Convergence rates in the law of large numbers and new kinds of convergence of random variables

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Abstract In this paper, we first study convergence rates in the law of large numbers for independent and identically distributed random variables. We obtain a strong L^p -convergence version and a strongly almost sure convergence version of the law of large numbers. Second, we investigate several new kinds of convergence of random variables and discuss their relations and properties.

Keywords Law of large numbers, strongly almost sure convergence, strong convergence in distribution, strong L^p -convergence.

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1 Introduction

It is well known that limit theorems play an important role in the probability theory and statistics. Let (Ω, \mathcal{F}, P) be a probability space and $\{X, X_n, n \ge 1\}$ be a sequence of random variables. We have different kinds of convergence:

- $\{X_n, n \ge 1\}$ is said to almost surely converge to X, if there exists a set $N \in \mathcal{F}$ such that P(N) = 0 and $\forall \omega \in \Omega \setminus N$, $\lim_{n \to \infty} X_n(\omega) = X(\omega)$, which is denoted by $X_n \xrightarrow{a.s.} X$ or $X_n \to X$ a.s..
- $\{X_n, n \ge 1\}$ is said to converge to X in probability, if for any $\varepsilon > 0$, $\lim_{n\to\infty} P(\{|X_n X| \ge \varepsilon\}) = 0$, which is denoted by $X_n \xrightarrow{P} X$.
- $\{X_n, n \ge 1\}$ is said to L^p -converge to X (p > 0) if $\lim_{n\to\infty} E[|X_n X|^p] = 0$, which is denoted by $X_n \xrightarrow{L^p} X$.
- $\{X_n, n \ge 1\}$ is said to L^{∞} -converge to X if $\lim_{n\to\infty} ||X_n X||_{\infty} = 0$, which is denoted by $X_n \xrightarrow{L^{\infty}} X$.
- $\{X_n, n \ge 1\}$ is said to converge to X in distribution, if for any bounded continuous function f, $\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$, which is denoted by $X_n \xrightarrow{d} X$.
- $\{X_n, n \ge 1\}$ is said to completely converge to X, if for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(\{|X_n X| \ge \varepsilon\}) < \infty$, which is denoted by $X_n \xrightarrow{c.c.} X$ (see [13]).
- $\{X_n, n \ge 1\}$ is said to S-L^p converge to X (p > 0) if $\sum_{n=1}^{\infty} E[|X_n X|^p] < \infty$, which is denoted by $X_n \xrightarrow{S-L^p} X$ (see [17, Definition 1.4]).

The relations among the different kinds of convergence can be described as follows.

$$X_n \xrightarrow{S-L^p} X \implies X_n \xrightarrow{c.c.} X \implies X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X \implies X_n \xrightarrow{d} X,$$
$$\bigwedge^{\uparrow} X_n \xrightarrow{L^\infty} X \implies X_n \xrightarrow{L^p} X$$

and

• if $X_n \xrightarrow{P} X$, then there exists a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $X_{n_k} \xrightarrow{a.s.} X$ as $k \to \infty$;

- if $X_n \xrightarrow{d} C$, where C is a constant, then $X_n \xrightarrow{P} C$;
- if $X_n \xrightarrow{d} X$, then by Skorokhod's theorem, there exists a sequence of random variables $\{Y, Y_n, n \ge 1\}$ such that for any $n \ge 1$, X_n and Y_n have the same distribution, X and Y have the same distribution, and $Y_n \xrightarrow{a.s.} Y$.

In virtue of the relation between convergence in probability and complete convergence, the relation between L^p convergence and S- L^p convergence, we introduce two new kinds of convergence of random variables, which are stronger versions of a.s. convergence and L^{∞} convergence, respectively.

Definition 1.1 Let $\alpha > 0$. $\{X_n, n \ge 1\}$ is said to strongly almost surely converge to X with order α , if

$$\sum_{n=1}^{\infty} |X_n - X|^{\alpha} < \infty \ a.s.,$$

which is denoted by $X_n \stackrel{S_{\alpha} \text{-a.s.}}{\longrightarrow} X$.

Definition 1.2 $\{X_n, n \ge 1\}$ is said to strongly L^{∞} -converge to X if

$$\sum_{n=1}^{\infty} \|X_n - X\|_{\infty} < \infty,$$

which is denoted by $X_n \xrightarrow{S-L^{\infty}} X$.

We now introduce two new kinds of convergence which are stronger versions of convergence in distribution.

Definition 1.3 $\{X_n, n \ge 1\}$ is said to S_1 -d converge to X, if for any bounded Lipschitz continuous function f,

$$\sum_{n=1}^{\infty} |E[f(X_n) - f(X)]| < \infty,$$

which is denoted by $X_n \xrightarrow{S_1 - d} X$.

Definition 1.4 Let F_n and F be the distribution functions of X_n and X, respectively. $\{X_n, n \ge 1\}$ is said to S_2 -d converge to X, if for any continuous point x of F,

$$\sum_{n=1}^{\infty} |F_n(x) - F(x)| < \infty,$$

which is denoted by $X_n \xrightarrow{S_2 - d} X$.

