

# CRITERIA FOR EMBEDDED EIGENVALUES FOR DISCRETE SCHRÖDINGER OPERATORS

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ABSTRACT. In this paper, we consider discrete Schrödinger operators of the form,

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n).$$

We view  $H$  as a perturbation of the free operator  $H_0$ , where  $(H_0u)(n) = u(n+1) + u(n-1)$ . For  $H_0$  (no perturbation),  $\sigma_{\text{ess}}(H_0) = \sigma_{\text{ac}}(H) = [-2, 2]$  and  $H_0$  does not have eigenvalues embedded into  $(-2, 2)$ . It is an interesting and important problem to identify the perturbation such that the operator  $H_0 + V$  has one eigenvalue (finitely many eigenvalues or countable eigenvalues) embedded into  $(-2, 2)$ . We introduce the *almost sign type potential* and develop the Prüfer transformation to address this problem, which leads to the following five results.

- 1: We obtain the sharp spectral transition for the existence of irrational type eigenvalues or rational type eigenvalues with even denominator.
- 2: Suppose  $\limsup_{n \rightarrow \infty} n|V(n)| = a < \infty$ . We obtain a lower/upper bound of  $a$  such that  $H_0 + V$  has one rational type eigenvalue with odd denominator.
- 3: We obtain the asymptotical behavior of embedded eigenvalues around the boundaries of  $(-2, 2)$ .
- 4: Given any finite set of points  $\{E_j\}_{j=1}^N$  in  $(-2, 2)$  with  $0 \notin \{E_j\}_{j=1}^N + \{E_j\}_{j=1}^N$ , we construct potential  $V(n) = \frac{O(1)}{1+|n|}$  such that  $H = H_0 + V$  has eigenvalues  $\{E_j\}_{j=1}^N$ .
- 5: Given any countable set of points  $\{E_j\}$  in  $(-2, 2)$  with  $0 \notin \{E_j\} + \{E_j\}$ , and any function  $h(n) > 0$  going to infinity arbitrarily slowly, we construct potential  $|V(n)| \leq \frac{h(n)}{1+|n|}$  such that  $H = H_0 + V$  has eigenvalues  $\{E_j\}$ .

## 1. INTRODUCTION

We consider the discrete Schrödinger equation,

$$(1) \quad (Hu)(n) = u(n+1) + u(n-1) + V(n)u(n) = Eu(n),$$

where  $V(n)$  is the perturbation. Denote by  $H_0$  the free Schrödinger operator, namely,

$$(H_0u)(n) = u(n+1) + u(n-1).$$

Without loss of generality, we only consider the discrete Schrödinger operator in the half line  $\mathbb{N}$ .

$$(2) \quad (Hu)(n) = u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \quad (n \geq 1)$$

with boundary condition

$$(3) \quad \frac{u(1)}{u(0)} = \tan \theta.$$

Similarly, we also have the continuous Schrödinger operator  $H = H_0 + V(x)$ .

Our goal is to identify the asymptotical behavior of the potentials  $V$  such that there is one eigenvalue (finitely many eigenvalues, or infinitely many eigenvalues) embedded into the absolutely continuous and essential spectra. The identification of eigenvalues/singular continuous

spectrum embedded into ac (ess) spectrum attracted much attention from different viewpoints, for example [8, 10, 13–16, 20–23, 26, 32, 33].

Suppose  $\limsup_{x \rightarrow \infty} x|V(x)| = a$ . Our first interest is the study of the sharp transition for the single eigenvalue embedded into the ac spectrum or ess spectrum.

Let us introduce the history of the continuous case first. By a result of Kato [11], there is no eigenvalue  $E$  with  $E > a^2$ , which holds in any dimension. From the classical Wigner-von Neumann type functions

$$V(x) = \frac{c}{1+x} \sin(kx + \phi),$$

we know that one can not do better than  $\frac{a^2}{4}$ . For the one dimensional case, Atkinson and Everitt [1] obtained the optimal bound  $\frac{4a^2}{\pi^2}$ , that is there is no eigenvalue if  $E > \frac{4a^2}{\pi^2}$  and there are examples with eigenvalues arbitrarily close to this bound. The proof is based on Prüfer transformation and sign type potentials, that is

$$V(x) = \frac{c}{1+x} \operatorname{sgn}(\sin(kx + \phi)).$$

We refer the readers to Simon's paper for the full history [30]. Recently, we constructed examples such that the optimal bound  $\frac{4a^2}{\pi^2}$  can be achieved [17].

One purpose of this paper is to obtain a similar sharp result for the discrete case. In the following, all the potentials satisfy

$$|V(n)| = \frac{O(1)}{1+n^{\frac{3}{4}}}.$$

By Weyl's theorem, the essential spectrum of  $H_0 + V$  equals  $[-2, 2]$ . Moreover, the interval  $(-2, 2)$  is covered with absolutely continuous spectrum, cf. [6]. For any  $E \in (-2, 2)$ , let  $E = 2 \cos \pi k(E)$  with  $k \in (0, 1)$ . Sometimes, we use  $k$  for simplicity.

**Definition.** We say  $E \in (-2, 2)$  is of rational (resp. irrational) type if  $k(E)$  is rational (resp. irrational). We say  $E \in (-2, 2)$  is of rational type with even (odd) denominator if the denominator of rational number  $k(E)$  is even (odd).

