THE REGULARITY THEORY FOR THE DOUBLE OBSTACLE PROBLEM FOR FULLY NONLINEAR OPERATOR

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ABSTRACT. In this paper, we prove the existence and uniqueness of $W^{2,p}$ $(n solutions of a double obstacle problem with <math>C^{1,1}$ obstacle functions. Moreover, we show the optimal regularity of the solution and the local C^1 regularity of the free boundary. In the study of the regularity of the free boundary, we deal with a general problem, the no-sign reduced double obstacle problem with an upper obstacle ψ , $F(D^2u, x) = f_{\chi_{\Omega(u)} \cap \{u < \psi\}} + F(D^2\psi, x)\chi_{\Omega(u) \cap \{u = \psi\}}, u \le \psi$ in B_1 , where $\Omega(u) = B_1 \setminus (\{u = 0\} \cap \{\nabla u = 0\})$.

1. INTRODUCTION

Obstacle problems with a single obstacle appear in various fields of study such as porous media, elasto-plasticity, optimal control, and financial mathematics, see [8, 4]. The regularity of the solutions and the free boundaries of the problems have been actively studied by [3, 6, 11, 13, 2].

The double obstacle problem, which is the obstacle problem with two obstacles, originates in the study of optimal investment problems with transaction costs, the game of tug-of-war, and semiconductor devices, (see [18] and the references therein). Recently, global homogeneous solutions to the double obstacle problem, with homogeneous obstacles was considered by [1] and the regularity of the free boundaries of the double obstacle problem for Laplacian was obtained by [15].

In this paper, we discuss the regularity of the solution and the free boundary for the double obstacle problem of the fully nonlinear operator. Precisely, we prove the existence and uniqueness of $W^{2,p}$ (n) solutions of*double obstacle problem* $for the fully nonlinear operator in a domain <math>D \subset \mathbb{R}^n$,

$$\begin{cases}
F(D^{2}u, x) \ge 0, & \text{in } \{u > \phi_{1}\} \cap D, \\
F(D^{2}u, x) \le 0, & \text{in } \{u < \phi_{2}\} \cap D, \\
\phi_{1}(x) \le u(x) \le \phi_{2}(x) & \text{in } D, \\
u(x) = g(x) & \text{on } \partial D,
\end{cases}$$
(FB)

with $\phi_1, \phi_2 \in C^{1,1}(\overline{D}), \partial D \in C^{2,\alpha}, g \in C^{2,\alpha}(\overline{D})$ and $\phi_1 \leq g \leq \phi_2$ in ∂D . The optimal $(C^{1,1})$ regularity of the solution u to (FB) is also obtained. Moreover, we have C^1 regularity of the free boundary $\partial \{u = \psi_1\}$ of (FB), by studying the regularity of the free boundary for a general problem $(FB_{nosign local})$, which contains a reduced version (FB_{local}) of (FB).

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Specifically, by subtracting the lower obstacle ϕ_1 from the solution *u*, the problem (*FB*) is reduced to the double obstacle problem with the zero lower obstacle:

$$F(D^{2}u, x) = f\chi_{\{0 < u < \psi\}} + F(D^{2}\psi, x)\chi_{\{0 < u = \psi\}}, \quad 0 \le u \le \psi \quad \text{in } B_{1}, \tag{FB}_{local}$$

with $\psi \in C^{1,1}(\overline{B_1}) \cap C^{2,1}(\overline{\{\psi > 0\}})$, $f \in C^{0,1}(B_1)$, see Subsection 1.2 for more detail. Furthermore, we consider a general problem ($FB_{nosign \ local}$) of (FB_{local}), which relaxes the sign condition of u in (FB_{local}) (i.e., the solution could be below the lower zero obstacle):

$$F(D^{2}u, x) = f\chi_{\Omega(u) \cap \{u < \psi\}} + F(D^{2}\psi, x)\chi_{\Omega(u) \cap \{u = \psi\}}, \quad u \le \psi \quad \text{in } B_{1}, \quad (FB_{nosign \ local})$$

where

$$\Omega(u) := B_1 \setminus (\{u = 0\} \cap \{\nabla u = 0\}) \text{ and } f \in C^{0,1}(B_1),$$

with the upper obstacle function

$$\psi \in C^{1,1}(B_1) \cap C^{2,1}(\overline{\Omega(\psi)}), \quad \Omega(\psi) := B_1 \setminus (\{\psi = 0\} \cap \{\nabla \psi = 0\}).$$

By obtaining the regularity of the free boundary $\Gamma(u) := \partial \Omega(u) \cap B_1$ of (*FB*_{nosign local}) Theorem 1.4, we have the regularity of the free boundary $\partial \{u = \psi_1\}$ for (*FB*) as a corollary, see Corollary 1.5.

The result of the problem ($FB_{nosign \ local}$) is a generalization of the theory for the no-sign single obstacle problem ($\psi = \infty$ in ($FB_{nosign \ local}$)) studied in [6, 7]. Moreover, it is an extension for the result of the problem for Laplacian in [15].

1.1. **Methodology and contents.** The main idea to have the regularity of the free boundary, $\Gamma(u)$ of (*FB*_{nosign local}), which corresponds to $\partial \{u = \psi_1\}$ in (*FB*), is considering the upper obstacle ψ as a solution of the single obstacle problem, (*FB*_{local}) with $\psi = \infty$. Additionally, we apply the method of blowup to the upper obstacle ψ with the thickness assumption of the zero set of ψ at 0, which means that the zero set near the free boundary point 0 is sufficiently large in some sense, see Subsection 1.5. Then, the blowup ψ_0 of the upper obstacle ψ is of the form $c(x_n^+)^2$, c > 0 and it is crucially used to have the regularity of the free boundary.

The main difficulty to have the regularity of the free boundary is the lack of monotonicity formulas, used in the problem for Laplacian in [15]. Precisely, in the paper, by using the formulas, we have the *classification of global solutions*, a global solution of (*FB*_{nosign local}) in whole domain \mathbb{R}^n is of the form $c(x_n^+)^2$, c > 0. However, it is not applicable for the fully nonlinear case due to the nonlinearity.

Hence, for the fully nonlinear operator, we focus on the fact that the global solution u is zero in a half-space $\{x_n \leq 0\}$. Then, the optimal $(C^{1,1})$ regularity for u implies that $\partial_e u/x_n$ is finite in \mathbb{R}^n . Therefore, we prove that $\partial_e u/x_n$ is identically zero in \mathbb{R}^n for any direction $e \in \mathbb{S}^{n-1} \cap e_n^{\perp}$, which implies that u is a one-dimensional function and it is of the form $\frac{c}{2}(x_n^+)^2$, for a positive constant c. It is noticeable that similar arguments for the second derivative have been introduced in [16], and the one for the first derivative as above has been considered in [10] in the study of the free boundary near the fixed boundary.

Now, we summarize the contents of this paper. In Subsection 2.1, we have the existence and uniqueness of the $W^{2,p}$ (n) solution of (*FB* $) by using a penalization method, Proposition 2.1. Since the obstacles <math>\phi_1$ and ϕ_2 have $C^{1,1}$ regularity, we consider the penalization method with bounded penalty term. In

Subsection 2.2, we have the optimal regularity of the solution of (*FB*) Proposition 2.3 by using the quadratic growth of the solution of (*FB*_{local}), Proposition 2.2.

In Subsection 3.2, we obtain the classification of global solutions Proposition 3.5, the global solution *u* to (*FB*_{nosign local}) with the upper obstacle $\psi = c(x_n^+)^2$ is also of the form $c_1(x_n^+)^2$, for some $c, c_1 > 0$, by using the argument discussed in the previous paragraph.

Therefore, in Subsection 3.3, we prove the directional monotonicity and the proof of the regularity for the free boundaries, Theorem 1.4, using the methods considered in [13, 19, 9, 15] and references therein.

Remark 1.1. The reason to set the regularity of the obstacle functions ϕ_1 , ϕ_2 in (FB), and ψ in (FB_{local}), (FB_{nosign local}) to C^{1,1} is closely related to the main idea introduced in the first paragraph of the previous subsection. Indeed, to apply the method of blowup to the upper obstacle ψ , the regularity of ψ should be at least C^{1,1}. Furthermore, the thickness assumption of the zero set of ψ means that the region where the equation $F(D^2\psi) = 0$ satisfies ({ $\psi = 0$ }) is sufficiently large. Hence, D² ψ should not be continuous, and therefore, the regularity of the upper obstacle ψ should not be better than C^{1,1}.

1.2. **Reduction of** (*FB*). By subtracting the lower obstacle ϕ_1 from the solution u, we reduce the problem (*FB*) to the double obstacle problem with zero lower obstacle. Specifically, we define $\tilde{F}(\mathcal{M}, x) := F(\mathcal{M} + D^2\phi_1, x) - F(D^2\phi_1, x)$ and $v := u - \phi_1$, where u is a $W^{2,p}$ (n) solution of (*FB*). Then,

$$\begin{split} \tilde{F}(D^2v,x) &= F(D^2u,x) - F(D^2\phi_1,x) \\ &= -F(D^2\phi_1,x)\chi_{\{\phi_1 < u < \phi_2\}} + \left(F(D^2\phi_2,x) - F(D^2\phi_1,x)\right)\chi_{\{\phi_1 < u = \phi_2\}} \\ &= -F(D^2\phi_1,x)\chi_{\{0 < v < \phi_2 - \phi_1\}} + \tilde{F}(D^2(\phi_2 - \phi_1),x)\chi_{\{0 < v = \phi_2 - \phi_1\}}. \end{split}$$

By replacing $f = -F(D^2\phi_1, x)$, $\psi = \phi_2 - \phi_1$ and reusing $v = u - \phi_1$ by u, $\tilde{F}(\mathcal{M}, x) = F(\mathcal{M} + D^2\phi_1, x) - F(D^2\phi_1, x)$ by $F(\mathcal{M}, x)$, u is a $W^{2,p}$ solution of

$$F(D^{2}u, x) = f\chi_{\{0 < u < \psi\}} + F(D^{2}\psi, x)\chi_{\{0 < u = \psi\}} \quad \text{a.e. in } D,$$
(1)

with $0 \le u \le \psi$ in D, $f \in L^{\infty}(D)$, and $\psi \in C^{1,1}(\overline{D})$. Since we discuss the local regularity of the free boundaries, we consider a local form (*FB*_{local}) of (1).