The rest of this paper is organized as follows. In Section 2, we study the law of large numbers for independent and identically distributed (i.i.d.) random variables. In particular, we obtain a strong L^p -convergence version and a strongly almost sure convergence version of the law of large numbers. In Section 3, we discuss the relations among several kinds of convergence. In Section 4, we present some open questions for further research.

2 Convergence rates in the law of large numbers

Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables. Define $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Hsu and Robbins ([13]) proved that if $E[X^2] < \infty$ and $E[X] = \mu$, then $\frac{S_n}{n} \xrightarrow{c.c.} \mu$. Erdös ([8]) proved the converse result. Baum and Katz ([1]) extended the Hsu-Robbins-Erdös theorem. Below is a special case of the Baum-Katz theorem.

Theorem 2.1 (Baum and Katz [1]). Let $\alpha \geq 1$. Suppose that $\{X, X_n, n \geq 1\}$ is a sequence of *i.i.d.* random variables with partial sum $S_n = \sum_{i=1}^n X_i$, $n \in \mathbb{N}$. Then, the condition $E|X|^{\alpha} < \infty$ and EX = 0 is equivalent to

$$\sum_{n=1}^{\infty} n^{\alpha-2} P(|S_n| > n\epsilon) < \infty, \quad \forall \epsilon > 0.$$

Lanzinger ([16]), Gut and Stadtmüller ([12]), Chen and Sung ([4]) extended the results of Baum and Katz.

Chow ([6]) first investigated the complete moment convergence and obtained the following result. Let $\alpha \ge 1$, $p \le \alpha$ and p < 2. Suppose that $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables with E[X] = 0. If $E[|X|^{\alpha} + |X|\log^{+}|X|] < \infty$, then

$$\sum_{n=1}^{\infty} n^{\frac{\alpha}{p} - \frac{1}{p} - 2} E\left[\left(|S_n| - \varepsilon n^{\frac{1}{p}}\right)^+\right] < \infty \text{ for all } \varepsilon > 0,$$

where $x^{+} = \max\{0, x\}.$

Chow's result has been generalized in various directions. Wang and Su ([25]), Wang et al. ([27]), Chen ([2]), Guo and Xu ([11]), Rosalsky et al. ([24]), Ye and Zhu ([28]), and Qiu et al. ([22]) studied the complete moment convergence for sums of Banach space valued random elements. Li and Zhang ([19]), Chen et al. ([3]), Kim et al. ([15]), and Zhou ([31]) considered the complete moment convergence for moving average processes. Jiang and Zhang ([14]), Li ([18]), Liu and Lin ([21]), Ye et al. ([29]), Fu and Zhang ([9]), Zhao and Tao ([30]), and Chen and Zhang ([5]) studied precise asymptotics for the complete moment convergence. Wang and Zhao ([26]), Liang et al. ([20]), and Guo ([10]) considered the complete moment convergence for negatively associated random variables. Qiu and Chen ([22]) studied the complete moment convergence for i.i.d. random variables and extended two results in Gut and Stadtmüller ([12]) to the complete moment convergence.

In the following of this section, we study convergence rates in the law of large numbers for i.i.d. random variables. In particular, we obtain a strong L^p -convergence version and a strongly almost sure convergence version of the law of large numbers.

2.1 Strong L^p -convergence version of the law of large numbers

Let $\{Y, Y_n, n \ge 1\}$ be a sequence of random variables and p > 0. We have

$$Y_n \stackrel{S-L^p}{\longrightarrow} Y \Rightarrow Y_n \stackrel{c.c.}{\longrightarrow} Y.$$
(2.1)

In this subsection, we consider the following question:

Does it hold that $\frac{S_n}{n} \xrightarrow{S-L^p} \mu$ for some p > 0 under some condition?

By the Hsu-Robbins-Erdös theorem and (2.1), we know that the condition $E[X^2] < \infty$ is needed in order that $\frac{S_n}{n} \xrightarrow{S-L^p} \mu$.

Theorem 2.2 (1) If $E[X^2] < \infty$ and $X \neq \mu$ a.s., then $\frac{S_n}{n} \stackrel{S-L^p}{\nrightarrow} \mu$ for any 0 . $(2) If <math>\alpha > 2$ and $E[|X|^{\alpha}] < \infty$, then $\frac{S_n}{n} \stackrel{S-L^p}{\to} \mu$ for any 2 .

Proof. We assume without loss of generality that $\mu = 0$ and $E[X^2] = 1$.

(1) We have

$$E\left[\left|\frac{S_n}{n}\right|^p\right] = \frac{1}{n^{p/2}}E\left[\left|\frac{S_n}{\sqrt{n}}\right|^p\right].$$

Denote the distribution function of $\frac{S_n}{\sqrt{n}}$ by F_n . Let $f \in C_c(\mathbb{R})$ satisfying $|f(x)| = |x|^p$ for $|x| \le 1$ and $|f(x)| \le |x|^p$ for |x| > 1. Then, by the central limit theorem, we have

$$E\left[\left|\frac{S_n}{\sqrt{n}}\right|^p\right] \ge \int_{-\infty}^{\infty} |f(x)| dF_n(x) \to \int_{-\infty}^{\infty} |f(x)| \frac{1}{\sqrt{2\pi}} e^{-|x|^2/2} dx > 0.$$