However, the question of sharp bounds for embedded eigenvalues in discrete case is much more delicate than in the continuous case. The bounds heavily depend on the arithmetic property of  $k(E)$ . If  $k(E)$  is irrational, Remling's arguments imply  $H = H_0 + V$  does not have eigenvalue in  $(-2\sqrt{1-A^2a^2}, 2\sqrt{1-A^2a^2})$  with  $A = \frac{2}{\pi}$  [26]. Like the continuous case, Wigner-von Neumann type functions

$$V(n) = \frac{c}{1+n} \sin(kn + \phi),$$

can only give the bound  $A = \frac{1}{2}$  (see Lemma 7.1). Thus there is a gap between  $\frac{1}{2}$  and  $\frac{2}{\pi}$ . We use the Prüfer transformation (cf. [14, 15, 26]) and sign type potentials for the discrete case to show that  $\frac{2}{\pi}$  is sharp. See Theorems 2.1–2.4.

The most important contribution of the present paper is to study the sharp bounds for rational  $k(E)$ , which is missing in previous literature. Suppose the denominator of  $k(E)$  is  $q$ . The average of  $|\sin(2\pi k(E)n + \phi)|$  with respect to  $n$  over each period  $q$  depends on the initial phase  $\phi$ , which is different from the irrational  $k(E)$ . Predicting the sharp bounds is the first challenge since there is no ergodic theorem at hand. More importantly, it is very easy to break the initial phase in each period. Another issue is that there are two transition lines for the discrete case (see Theorems 2.3 and 2.4). For the continuous Schrödinger operator, the average does not depend on energies  $E$ , so it is easier to deal with. Thus the problem of embedded eigenvalues for the discrete case has significant new challenges.

We distinguish the eigenvalues by the denominator of  $k(E)$  is even and the denominator of  $k(E)$  is odd. Obtaining the sharp transitions for the rational type eigenvalues depends on whether we can construct potentials in each period and control the initial phase  $\phi$  after each period  $q$ .

For even denominator case, our idea is to choose a good potential  $V$  (half of the values of  $V(n)$  are positive and the remaining half are negative in each period) to create some cancellation so that we can almost keep the initial phase  $\phi$  after each period. It is very difficult to construct a potential which creates the cancellation, does not change the initial  $\phi$  and achieves the optimal bounds at the same time. We address this problem by taking the second leading entry of the equation of the Prüfer angle (see (16)) into consideration, which is more delicate than the usual method. Even when using this way, there are two issues to be addressed. The first issue is that  $\cot \pi x$  is a singular function so it is difficult to control the derivative. Luckily, the trajectories of  $\{\phi + jk\}_{j \in \mathbb{N}}$  that we must choose (in order to achieve the optimal bound) can be shown to be far away from the singular points of functions  $\cot \pi x$  (see (56)). The second issue is that the usual sign type potentials cannot change the initial phase much after one period. However, it will destroy the initial phase after plenty of periods since there is no full cancellation. We adapt the potentials a little bit (we call the result *almost sign type potentials*) to create the full cancellation by solving an algebraic equation (see (59)).

For the odd case, it is impossible to create such cancellation. Thus the initial phases will change in every period. By some delicate estimates, we get two nice bounds  $B_q$  and  $A_q$ , where  $q$  is the denominator of  $k(E)$ . See (5), (6) and (7) for the definitions of  $B_q$  and  $A_q$ . However, there is still a gap between  $B_q$  and  $A_q$ . We should mention that  $B_q$  and  $A_q$  are close. In particular, they share the same asymptotic  $-\frac{2}{\pi}$  as  $q$  goes to infinity, which is exactly the bound for irrational case. Also,  $B_q > \frac{1}{2}$  (see (27)). It means that the bound we get is better than that given by Wigner-von Neumann type functions.

Another interest of ours is to investigate the distribution of eigenvalues embedded into  $(-2, 2)$ . Under the assumption that  $\limsup_{n \rightarrow \infty} n|V(n)| < \infty$ , the possible limits of the embedded eigenvalues for all the boundary conditions are  $-2$  and  $2$ . We obtained the asymptotical behaviors of  $|E_i \pm 2|$ . See Theorem 2.11.

Our last result is to construct finitely or infinitely many eigenvalues embedded into  $(-2, 2)$ . For the continuous case, Simon [29] and Naboko [24] constructed potentials such that dense eigenvalues can be embedded in absolutely continuous spectrum (or essential spectrum). For the discrete case, Naboko and Yakovlev [25] constructed potentials  $V$  such that  $H_0 + V$  has the given eigenvalues. But the rational independence of  $k(E)$  was needed in their construction. Remling [27] constructed power decaying potentials  $V$  such that (2) has an  $\ell^2(\mathbb{N})$  for a full Lebesgue measure set of  $E \in (-2, 2)$  (or singular continuous spectrum). Recently, Jitomirskaya and Liu [9] introduced piecewise functions to construct potentials such that  $H_0 + V$  has the given eigenvalues without any rationally independent assumption, which works for manifolds [9] and perturbed periodic operators [19]. We develop similar ideas to deal with the discrete Schrödinger operator. See Theorems 2.12 and 2.13. In our other two papers, we will use piecewise functions to construct perturbed periodic Jacobi operators with embedded eigenvalues [18] and perturbed Stark type operators with embedded eigenvalues [17]. Although some ideas of corresponding construction in this paper are from [9, 18, 19, 29], there are several new ingredients. Comparing to [18] and [19], the potentials here are given in an explicit way (piecewise Wigner-von Neumann type functions). Since in the discrete case, potentials with support in any interval  $[a, b]$  is a space of finite dimension, it is difficult to apply Simon's construction to the discrete case [29]. See Remarks 2.15 and 2.16 for more details.

## 2. MAIN RESULTS

For any  $E \in (-2, 2)$ , let  $E = 2 \cos \pi k$  with  $k \in (0, 1)$ . Sometimes, we use  $k(E)$  to indicate the dependence. In the rest of this paper,  $E$  is always in  $(-2, 2)$  and  $k$  is in  $(0, 1)$ .

