{8}$, there exists a positive constant C_2 independent of x such that

$$|I_2| < \infty. \tag{3.9}$$

It remains to estimate the term I_1. By virtue of $0 < x < \frac{1}{8}$ and $0 < y < \frac{1}{2}$, we see $|x-y| < 1$ and $|x+y| < 1$, which together with the properties of η yields

$$f(x,y) + f(x,-y) = 2y|y^2 - 2xy|^{p-2} - 2y|y^2 + 2xy|^{p-2}.$$

Therefore, for any $x \in (0, \frac{1}{8})$, there hold

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \frac{2y|y^2 - 2xy|^{p-2} - 2y|y^2 + 2xy|^{p-2}}{|y|^{1+sp}} dy \\
&= \int_0^{\frac{1}{2}} \frac{2}{y^{2+sp-p}} (|y-2x|^{p-2} - |y+2x|^{p-2}) dy \\
&= 2\int_0^{\frac{1}{2x}} \frac{x}{(xz)^{2+sp-p}} (|xz-2x|^{p-2} - |xz+2x|^{p-2}) dz \\
&= \frac{2}{x^{sp-2p+3}} \int_0^{\frac{1}{2x}} \frac{|z-2|^{p-2} - |z+2|^{p-2}}{z^{2+sp-p}} dz \\
&= \frac{2}{x^{sp-2p+3}} \int_0^2 \frac{(2-z)^{p-2} - (z+2)^{p-2}}{z^{2+sp-p}} dz \\
&\quad + \frac{2}{x^{sp-2p+3}} \int_2^{\frac{1}{2x}} \frac{(z-2)^{p-2} - (z+2)^{p-2}}{z^{2+sp-p}} dz \\
&= \frac{2}{x^{sp-2p+3}} \int_0^2 \frac{(2-z)^{p-2} - 2^{p-2} + 2^{p-2} - (z+2)^{p-2}}{z^{2+sp-p}} dz \\
&\quad + \frac{2}{x^{sp-2p+3}} \int_2^{\frac{1}{2x}} \frac{(z-2)^{p-2} - (z+2)^{p-2}}{z^{2+sp-p}} dz \\
&= \frac{2}{x^{sp-2p+3}} \int_0^2 \frac{2(2-p)2^{p-3}z + O(|z|^2)}{z^{2+sp-p}} dz \\
&\quad + \frac{2}{x^{sp-2p+3}} \int_2^{\frac{1}{2x}} \frac{(z-2)^{p-2} - (z+2)^{p-2}}{z^{2+sp-p}} dz \\
&< \infty,
\end{aligned} \tag{3.10}$$

which along with (3.7), (3.8) and (3.9) shows that for any fixed $x \in (0, \frac{1}{8})$, there holds

$$\left| \int_{-\infty}^{+\infty} \frac{\partial F(x,y)}{\partial x} dy \right| < \infty.$$

By exchanging the order of integration and differentiation, we derive that $((-\triangle)_p^s u)'$ is well-defined for any $x \in (0, \frac{1}{8})$, and

$$((-\triangle)_p^s u)'(x) = \frac{1}{2} C_{s,p} \int_{-\infty}^{+\infty} \frac{\partial F(x,y)}{\partial x} dy.$$

If $p < \frac{3}{2-s}$, then (3.10) implies that

$$\begin{aligned}
I_1 &= \frac{2}{x^{sp-2p+3}} \int_0^2 \frac{(2-z)^{p-2} - (z+2)^{p-2}}{z^{2+sp-p}} dz \\
&\quad + \frac{2}{x^{sp-2p+3}} \int_2^{\frac{1}{2x}} \frac{(z-2)^{p-2} - (z+2)^{p-2}}{z^{2+sp-p}} dz \\
&\le \frac{2}{x^{sp-2p+3}} \int_0^2 \frac{(2-z)^{p-2} - 2^{p-2}}{z^{2+sp-p}} dz \\
&= \frac{2}{x^{sp-2p+3}} \int_0^2 \frac{(2-p)2^{p-3} z + O(z^2)}{z^{2+sp-p}} dz \\
&\le -C x^{2p-sp-3},
\end{aligned}$$

which verifies that

$$\lim_{x \to 0+} I_1 = -\infty.$$

Consequently, we conclude that

$$\begin{aligned}
\lim_{x \to 0+} ((-\triangle)_p^s u)'(x) &= \lim_{x \to 0+} \frac{1}{2} C_{s,p} \int_{-\infty}^{+\infty} \frac{\partial F(x,y)}{\partial x} dy \\
&= \lim_{x \to 0+} C_{s,p}(p-1)(I_1 + I_2 + I_3) \\
&= -\infty,
\end{aligned}$$

that is, (1.9) holds.