Define

$$c = \int_{-\infty}^{\infty} |f(x)| \frac{1}{\sqrt{2\pi}} e^{-|x|^2/2} dx.$$

Then, there exists $N \in \mathbb{N}$ such that

$$E\left[\left|\frac{S_n}{\sqrt{n}}\right|^p\right] \ge \frac{c}{2}, \quad \forall n \ge N.$$

Therefore,

$$\sum_{n=1}^{\infty} E\left[\left|\frac{S_n}{n}\right|^p\right] \ge \sum_{n=N}^{\infty} \frac{1}{n^{p/2}} E\left[\left|\frac{S_n}{\sqrt{n}}\right|^p\right] \ge \frac{c}{2} \sum_{n=N}^{\infty} \frac{1}{n^{p/2}} = \infty.$$

(2) If Y is a random variable, we denote $||Y||_{L^r} := (E[|Y|^r])^{1/r}$ for $r \ge 1$. By the Burkholder-Davis-Gundy inequality and Minkowski's inequality, we have

$$E[|S_{n}|^{\alpha}] \leq cE[(X_{1}^{2} + \dots + X_{n}^{2})^{\alpha/2}]$$

= $c||X_{1}^{2} + \dots + X_{n}^{2}||_{L^{\alpha/2}}^{\alpha/2}$
 $\leq c(||X_{1}^{2}||_{L^{\alpha/2}} + \dots + ||X_{n}^{2}||_{L^{\alpha/2}})^{\alpha/2}$
= $cn^{\alpha/2}E[|X|^{\alpha}],$ (2.2)

where c > 0 is a constant, which is independent of n. Then,

$$\sum_{n=1}^{\infty} E\left[\left|\frac{S_n}{n}\right|^{\alpha}\right] \le c \sum_{n=1}^{\infty} n^{-\alpha/2} E[|X|^{\alpha}] < \infty.$$

For 2 , we obtain by (2.2) that

$$\sum_{n=1}^{\infty} E\left[\left|\frac{S_n}{n}\right|^p\right] \leq \sum_{n=1}^{\infty} \frac{1}{n^p} \left(E\left[|S_n|^{\alpha}\right]\right)^{p/\alpha}$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{n^p} (cn^{\alpha/2} E[|X|^{\alpha}])^{p/\alpha}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} (cE[|X|^{\alpha}])^{p/\alpha}$$
$$< \infty.$$

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In [6], Chow also obtained the following result. Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with E[X] = 0. Suppose that $1 < \alpha < 2$. If $E[|X|^{\alpha} \log^+ |X|] < \infty$, then

$$\sum_{n=1}^{\infty} n^{-2} E[|S_n|^{\alpha}] < \infty.$$

As a direct consequence of Theorem 2.2 and its proof, we have the following corollaries.

Corollary 2.3 Suppose that $\alpha > 2$, $E[|X|^{\alpha}] < \infty$ and E[X] = 0. Then, for any $2 and <math>\beta > (p+2)/2$, we have

$$\sum_{n=1}^{\infty} n^{-\beta} E[|S_n|^p] < \infty.$$

Corollary 2.4 Suppose that $X \not\equiv \mu$ a.s. and $E[|X|^{\alpha}] < \infty$ for any $\alpha > 0$. Then $\frac{S_n}{n} \xrightarrow{S-L^p} \mu$ if and only if p > 2.

2.2 Strongly almost sure convergence version of the law of large numbers

In this subsection, we consider the following question:

Does it hold that $\frac{S_n}{n} \xrightarrow{S_\alpha \text{-}a.s.} \mu$ for some $\alpha > 0$ under some condition?

Theorem 2.5 (1) If $E[X^4] < \infty$ and $X \not\equiv \mu$ a.s., then $\sum_{n=1}^{\infty} \left| \frac{S_n}{n} - \mu \right|^{\alpha} = \infty$ a.s. for any $0 < \alpha \leq 2$.

(2) If $E[|X|^2] < \infty$, then for any $\alpha > 2$ we have

$$\frac{S_n}{n} \stackrel{S_{\alpha} \text{-}a.s.}{\longrightarrow} \mu.$$

Proof. We assume without loss of generality that $\mu = 0$ and $E[X^2] = 1$. (1) For $N \in \mathbb{N}$, we have

$$E\left[\left|\frac{S_n}{n}\right|^2\right] = \frac{1}{n^2} E\left[\sum_{i=1}^n X_i^2 + 2\sum_{1 \le i < j \le n} X_i X_j\right] = \frac{1}{n}.$$

Define

$$e_N = \sum_{n=1}^N \frac{1}{n}, \qquad W_N = e_N + 2\sum_{n=2}^N \frac{\sum_{1 \le i < j \le n} X_i X_j}{n^2},$$

and

$$R_N = \sum_{n=1}^N \left(\frac{X_1^2 + \dots + X_n^2}{n^2} - \frac{1}{n} \right).$$

Then,

$$\sum_{n=1}^{N} \left| \frac{S_n}{n} \right|^2 = W_N + R_N.$$
(2.3)