Define  $S_q \subset (-2, 2)$  for  $q \in \mathbb{N} \setminus \{1\}$ ,

$$(4) \quad S_q = \{E \in (-2, 2) : k(E) = \frac{p}{q}, \text{ for some } p = 1, 2, \dots, q-1 \text{ and } \gcd(p, q) = 1\}.$$

For even  $q \geq 2$ , let

$$(5) \quad A_q = \frac{2}{q \sin \frac{\pi}{q}}.$$

For odd  $q \geq 3$ , let

$$(6) \quad A_q = \frac{2 \cos \frac{\pi}{2q}}{q \sin \frac{\pi}{q}},$$

and

$$(7) \quad B_q = \frac{1 + \cos \frac{\pi}{q}}{q \sin \frac{\pi}{q}}.$$

Denote by

$$(8) \quad S_0 = \{E \in (-2, 2) : k(E) \text{ is irrational}\},$$

and

$$(9) \quad A_0 = \frac{2}{\pi}.$$

We should mention that there are no definitions for  $S_1$  and  $A_1$ . In the following, we always assume  $q \neq 1$ .

**Remark:**

- By the definition of  $A_q$  and  $B_q$ , one has

$$(10) \quad \lim_{q \rightarrow \infty} B_q = \lim_{q \rightarrow \infty} A_q = \frac{2}{\pi}.$$

- We also have for odd  $q \geq 3$  (see (27)),

$$(11) \quad B_q > \frac{1}{2}.$$

**Theorem 2.1.** *Suppose potential  $V$  satisfies*

$$\limsup_{n \rightarrow \infty} |nV(n)| = a < \frac{1}{A_q}.$$

*Let  $E_q = 2\sqrt{1 - a^2 A_q^2}$ . Then for any boundary condition (3), the operator  $H = H_0 + V$  given by (2) does not admit any eigenvalue in  $(-E_q, E_q) \cap S_q$ .*

**Remark 2.2.** Remling's argument implies Theorem 2.1 for the case  $q = 0$  [26]. We list the result and also give the proof in this paper for completeness.

**Theorem 2.3.** *Suppose  $q$  is even and  $a > \frac{1}{A_q}$ . Then for any  $\theta \in [0, \pi]$  and  $E \in S_q$ , there exist potentials  $V$  such that  $\limsup_{n \rightarrow \infty} |nV(n)| = a$  and  $E$  is an eigenvalue of the associated Schrödinger operator  $H = H_0 + V$  with boundary condition (3).*

**Theorem 2.4.** *Suppose  $q$  is even. For any  $0 < a < \frac{1}{A_q}$ , let  $E_q = 2\sqrt{1 - a^2 A_q^2}$ . Suppose  $E \in S_q$  and  $E \in (-2, -E_q) \cup (E_q, 2)$ . Then for any  $\theta \in [0, \pi]$ , there exist potentials  $V$  such that  $\limsup_{n \rightarrow \infty} |nV(n)| = a$  and  $E$  is an eigenvalue of the associated Schrödinger operator  $H = H_0 + V$  with boundary condition (3).*

From Theorems 2.1, 2.3 and 2.4, we see that for even  $q$ ,  $\frac{1}{A_q}$  and  $E_q$  are the sharp transitions of eigenvalues embedded into ac spectrum for  $E \in S_q$ . Moreover, we have the following two interesting corollaries,

**Corollary 2.5.** *Suppose potential  $V$  satisfies*

$$\limsup_{n \rightarrow \infty} |nV(n)| = a < 1.$$

*Then for any boundary condition (3),  $H = H_0 + V$  does not have any eigenvalue in  $(-2\sqrt{1 - a^2}, 2\sqrt{1 - a^2})$ .*

**Corollary 2.6.** *Suppose  $a > 1$ . Then for any boundary condition (3), there exist potentials  $V$  such that*

$$\limsup_{n \rightarrow \infty} |nV(n)| = a,$$

*and  $H = H_0 + V$  has some eigenvalue  $E \in (-2, 2)$ .*

We now turn to the case of odd  $q$ .

**Theorem 2.7.** *Suppose  $q$  is odd and  $a > \frac{1}{B_q}$ . Then for any boundary condition (3) and any  $E \in S_q$ , there exist potentials  $V$  such that  $\limsup_{n \rightarrow \infty} |nV(n)| = a$  and the associated Schrödinger operator  $H = H_0 + V$  has eigenvalue  $E$ .*

**Theorem 2.8.** *Suppose  $q$  is odd. For any  $0 < a < \frac{1}{B_q}$ , let  $\tilde{E}_q = 2\sqrt{1 - a^2 B_q^2}$ . Suppose  $E \in S_q$  and  $E \in (-2, -\tilde{E}_q) \cup (\tilde{E}_q, 2)$ . Then for any  $\theta \in [0, \pi]$ , there exist potentials  $V$  such that  $\limsup_{n \rightarrow \infty} |nV(n)| = a$  and the associated Schrödinger operator  $H = H_0 + V$  with boundary condition (3) has eigenvalue  $E$ .*

By Theorems 2.1, 2.7 and 2.8, there is a gap between  $\frac{1}{A_q}$  and  $\frac{1}{B_q}$  for odd  $q$ .

Denote by

$$P = \{E \in (-2, 2) : (2) \text{ has an } \ell^2(\mathbb{N}) \text{ solution}\}.$$

$P$  is the collections of the eigenvalues of  $H_0 + V$  with all the possible boundary conditions at 0.