1.3. Notations. We will use the following notations throughout the paper.

C, C_0, C_1	generic constants
Xε	the characteristic function of the set E , ($E \subset \mathbb{R}^n$)
Ē	the closure of <i>E</i>
∂E	the boundary of a set <i>E</i>
E	<i>n</i> -dimensional Lebesgue measure of the set <i>E</i>
$B_r(x), B_r$	$\{y \in \mathbb{R}^n : y - x < r\}, B_r(0)$
$\Omega(u), \Omega(\psi)$	see Equation (<i>FB</i> _{nosign local})
$\Lambda(u), \Lambda(\psi)$	$B_1 \setminus \Omega(u), B_1 \setminus \Omega(\psi)$
$\Gamma(u), \Gamma^{\psi}(u)$	$\partial \Lambda(u) \cap B_1, \partial \{u = \psi\} \cap B_1$
$\Gamma^d(u)$	$\Gamma(u) \cap \Gamma^{\psi}(u)$ (the intersection of free boundaries)
$\partial_{\nu}, \partial_{\nu e}$	first and second directional derivatives
$P_r(M), P_\infty(M)$	see Definition 1.2, 1.3
$\delta_r(u, x), \delta_r(u)$	see Definition 1.1
₽ + ,₽ [−]	Pucci operators
S, <u>S</u> , <u>S</u> , S*	the viscosity solution spaces for the Pucci operators

We refer to the book of Caffarelli-Cabré [5], for the definitions of the viscosity solution, Pucci operators \mathcal{P}^{\pm} and the spaces of viscosity solutions of the Pucci operators $\mathcal{S}(\lambda_0, \lambda_1, f)$, $\overline{\mathcal{S}}(\lambda_0, \lambda_1, f)$, $\underline{\mathcal{S}}(\lambda_0, \lambda_1, f)$, and $\mathcal{S}^*(\lambda_0, \lambda_1, f)$.

1.4. **Conditions on** $F = F(\mathcal{M}, x)$. We assume that the fully nonlinear operator $F(\mathcal{M}, x)$ satisfies the following conditions:

(F1) F(0, x) = 0 for all $x \in \mathbb{R}^n$.

(F2) *F* is uniformly elliptic with ellipticity constants $0 < \lambda_0 \le \lambda_1 < +\infty$, that is

$$\lambda_0 \|\mathcal{N}\| \le F(\mathcal{M} + \mathcal{N}, x) - F(\mathcal{M}, x) \le \lambda_1 \|\mathcal{N}\|,$$

for any symmetric $n \times n$ matrix \mathcal{M} and positive definite symmetric $n \times n$ matrix \mathcal{N} .

(F3) $F(\mathcal{M}, x)$ is convex in \mathcal{M} for all $x \in \mathbb{R}^n$.

(F4)

(F4)'

$$|F(\mathcal{M}, x) - F(\mathcal{M}, y)| \le C ||\mathcal{M}|| ||x - y|^{\alpha},$$

for some $0 < \alpha \leq 1$.

$$|F(\mathcal{M}, x) - F(\mathcal{M}, y)| \le C(||\mathcal{M}|| + C_1)|x - y|^{\alpha}$$
, for some $0 < \alpha \le 1$.

Remark 1.2. We define oscillations of the fully nonlinear operator F in the variable x by

$$\beta_F(x, x_0) := \sup_{\mathcal{M} \in S \setminus \{0\}} \frac{|F(\mathcal{M}, x) - F(\mathcal{M}, x_0)|}{||\mathcal{M}||}$$

and

$$\tilde{\beta}_F(x, x_0) := \sup_{\mathcal{M} \in S} \frac{|F(\mathcal{M}, x) - F(\mathcal{M}, x_0)|}{||\mathcal{M}|| + 1}$$

For any fixed x_0 , the condition (F4) implies that β_F and $\tilde{\beta}_F$ are C^{α} at x_0 . Then, β_F and $\tilde{\beta}_F$ satisfy the conditions for the $W^{2,p}$ and $C^{2,\alpha}$ estimates of viscosity solutions v to $F(D^2v, x) = f(x)$, respectively (see Chapter 7 and 8 in [5] and [20]).

Hence, in Section 2, we assume that F satisfies (F4) and the $W^{2,p}$ estimate is used in the proof of the existence and uniqueness of $W^{2,p}$ solution (FB_{nosign local}), Proposition 2.1 and $C^{2,\alpha}$ estimate is used in the proof of optimal regularity of solution, Proposition 2.3.

In Section 3, we study the regularity of the free boundary for the reduced forms, (FB_{local}) and $(FB_{nosign \ local})$. If F is the fully nonlinear operator of (FB), then $\tilde{F} = F(\mathcal{M} + D^2\phi_1, x) - F(D^2\phi_1, x)$, introduced in Subsection 1.2, is the fully nonlinear operator in (FB_{local}) and $(FB_{nosign \ local})$. If $F(\mathcal{M}, x)$ satisfies (F4), then $\tilde{F}(\mathcal{M}, x)$ satisfies (F4)' and $\beta_{\tilde{F}}(x, x_0)$ is C^{α} for the variable x at fixed $x_0 \in \mathbb{R}^n$. Hence, we have the $C^{2,\alpha}$ estimate of viscosity solutions v to $\tilde{F}(D^2v, x) = f(x)$, and it is used in Lemma 3.4, to have that the blowup u_0 of u of $(FB_{nosign \ local})$ is a global solution.

We note that, in Section 3, when we study the regularity of the free boundary for the reduced forms, (FB_{local}) and $(FB_{nosign \ local})$, we denote the fully nonlinear operator by F, instead of \tilde{F} . Hence, we assume (F4)' for a fully nonlinear operator F, in Section 3.

1.5. **Definitions.** In this subsection, we define the rescaling, blowup, thickness of coincidence sets $\Lambda(u)$ and $\Lambda(u) \cap \Lambda(\psi)$, and solution spaces. These concepts are already discussed in the literature of the obstacle problem, e.g. [3, 4, 13, 19, 7, 15]. We introduce the concepts for (*FB*_{nosign local}), for the reader's convenience.

In order to find the possible configuration of the solution near the free boundary, the following blowup concept has been used heavily at [3, 8] and other references. For a $W^{2,n}$ solution, u, of ($FB_{nosign \ local}$) in B_r , we define the *rescaling function* of u at $x_0 \in \partial \Lambda(u) \cap B_r$ with $\rho > 0$ as

$$u_{\rho}(x) = u_{\rho,x_0}(x) := \frac{u(x_0 + \rho x) - u(x_0)}{\rho^2}, \quad \text{for } x \in (B_r - x_0)/\rho.$$

By optimal ($C^{1,1}$) regularity of solution u (Theorem 1.3), for any sequence $\rho_i \to 0$, there exists a subsequence ρ_{i_i} of ρ_i and $u_0 \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that

 $u_{\rho_{i_i}} \to u_0$ uniformly in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$ for any $0 < \alpha < 1$.

The limit function u_0 is a *blowup of u at* x_0 .

Definition 1.1. (*Thickness of the coincident set* $\Lambda(u)$) *We denote by* $\delta_r(u, x)$ *the thickness of* $\Lambda(u)$ *in* $B_r(x)$, *i.e.*,

$$\delta_r(u,x) := \frac{MD(\Lambda(u) \cap B_r(x))}{r}$$

where MD(A), the minimal diameter of subset A of \mathbb{R}^n , is the least distance between two parallel hyperplanes containing A. We will use the abbreviated notation $\delta_r(u)$ for $\delta_r(u, 0)$.

To briefly explain the theory of the regularity of free boundary in Section 3, we define classes of local and global solutions of the problem.

Definition 1.2. (Local solutions) We say a $W^{2,n}$ function u belongs to the class $P_r(M)$ $(0 < r < \infty)$, if u satisfies

(i)
$$F(D^2u, x) = f\chi_{\Omega(u) \cap \{u < \psi\}} + F(D^2\psi, x)\chi_{\Omega(u) \cap \{u=\psi\}}, \quad u \le \psi \quad in B_r,$$

(ii) $||D^2u||_{L^{\infty}, B_r} \le M,$
(iii) $0 \in \Gamma^d(u),$
where $f \in C^{0,\alpha}(B_r)$ and $\psi \in C^{1,1}(B_r) \cap C^{2,\alpha}(\overline{\Omega(\psi)}).$

Definition 1.3. (Global solutions) We say a $W^{2,n}$ function u belongs to the class $P_{\infty}(M)$, *if* u satisfies

- (i) $F(D^2u) = \chi_{\Omega(u) \cap \{u < \psi\}} + F(D^2\psi)\chi_{\Omega(u) \cap \{u = \psi\}}, \quad u \le \psi \quad in \mathbb{R}^n,$
- (*ii*) $F(D^2\psi) = a\chi_{\Omega(\psi)}$ in \mathbb{R}^n , for a constant a > 1,
- (iii) $||D^2u||_{\infty,\mathbb{R}^n} \leq M$,
- (*iv*) $0 \in \Gamma(u)$.

1.6. **Main Theorems.** The purposes of the paper are to obtain the existence, uniqueness, and optimal regularity of the solution for the double obstacle problem and the regularity of the free boundary. The main theorems are as follows:

Theorem 1.3 (Existence, uniqueness and optimal regularity). *Assume F satisfies* (*F1*)-(*F4*). *Then the following holds:*

- (*i*) For $n , there exist <math>W^{2,p}$ solution u of (FB) with $\phi_1, \phi_2 \in C^{1,1}(\overline{D}), \partial D \in C^{2,\alpha}$, $g \in C^{2,\alpha}(\overline{D})$, and $\phi_1 \le g \le \phi_2$ in D.
- (ii) For any compact set K in D, we have

 $\|u\|_{C^{1,1}(K)} \le M < \infty,$

for some constant $M = M(||u||_{L^{\infty}(D)}, ||\phi_1||_{C^{1,1}(D)}, ||\phi_2||_{C^{1,1}(D)}, dist(K, \partial D)) > 0.$

Theorem 1.4 (Regularity of free boundary). Assume $F \in C^1$ satisfies (F1)-(F3) and (F4)' and let $u \in P_1(M)$ with an upper obstacle ψ such that

$$0 \in \partial \Omega(\psi), \quad \lim_{x \to 0, x \in \Omega(\psi)} F(D^2 \psi(x), x) > f(0), \quad f \ge c_0 > 0 \text{ in } B_1,$$

and

$$\inf\left\{F(D^2\psi, x), F(D^2\psi, x) - f\right\} \ge c_0 > 0 \text{ in } \Omega(\psi).$$

Suppose

$$\delta_r(u,\psi) := \frac{MD(\Lambda(u) \cap \Lambda(\psi) \cap B_r)}{r} \ge \epsilon_0 \quad \text{for all } r < 1/4.$$
(2)

Then there is $r_0 = r_0(u, c_0, \|\nabla F\|_{L^{\infty}(B_{M+\|\phi_1\|_{C^{1,1}(D)}} \times B_1)}, \|F\|_{L^{\infty}(B_M+\|\phi_1\|_{C^{1,1}(D)} \times B_1)}, \|\nabla f\|_{L^{\infty}(B_1)}) > 0,$ such that $\Gamma(u) \cap B_{r_0}$ is a C^1 graph.