For $M, N \in \mathbb{N}$ with M < N, we have

$$E[(R_N - R_M)^2]$$

$$= E\left[\left(\sum_{n=M+1}^N \frac{(X_1^2 - 1) + \dots + (X_n^2 - 1)}{n^2}\right)^2\right]$$

$$= \sum_{n=M+1}^N \frac{E[(X^2 - 1)^2]}{n^3} + 2\sum_{M+1 \le k < l \le N} \frac{E[(\sum_{i=1}^k (X_i^2 - 1))(\sum_{j=1}^l (X_j^2 - 1))]}{k^2 l^2}$$

$$= \sum_{n=M+1}^N \frac{E[(X^2 - 1)^2]}{n^3} + 2\sum_{M+1 \le k < l \le N} \frac{E[(X^2 - 1)^2]}{k l^2}$$

$$\leq (E[(X^2 - 1)^2]) \left\{ \sum_{n=M+1}^N \frac{1}{n^3} + 2 \sum_{l=M+1}^N \frac{e_l}{l^2} \right\} \\ \to 0 \quad \text{as} \quad M \to \infty.$$

Hence $\{R_N\}_{N=1}^{\infty}$ is a Cauchy sequence in L^2 , which implies that $\{R_N\}_{N=1}^{\infty}$ converges to some $R \in L^2$ in probability.

Suppose that $\frac{S_n}{n} \xrightarrow{S_2\text{-}a.s.} 0$. Dividing both sides of (2.3) by e_N and letting $N \to \infty$, we get

$$1 + \frac{2}{e_N} \sum_{n=2}^N \frac{\sum_{1 \le i < j \le n} X_i X_j}{n^2} \to 0 \quad \text{in probability} \quad \text{as } N \to \infty,$$

which implies that

$$\frac{1}{e_N} \sum_{n=2}^{N} \frac{\sum_{1 \le i < j \le n} X_i X_j}{n^2} \to -\frac{1}{2} \quad \text{in probability as } N \to \infty.$$
(2.4)

We have

$$E\left[\left(\frac{1}{e_{N}}\sum_{n=2}^{N}\frac{\sum_{1\leq i< j\leq n}X_{i}X_{j}}{n^{2}}\right)^{2}\right]$$

$$=\frac{1}{(e_{N})^{2}}\left(\sum_{n=2}^{N}\frac{E\left[\left(\sum_{1\leq i< j\leq n}X_{i}X_{j}\right)^{2}\right]}{n^{4}}\right)$$

$$+2\sum_{2\leq k< l\leq N}\frac{1}{k^{2}l^{2}}E\left[\left(\sum_{1\leq i< j\leq k}X_{i}X_{j}\right)\left(\sum_{1\leq i< j\leq l}X_{i}X_{j}\right)\right]\right)$$

$$=\frac{1}{(e_{N})^{2}}\left(\sum_{n=2}^{N}\frac{n-1}{2n^{3}}+\sum_{2\leq k< l\leq N}\frac{k-1}{kl^{2}}\right)$$

$$\leq\frac{1}{(e_{N})^{2}}\left(\sum_{n=2}^{N}\frac{1}{n^{2}}+e_{N}\right)$$

$$\leq\sum_{n=1}^{\infty}\frac{1}{n^{2}}.$$
(2.5)

By (2.4) and (2.5), we get

$$\lim_{N \to \infty} E\left[\frac{1}{e_N} \sum_{n=2}^N \frac{\sum_{1 \le i < j \le n} X_i X_j}{n^2}\right] = -\frac{1}{2},$$

which contradicts with

$$E\left[\frac{1}{e_N}\sum_{n=2}^N \frac{\sum_{1 \le i < j \le n} X_i X_j}{n^2}\right] = 0, \quad \forall N \ge 2.$$

Then, $\frac{S_n}{n} \xrightarrow{S_2\text{-}a.s.} 0$. Therfore, we obtain by the Hewitt-Savage 0-1 law that $\sum_{n=1}^{\infty} \left| \frac{S_n}{n} \right|^2 = \infty$ a.s..

By the strong law of large numbers, there exists a set $N \in \mathcal{F}$ satisfying P(N) = 0 and for any $\omega \in \Omega \setminus N$, there exists $M(\omega) \in \mathbb{N}$ such that for any $n \geq M(\omega)$,

$$\frac{|S_n(\omega)|}{n} < 1$$

It follows that for any $0 < \alpha < 2$ and $\omega \in \Omega \setminus N$,

$$\sum_{n=M(\omega)}^{\infty} \left| \frac{S_n(\omega)}{n} \right|^2 \le \sum_{n=M(\omega)}^{\infty} \left| \frac{S_n(\omega)}{n} \right|^{\alpha}.$$