In the case $p = \frac{3}{2-s}$, we first prove that $\left((-\triangle)_p^s u\right)'(0) = 0$. In fact,

$$\begin{aligned}
\left((-\triangle)_p^s u\right)'_+(0) &= \lim_{x \to 0+} \frac{(-\triangle)_p^s u(x) - (-\triangle)_p^s u(0)}{x} \\
&= \frac{1}{2} C_{s,p} \lim_{x \to 0+} \frac{1}{x} \int_{-\infty}^{+\infty} F(x,y) - F(0,y) dy \\
&= C_{s,p} \lim_{x \to 0+} \frac{1}{x} \int_0^{+\infty} F(x,y) - F(0,y) dy \\
&= C_{s,p} \Big(\lim_{x \to 0+} \int_0^{\frac{1}{2}} \frac{F(x,y) - F(0,y)}{x} dy \\
&\qquad + \lim_{x \to 0+} \int_{\frac{1}{2}}^{\frac{5}{2}} \frac{F(x,y) - F(0,y)}{x} dy \\
&\qquad + \lim_{x \to 0+} \int_{\frac{5}{2}}^{+\infty} \frac{F(x,y) - F(0,y)}{x} dy \Big) \\
&:= C_{s,p}(J_1 + J_2 + J_3)
\end{aligned}$$

For J_3, in view of $y > \frac{5}{2}$ and $0 < x < \frac{1}{8}$, there hold $|x-y| > 2$ and $|x+y| > 2$, which along with the properties of η and $sp = 2p-3$ implies that

$$F(x,y) = \frac{2x^{2p-2}}{y^{2p-2}} \text{ and } F(0,y) = 0.$$

It then follows that

$$J_3 = 2\lim_{x\to 0+} x^{2p-3}\int_{\frac{5}{2}}^{+\infty} \frac{1}{y^{2p-2}}dy = 0. \tag{3.11}$$

For J_2, by exchanging the order of integration and limit, we have

$$\begin{aligned}
J_2 &= \int_{\frac{1}{2}}^{\frac{5}{2}} \frac{\partial F}{\partial x}(0,y)dy \\
&= (p-1)\int_{\frac{1}{2}}^{\frac{5}{2}} \Bigg(\frac{|\eta(y)y^2|^{p-2}\left(-y^2\eta'(y) - 2y\eta(y)\right)}{y^{2p-2}} \\
&\qquad + \frac{|\eta(-y)y^2|^{p-2}\left(-y^2\eta'(-y) + 2y\eta(-y)\right)}{y^{2p-2}} \Bigg) dy
\end{aligned} \tag{3.12}$$

Since $\eta(y) = \eta(-y)$ in $\mathbb{R}$, there holds $\eta'(y) = -\eta'(-y)$, which along with (3.12) implies

$$J_2 = 0.$$

As for J_1, we see

$$\begin{aligned}
J_1 &= \lim_{x\to 0+} \frac{1}{x}\int_0^{\frac{1}{2}} F(x,y) - F(0,y)dy \\
&= \lim_{x\to 0+} \frac{1}{x}\int_0^{\frac{1}{2}} \frac{\partial F}{\partial x}(0,y)x + O(x^2)dy \\
&= \int_0^{\frac{1}{2}} \frac{\partial F}{\partial x}(0,y)dy \\
&= (p-1)\int_0^{\frac{1}{2}} \frac{|y^2|^{p-2}(-2y) + |y^2|^{p-2}(2y)}{y^{2p-2}}dy \\
&= 0.
\end{aligned}$$