Since Theorem 1.4 is for the reduced forms (FB_{local}) and ($FB_{nosign \ local}$) with $\tilde{F}(M, x) = F(M + D^2\psi, x) - F(D^2\psi, x)$, where F is the fully nonlinear operator in (FB), the local regularity of the free boundary for (FB) is obtained as a corollary.

Corollary 1.5. Let u, ϕ_1 , and ϕ_2 be as in Theorem 1.3 and we assume that $\phi_2 - \phi_1 \in C^{2,1}(\overline{\{\phi_1 < \phi_2\}})$ and $\phi_1 \in C^{2,1}(D)$. Suppose that $0 \in \partial \{u > \phi_1\} \cap \partial \{u < \phi_2\}$,

$$0 \in \partial \{\phi_1 < \phi_2\}, \quad \lim_{x \to 0, x \in \{\phi_1 < \phi_2\}} F(D^2 \phi_2(x), x) > 0, \quad -F(D^2 \phi_1, x) \ge c_0 > 0 \text{ in } B_1,$$

and

$$\inf \left\{ F(D^2\phi_2, x) - F(D^2\phi_1, x), F(D^2\phi_2, x) \right\} \ge c_0 > 0 \text{ in } \{\phi_1 < \phi_2\}.$$

Suppose

$$\delta_r(\phi_2 - \phi_1, z) \ge \epsilon_0$$
 for all $r < 1/4, z \in \partial \{\phi_1 < \phi_2\}$

and

 $\delta_r(u - \phi_1, \phi_2 - \phi_1) \ge \epsilon_0$ for all r < 1/4.

Then, there is $r_0 = r_0(v - \phi_1, c_0, \|\nabla F\|_{L^{\infty}(B_M \times B_1)}, \|F\|_{L^{\infty}(B_M \times B_1)}, \|\nabla F(D^2\phi_1(x), x)\|_{L^{\infty}(B_1)}) > 0$ such that $\partial \{u = \phi_1\} \cap B_{r_0}$ is a C^1 graph.

Remark 1.6. We assume the thickness of $\Lambda(\psi)$ and $\Lambda(u)$ satisfies the assumption (2) in Theorem 1.4. Then, the assumption implies that

$$\delta_r(u_0,\psi_0) \ge \epsilon_0 \quad \text{for all } r > 0, \tag{3}$$

for any blowups u_0 and ψ_0 of u and ψ at 0, respectively. By (3), we have that the blowups ψ_0 of ψ are the half-space type upper obstacle, $\psi_0 = \frac{a}{2}(x_n^+)^2$, in an appropriate system of coordinates, see e.g. Proposition 4.7 of [14]. Furthermore, (3) implies that the blowup

 u_0 of $u \in P_1(M)$ is a nonnegative function and moreover, u is also nonnegative in a neighborhood of 0, see Section 3.2 and 3.3. Thus, the solution $u \in P_1(M)$ to the general problem (FB_{nosign local}) becomes a solution of (FB_{local}).

In contrast to the Laplacian case in [15], we have the regularity of one of the two free boundaries. Precisely, we only have the regularity of the free boundary $\partial \{u = \phi_1\}$ which is emerged by the lower coincident set $\{u = \phi_1\}$, not $\partial \{u = \phi_2\}$, for (FB) in Corollary 1.5. The free boundary $\partial \{u = \phi_1\}$ is corresponding to $\Gamma(u)$ in Theorem 1.4, for the general problem (FB_{nosign local}). The reason for this result is the lack of the regularity theory for the free boundary of "single" obstacle problem for "concave" fully nonlinear operator, see Remark 3.3.

2. Existence, Uniqueness and Optimal Regularity

2.1. **Existence, uniqueness of** $W^{2,p}$ **solution.** For the single obstacle problem in [8, 13], the authors used an unbounded penalization term $\beta_{\epsilon}(z)$, such that $\beta_{\epsilon}(z)$ to $-\infty$, for z < 0, $\epsilon \to 0$. Then, C^2 regularity for obstacle function ϕ is needed to show that $\beta_{\epsilon}(u_{\epsilon} - \phi)$ is bounded, where u_{ϵ} is a solution of the penalization problem for the single obstacle problem with the obstacle function ϕ .

On the other hand, in this subsection, we consider a penalization problem (4) with a new penalty term β_{ϵ} , whose L^{∞} norms are uniformly bounded by a constant that depends only on $C^{1,1}$ norms of the obstacle functions ϕ_1 and ϕ_2 .

Then, we have solutions u_{ϵ} of the penalization problem (4) such that $W^{2,\infty}$ norms of u_{ϵ} are uniformly bounded. Hence, there is a limit function u_0 of u_{ϵ} as $\epsilon \to 0$ in $W^{2,p}$ sense. Finally, we prove that the limit function u_0 of u_{ϵ} is the unique solution of (*FB*) with the obstacle functions $\phi_1 \in C^{1,1}$ and $\phi_2 \in C^{1,1}$.

Proposition 2.1. Assume F satisfies (F1)-(F4). For $n , there is a unique viscosity solution <math>u \in W^{2,p}(D)$ of (FB) with

$$||u||_{W^{2,p}(D)} \le C\left(||F(D^2\phi_1, x)||_{L^{\infty}(D)}, ||F(D^2\phi_2, x)||_{L^{\infty}(D)}\right),$$

where $\phi_1, \phi_2 \in C^{1,1}(\overline{D}), \partial D \in C^{2,\alpha}, g^{2,\alpha} \in C(\overline{D})$, and $\phi_1 \leq g \leq \phi_2$ on ∂D .

Proof. Let $\beta_1(z) \in C^{\infty}(\mathbb{R})$ be a function satisfying

$$\begin{cases} \beta_1(z) = -\max\left\{ ||F(D^2\phi_1, x)||_{L^{\infty}(D)}, ||F(D^2\phi_2, x)||_{L^{\infty}(D)} \right\} & \text{if } z < -1, \\ \beta_1(z) = 0 & \text{if } z > 1, \\ \beta_1(z) \le 0 & \text{in } z \in \mathbb{R}, \end{cases}$$

and define $\beta_{\epsilon}(z) := \beta_1(z/\epsilon)$, for $\epsilon > 0$. We consider a penalization problem,

$$\begin{cases} F(D^2u, x) = \beta_{\epsilon}(u - \phi_1) - \beta_{\epsilon}(\phi_2 - u) & \text{in } D, \\ u(x) = g(x) & \text{on } \partial D. \end{cases}$$
(4)

By the $W^{2,p}$ regularity in [5] and [20], for each $v \in C^{0,\alpha}(D)$ ($0 < \alpha < 1$) there is a unique solution $w \in W^{2,p}(D)$ (n), of

$$\begin{cases} F(D^2w, x) = \beta_{\varepsilon}(v - \phi_1) - \beta_{\varepsilon}(\phi_2 - v) & \text{in } D, \\ w(x) = g(x) & \text{on } \partial D, \end{cases}$$

with

By

$$\|w\|_{W^{2,p}(D)} \leq \|w\|_{L^{\infty}(D)} + \|g\|_{W^{2,p}(D)} + \|\beta_{\epsilon}(v-\phi_1) - \beta_{\epsilon}(\phi_2 - v)\|_{L^p(D)},$$

the boundedness of β_{ϵ} , we have

$$\|w\|_{W^{2,p}(D)} \le C_0,\tag{5}$$

where C_0 is a constant which is independent for ϵ and v.

Let us consider a map *S* such that w = Sv for $v \in C^{0,\alpha}(D)$. Since $W^{2,p}$ space is compactly embedded in $C^{0,\alpha}$, the boundedness of $W^{2,p}$ norm of w, (5) implies that $S|_{B_{C_0}} : B_{C_0} \to B_{C_0}$ is a compact map, where B_{C_0} is the C_0 ball centered at 0 in $C^{0,\alpha}(D)$ and $S|_{B_{C_0}}$ is the function *S* from B_{C_0} to B_{C_0} defined by $S|_{B_{C_0}}(v) = S(v)$. Furthermore, the $W^{2,p}$ estimate implies that $S|_{B_{C_0}}$ is continuous. Hence, by Schauder's fixed-point theorem, there is a function $u_{\epsilon} \in B_{C_0}$ such that $S|_{B_{C_0}}u_{\epsilon} = u_{\epsilon}$, i.e., there is $u_{\epsilon} \in W^{2,p}(D)$ such that u_{ϵ} is a solution of (4) and $||u_{\epsilon}||_{W^{2,p}(D)} \leq C_0$, where C_0 does not depend on ϵ . Then, there is a sequence $\epsilon = \epsilon_i \to 0$ and $u \in W^{2,p}(D)$ such that

 $u_{\epsilon} \to u$ weakly in $W^{2,p}(D)$, n .

Thus, we have that $||u||_{W^{2,p}(D)} \leq C_0$ and

 $u_{\epsilon} \rightarrow u$ uniformly in *D*.

We claim that *u* is a solution of the double obstacle problem (*FB*). First, we are going to prove that $F(D^2u, x) \ge 0$ in $\{u > \phi_1\} \cap D$. Let x_0 be a point in $\{u > \phi_1\} \cap D$ and let $\delta = (u(x_0) - \phi_1(x_0))/2$. Then, by the uniform convergence of u_{ϵ} to *u*, there is a ball $B_r(x_0) \Subset \{u > \phi_1\} \cap D$ and $\epsilon_0 > 0$ such that $u_{\epsilon} - \phi_1 \ge \delta$ in $B_r(x_0)$, for $\epsilon < \epsilon_0$. By the definition of β_{ϵ} , for $\epsilon \le \min\{\epsilon_0, \delta\}$, we have

$$\beta_{\epsilon}(u_{\epsilon} - \phi_1) \equiv 0$$
 and $F(D^2 u_{\epsilon}, x) \ge 0$ in $B_r(x_0)$.