Therefore, $\sum_{n=1}^{\infty} \left| \frac{S_n}{n} \right|^2 = \infty$ a.s. implies that $\sum_{n=1}^{\infty} \left| \frac{S_n}{n} \right|^{\alpha} = \infty$ a.s. for any $0 < \alpha < 2$. (2) By the Hartman-Wintner law of iterated logarithm, we have

$$\limsup_{n \to \infty} \frac{|S_n|}{\sqrt{2n \log \log n}} = 1 \ a.s..$$

Then, there exists a set $N \in \mathcal{F}$ satisfying P(N) = 0 and for any $\omega \in \Omega \setminus N$, there exists $M(\omega) \in \mathbb{N}$ such that for any $n \geq M(\omega)$,

$$\frac{|S_n(\omega)|}{\sqrt{2n\log\log n}} < 2.$$

It follows that for any $\alpha > 2$ and $\omega \in \Omega \setminus N$,

$$\sum_{n=1}^{\infty} \left| \frac{S_n}{n} \right|^{\alpha} < \infty,$$

i.e., $\frac{S_n}{n} \xrightarrow{S_{\alpha} - a.s.} 0.$

Remark 2.6 By analogues of the Hartman-Wintner law of iterated logarithm in the infinite variance case, we can show that $\frac{S_n}{n} \xrightarrow{S_{\alpha} - a.s.} \mu$ for any $\alpha > 2$ under weaker conditions. Define $L(x) = \log \max\{e, x\}$ and LL(x) = L(Lx) for $x \in \mathbb{R}$. By Einmahl and Li [7, Corollaries 1 and 2], we have $\frac{S_n}{n} \xrightarrow{S_{\alpha} - a.s.} \mu$ for any $\alpha > 2$ if one of the following conditions is fulfilled.

(i) For some $p \geq 1$,

$$E\left[\frac{(X-\mu)^2}{(LL(|X-\mu|))^p}\right] < \infty, \quad \limsup_{x \to \infty} (LL(x))^{1-p} E[(X-\mu)^2 \mathbf{1}_{\{|X-\mu| \le x\}}] < \infty.$$

(ii) For some r > 0,

$$E\left[\frac{(X-\mu)^2}{(L(|X-\mu|))^r}\right] < \infty, \quad \limsup_{x \to \infty} \frac{LL(x)}{(L(x))^r} E[(X-\mu)^2 \mathbf{1}_{\{|X-\mu| \le x\}}] < \infty.$$

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3 Relations among several kinds of convergence

3.1 Main results

Proposition 3.1 Let $\{X, X_n, n \ge 1\}$ be a sequence of random variables. Then

(i) For $p \ge 1$, we have $X_n \xrightarrow{S-L^{\alpha}} X \Rightarrow X_n \xrightarrow{S-L^p} X$; (ii) For $\alpha > 0$, we have $X_n \xrightarrow{S-L^{\alpha}} X \Rightarrow X_n \xrightarrow{S_{\alpha} - a.s.} X$; (iii) For any $\alpha \ge 1$, we have $X_n \xrightarrow{S-L^{\infty}} X \Rightarrow X_n \xrightarrow{S_{\alpha} - a.s.} X$.

It is well known that $X_n \xrightarrow{P} X$ if and only if for any subsequence $\{X_{n'}\}$ of $\{X_n\}$, there exists a subsequence $\{X_{n'_k}\}$ of $\{X_{n'}\}$ such that $X_{n'_k} \xrightarrow{a.s.} X$. In the following, we strengthen this result to the strongly almost sure convergence.

Theorem 3.2 $X_n \xrightarrow{P} X$ if and only if for any subsequence $\{X_{n'}\}$ of $\{X_n\}$ and some (hence all) $\alpha > 0$, there exists a subsequence $\{X_{n'_k}\}$ of $\{X_{n'}\}$ such that $X_{n'_k} \xrightarrow{S_{\alpha} - a.s.} X$.

As a direct consequence of Theorem 3.2, we have the following corollary.

Corollary 3.3 If $X_n \xrightarrow{c.c.} X$, or $X_n \xrightarrow{a.s.} X$, or $X_n \xrightarrow{L^{\infty}} X$, or $X_n \xrightarrow{L^p} X$, then for any $\alpha > 0$, there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $X_{n_k} \xrightarrow{S_{\alpha}\text{-}a.s.} X$.

Proposition 3.4 $X_n \xrightarrow{S-L^1} X \Rightarrow X_n \xrightarrow{S_1-d} X$.

Now we have the following diagram:

Theorem 3.5 Let C be a constant. Then $X_n \xrightarrow{S_2 - d} C \Leftrightarrow X_n \xrightarrow{c.c.} C$.

If C is a constant and $Y_n \xrightarrow{S_2 - d} C$, then by Theorem 3.5 we know that $Y_n \xrightarrow{c.c.} C$. In the following Slutsky-type theorem, we need a stronger condition than $Y_n \xrightarrow{c.c.} C$.

Proposition 3.6 Suppose that $X_n \xrightarrow{S_1 - d} X$ and $Y_n \xrightarrow{S - L^1} C$. Then

(i) $X_n + Y_n \xrightarrow{S_1 - d} X + C;$

(ii) if $\{X_n\}$ is a sequence of bounded random variables, then $X_n Y_n \xrightarrow{S_1 - d} CX$;

(iii) if $\{X_n\}$ and $\{1/Y_n\}$ are two sequences of bounded random variables and $C \neq 0$, then $\frac{X_n}{Y_n} \xrightarrow{S_1 - d} \frac{X}{C}$.

Proposition 3.7 Let $\{X, X_n, n \ge 1\}$ be a sequence of random variables and $\{F, F_n, n \ge 1\}$ be the corresponding sequence of distribution functions. Then $X_n \xrightarrow{S_2-d} X$ if one of the following conditions is fulfilled.