**Corollary 2.9.** *Suppose potential  $V$  satisfies*

$$\limsup_{n \rightarrow \infty} |nV(n)| = a < \frac{\pi}{2}.$$

*Then for any  $\epsilon > 0$ ,  $P \cap (-2\sqrt{1 - \frac{4a^2}{\pi^2}} + \epsilon, 2\sqrt{1 - \frac{4a^2}{\pi^2}} - \epsilon)$  is a finite set.*

**Remark 2.10.** Under the assumption of Corollary 2.9, Remling [26] showed that  $H_0 + V$  has no singular continuous spectrum in  $(-2\sqrt{1 - \frac{4a^2}{\pi^2}}, 2\sqrt{1 - \frac{4a^2}{\pi^2}})$ .

The asymptotical behaviors of eigenvalues lying outside  $[-2, 2]$  has been well studied since it is related to a lot of topics in spectral theory, for example the regular behavior of the spectral measure near the points  $-2$  and  $2$  and the problems of purely ac spectrum on  $[-2, 2]$  [2-5, 7, 12, 31]. Our next result is to investigate the asymptotical behaviors of eigenvalues close to the boundaries  $-2$  and  $2$  lying in  $(-2, 2)$ .

**Theorem 2.11.** *Suppose*

$$\limsup_{n \rightarrow \infty} n|V(n)| = a.$$

*Then  $P$  is a countable set with two possible accumulation points 2 and  $-2$ . Moreover, the following estimate holds,*

$$\sum_{E_i \in P} (4 - E_i^2) \leq 4a^2 + 4 \min\{1, a\}.$$

Theorem 2.11 implies the speed of  $E_i \in P$  going to the boundaries  $\pm 2$  behaves  $|E_i - 2| \approx \frac{1}{1+i}$  ( $|E_i + 2| \approx \frac{1}{1+i}$ ). Remling [28] showed that in the continuous case, it is impossible to improve it to  $|E_i \pm 2| \approx \frac{1}{1+i^{1+\epsilon}}$ . This means that the bound in Theorem 2.11 is optimal in some sense.

The proof of Theorem 2.11 is motivated by [15]. The key idea of [15] is to show the almost orthogonality of  $\theta(n, k(E_1))$  and  $\theta(n, k(E_2))$ , where  $\theta(n, k(E_1))$  ( $\theta(n, k(E_2))$ ) is the Prüfer angle with respect to energy  $E_1$  ( $E_2$ ). For the discrete case, it is hard to verify the almost orthogonality. Luckily, a weaker version of almost orthogonality in the discrete setting has been obtained in [18], which is enough to hand our problem here.

Our next two results are to construct potentials with finitely many (countable) eigenvalues embedded into  $(-2, 2)$ . For a set  $A \subset \mathbb{R}$ , denote by  $A + A = \{x + y : x \in A \text{ and } y \in A\}$ .

**Theorem 2.12.** *Given any finite set of points  $A = \{E_j\}_{j=1}^N$  in  $(-2, 2)$  with  $0 \notin A + A$  and  $\{\theta_j\}_{j=1}^N \subset [0, \pi]$ , there exist potentials  $V(n) = \frac{O(1)}{1+n}$  such that for each  $E_j \in A$ , (1) has an  $\ell^2(\mathbb{N})$  solution with boundary condition  $\frac{u(1)}{u(0)} = \tan \theta_j$ .*

**Theorem 2.13.** *Given any countable set of points  $B = \{E_j\}$  in  $(-2, 2)$  with  $0 \notin B + B$ , any sequence  $\{\theta_j\}_j \subset [0, \pi]$  and any function  $h(n) > 0$  going to infinity arbitrarily slowly, there exist potentials  $|V(n)| \leq \frac{h(n)}{1+n}$  such that for each  $E_j \in A$ , (1) has an  $\ell^2(\mathbb{N})$  solution with boundary condition  $\frac{u(1)}{u(0)} = \tan \theta_j$ .*

**Remark 2.14.** In [18], Theorems 2.12 and 2.13 have been proved for perturbed periodic Jacobi operators. However, the explicit formula for the potentials can not be given. We will use the piecewise Wigner-von Neumann type functions to complete our construction in this paper, which we believe to be of independent interest.

**Remark 2.15.** For the continuous case, Simon [29] used Wigner-von Neumann type functions  $V(x) = \frac{\alpha}{1+x} \sum_j \sin(2\lambda_j x + 2\phi_j) \chi_{[a_j, \infty)}$ , and functions  $W$  with support in  $(1, 2)$  to do the construction. For the continuous case, we can adapt potential  $W$  to match the boundary condition  $\theta_j$ . In the discrete case, this is impossible.

The rest of the paper is organized in the following way. In Section 3, we will show the absence of embedded eigenvalues if our perturbation is small, and finish the proof of Theorem 2.1 and Corollary 2.5. In Section 4, we give some preparations for the proof of the rational type eigenvalues with even denominators. In Section 5, we will construct potentials such that the associated operators have one embedded eigenvalue, and prove Theorems 2.3, 2.4, 2.7, 2.8, and Corollaries 2.6 and 2.9. In Section 6, we will prove Theorem 2.11. In Section 7, we will construct potentials such that the associated operators have finitely (countably) many embedded eigenvalues.