By the closedness of the family of viscosity solutions, Proposition 2.9 of [5], the uniform convergence of u_{ϵ} to u implies that $F(D^2u, x) \ge 0$ in $B_r(x_0)$. Since $x_0 \in \{u > \phi_1\} \cap D$ is arbitrary, we obtain $F(D^2u, x) \ge 0$ in $\{u > \phi_1\} \cap D$. We also have $F(D^2u, x) \le 0$ in $\{u < \phi_2\} \cap D$, from the same argument as above.

Next, we prove that $\phi_1 \le u \le \phi_2$ in *D*. Suppose that $\{u < \phi_1\} \cap D$ is not empty and let x_0 be a point in $\{u < \phi_1\} \cap D$. Then, by the uniform convergence of u_{ϵ} , there is a ball $B_r(x_0)$ such that

$$\beta_{\epsilon}(u_{\epsilon} - \phi_1) = -\max\left\{ \|F(D^2\phi_1, x)\|_{L^{\infty}(D)}, \|F(D^2\phi_2, x)\|_{L^{\infty}(D)} \right\}, \beta_{\epsilon}(\phi_2 - u_{\epsilon}) \equiv 0 \text{ in } B_r(x_0)$$

and

 $F(D^2u_{\epsilon}, x) \leq F(D^2\phi_1)$ in $B_r(x_0)$, for sufficiently small ϵ .

Consequently, $F(D^2u, x) \leq F(D^2\phi_1)$ in $\{u < \phi_1\} \cap D$. Moreover, the boundary condition $\psi_1 \leq u = g$ on ∂D implies $\{u < \phi_1\} \cap D \Subset D$ and $u \equiv \phi_1$ on $\partial(\{u < \phi_1\} \cap D)$. Hence, by the maximum principle, we have $u \geq \phi_1$ in $\{u < \phi_1\} \cap D$ and it is a contradiction. The same method implies that $\{u > \phi_2\} \cap D = \emptyset$ and $\phi_1 \leq u \leq \phi_2$ in D. Hence, u is a solution of (*FB*).

In order to prove the uniqueness, we suppose that there are two solutions u_1 and u_2 of (*FB*) and $\{u_1 < u_2\} \cap D$ is not empty. In $\{u_1 < u_2\} \cap D$, the conditions $\phi_1 \le u_1 \le \phi_2$ and $\phi_1 \le u_2 \le \phi_2$ in D imply that $\phi_2 > u_1$ and $u_2 > \phi_1$ and we have $F(D^2u_1, x) \le 0 \le F(D^2u_2, x)$ in $\{u_1 < u_2\} \cap D$. Furthermore, by the boundary condition for u_1 and u_2 , we have that $u_1 \equiv u_2$ on $\partial(\{u_1 < u_2\} \cap D)$. Therefore, by the comparison principle, we have that $u_1 \ge u_2$ in $\{u_1 < u_2\} \subset D$ and we arrive at a contradiction.

2.2. **Optimal Regularity.** In this subsection, we prove the optimal regularity of the double obstacle problem (*FB*) with $C^{1,1}$ obstacles by using the reduced form of (*FB*_{local}). We will first prove the quadratic growth of the solution at the free boundary point.

Definition 2.1. For a positive constant c', let $\mathcal{G}(c')$ be a class of solutions $u \in W^{2,n}(B_1)$ of

$$F(D^{2}u, x) = f(x)\chi_{\{0 < u < \psi\}} + F(D^{2}\psi, x)\chi_{\{0 < u = \psi\}}, \quad 0 \le u \le \psi \text{ in } B_{1},$$
(6)

with $|f(x)|, |F(D^2\psi, x)|, |\psi| \le c'$ in B_1 and $0 \in \Gamma(u)$.

Proposition 2.2 (Quadratic growth). *Assume F satisfies* (F1) and (F2). For any $u \in G(c')$, we have

$$S(r, u) := \sup_{x \in B_r} u(x) \le C_0 r^2,$$
(7)

for a positive constant $C_0 = C_0(c')$.

Proof. First, we show that there is a positive constant C_0 such that

$$S(2^{-j-1}, u) \le \max(C_0 2^{-2j}, 2^{-2} S(2^{-j}, u)) \quad \text{for all } j \in \mathbb{N} \cup \{0, -1\}.$$
(8)

Suppose it fails, then, for each $j \in \mathbb{N} \cup \{0, -1\}$, there exists $u_j \in \mathcal{G}$ such that

$$S(2^{-j-1}, u_j) > max(j2^{-2j}, 2^{-2}S(2^{-j}, u_j)).$$
(9)

We consider

$$\tilde{u}_j(x) := \frac{u(2^{-j}x)}{S(2^{-j-1},u)} \quad x \in B_{2^j}.$$

Then, by the definition of \tilde{u} and (9),

$$S(\tilde{u}_j, 1/2) = 1$$
, $S(\tilde{u}_j, 1) = 4$, and $\tilde{u}_j(0) = 0$.

Since $u \in \mathcal{G}(c')$, by the condition (F1) and Proposition 2.13 of [5], we know that $u \in S^*(\frac{\lambda}{n}, \Lambda, c')$. Thus, the inequality (9) implies

$$\mathcal{P}^+(D^2\tilde{u}(x)) = \frac{2^{-2j}}{S(2^{-j-1},u)} \cdot \mathcal{P}^+(D^2u(2^{-j}x)) \ge -\frac{c'}{j}$$

and

$$\mathcal{P}^{-}(D^{2}\tilde{u}(x)) = \frac{2^{-2j}}{S(2^{-j-1}, u)} \cdot \mathcal{P}^{-}(D^{2}u(2^{-j}x)) \le \frac{c'}{j},$$

where P^{\pm} are Pucci operators, i.e., we obtain that $\tilde{u} \in S^*(\lambda/n, \Lambda, c'/j)$. By Harnack inequality (Theorem 4.3 of [5]) and C^{α} regularity (Proposition 4.10 of [5]), we know that $\tilde{u}_j \rightarrow \tilde{u}$ in B_1 , up to subsequence and

$$\tilde{u} \in S^*(\lambda/n, \Lambda, 0)$$
 in B_1 ,

 $\tilde{u} \neq 0$ in $B_{1/2}$, and $\tilde{u}(0) = 0$. In other words, a nontrivial viscosity solution \tilde{u} has its minimum at an interior point. Hence, by the strong maximum principle, it is a contradiction.

Next, we claim that

$$S(2^{-j}, u) \le C_0 2^{-2j+2}$$
 for all $j \in \mathbb{N} \cup \{0\}.$ (10)

We may assume that $C_0 > c'/4$. Then, (10) holds for j = 0. Assume that (10) holds for $j = j_0$. By (8), we have the inequality (10) for $j_0 + 1$,

$$S(2^{-(j_0+1)}, u) \le max(C_0 2^{-2j_0}, 2^{-2}S(2^{-j_0}, u)) \le C_0 2^{-2j_0}.$$

Thus, by the mathematical induction, we have (10) for all $j \in \mathbb{N} \cup \{0\}$.

Now, take a positive number *r*, and take a natural number *j* such that $2^{-j-1} \le r \le 2^{-j}$. Then, by (10), we have

$$S(r, u) \le S(2^{-j}, u) \le C_0 2^{-2j+2} = C_0 2^4 2^{-2j-2} \le C_0 2^4 r^2.$$

Thus, we have the quadratic growth rate (7) of u at 0.

Proposition 2.3 (Optimal regularity). Assume *F* satisfies (F1)-(F4). Let $u \in W^{2,n}(D)$ be a solution of (FB), with $\phi_1, \phi_2 \in C^{1,1}(\overline{D}), \partial D \in C^{2,\alpha}, g \in C^{2,\alpha}(\overline{D}), and \phi_1 \leq g \leq \phi_2$ on ∂D . Then $u \in W^{2,\infty}_{loc}(D)$.

Proof. Let *K* be a compact set in *D* and $\delta = dist(K, \partial D)$. Since $u \in W^{2,p}(D)$, $D^2 u = D^2 \phi_1$ a.e. on $\{u = \phi_1\}$ and $D^2 u = D^2 \phi_2$ a.e. on $\{u = \phi_2\}$. Thus, it suffice to show that $\|u\|_{W^{2,\infty}(\{\phi_1 < u < \phi_2\} \cap K)} < +\infty$. Let x_0 be a point in $\{\phi_1 < u < \phi_2\} \cap K$ and denote $d(x_0) := dist(x_0, \partial \{u = \phi_1\} \cup \partial \{u = \phi_2\})$. We may assume that $d(x_0) = dist(x_0, \partial \{u = \phi_1\})$.

Case 1) $5d(x_0) < \delta$.

For $v := u - \phi_1$, we have that

$$\tilde{F}(D^2v, x) = -F(D^2\phi_1, x)\chi_{\{0 < v < \phi_2 - \phi_1\}} + \tilde{F}(D^2(\phi_2 - \phi_1), x)\chi_{\{0 < v = \phi_2 - \phi_1\}} \quad \text{in } D,$$

where $\tilde{F}(\mathcal{M}, x) = F(\mathcal{M} + D^2\phi_1, x) - F(D^2\phi_1, x)$, see Subsection 1.2.

Let $y_0 \in \partial B_{d(x_0)}(x_0) \cap \{u = \phi_1\}$. Then $B_{4d(x_0)}(y_0) \subset B_{5d(x_0)}(x_0) \in D$. Since $\phi_1, \phi_2 \in C^{1,1}(\overline{D})$, we know that $v(4dx + y^0)/(4d)^2$ is in the solution space $\mathfrak{G}(c')$ for a positive number c' for the fully nonlinear operator \tilde{F} which also satisfies (F1) and (F2). Then, by Proposition 2.2, we obtain

$$\|u - \phi_1\|_{L^{\infty}(B_{2d}(y_0))} \le C(\|\phi_1\|_{C^{1,1}(\overline{D})}, \|\phi_2\|_{C^{1,1}(\overline{D})})d^2.$$

Since $F(D^2u, x) = 0$ in $B_{d(x_0)}(x_0) \subset \{\phi_1 < u < \phi_2\}$, by $C^{2,\alpha}$ estimate, we have that

$$||D^{2}(u-\phi_{1})||_{L^{\infty}(B_{d/2}(x_{0}))} \leq C \frac{||u-\phi_{1}||_{L^{\infty}(B_{d}(x_{0}))}}{d^{2}}.$$

Thus, $B_d(x_0) \subset B_{2d}(y_0)$ implies

$$\|D^{2}(u-\phi_{1})\|_{L^{\infty}(B_{d/2}(x_{0}))} \leq C(\|\phi_{1}\|_{C^{1,1}(\overline{D})}, \|\phi_{2}\|_{C^{1,1}(\overline{D})}),$$

and

$$||D^{2}u||_{L^{\infty}(B_{d/2}(x_{0}))} \leq C(||\phi_{1}||_{C^{1,1}(\overline{D})}, ||\phi_{2}||_{C^{1,1}(\overline{D})})$$

Case 2) $5d(x_0) > \delta$.