(1) X is a discrete random variable such that $\{x \in \mathbb{R} : P(X = x) = 0\}$ is an open subset of \mathbb{R} and $X_n \xrightarrow{c.c.} X$.

(2) X has a bounded density function and $\sum_{n=1}^{\infty} P\{n(\log n)^{1+\beta}|X_n - X| \ge \delta\} < \infty$ for two positive constants β and δ .

3.2 Proofs

Proof of Proposition 3.1.

(i) If $||X_n - X||_{\infty} < 1$ and $p \ge 1$, we have

$$E[|X_n - X|^p] \le ||X_n - X||_{\infty}^p \le ||X_n - X||_{\infty},$$

which together with the definitions of $X_n \xrightarrow{S-L^{\infty}} X$ and $X_n \xrightarrow{S-L^p} X$ implies that $X_n \xrightarrow{S-L^{\infty}} X \Rightarrow X_n \xrightarrow{S-L^p} X$.

(ii) Let $\alpha > 0$. If $X_n \xrightarrow{S-L^{\alpha}} X$, then $\sum_{n=1}^{\infty} E[|X_n - X|^{\alpha}] < \infty$. By the monotone convergence theorem, we have

$$\sum_{n=1}^{\infty} E[|X_n - X|^{\alpha}] = E\left[\sum_{n=1}^{\infty} |X_n - X|^{\alpha}\right].$$

It follows that $E[\sum_{n=1}^{\infty} |X_n - X|^{\alpha}] < \infty$ and thus $\sum_{n=1}^{\infty} |X_n - X|^{\alpha} < \infty$ a.s., i.e., $X_n \xrightarrow{S_{\alpha} - a.s.} X$.

(iii) It is a direct consequence of (i) and (ii).

Proof of Theorem 3.2. The sufficiency is obvious. We only prove the necessity. Suppose that $X_n \xrightarrow{P} X$ and $\{X_{n'}\}$ is a subsequence of $\{X_n\}$. Then $X_{n'} \xrightarrow{P} X$. Thus, for any $k \in \mathbb{N}$, we have

$$\lim_{n' \to \infty} P\left\{ |X_{n'} - X|^{\alpha} \ge \frac{1}{k^2} \right\} = 0.$$

It follows that there exists a sequence $\{X_{n'_{k}}\}$ of $\{X_{n'_{k}}\}$ such that for any $k \in \mathbb{N}$,

$$P\left\{|X_{n'_k} - X|^{\alpha} \ge \frac{1}{k^2}\right\} \le \frac{1}{k^2},$$

which implies that

$$\sum_{k=1}^{\infty} P\left\{ |X_{n'_k} - X|^{\alpha} \ge \frac{1}{k^2} \right\} < \infty.$$

By the Borel-Cantelli lemma, we get

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left\{|X_{n'_k}-X|^{\alpha} \ge \frac{1}{k^2}\right\}\right) = 0,$$

which implies that

$$P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\left\{|X_{n'_{k}}-X|^{\alpha}<\frac{1}{k^{2}}\right\}\right)=1.$$

Therefore, $X_{n'_k} \xrightarrow{S_{\alpha} \text{-}a.s.} X$. The proof is complete.

Proof of Proposition 3.4. Suppose that f is a bounded Lipschitz continuous function. Then there a positive constant C such that

$$|f(x) - f(y)| \le C|x - y|, \ \forall x, y \in \mathbb{R}.$$

It follows that

$$|E[f(X_n) - f(X)]| \le E[|f(X_n) - f(X)|] \le CE[|X_n - X|],$$

which together with the definitions of $X_n \xrightarrow{S-L^1} X$ and $X_n \xrightarrow{S_1-d} X$ implies that $X_n \xrightarrow{S-L^1} X \Rightarrow X_n \xrightarrow{S_1-d} X$.

Proof of Theorem 3.5.

" \Rightarrow " For any $\epsilon > 0$, we have

$$P\{|X_n - C| \ge \epsilon\} = 1 - P\{X_n < C + \epsilon\} + P\{X_n \le C - \epsilon\}$$

$$\le 1 - F_n\left(C + \frac{\epsilon}{2}\right) + F_n(C - \epsilon).$$
(3.1)

If $X_n \xrightarrow{S_2 - d} C$, then for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \left| F_n\left(C + \frac{\epsilon}{2}\right) - 1 \right| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left| F_n\left(C - \epsilon\right) - 0 \right| < \infty,$$

which together with (3.1) implies that for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\{|X_n - C| \ge \epsilon\} < \infty.$$

Hence $X_n \xrightarrow{c.c.} C$.