### 3. PROOF OF THEOREM 2.1 AND COROLLARY 2.5

Let us introduce the Prüfer transformation first (cf. [14, 15, 26]). Suppose  $u(n, E)$  is a solution of (2) with  $u(0, E) = 0$  and  $u(1, E) = 1$ . We do not make the difference between

$u(n, k(E))$ ,  $u(n, k)$  and  $u(n, E)$ . In Sections 3, 4 and 5, all the potential  $V$  satisfy

$$(12) \quad |V(n)| = \frac{O(1)}{1+n}.$$

Let

$$(13) \quad Y(n, k) = \frac{1}{\sin \pi k} \begin{pmatrix} \sin \pi k & 0 \\ -\cos \pi k & 1 \end{pmatrix} \begin{pmatrix} u(n-1, k) \\ u(n, k) \end{pmatrix}.$$

Define the Prüfer variables  $R(n, k)$  and  $\theta(n, k)$  as

$$(14) \quad Y(n, k) = R(n, k) \begin{pmatrix} \sin(\pi\theta(n, k) - \pi k) \\ \cos(\pi\theta(n, k) - \pi k) \end{pmatrix}.$$

It is well known that  $R$  and  $\theta$  obey the equations

$$(15) \quad \frac{R(n+1, k)^2}{R(n, k)^2} = 1 - \frac{V(n)}{\sin \pi k} \sin 2\pi\theta(n, k) + \frac{V(n)^2}{\sin^2 \pi k} \sin^2 \pi\theta(n, k)$$

and

$$(16) \quad \cot(\pi\theta(n+1, k) - \pi k) = \cot \pi\theta(n, k) - \frac{V(n)}{\sin \pi k}.$$

We will give some Lemmas, which are useful in the proof of main Theorems.

**Lemma 3.1.** [14, Prop.2.4] Suppose  $\theta(n, k)$  satisfies (16) and  $|\frac{V(n)}{\sin \pi k}| < \frac{1}{2}$ . Then we have

$$(17) \quad |\theta(n+1, k) - k - \theta(n, k)| \leq \left| \frac{V(n)}{\sin \pi k} \right|.$$

The following Lemma is an improvement of Lemma 3.1 if  $\theta(n, k)$  is far way from the singular points of  $\cot \pi x$ .

**Lemma 3.2.** Suppose

$$(18) \quad \frac{1}{|\sin \pi\theta(n, k)|} = O(1).$$

Then under the condition of (12), we have for large  $n$ ,

$$\theta(n+1, k) = k + \theta(n, k) + \sin^2 \pi\theta(n, k) \frac{V(n)}{\pi \sin \pi k} + \frac{O(1)}{1+n^2}.$$

*Proof.* Let  $\theta_0 = \theta(n, k)$  and  $\theta_1 = \theta(n+1, k)$ . By Lemma 3.1, one has

$$(19) \quad \theta_1 = k + \theta_0 + \frac{O(1)}{1+n}.$$

Let  $f(x) = \cot \pi x$ . By the assumption (18) and (19), one has for large  $n$ ,

$$(20) \quad f''(x) = O(1).$$

for all  $x \in [\theta_0, \theta_1 - k]$  or  $x \in [\theta_1 - k, \theta_0]$ .

By (16), one has

$$(21) \quad f(\theta_1 - k) = f(\theta_0) - \frac{V(n)}{\sin \pi k}.$$

Using the Taylor series and (20), one has

$$(22) \quad f(\theta_1 - k) = f(\theta_0) + f'(\theta_0)(\theta_1 - k - \theta_0) + O(1)(\theta_1 - k - \theta_0)^2.$$

Notice that

$$(23) \quad f'(x) = -\frac{\pi}{\sin^2 \pi x}.$$

The Lemma follows (19), (21), (22) and (23).  $\square$

**Lemma 3.3.** *Let*

$$(24) \quad \tilde{A}_q = \frac{1}{q} \max_{\phi \in [0, 2\pi]} \sum_{j=0}^{q-1} \left| \sin\left(\frac{2\pi}{q}j + \phi\right) \right|.$$

Then for all  $q \geq 2$ ,

$$(25) \quad \tilde{A}_q = A_q.$$

Moreover, for even  $q$ ,

$$(26) \quad A_q = \frac{2}{q} \sum_{j=0}^{\frac{q}{2}-1} \sin\left(\frac{2\pi}{q}j + \frac{\pi}{q}\right).$$

For odd  $q$ , we have

$$(27) \quad B_q = \frac{1}{q} \min_{\phi \in [0, 2\pi]} \sum_{j=0}^{q-1} \left| \sin\left(\frac{2\pi}{q}j + \phi\right) \right| > \frac{1}{2}.$$

*Proof.* It is well known that

$$(28) \quad \sin(a) + \sin(a+x) + \cdots + \sin(a+(n-1)x) = \frac{\cos(a - \frac{x}{2}) - \cos(a + (n - \frac{1}{2})x)}{2 \sin \frac{x}{2}}.$$

Let us consider the even case first. By the definition of  $\tilde{A}_q$ , one has for even  $q$ ,

$$(29) \quad \tilde{A}_q = \frac{2}{q} \max_{\phi \in [0, \frac{2\pi}{q}]} \sum_{j=0}^{\frac{q}{2}-1} \sin\left(\frac{2\pi}{q}j + \phi\right),$$

and for odd  $q$ ,

$$(30) \quad \tilde{A}_q = \frac{1}{q} \max_{\phi \in [0, \frac{\pi}{q}]} \left( \sum_{j=0}^{\frac{q-1}{2}} \sin\left(\frac{2\pi}{q}j + \phi\right) - \sum_{j=\frac{q+1}{2}}^{q-1} \sin\left(\frac{2\pi}{q}j + \phi\right) \right).$$

Applying (28), one has

$$(31) \quad \sum_{j=0}^{\frac{q}{2}-1} \sin\left(\frac{2\pi}{q}j + \phi\right) = \frac{\cos(\phi - \frac{\pi}{q})}{\sin \frac{\pi}{q}}.$$

(25) and (26) follows from (29) and (31).