The interior derivative estimate for *u* in $B_{\delta/4}(x_0)$ gives

$$||D^{2}u||_{L^{\infty}(B_{\delta/10}(x_{0}))} \leq C\frac{4^{2}}{\delta^{2}}||u||_{L^{\infty}(D)}.$$

For the case $d(x_0) = dist(x_0, \partial \{u = \phi_2\})$, the same argument as above with $\phi_2 - u$ implies the boundedness of the Hessian matrix of u. Therefore, we obtain the optimal regularity of the solution u of (*FB*).

We note that the property for the classical single obstacle problem (i.e., (*FB*) with $\phi_2 = \infty$ and $F = \Delta$) is obtained in [3, 4]. The growth rate for the reduced single obstacle problem ((*FB*_{local}) with $\phi_2 = \infty$) is discussed in [16], and the optimal (p/p - 1) growth rate for the p-Laplacian case is obtained in [12, 17].

For the case (*FB*_{nosign local}), with $f \in C^{0,\alpha}$ and $\psi \in C^{2,\alpha}$, the optimal regularity of the solutions is obtained by the theory in [9] (see [7, 9] and Theorem 2.1 of [15] for more detail).

3. Regularity of the Free Boundary $\Gamma(u)$

In this section, we discuss the regularity of the free boundary of the double obstacle problem, $(FB_{nosign \, local})$. In Subsection 3.2, we show the classification of the global solution, which means that the global solution $u \in P_{\infty}(M)$ with the upper obstacle $\psi(x) = \frac{a}{2}(x_n^+)^2$ (a > 1) is $u = \frac{1}{2}(x_n^+)^2$ or $u = \frac{a}{2}(x_n^+)^2$, see Proposition 3.5. In Subsection 3.3, we prove the directional monotonicity of the local solution $u \in P_1(M)$, Lemma 3.7. Then, we have that $u \in P_1(M)$ is nonnegative in a small neighborhood *B* of 0, and *u* is a solution of the simple problem (FB_{local}) in *B*. Therefore, the blowup u_0 of *u* should be $\frac{1}{2}(x_n^+)^2$, not $\frac{a}{2}(x_n^+)^2$ of $u \in P_1(M)$. Finally, we prove the regularity of the free boundary $\Gamma(u)$, by using the directional monotonicity.

3.1. **Non-degeneracy.** In this subsection, we study the non-degeneracy of the solution $u \in P_1(M)$, which is one of the important properties for solutions of obstacle problems. This property implies that 0 is also on the free boundary $\Gamma(u_0)$, where u_0 is a blow-up of u at $0 \in \Gamma(u)$, and $\Gamma(u)$ has a Lebesgue measure zero.

Lemma 3.1. Assume *F* satisfies (F1) and (F2). Let $u \in P_1(M)$. If $f \ge c_0 > 0$ in B_1 and $F(D^2\psi, x) \ge c_0 > 0$ in $\Omega(\psi)$, then

$$\sup_{B_r(x)} u \ge u(x) + \frac{c}{8\lambda_1 n} r^2 \quad x \in \overline{\Omega(u)} \cap B_1,$$
(11)

where $B_r(x) \subseteq B_1$.

Proof. Let $x_0 \in \Omega(u) \cap B_1$ and $u(x_0) > 0$. Since $u \le \psi$, we know that $\{u = \psi\} = \{u = \psi\} \cap \{\nabla u = \nabla \psi\}$ and therefore, $\Omega(u) \cap \{u = \psi\} \subset \Omega(\psi)$. By the assumptions for f and $F(D^2\psi, x)$, we obtain $F(D^2u, x) = f\chi_{\Omega(u) \cap \{u < \psi\}} + F(D^2\psi, x)\chi_{\Omega(u) \cap \{u = \psi\}} \ge c_0$ in $\Omega(u)$. Thus, the uniformly ellipticity, (F2) in Definition 1.4 implies

$$F(D^2w, x) \ge F(D^2u, x) - c \ge 0 \text{ on } B_r(x_0) \cap \Omega(u),$$

where

$$w(x) := u(x) - u(x_0) - \frac{c}{2\lambda_1 n} |x - x_0|^2.$$

Since $w(x_0) = 0$ and w(x) < 0 on $\partial \Omega(u)$, the maximum principle on $B_r(x_0) \cap \Omega(u)$ implies

$$\sup_{\partial B_r(x_0)\cap\Omega(u)} w > 0 \quad \text{and} \quad \sup_{\partial B_r(x)} u \ge u(x) + \frac{c}{2\lambda_1 n} r^2.$$

Let $x_0 \in \Omega(u) \cap B_1$ and $u(x_0) \le 0$. If there is a point $x_1 \in B_{r/2}(x_0)$ such that $u(x_1) > 0$. Then by the first case in the previous paragraph for x_1 implies the non-degeneracy for x_0 .

If $u(x) \le 0$ in $B_{r/2}(x_0)$, then $u(x) \equiv 0$ in $B_{r/2}(x_0)$ or u(x) < 0 in $B_{r/2}(x_0)$, by the maximum principle. Since $x^0 \in \Omega(u)$, the second case is only possible and in the case, $F(D^2u, x) \ge c$ in $B_{r/2}(x_0)$. Then, it implies the non-degeneracy of u at x_0 .

For the case of $x_0 \in \partial \Omega(u) \cap B_1$, we take a sequence of points $x_j \in \Omega(u)$ such that $x_j \to x_0$ as $j \to \infty$. By passing to the limit as j goes to ∞ , we have the desired inequality for $x_0 \in \overline{\Omega(u)} \cap B_1$.

By the non-degeneracy and of the solution $u \in P_1(M)$, we have the local porosity of $\Gamma(u)$ and $\Gamma(u)$ has Lebesgue measure zero, e.g. Section 3.2.1 of [19].

Remark 3.2. For $u \in P_1(M)$, by the definition of the rescalings and blowups and the non-degeneracy, we have $\sup_{\partial B_r} u_0 \ge \frac{c}{8\lambda_{1n}}r^2$, for r > 0, where u_0 is a blowup of u at 0. Thus, the origin 0 is on the free boundary $\Gamma(u_0)$ of u_0 . On the other hand, in general, we do not know that $0 \in \Gamma^{\psi}(u)$ implies $0 \in \Gamma^{\psi_0}(u_0)$, for $u \in P_1(M)$.

However, in the same manner as the linear case in Remark 2.4 of [15], if we assume that u is a non-negative function, then $v := \psi - u$ is a solution of

$$\tilde{F}(D^2v, x) = \left(F(D^2\psi, x) - f\right)\chi_{\{0 < v < \psi\}} + F(D^2\psi, x)\chi_{\{0 < v = \psi\}} \quad in \ B_1.$$

where $\tilde{F}(M, x) := F(D^2\psi, x) - F(D^2\psi - M, x)$. Then, if we assume that $F(D^2\psi, x) - f \ge c$ and $F(D^2\psi, x) \ge c$ in $\{\psi > 0\}$, we have the non-degeneracy for v which implies $0 \in \Gamma(v_0) = \Gamma^{\psi_0}(u_0)$ and $|\Gamma(v)| = |\Gamma^{\psi}(u)| = 0$.

Remark 3.3. We also note that

$$\tilde{F}(M, x) = F(D^2\psi, x) - F(D^2\psi - M, x)$$

is a concave fully nonlinear operator. Thus, we can not apply the theory of the obstacle problem for the convex fully nonlinear operator in [13] to $\tilde{F}(M, x)$.

Precisely, it is uncertain that we can have Lemma 3.10 for v, which is that the blowup of v at $x \in \Gamma(v) = \Gamma^{\psi}(u)$ near 0 is of the form $\frac{c}{2}(x_n^+)^2$, for a positive constant c. Hence, in contrast with linear theory [15], we only have the regularity of the free boundary $\Gamma(u)$, not $\Gamma^{\psi}(u)$, see Theorem 1.3 and Corollary 1.5.

Lemma 3.4. Assume F satisfies (F1)-(F3) and (F4)'. Let $u \in P_1(M)$ with an upper obstacle ψ such that

$$0 \in \partial \Omega(\psi), \quad \lim_{x \to 0, x \in \Omega(\psi)} F(D^2\psi(x), x) > f(0), \quad f \ge c_0 > 0 \text{ in } B_1,$$

and

$$F(D^2\psi, x) \ge c_0 > 0$$
 in $\Omega(\psi)$.

Then $u_0 \in P_{\infty}(M)$.