To summary, we conclude that

$$\left((-\triangle)_p^s u\right)'_+(0) = 0.$$

Similarly, we can prove

$$\left((-\triangle)_p^s u\right)'_-(0) = 0.$$

It then follows that

$$\left((-\triangle)_p^s u\right)'(0) = 0.$$

On the other hand, (3.8) implies that

$$\lim_{x\to 0+} I_3 = 0. \tag{3.13}$$

Thanks to (3.9), by exchanging the order of integration and limit, we obtain

$$\lim_{x\to 0+} I_2 = 0. \tag{3.14}$$

A similar calculation to (3.10) implies that for any $x \in (0, \frac{1}{8})$,

$$\begin{aligned}
I_1 &= 2\int_0^{\frac{1}{2x}} \frac{|z-2|^{p-2} - |z+2|^{p-2}}{z^{p-1}} dz \\
&= 2\int_0^{2} \frac{(2-z)^{p-2} - (z+2)^{p-2}}{z^{p-1}} dz \\
&\quad + 2\int_2^{\frac{1}{2x}} \frac{(z-2)^{p-2} - (z+2)^{p-2}}{z^{p-1}} dz \\
&< 2\int_0^{2} \frac{(2-z)^{p-2} - (z+2)^{p-2}}{z^{p-1}} dz \\
&< 2\int_1^{2} \frac{(2-z)^{p-2} - (z+2)^{p-2}}{z^{p-1}} dz \\
&\le \frac{2(1-3^{p-2})}{2^{p-1}} \\
&< 0.
\end{aligned} \tag{3.15}$$

To summary, we derive that

$$\begin{aligned}
\lim_{x\to 0+}((-\triangle)_p^s u)'(x) &= \frac{1}{2} C_{s,p} \lim_{x\to 0+} \int_{-\infty}^{+\infty} \frac{\partial F(x,y)}{\partial x} dy \\
&= C_{s,p}(p-1) \lim_{x\to 0+} (I_1 + I_2 + I_3) \\
&< C_{s,p}(p-1) \frac{2(1-3^{p-2})}{2^{p-1}} \\
&< 0.
\end{aligned}$$

The proof is complete. □

Appendix A

In this Appendix, we list some results in [21] that were used in the proof of Theorem 1.1. The first one is the weak comparison principle.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume $u, v \in \widetilde{W}^{s,p}(\Omega)$ satisfy, in the weak sense,*

$$\begin{cases} (-\triangle)_p^s u \ge (-\triangle)_p^s v & \text{in } \Omega, \\ u \ge v & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases}$$

Then

$$u \ge v \quad a.e. \text{ in } \Omega.$$

Another key ingredient is the following "change of variables" lemma.

Lemma A.2. *Let* Ψ *be a* $C^{1,1}$ *diffeomorphism of* $\mathbb{R}^n$ *such that* $\Psi = I$ *in* $B_r^c(0)$, $r > 0$. *Then the function* $v(x) = (\Psi^{-1}(x) \cdot e_n)_+^s$ *belongs to* $\widetilde{W}_{loc}^{s,p}(\mathbb{R}^n)$ *and is a weak solution of*

$$(-\triangle)_p^s v = f \quad in\ \Psi(\mathbb{R}_+^n),$$

with

$$||f||_\infty \leq C\left(||D\Psi||_\infty, ||D\Psi^{-1}||_\infty, r\right)||D^2\Psi||_\infty,$$

where $C\left(||D\Psi||_\infty, ||D\Psi^{-1}||_\infty, r\right)$ *is a positive constant. Moreover,*

$$\lim_{\epsilon\to 0} C_{n,s,p}\int_{\{|\Psi^{-1}(x)-\Psi^{-1}(y)|>\epsilon\}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{n+sp}}dy = f \ \ in\ L_{loc}^1(\Psi(\mathbb{R}_+^n)). \quad \text{(A.1)}$$

Remark A.1. *The equality* (A.1) *follows from the proof of "change of variables" lemma.*

The following lemma implies that the point-wise solution is also a weak solution.

Lemma A.3. *Let* $u \in \widetilde{W}_{loc}^{s,p}(\Omega)$ *and* D *denote the diagonal of* $\mathbb{R}^n \times \mathbb{R}^n$. *For any* $\epsilon > 0$, *assume* $A_\epsilon \subset \mathbb{R}^n \times \mathbb{R}^n$ *is a neighborhood of* D *and satisfies*

(i) $(x, y) \in A_\epsilon$ *for all* $(y, x) \in A_\epsilon$,

(ii) $\sup_{x\in A_\epsilon} dist(x, D) \to 0$ *as* $\epsilon \to 0$.

For any $x \in \mathbb{R}^n$, *we set* $A_\epsilon(x) = \{y \in \mathbb{R}^n | (x, y) \in A_\epsilon\}$ *and*

$$g_\epsilon(x) = C_{n,s,p}\int_{A_\epsilon^c(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}dy.$$

If $g_\epsilon \to f$ *in* $L_{loc}^1(\Omega)$, *then* u *is a weak solution of*

$$(-\triangle)_p^s u = f \quad in\ \Omega.$$

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