Now let us consider the odd  $q$ . Applying (28), one has

$$(32) \quad \sum_{j=0}^{\frac{q-1}{2}} \sin\left(\frac{2\pi}{q}j + \phi\right) - \sum_{j=\frac{q+1}{2}}^{q-1} \sin\left(\frac{2\pi}{q}j + \phi\right) = \frac{\cos \phi + \cos(\phi - \frac{\pi}{q})}{\sin \frac{\pi}{q}}.$$

(32) achieves the maximum at  $\phi = \frac{\pi}{2q}$  and the minimum at  $\phi = 0$ . It leads to (25) and the equality part of (27).



We will prove the inequality part of (27). It immediately follows from

$$\begin{aligned} B_q &= \frac{1}{q} \sum_{j=0}^{q-1} \left| \sin \frac{2\pi}{q} j \right| \\ &> \frac{1}{q} \sum_{j=0}^{q-1} \sin^2 \frac{2\pi}{q} j = \frac{1}{2}. \end{aligned}$$

□

**Lemma 3.4.** *Suppose  $k$  is irrational. Then for any  $\varepsilon > 0$ , there exists some  $N > 0$  such that for large  $n_0$ ,*

$$(33) \quad \left| \left( \frac{1}{N} \sum_{n=n_0}^{n=n_0+N-1} |\sin 2\pi\theta(n, k)| \right) - \frac{2}{\pi} \right| \leq \varepsilon.$$

*Proof.* Notice that

$$\int_0^1 |\sin 2\pi\varphi| d\varphi = \frac{2}{\pi}.$$

By the Ergodicity of irrational rotation  $k$ , for any  $\varepsilon > 0$ , there exists some  $N > 0$  such that for any  $n_0$  and  $\phi \in [0, 2\pi)$ , we have

$$(34) \quad \left| \left( \frac{1}{N} \sum_{n=n_0}^{n=n_0+N-1} |\sin(2\pi kn + \phi)| \right) - \frac{2}{\pi} \right| \leq \varepsilon.$$

By (12), (16) and (17), one has for large  $n_0$ ,

$$(35) \quad |\theta(n', k) - (n' - n_0)k - \theta(n_0, k)| \leq \varepsilon$$

for all  $n_0 \leq n' \leq n_0 + N$ . Now the Lemma follows from (34) and (35).

□

**Lemma 3.5.** *Suppose  $k \in S_q$  with  $q \geq 2$ . Then for any  $\varepsilon > 0$  and large  $n_0$ ,*

$$(36) \quad \frac{1}{q} \sum_{n=n_0}^{n=n_0+q-1} |\sin 2\pi\theta(n, k)| \leq A_q + \varepsilon,$$

and for odd  $q \geq 3$ ,

$$(37) \quad \frac{1}{q} \sum_{n=n_0}^{n=n_0+q-1} |\sin 2\pi\theta(n, k)| \geq B_q - \varepsilon.$$

*Proof.* By the definition of  $A_q$  and (25), for any  $n_0$  and  $\phi \in [0, 2\pi)$ , we have

$$(38) \quad \frac{1}{q} \sum_{n=n_0}^{n=n_0+q-1} |\sin(2\pi kn + \phi)| \leq A_q.$$

By (12), (16) and (17) again, one has for large  $n_0$ ,

$$(39) \quad |\theta(n', k) - (n' - n_0)k - \theta(n_0, k)| \leq \varepsilon$$

for all  $n_0 \leq n' \leq n_0 + q$ . Now (36) follows from (38) and (39). Similarly, (37) follows from (27).

□

**Proof of Theorem 2.1.** By the assumption of Theorem 2.1, we have for any  $\varepsilon > 0$ ,

$$(40) \quad |V(n)| \leq \frac{a + \varepsilon}{1 + n} \text{ for large } n.$$

We first consider  $E \in S_0$ , i.e.,  $k(E)$  is irrational. By (15) and (12), one has

$$(41) \quad \ln R(n+1, k)^2 - \ln R(n, k)^2 = -\frac{V(n)}{\sin \pi k} \sin 2\pi\theta(n, k) + \frac{O(1)}{n^2 + 1}.$$

Denote by  $[x]$  be the largest integer less or equal than  $x$ . Assume  $n_0$  is large enough. By (33), we have for all  $n > n_0$ ,

$$\begin{aligned} \ln R(n+1, k)^2 &\geq \ln R(n_0, k)^2 - \sum_{j=1}^n \frac{(a + \varepsilon)}{(1 + j) \sin \pi k} |\sin 2\pi\theta(j, k)| - \sum_{j=n_0}^n \frac{O(1)}{j^2 + 1} \\ &\geq -C(k, n_0, a) - \frac{(a + \varepsilon)}{\sin \pi k} \sum_{i=1}^{[\frac{n-n_0}{N}]} \sum_{m=n_0+iN-1}^{m=n_0+iN-1} \frac{|\sin 2\pi\theta(m, k)|}{m + 1} \\ &\geq -C(k, n_0, a) - \frac{(a + \varepsilon)}{\sin \pi k} \sum_{i=1}^{[\frac{n-n_0}{N}]} \frac{1}{n_0 + (i-1)N} \sum_{m=n_0+(i-1)N}^{m=n_0+iN-1} |\sin 2\pi\theta(m, k)| \\ &\geq -C(k, n_0, a) - \frac{(a + \varepsilon)}{\sin \pi k} \left(\frac{2}{\pi} + \varepsilon\right) \sum_{i=1}^{[\frac{n-n_0}{N}]} \frac{N}{n_0 + (i-1)N} \\ (42) \quad &\geq -C(k, n_0, a) - \frac{(a + \varepsilon)}{\sin \pi k} \left(\frac{2}{\pi} + \varepsilon\right) \ln n. \end{aligned}$$

Since  $E = 2 \cos \pi k$  and  $|E| < E_0$  with  $E_0 = 2\sqrt{1 - a^2 A_0^2}$ , we have for small enough  $\varepsilon > 0$ ,

$$(43) \quad \frac{(a + \varepsilon)}{\sin \pi k} \left(\frac{2}{\pi} + \varepsilon\right) < 1.$$

Thus by (42) and (43), we have for large  $n$ ,

$$R^2(n, k) \geq \frac{1}{Cn}.$$

This implies  $R(n, k)$  is not in  $\ell^2(\mathbb{N})$ . By (13) and (14), we have that  $u(n, k)$  is not in  $\ell^2(\mathbb{N})$ . We finish the proof for the irrational  $k$ .