Proof. Let u_{r_i} and ψ_{r_i} be sequences of the rescaling functions converging to blowups, u_0 and ψ_0 , respectively. First, we claim that

$$F(D^2\psi_0, 0) = F(D^2\psi(0), 0)\chi_{\Omega(\psi_0)} \quad \text{in } \mathbb{R}^n,$$

where $\Omega(\psi_0) = \mathbb{R}^n \setminus \{\{\psi_0 = 0\} \cap \{\nabla \psi_0 = 0\}\}$ and $F(D^2 \psi(0), 0) := \lim_{x \to 0, x \in \Omega(\psi)} F(D^2 \psi(x), x)$. Let x be a point in $\Omega(\psi_0)$. Then, by $C_{loc}^{1,\alpha}$ convergence of ψ_{r_i} to ψ_0 , we know that there exist $\delta > 0$ and i_0 such that $B_{\delta}(x) \subset \Omega(\psi_{r_i})$, for all $i \ge i_0$. Then, by the definition of rescalings ψ_{r_i} , we have $r_i x \in \Omega(\psi)$. Furthermore, $\psi \in C^{2,\alpha}(\Omega(\psi))$ implies strong convergence of ψ_{r_i} to ψ_0 in $C^{2,\beta}(B_{\delta}(x))$ for some $0 < \beta < \alpha$. Thus,

$$F(D^{2}\psi_{0}(x),0) = \lim_{i \to \infty} F(D^{2}\psi_{r_{i}}(x),r_{i}x) = \lim_{i \to \infty} F(D^{2}\psi(r_{i}x),r_{i}x) = F(D^{2}\psi(0),0).$$

Next, we prove that u_0 is a solution of

$$F(D^2 u_0, 0) = f(0)\chi_{\Omega(u_0) \cap \{u_0 < \psi_0\}} + F(D^2 \psi(0), 0)\chi_{\Omega(u_0) \cap \{u_0 = \psi_0\}}, \quad u_0 \le \psi_0 \quad \text{in } \mathbb{R}^n$$

The rescaling u_{r_i} is a solution of

$$F(D^{2}u_{r_{i}}, r_{i}x) = f(r_{i}x)\chi_{\Omega(u_{r_{i}}) \cap \{u_{r_{i}} < \psi_{r_{i}}\}} + F(D^{2}\psi_{r_{i}}, r_{i}x)\chi_{\Omega(u_{r_{i}}) \cap \{u_{r_{i}} = \psi_{r_{i}}\}}, \quad u_{r_{i}} \le \psi_{r_{i}} \quad \text{in } B_{1/r_{i}},$$

where $\Omega(u_{r_i}) := B_{1/r_i} \setminus (\{u_{r_i} = 0\}) \cap \{\nabla u_{r_i} = 0\})$. Let *x* be a point in $\Omega(u_0) \cap \{u_0 < \psi_0\}$. Then, by $C_{loc}^{1,\alpha}$ convergence of u_{r_i} to u_0 , there exist $\delta > 0$ and i_0 such that $B_{\delta}(x) \subset \Omega(u_{r_i}) \cap \{u_{r_i} < \psi_{r_i}\}$, for all $i \ge i_0$. Then

$$F(D^2 u_{r_i}(y)) = f(r_i x) \quad \text{in } B_{\delta}(x)$$

Since $f \in C^{0,\alpha}(B_1)$, we have $C^{2,\alpha}$ estimates for u_{r_i} and we may assume strong convergence of u_{r_i} to u_0 in $C^{2,\beta}(B_{\delta}(x))$ for some $0 < \beta < \alpha$. Thus we have $|D^2u_0(x)| \le M$ and

$$F(D^2u_0(x), 0) = \lim_{i \to \infty} F(D^2u_{r_i}(x), r_i x) = \lim_{i \to \infty} f(r_i x) = f(0) \ge c_0 > 0.$$

Since F(0, 0) = 0, we know that $F(D^2u_0, 0) = 0$ a.e. on $\{u_0 = 0\}$. Moreover, $\Omega(u_0) \cap \{u_0 = \psi_0\} \subset \Omega(\psi_0)$ implies that $F(D^2u_0, 0) = F(D^2\psi_0, 0)$ in $\Omega(u_0) \cap \{u_0 = \psi_0\}$.

Therefore, u_0 is a solution of

$$F(D^2 u_0, 0) = f(0)\chi_{\Omega(u_0) \cap \{u_0 < \psi_0\}} + F(D^2 \psi(0), 0)\chi_{\Omega(u_0) \cap \{u_0 = \psi_0\}}, \quad u_0 \le \psi_0 \quad \text{ in } \mathbb{R}^n,$$

with $f(0) < F(D^2\psi(0), 0)$. Furthermore, by the non-degeneracy, Lemma 3.1, we have $0 \in \Gamma(u_0)$, see Remark 3.2. Consequently, u_0 is in $P_{\infty}(M)$ for the fully nonlinear operator G(M) = F(M, 0)/f(0).

3.2. **Classification of Global Solutions.** In this subsection, we discuss the classification of global solutions, which means that the global solution $u \in P_{\infty}(M)$ with the upper obstacle $\psi = \frac{a}{2}(x_n^+)^2$ (a > 1) and the thickness assumption (2), is $\frac{a}{2}(x_n^+)^2$ or $\frac{1}{2}(x_n^+)^2$.

First, we observe that the zero set of u contains a half plain $\{x_n < 0\}$, by the thickness assumption (2) and the non-degeneracy, Lemma 3.1. Thus, the optimal $(C^{1,1})$ regularity for the solution $u \in P_{\infty}(M)$ implies that $\partial_e u/x_n$ is finite. Moreover, by considering the rescaling functions of u and ψ with a distance from $x \in \{x_n > 0\}$ to $\{x_n = 0\}$, (12), we show that $\partial_e u \equiv 0$, for any direction $e \in \mathbb{S}^{n-1} \cap e_n^{\perp}$. It implies that u is one-dimensional, and it is $\frac{a}{2}(x_n^+)^2$ or $\frac{1}{2}(x_n^+)^2$.

Proposition 3.5. Assume F = F(M) satisfies (F1)-(F2). Let $u \in P_{\infty}(M)$ be a solution of

$$F(D^2u) = \chi_{\Omega(u) \cap \{u < \psi\}} + a\chi_{\Omega(u) \cap \{u = \psi\}}, \quad u \le \psi \quad a.e. \text{ in } \mathbb{R}^n,$$

with the upper obstacle

$$\psi(x) = \frac{a}{2}(x_n^+)^2,$$

for a constant a > 1. Suppose

$$\delta_r(u,\psi) \ge 0$$
, for all $r > 0$.

Then, we have

$$u(x) = \frac{1}{2}(x_n^+)^2$$
 or $u(x) = \frac{a}{2}(x_n^+)^2$.

Proof. The condition $u \leq \psi = \frac{a}{2}(x_n^+)^2$ on \mathbb{R}^n implies that $u(x) \leq 0$ on $\{x_n \leq 0\}$. We claim that $\{x_n < 0\} \subset \Lambda(u)$. First, we suppose that $\partial \Omega(u) \cap \{x_n < 0\} \neq \emptyset$. Then, by non-degeneracy, (Lemma 3.1), we have that $\{u > 0\} \cap \{x_n < 0\} \neq \emptyset$ and we arrive at a contradiction. Next, we suppose that $\{x_n < 0\} \subset \Omega(u)$. Since $\{\psi = 0\} = \Lambda(\psi) = \{x_n \leq 0\}$, it is a contraction to $\delta_r(u, \psi) \geq 0$, for all r > 0.

Therefore, we have that $\{x_n < 0\}$ is contained in $\Lambda(u)$. Hence, u = 0 on $\{x_n \le 0\}$ and $\partial_e u = 0$ on $\{x_n \le 0\}$ for all $e \in \mathbb{S}^{n-1} \cap e_n^{\perp}$, where $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ and $e^{\perp} := \{x \in \mathbb{R}^n : x \perp e\}$ for $e \in \mathbb{S}^{n-1}$.

In order to have the conclusion, it suffices to show that $\partial_e u \equiv 0$ on \mathbb{R}^n , for any direction $e \in \mathbb{S}^{n-1} \cap e_n^{\perp}$. Thus, we fix $e_1 \in \mathbb{S}^{n-1} \cap e_n^{\perp}$ and define

$$0 \le \sup_{x \in \{x_n > 0\}} \frac{\partial_1 u(x)}{x_n} =: M_0$$

By the optimal regularity and $\partial_1 u = 0$ on $\{x_n \le 0\}$, we know that M_0 is finite. If we prove that M_0 is 0, we have that $\partial_1 u \equiv 0$ on \mathbb{R}^n . Since the direction e_1 is arbitrary, we have that $\partial_e u = 0$ on $\{x_n \le 0\}$, for all $e \in \mathbb{S}^{n-1} \cap e_n^{\perp}$.

Arguing by contradiction, suppose $M_0 > 0$. Since $\partial_1 u \equiv 0$ on $(\Omega(u) \cap \{u < \psi\})^c$, we can take a sequence $x^j \in \Omega(u) \cap \{u < \psi\} \subset \{x_n > 0\}$ such that

$$\lim_{j\to\infty}\frac{1}{x_n^j}\partial_1 u(x^j) = M_0$$

Let $r_i := x_n^j = |x_n^j|$ and consider rescaling functions

$$u_{r_j}(x) := \frac{u\left(((x^j)', 0) + r_j x\right)}{(r_j)^2} \quad \text{and} \quad \psi_{r_j}(x) := \frac{\psi\left(((x^j)', 0) + r_j x\right)}{(r_j)^2} = \psi(x).$$
(12)

Then, $D^2 u_{r_i}$ are uniformly bounded and $u_{r_i} \equiv 0$ on $\{x_n \le 0\}$. Thus,

$$u_{r_j}(x) \to \tilde{u}(x) \text{ in } C_{loc}^{1,\alpha}(\mathbb{R}^n) \text{ for any } \alpha \in [0,1),$$
$$\tilde{u} \equiv 0 \quad \text{ on } \{x_n \le 0\}$$
(13)

and \tilde{u} is a solution of

$$F(D^2 u) = \chi_{\Omega(u) \cap \{u < \psi\}} + a \chi_{\Omega(u) \cap \{u = \psi\}}, \quad u \le \psi \quad \text{a.e. in } \mathbb{R}^n,$$

with the upper obstacle

$$\psi(x) = \frac{a}{2}(x_n^+)^2.$$

By the definition of M_0 , for $x \in \{x_n > 0\}$,

$$\partial_1 u_{r_j}(x) = \frac{\partial_1 u\left(((x^j)', 0) + r_j x\right)}{r_j x_n} \cdot x_n \le M_0 x_n.$$

Hence, we have $\partial_1 \tilde{u}(x) \leq M_0 x_n$ on $\{x_n > 0\}$. Moreover,

$$\partial_1 \tilde{u}(e_n) = \lim_{j \to \infty} \partial_1 u_{r_j}(e_n) = \lim_{j \to \infty} \frac{\partial_1 u \left(((x^j)', 0) + r_j e_n \right)}{r_j} = \lim_{j \to \infty} \frac{\partial_1 u(x^j)}{x_n^j} = M_0.$$

If $e_n \in (\Omega(\tilde{u}) \cap \{\tilde{u} < \psi\})^c$, then $\partial_1 \tilde{u}(e_n) = 0$ and we arrive at a contradiction. Thus, $e_n \in \Omega(\tilde{u}) \cap \{\tilde{u} < \psi\}$. Let $\tilde{\Omega}(\tilde{u})$ be the connected component of $\Omega(\tilde{u}) \cap \{\tilde{u} < \psi\}$ containing e_n . By (13), we know that $\tilde{\Omega}(\tilde{u}) \subset \Omega(\tilde{u}) \subset \{x_n > 0\}$.