" \Leftarrow " Let $X \equiv C$ and denote by F the distribution of X. For any $\epsilon > 0$ and $x \in \mathbb{R}$, we have

$$F(x-\epsilon) - P\{|X_n - X| \ge \epsilon\} \le F_n(x) \le P\{|X_n - X| \ge \epsilon\} + F(x+\epsilon).$$
(3.2)

If x > C, set $\epsilon = (x - C)/2$. By (3.2), we have

$$1 - P\{|X_n - C| \ge \epsilon\} \le F_n(x) \le P\{|X_n - C| \ge \epsilon\} + 1,$$

i.e.,

$$-P\{|X_n - C| \ge \epsilon\} \le F_n(x) - 1 \le P\{|X_n - C| \ge \epsilon\}.$$
(3.3)

If x < C, set $\epsilon = (C - x)/2$. By (3.2), we have

$$0 - P\{|X_n - C| \ge \epsilon\} \le F_n(x) \le P\{|X_n - C| \ge \epsilon\} + 0,$$

i.e.,

$$-P\{|X_n - C| \ge \epsilon\} \le F_n(x) - 0 \le P\{|X_n - C| \ge \epsilon\}.$$
(3.4)

By $X_n \xrightarrow{c.c.} C$, (3.3) and (3.4), we obtain that for any $x \neq C$,

$$\sum_{n=1}^{\infty} |F_n(x) - F(x)| < \infty.$$

Hence $X_n \xrightarrow{S_2 - d} C$.

Proof of Proposition 3.6. Suppose that f is a bounded Lipschitz continuous function. Then there exits a positive constant K such that

$$|f(x) - f(y)| \le K|x - y|, \quad \forall x, y \in \mathbb{R}.$$

(i) We have

$$|E[f(X_n + Y_n)] - E[f(X + C)]| \le |E[f(X_n + Y_n)] - E[f(X_n + C)]| + |E[f(X_n + C)] - E[f(X + C)]| \le KE[|Y_n - C|] + |E[f(X_n + C)] - E[f(X + C)]|.$$
(3.5)

By the assumption that $Y_n \xrightarrow{S-L^1} C$, we have

$$\sum_{n=1}^{\infty} E[|Y_n - C|] < \infty.$$
(3.6)

Define g(x) = f(x+c). Then, g is a bounded Lipschitz continuous function and thus by the assumption that $X_n \xrightarrow{S_1-d} X$, we get

$$\sum_{n=1}^{\infty} |E[f(X_n + C)] - E[f(X + C)]| < \infty.$$
(3.7)

By (3.5)-(3.7) and the definition of S_1 -d convergence, we obtain that $X_n + Y_n \xrightarrow{S_1 - d} X + C$.

The proofs for (ii) and (iii) are similar, so we omit the details.

Proof of Proposition 3.7.

(1) Suppose that $x \in \mathbb{R}$ with P(X = x) = 0. Then, by the assumption that $\{x \in \mathbb{R} : P(X = x) = 0\}$ is an open subset of \mathbb{R} , there exists $\epsilon > 0$ such that

$$F(x) = F(x + \epsilon) = F(x - \epsilon),$$

which together with (3.2) implies that

$$|F_n(x) - F(x)| \leq P\{|X_n - X| \geq \epsilon\} + |F(x + \epsilon) - F(x)| + |F(x - \epsilon) - F(x)| \\ = P\{|X_n - X| \geq \epsilon\}.$$

It follows that

$$\sum_{n=1}^{\infty} |F_n(x) - F(x)| \le \sum_{n=1}^{\infty} P\{|X_n - X| \ge \epsilon\},\$$

which together with the definitions of complete convergence and S_2 -d implies that $X_n \xrightarrow{S_2-d} X$.

(2) By the assumption, we know that there exists a positive constant C such that $|f(x)| \leq C, \forall x \in \mathbb{R}$. It follows that for any $x, y \in \mathbb{R}$,

$$|F(x) - F(y)| = \left| \int_{x}^{y} f(u) du \right| \le C|y - x|.$$
(3.8)

By (3.2) and (3.8), we have

$$|F_n(x) - F(x)| \leq P\{|X_n - X| \geq \epsilon\} + |F(x + \epsilon) - F(x)| + |F(x - \epsilon) - F(x)|$$

$$\leq P\{|X_n - X| \geq \epsilon\} + 2C\epsilon.$$

Then,

$$\begin{split} \sum_{n=1}^{\infty} |F_n(x) - F(x)| &\leq \sum_{n=1}^{\infty} \left(P\left\{ |X_n - X| \ge \frac{\delta}{n(\log n)^{1+\beta}} \right\} + 2C \frac{\delta}{n(\log n)^{1+\beta}} \right) \\ &= \sum_{n=1}^{\infty} P\left\{ n(\log n)^{1+\beta} |X_n - X| \ge \delta \right\} + 2C\delta \sum_{n=1}^{\infty} \frac{1}{n(\log n)^{1+\beta}} \\ &< \infty, \end{split}$$

and thus $X_n \xrightarrow{S_2 - d} X$.

3.3 Examples and remarks

The following example shows that $X_n \xrightarrow{S-L^{\infty}} X$ is stronger than $X_n \xrightarrow{S-L^p} X$ in general.