Assume  $E \in S_q$  for  $q \geq 2$ . The proof is similar. We only need to replace  $N$  with  $q$  and (33) with (36). □

**Proof of Corollary 2.5.** By the definition of  $A_q$ , one has

$$A_q \leq 1 \text{ for all possible } q.$$

Now Corollary 2.5 follows from Theorem 2.1. □

#### 4. PREPARATIONS FOR THE RATIONAL TYPE EIGENVALUES WITH EVEN DENOMINATORS

In this section, we consider  $k \in S_q$  with even  $q \geq 2$ .

By the definition of  $A_q$  ( $q \geq 2$ ) and Lemma (3.3), for any  $\varepsilon > 0$ , there exists  $\delta$  such that

$$(44) \quad \frac{1}{q} \sum_{j=0}^{q-1} \left| \sin\left(\frac{2\pi}{q}j + \phi\right) \right| \geq A_q - \varepsilon$$

holds for all  $\phi \in (\frac{\pi}{q} - 2\pi\delta, \frac{\pi}{q} + 2\pi\delta) \subset (0, \frac{2\pi}{q})$ , where  $\delta$  is small enough (will be determined soon).

Suppose  $k = \frac{p}{q}$  with coprime  $p$  and  $q$ . Let (by the fact  $q$  is even)  $p_1^+, p_2^+, \dots, p_{\frac{q}{2}}^+$  and  $p_1^-, p_2^-, \dots, p_{\frac{q}{2}}^-$  be a permutation of  $0, 1, 2, \dots, p-1$  such that for all  $\phi \in (\frac{\pi}{q} - 2\pi\delta, \frac{\pi}{q} + 2\pi\delta)$ ,

$$(45) \quad \sin\left(\frac{2\pi p}{q} p_j^+ + \phi\right) > 0$$

and

$$(46) \quad \sin\left(\frac{2\pi p}{q} p_j^- + \phi\right) < 0$$

for  $j = 1, 2, \dots, \frac{q}{2}$ . Actually,

$$(47) \quad p_j^+ \frac{p}{q} = \frac{j-1}{q} \pmod{\mathbb{Z}}$$

and

$$(48) \quad p_j^- \frac{p}{q} = \frac{1}{2} + \frac{j-1}{q} \pmod{\mathbb{Z}}$$

for  $j = 1, 2, \dots, \frac{q}{2}$ .

Now we are in the position to construct  $V$ . Let  $n_0$  be a large fixed positive integer. Define  $V(n) = 0$  for all  $n \leq n_0 - 2$ . Let  $V(n_0 - 1)$  be such that

$$(49) \quad \theta(n_0, k) = \frac{1}{2q}.$$

Suppose  $a > 0$ . We will define for  $m \geq 0$ ,

$$(50) \quad V(n_0 + qm + p_j^+) = \frac{a_{m,j}^+}{1 + n_0 + qm}$$

and

$$(51) \quad V(n_0 + qm + p_j^-) = -\frac{a_{m,j}^-}{1 + n_0 + qm},$$

where  $a_{m,j}^\pm > 0$  is close to  $a$ . We will give the values of  $a_{m,j}^\pm$  later.

**Theorem 4.1.** *Let  $k \in S_q$  with even  $q \geq 2$ . Let  $m \geq 0$ . Suppose  $V(n-1)$  and  $\theta(n, k)$  are defined for all  $n \leq n_0 + mq$ . Suppose  $\theta(n_0 + mq, k) \in (\frac{1}{2q} - \delta, \frac{1}{2q} + \delta)$ . Then there exists  $a_{m,j}^\pm$ ,  $j = 1, 2, \dots, \frac{q}{2}$  and  $\tilde{\delta}$  (small enough depending on  $\delta$  and  $\tilde{\delta} \rightarrow 0$  as  $\delta \rightarrow 0$ ) such that the following statements hold,*

**Nominators  $a_{m,j}^\pm$ :** for all  $j = 1, 2, \dots, \frac{q}{2}$ ,

$$|a_{m,j}^\pm - a| \leq \tilde{\delta}.$$

**Potentials  $V(n)$  on  $(n_0 + mq - 1, n_0 + mq + q)$ :** Let

$$V(n_0 + mq + p_j^\pm) = \pm \frac{a_{m,j}^\pm}{1 + n_0 + mq}$$

and define  $\theta(n, k)$  by (16) for  $n_0 + mq + 1 \leq n \leq n_0 + mq + q$ . Then

$$(52) \quad \theta(n_0 + mq + q, k) = qk + \theta(n_0 + mq, k) + \frac{O(1)}{1 + n_0 + m^2}.$$

*Proof.* Recall that  $k = \frac{p}{q}$  and the potential  $V$  we constructed will satisfy (12).

By (12) and (17)

$$(53) \quad \theta(n+j, k) = jk + \theta(n, k) + \frac{O(1)}{1+n},$$

for any  $n$  and  $1 \leq j \leq q$ .