By differentiating $F(D^2\tilde{u}) = 1$ on $\tilde{\Omega}(\tilde{u})$ with respect to e_1 , we have $F_{ij}(D^2\tilde{u})\partial_{ij}\partial_1\tilde{u} = 0$ and $F_{ij}(D^2\tilde{u})\partial_{ij}\partial_1(\tilde{u} - M_0x_n) = 0$ on $\tilde{\Omega}(\tilde{u})$. Thus, the strong maximum principle implies that

$$\partial_1 \tilde{u} = M_0 x_n \quad \text{ on } \tilde{\Omega}(\tilde{u}) \subset \{x_n > 0\}.$$

If there exists $x \in \partial \tilde{\Omega}(\tilde{u}) \cap \{x_n > 0\}$, then $\partial_1 \tilde{u}(x) = 0 = Mx_n$ and we have a contradiction, i.e., we have $\partial \tilde{\Omega}(\tilde{u}) \cap \{x_n > 0\} = \emptyset$. Then, $\tilde{\Omega}(\tilde{u}) \subset \{x_n > 0\}$ implies $\tilde{\Omega}(\tilde{u}) = \{x_n > 0\}, \partial_1 \tilde{u} \equiv Mx_n$ on $\{x_n > 0\}$ and

$$\tilde{u}(x) = M_0 x_1 x_n + g(x_2, ..., x_n)$$
 on $\{x_n > 0\}$.

Since \tilde{u} is in $C_{loc}^{1,1}(\mathbb{R}^n)$ and $\tilde{u} \equiv 0$ on $\{x_n \leq 0\}$, we have

$$\partial_n \tilde{u}(x) = M_0 x_1 + \partial_n g(x_2, ..., x_n) = 0$$
 on $\{x_n = 0\}$

and it does not hold unless $M_0 = 0$ and $\partial_n g(x_2, ..., 0) = 0$. Hence, we arrive at a contradiction.

3.3. **Directional Monotonicity and proof of Theorem 1.4.** In this subsection, we show the directional monotonicity for solutions of $(FB_{nosign \ local})$ and the regularity of the solutions $u \in P_1(M)$. We note that the argument for the linear case is discussed in [15].

Lemma 3.6. Assume $F \in C^1$ satisfies (F1)-(F3) and (F4)' and let u be a solution of

 $F(D^2u, rx) = f(rx)\chi_{\Omega(u) \cap \{u < \psi\}} + F(D^2\psi(rx), rx)\chi_{\Omega(u) \cap \{0 < u = \psi\}}, \quad u \le \psi \quad in B_1$

and assume that $f(x) \ge c_0 > 0$ in B_1 , Suppose that we have

$$C\partial_e \psi - \psi \ge 0$$
, $C\partial_e u - u \ge -\epsilon_0$ in B_1 ,

for a direction e and $\epsilon_0 < c/64\lambda_1 n$. Then we obtain

$$C\partial_e u - u \ge 0$$
 in $B_{1/4}$

if $0 < r \le r'_0$, *for some*

$$r'_{0} = r'_{0}(C, c_{0}, ||\nabla F||_{L^{\infty}(B_{M} \times B_{1})}, ||\nabla f||_{L^{\infty}(B_{1})}).$$

Proof. By differentiating $F(D^2u, rx) = f(rx)$ with respect to the direction *e*, we have

$$F_{ij}(D^2u(x), x)\partial_{ij}\partial_e u(x) = r\partial_e f(rx) - r(\partial_e F)(D^2u(x), rx) \quad \text{in } \Omega(u) \cap \{u < \psi\},$$
(14)

where $\partial_e F$ is the spatial directional derivative of F(M, x) in the direction *e*.

Arguing by contradiction, suppose there is a point $y \in B_{1/2} \cap \Omega(u) \cap \{u < \psi\}$ such that $C\partial_e u(y) - u(y) < 0$. We consider the auxiliary function

$$\phi(x) = C\partial_e u(x) - u(x) + \frac{c_0}{4\lambda_1 n} |x - y|^2.$$

By Proposition 2.13 of [5] and the condition (*F*1) (*F*(0, *x*) = 0 for all $x \in \mathbb{R}^n$), we have $u \in S(\lambda_0/n, \lambda_1, f(rx))$ in $\Omega(u) \cap \{u < \psi\}$ and moreover (14) implies $\partial_e u \in S(\lambda_0/n, \lambda_1, r\partial_e f(rx) - r(\partial_e F)(D^2u(x), rx))$. Hence, we have

$$\phi \in \overline{S}(\lambda_0/n, \lambda_1, r\partial_e f(rx) - r(\partial_e F)(D^2 u(x), rx) - f(rx) + c_0/2) \text{ in } \Omega(u) \cap \{u < \psi\}.$$

Furthermore, since there is a sufficiently small constant $\tilde{r}_0 = \tilde{r}_0(C, c_0, \|\nabla f\|_{L^{\infty}(B_1)}, \|\nabla F\|_{L^{\infty}(B_M \times B_1)}) > 0$ such that

$$Cr\partial_e f(rx) - Cr(\partial_e F)(D^2u(x), rx) - f(rx) + c_0/2$$

$$\leq Cr||\nabla f||_{L^{\infty}(B_1)} + Cr||\nabla_x F||_{L^{\infty}(B_M \times B_1)} - f(rx) + c_0/2 \leq 0,$$

for all $r < \tilde{r}_0$, we have

 $\phi(x) \in \overline{S}(\lambda_0/n, \lambda_1, 0)$ in $\Omega(u) \cap \{u < \psi\}$, for all $r \le \tilde{r}_0$. By the minimum principle of ϕ in $B_{1/4}(y) \cap \Omega(u) \cap \{u < \psi\}$, we have

$$\inf_{B_{1/4}(y)\cap\Omega(u)\cap\{u<\psi\}\}}\phi\leq\phi(y)<0$$

Moreover, $C\partial_e \psi - \psi \ge 0$ in B_1 implies that $\phi \ge 0$ on $\partial (\Omega(u) \cap \{u < \psi\})$. Thus, we obtain

$$\inf_{\partial B_{1/4}(y)\cap(\Omega(u)\cap\{u<\psi\})}\phi<0\quad\text{and}\quad\inf_{\partial B_{1/4}(y)\cap(\Omega(u)\cap\{u<\psi\})}(C\partial_e u-u)<-\frac{c_0}{128\lambda_1n}.$$

Since $\epsilon_0 < \frac{c_0}{64\lambda_1 n}$, we have a contradiction.

Lemma 3.7 (Directional monotonicity). *Let* u, ψ , F *be as in Theorem 1.4. Then for any* $\delta \in (0, 1]$ *, there exists*

$$r_{\delta} = r_{\delta}(u, \psi, c_0, \|\nabla F\|_{L^{\infty}(B_M \times B_1)}, \|F\|_{L^{\infty}(B_M \times B_1)}, \|\nabla f\|_{L^{\infty}(B_1)}) > 0$$

such that

$$u \ge 0$$
 in B_{r_1}
 $\partial_e u \ge 0$ in B_{r_δ} for any $e \in C_\delta \cap \partial B_1$,

where

$$C_{\delta} = \{ x \in \mathbb{R}^n : x_n > \delta | x' | \}, \quad x' = (x_1, ..., x_{n-1}).$$

Proof. We denote u_r and ψ_r by rescalings of u_{r_i} and ψ_{r_i} , respectively. The thickness assumption (3) implies that $\psi_0 = \frac{a}{2}(x_n^+)^2$, (a > 1), in an appropriate system of coordinates, see e.g. Proposition 4.7 of [14]. Then, by Proposition 3.5, we know that u_0 is $\frac{1}{2}(x_n^+)^2$ or $\frac{a}{2}(x_n^+)^2$. Hence, for any $e \in C_{\delta} = \{x \in \mathbb{R}^n : x_n > \delta | x' |\}, x' = (x_1, ..., x_{n-1})$, we obtain

$$\delta^{-1}\partial_e\psi_0 - \psi_0 \ge 0$$
 and $\delta^{-1}\partial_e u_0 - u_0 \ge 0$ in \mathbb{R}^n .

By the $C^{1,\alpha}$ convergence of ψ_r and u_r to ψ_0 and u_0 , respectively, we obtain that

 $\delta^{-1}\partial_e\psi_r - \psi_r \ge -\epsilon_0$ and $\delta^{-1}\partial_eu_r - u_r \ge -\epsilon_0$ in B_1 ,

for $\epsilon_0 < c/64\lambda_1 n$ and $r < \hat{r}_{\delta}(u, \psi)$. Then, by applying the directional monotonicity for the solution of the single obstacle, Lemma 13 of [9] to ψ , we have that $\delta^{-1}\partial_e\psi_r - \psi_r \ge 0$ in $B_{1/2}$ for all $r < r'_{\delta}(u, \psi, c_0, ||\nabla F||_{L^{\infty}(B_M \times B_1)}, ||\nabla f||_{L^{\infty}(B_1)})$. Furthermore, by Lemma 3.6, we have

$$\delta^{-1}\partial_e u_r - u_r \ge 0 \quad \text{in } B_{1/4},\tag{15}$$

for $0 < r \leq \tilde{r}_{\delta} = \tilde{r}_{\delta}(u, \psi, c_0, \|\nabla F\|_{L^{\infty}(B_M \times B_1)}, \|\nabla f\|_{L^{\infty}(B_1)}).$

We claim that $u_r = 0$ in $\{x_n < -1/8\} \cap B_{1/4}$. By the $C^{1,\alpha}$ convergence of u_r to u, we may assume that $||u_r - u_0||_{L^{\infty}(B_1)} \le \frac{c}{8\lambda_1 n} \times \frac{1}{128}$ for $0 < r \le \tilde{r}_{\delta}$. Let x_0 be a point in $\{x_n < 0\} \cap \Omega(u_r) \cap B_{1/4}$. By the non-degeneracy, Lemma 3.1, we have

$$\sup_{\partial B_{\rho}(x_0)} u_r \ge u_r(x_0) + \frac{c}{8\lambda_1 n} \rho^2, \tag{16}$$

where $\rho := |(x_0)_n|$. Since $u_0 = 0$ in $\{x_n \le 0\}$ implies $||u_r - u_0||_{L^{\infty}(\partial B_{\rho}(x_0))} = ||u_r||_{L^{\infty}(\partial B_{\rho}(x_0))} \le \frac{c}{8\lambda_1 n} \times \frac{1}{128}$, by (16), we have $\frac{c}{8\lambda_1 n} \rho^2 \le \frac{c}{4\lambda_1 n} \times \frac{1}{128}$ and $\rho \le \frac{1}{8}$. Therefore, $\{x_n < 0\} \cap \Omega(u_r) \cap B_{1/4} \subset \{-1/8 < x_n < 0\}$ and $u_r = 0$ in $\{x_n < -1/8\} \cap B_{1/4}$.