Example 3.8 Define $\Omega = (0, 1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ and P be the Lebesgue measure on Ω . For $n \in \mathbb{N}$, we define a random variable X_n by

$$X_n(\omega) = \begin{cases} 1, & \text{if } \omega \in (0, \frac{1}{n^2});\\ 0, & \text{if } \omega \in [\frac{1}{n^2}, 1). \end{cases}$$

Then, for any p > 0, we have

$$\sum_{n=1}^{\infty} E[|X_n - 0|^p] = \sum_{n=1}^{\infty} \int_0^{\frac{1}{n^2}} 1^p dP = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

i.e., $X_n \xrightarrow{S-L^p} 0$. Obviously, we have $||X_n - 0||_{\infty} = 1$ for any $n \in \mathbb{N}$. Hence $||X_n - 0||_{\infty} \neq 0$, which implies that $X_n \xrightarrow{S-L^{\infty}} 0$.

The following example shows that $X_n \xrightarrow{S-L^{\alpha}} X$ is stronger than $X_n \xrightarrow{S_{\alpha}-a.s.} X$ in general.

Example 3.9 Let $\alpha > 0$. Define $\Omega = (0, 1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ and P be the Lebesgue measure on Ω . For $n \in \mathbb{N}$, we define a random variable X_n by

$$X_n(\omega) = \begin{cases} 1, & \text{if } \omega \in (0, \frac{1}{n}); \\ 0, & \text{if } \omega \in [\frac{1}{n}, 1). \end{cases}$$

It is easy to check that $X_n \xrightarrow{S_{\alpha} \text{-}a.s.} 0$ but $X_n^{\alpha} \xrightarrow{c.c.} 0$ and hence $X_n \xrightarrow{S-L^{\alpha}} 0$.

Remark 3.10 (i) By Examples 3.8 and 3.9 we know that $X_n \xrightarrow{S-L^{\infty}} X$ is stronger than $X_n \xrightarrow{S_{\alpha}\text{-}a.s.} X$ in general.

(ii) By Example 3.9 and Theorem 3.5, we know that $X_n \xrightarrow{S_\alpha - a.s.} X \Rightarrow X_n \xrightarrow{S_2 - d} X$ in general.

Example 3.9 shows that $X_n \xrightarrow{S_{\alpha} \text{-}a.s.} X$ does not imply that $X_n \xrightarrow{c.c.} X$ in general. Conversely, the following example shows that $X_n \xrightarrow{c.c.} X$ does not imply $X_n \xrightarrow{S_{\alpha} \text{-}a.s.} X$ in general too.

Example 3.11 Let $\alpha > 0$. Define $\Omega = (0, 1)$, $\mathcal{F} = \mathcal{B}(\Omega)$ and P be the Lebesgue measure on Ω . For $n \in \mathbb{N}$, we define a random variable X_n by

$$X_n(\omega) = \begin{cases} 1, & \text{if } \omega \in (0, \frac{1}{n^2});\\ \frac{1}{n^{1/\alpha}}, & \text{if } \omega \in [\frac{1}{n^2}, 1). \end{cases}$$

For any $\epsilon > 0$, there exists N such that $\frac{1}{N^{1/\alpha}} < \epsilon$. Then, for any $n \ge N$, we have $\frac{1}{n^{1/\alpha}} \le \frac{1}{N^{1/\alpha}} < \epsilon$ and thus

$$\sum_{n=1}^{\infty} P\{|X_n - 0| \ge \epsilon\} \le \sum_{n=1}^{N-1} P\{|X_n - 0| \ge \epsilon\} + \sum_{n=N}^{\infty} \frac{1}{n^2} < \infty.$$

Hence $X_n \xrightarrow{c.c.} 0$.

Obviously, for any $\omega \in (0,1)$, we have $\sum_{n=1}^{\infty} |X_n - 0|^{\alpha} = \sum_{n=1}^{\infty} X_n^{\alpha} = \infty$. Thus $X_n \xrightarrow{S_{\alpha} \text{-a.s.}} 0$.

The following example shows that if X is nondegenerate, then for $i \in \{1, 2\}$, we do not have $X_n \xrightarrow{S_i - d} X \Rightarrow X_n \xrightarrow{P} X$.

Example 3.12 Let $\{X, X_n, n \ge 1\}$ be a sequence of *i.i.d.* random variables. Obviously, we have $X_n \xrightarrow{S_i - d} X$ for $i \in \{1, 2\}$. Suppose that X is nondegenerate. Then, there exists positive constants c_1 and c_2 such that $P\{|X_n - X| \ge c_1\} \equiv c_2$ for any $n \in \mathbb{N}$. Therefore, $X_n \xrightarrow{P} X$.

4 Some open questions

In this section, we present some open questions for further research.

Question 1. What is the relation between the S_1 -d convergence and the S_2 -d convergence?

Question 2. Does $X_n \xrightarrow{S-L^{\infty}} X$ imply that $X_n \xrightarrow{S_2 - d} X$?

Question 3. Does $X_n \xrightarrow{S-L^1} X$ imply that $X_n \xrightarrow{S_2-d} X$?

Question 4. Does $X_n \xrightarrow{S_{\alpha} - a.s.} X$ ($\alpha > 0$) imply that $X_n \xrightarrow{S_1 - d} X$?

Question 5. Does $X_n \xrightarrow{c.c.} X$ ($\alpha > 0$) imply that $X_n \xrightarrow{S_i - d} X$ for $i \in \{1, 2\}$?

Question 6. Can we give a Skorokhod-type theorem for the strong convergence in distribution and the S_{α} -a.s. convergence?

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