Let us fix $\delta = 1$ and let $v(x) = exp(-e \cdot x)u_r(x)$. Then, by (15), we have $\partial_e v \ge 0$ in $B_{1/4}$ for any $e \in C_{\delta}$. Thus, $u_r = 0$ in $\{x_n < -1/8\} \cap B_{1/4}$ implies that v = 0 in $\{x_n < -1/8\} \cap B_{1/4}, v \ge 0$ in $B_{1/4}$, and $u_r \ge 0$ in $B_{1/4}$ for all $r \le \tilde{r}_1$. By the definition of scaling functions u_r , we have $u \ge 0$ in B_{r_1} , for $r_1 = \frac{1}{4\tilde{r}_1}$. Furthermore, (15) implies that

$$\partial_e u \ge 0$$
 in B_{r_δ} for any $e \in C_\delta \cap \partial B_1$,
for $r_\delta = \frac{1}{4\tilde{r}_\delta} < r_1, \delta \in (0, 1]$.

Lemma 3.8. Let u, ψ, F be as in Theorem 1.4. Then there exists

$$r_1 = r_1(u, \psi, c_0, \|\nabla F\|_{L^{\infty}(B_M \times B_1)}, \|F\|_{L^{\infty}(B_M \times B_1)}, \|\nabla f\|_{L^{\infty}(B_1)}) > 0$$

such that u is a solution of

$$F(D^{2}u, x) = f\chi_{\{0 < u < \psi\}} + F(D^{2}\psi, x)\chi_{\{0 < u = \psi\}}, \quad 0 \le u \le \psi \quad in \ B_{r_{1}}.$$

Moreover, if u_0 and ψ_0 are blowup functions of u and ψ at 0, respectively, then in an appropriate system of coordinates,

$$\psi_0(x) = \frac{a}{2}(x_n^+)^2$$
 and $u_0(x) = \frac{1}{2}(x_n^+)^2$.

Proof. By Lemma 3.7, there is $r_1 = r_1(u, \psi, c_0, ||\nabla F||_{L^{\infty}(B_M \times B_1)}, ||F||_{L^{\infty}(B_M \times B_1)}, ||\nabla f||_{L^{\infty}(B_1)}) > 0$ such that $u \ge 0$ in B_{r_1} . Hence u is a solution of

$$F(D^2u, x) = f\chi_{\{0 < u < \psi\}} + F(D^2\psi, x)\chi_{\{0 < u = \psi\}}, \quad 0 \le u \le \psi \quad \text{in } B_{r_1}.$$

and $v := \psi - u$ is a solution of

$$\tilde{F}(D^2v, x) = \left(F(D^2\psi, x) - f\right)\chi_{\{0 < v < \psi\}} + F(D^2\psi, x)\chi_{\{0 < v = \psi\}} \quad 0 \le v \le \psi \quad \text{in } B_1,$$

where $\tilde{F}(M, x) := F(D^2\psi, x) - F(D^2\psi - M, x)$. Since $0 \le v \le \psi$, we have that $\{v > 0\} \subset \{\psi > 0\} = \Omega(\psi)$. Thus, $\min\{F(D^2\psi, x), F(D^2\psi, x) - f\} \ge c_0 > 0$ in $\Omega(\psi)$ implies

$$\tilde{F}(D^2v, x) = (F(D^2\psi, x) - f)\chi_{\{0 < v < \psi\}} + F(D^2\psi, x)\chi_{\{0 < v = \psi\}} \ge c_0 > 0 \quad \text{ in } \{v > 0\}.$$

Thus, by the same argument in Lemma 3.1, we have the non-degeneracy for v,

$$\sup_{\partial B_r(x)} v \ge v(x) + \frac{\lambda}{8n} r^2 \quad x \in \overline{\Omega(v)} \cap B_{r_0},$$

for $B_r(x) \in B_{r_0}$. This implies $0 \in \Gamma(v_0) = \Gamma^{\psi_0}(u_0)$, where v_0 is a blowup functions of v at 0 such that $v_0 = \psi_0 - u_0$, see Remark 3.2. Consequently, we have

$$\psi_0(x) = \frac{a}{2}(x_n^+)^2$$
 and $u_0(x) = \frac{1}{2}(x_n^+)^2$

in an appropriate system of coordinates.

By the uniqueness of the blowup for the single obstacle problem, we have the uniqueness of blowup for
$$\psi$$
, i.e., for any sequence $\lambda \rightarrow 0$,

$$\psi_{\lambda} \to \psi_0 = \frac{1}{2} (x_n^+)^2 \quad \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^n)$$

in an appropriate system of coordinates. Then, the uniqueness of the blowup for *u* directly follows from Lemma 3.8.

Proposition 3.9 (Uniqueness of blowup). Let u, ψ, F be as in Theorem 1.4. Then the blowup function of u at 0 is unique, i.e., in an appropriate system of coordinates, for any sequence $\lambda \to 0$,

$$u_{\lambda} \rightarrow u_0 = \frac{1}{2} (x_n^+)^2 \quad in \ C_{loc}^{1,\alpha}(\mathbb{R}^n)$$

as $\lambda \to 0$.

In the following lemma, we have that the blowups of *u* for any points *x* near 0 are also half-space functions.

Lemma 3.10. Let u, ψ, F and r_1 be as in Theorem 1.4. Then there is $r'_1 = r'_1(u, \psi) > 0$ such that the blowup function of u at $x \in \Gamma(u) \cap B_{r'_1}$ are half-space functions.

Proof. By the directional monotonicity for *u* and ψ (Lemma 3.7 and 3.8), we have that, for any $\delta \in (0, 1]$, there exists

$$r_{\delta} = r_{\delta}(u, c_0, \|\nabla F\|_{L^{\infty}(B_M \times B_1)}, \|F\|_{L^{\infty}(B_M \times B_1)}, \|\nabla f\|_{L^{\infty}(B_1)}) > 0$$

such that $r_1 \ge r'_{\delta} = r'_{\delta}(u, \psi) > 0$ and

$$\psi, u \ge 0$$
 in $B_{r'_1}$
 $e_{\psi}, \partial_e u \ge 0$ in $B_{r'_{\delta}}$ for any $e \in C_{\delta}$

Then, the free boundaries $\partial \{u = 0\} \cap B_{r'_1} = \Gamma(u) \cap B_{r'_1}, \partial \{\psi = 0\} \cap B_{r'_1}$ are represented by Lipschitz functions.

Let x^0 be a point in $\Gamma(u) \cap B_{r'_1}$ and assume that there exists $r_0 > 0$ such that

$$\{u = \psi\} \cap B_r(x^0) \neq \emptyset$$
 for all $r < r_0$

Then, there is a sequence of points x^j such that $x^j \in \{u = \psi\}$ and $x^j \to x^0$ as $j \to \infty$. Thus,

$$\psi(x^j) = u(x^j) \to 0$$
 as $j \to \infty$, and $x^0 \in \{\psi = 0\}$.

Since *u* is nonnegative in $B_{r'_1}$, we have $0 \le u \le \psi$ and $\{\psi = 0\} \subset \{u = 0\}$ in $B_{r'_1}$. Thus, $x^0 \in \Gamma(u) \cap B_{r'_1}$ implies $x^0 \in \partial \{\psi = 0\}$. Furthermore, by the Lipschitz regularity of the zero set of ψ , $\{\psi = 0\}$ and the positivity of *u*, $(0 \le u \le \psi)$, we obtain

$$\delta_r(u, \psi, x_0) = \delta_r(\psi, x_0) \ge \epsilon_0 \quad \text{for all } r < 1/4.$$

Then, by classification of blowup, Proposition 3.5, we know that the blowup of *u* at x^0 is a *half-space solution*, which means that it is of the form $\frac{c}{2}(x_n^+)^2$, for a positive constant *c*.

Next, we assume that, for $x^0 \in \Gamma(u) \cap B_{r'_1}$, there exists $r_0 > 0$ such that

$$\{u = \psi\} \cap B_{r_0}(x^0) = \emptyset.$$

Then *u* is a solution of

$$F(D^2u, x) = f\chi_{\{u>0\}}, \quad u \ge 0 \quad \text{in } B_{r_0}(x^0).$$

On the other hand, Lipschitz regularities of $\Gamma(u)$ implies the thickness assumption for *u* near x_0 . Then, the blowup function of *u* at 0 is a half-space solution.

By using lemmas in this subsection and arguments in [13, 19, 9, 15], we prove one of the main theorems of the paper, Theorem 1.4.

Proof of Theorem 1.4. The directional monotonicity for u, Lemma 3.7, implies that the free boundary $\Gamma(u) \cap B_{r\delta/2}$ is represented as a graph $x_n = f(x')$ for Lipschitz function f and the Lipschitz constant of f is less than δ in $B_{r\delta/2}$. Since $\delta > 0$ can be chosen arbitrary small, we have a tangent plane of $\Gamma(u)$ and the normal vector e_n at 0. By Lemma 3.10, for any $z \in \Gamma(u) \cap B_{r'_1}$, we know that the blowup is the half-space solution and there is a tangent plane for $z \in \Gamma(u) \cap B_{r'_1}$ with tangent vector v_z . By Lemma 3.7, for $z \in \Gamma(u) \cap B_{r_{\delta}}$, we have $v_z \cdot e \ge 0$ for any $e \in C_{\delta}$. Hence, v_z is in $C_{1/\delta}$ and then, for sufficiently small $\delta > 0$, v_z is close to e_n . Specifically, we have that

$$|v_z - e_n| \le C\delta, \quad z \in \Gamma(u) \cap B_{r_\delta}.$$

Therefore, $\Gamma(u)$ is C^1 at 0 and by the same argument, $\Gamma(u) \cap B_{r'_1}$ is C^1 .

DOUBLE OBSTACLE PROBLEMS

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