By the definitions of  $p_j^\pm$ , one has

$$(54) \quad \theta(n_0 + mq + p_j^+, k) = \frac{j-1}{q} + \theta(n_0 + mq, k) + \frac{O(1)}{1+n_0+m} \pmod{\mathbb{Z}},$$

and

$$(55) \quad \theta(n_0 + mq + p_j^-, k) = \frac{1}{2} + \frac{j-1}{q} + \theta(n_0 + mq, k) + \frac{O(1)}{1+n_0+m} \pmod{\mathbb{Z}},$$

for  $1 \leq j \leq \frac{q}{2}$ .

By the assumption that  $\theta(n_0 + mq, k) \in (\frac{1}{2q} - \delta, \frac{1}{2q} + \delta)$ , one has

$$(56) \quad \frac{1}{|\sin \pi \theta(n_0 + mq + p_j^\pm, k)|} = O(1),$$

for  $1 \leq j \leq \frac{q}{2}$ .

By Lemma 3.2, one has for  $j = 0, 1, 2, \dots, q-1$ ,

$$(57) \quad \begin{aligned} & \theta(n_0 + mq + j + 1, k) \\ &= k + \theta(n_0 + mq + j, k) + \sin^2 \pi \theta(n_0 + mq + j, k) \frac{V(n_0 + mq + j)}{\pi \sin \pi k} + \frac{O(1)}{1+n_0+m^2} \\ &= k + \theta(n_0 + mq + j, k) + \sin^2 \pi (\theta(n_0 + mq, k) + jk) \frac{V(n_0 + mq + j)}{\pi \sin \pi k} + \frac{O(1)}{1+n_0+m^2}, \end{aligned}$$

where the last inequality holds by (53).

Thus, one has

$$(58) \quad \begin{aligned} & \theta(n_0 + mq + q, k) \\ &= qk + \theta(n_0 + mq, k) + \sum_{j=0}^{q-1} \sin^2 \pi (\theta(n_0 + mq, k) + jk) \frac{V(n_0 + mq + j)}{\pi \sin \pi k} + \frac{O(1)}{1+n_0+m^2}. \end{aligned}$$

By (58), in order to guarantee (52), we only need to construct  $a_{m,j}^\pm$  such that

$$(59) \quad \sum_{j=0}^{q-1} \sin^2 \pi (\theta(n_0 + mq, k) + jk) V(n_0 + mq + j) = 0.$$

Let  $a_{m,j}^+ = a$  for all  $j = 1, 2, \dots, \frac{q}{2}$  and  $a_{m,j}^- = a$  for all  $j = 1, 2, \dots, \frac{q}{2} - 1$ . By (50) and (51), it suffices to determine  $a_{m, \frac{q}{2}}^-$  such that

$$(60) \quad \begin{aligned} & a \sum_{j=1}^{\frac{q}{2}} \sin^2 \pi (\theta(n_0 + mq, k) + p_j^+ k) \\ & - a \sum_{j=1}^{\frac{q}{2}-1} \sin^2 \pi (\theta(n_0 + mq, k) + p_j^- k) - a_{m, \frac{q}{2}}^- \sin^2 \pi (\theta(n_0 + mq, k) + \frac{q-1}{q}) = 0. \end{aligned}$$

By the definition of  $p_j^\pm$  ((47) and (48)) and  $k = \frac{q}{2}$ , it suffices to guarantee that

$$(61) \quad a \sum_{j=1}^{\frac{q}{2}} \sin^2 \pi \left( \theta(n_0 + mq, k) + \frac{j-1}{q} \right) - a \sum_{j=1}^{\frac{q}{2}-1} \sin^2 \pi \left( \theta(n_0 + mq, k) + \frac{1}{2} + \frac{j-1}{q} \right) - a_{m, \frac{q}{2}}^- \sin^2 \pi \left( \theta(n_0 + mq, k) + \frac{q-1}{q} \right) = 0.$$

Direct computation implies that

$$(62) \quad \sum_{j=1}^{\frac{q}{2}} \sin^2 \pi \left( \frac{1}{2q} + \frac{j-1}{q} \right) = \sum_{j=1}^{\frac{q}{2}} \sin^2 \pi \left( \frac{1}{2q} + \frac{1}{2} + \frac{j-1}{q} \right).$$

This shows that if  $\theta(n_0 + mq, k) = \frac{1}{2q}$ , (61) holds for  $a_{m, \frac{q}{2}}^- = a$ .

In our case,  $\theta(n_0 + mq, k) \in (\frac{1}{2q} - \delta, \frac{1}{2q} + \delta)$ . Then there exist  $\tilde{\delta} > 0$  (small) and  $a_{m, \frac{q}{2}}^-$  such that  $|a_{m, \frac{q}{2}}^- - a| \leq \tilde{\delta}$  and (61) holds. □

## 5. PROOF OF THEOREMS 2.3, 2.4, 2.7, 2.8, COROLLARIES 2.6 AND 2.9

**Proof of Theorems 2.3 and 2.4 for  $q = 0$ .** Solve the following equation with initial condition  $\theta(0)$ ,

$$(63) \quad \cot(\pi\theta(n+1, k) - \pi k) = \cot \pi\theta(n, k) - \frac{V(n)}{\sin \pi k},$$

with

$$V(n) = \frac{a}{1+n} \operatorname{sgn}(\sin 2\pi\theta(n, k)),$$

where  $\operatorname{sgn}(\cdot)$  is the sign function. Thus equation (15) becomes

$$(64) \quad \frac{R(n+1, k)^2}{R(n, k)^2} = 1 - \frac{a}{\sin \pi k} \frac{|\sin 2\pi\theta(n, k)|}{1+n} + \frac{O(1)}{1+n^2}.$$

Applying (33) and following the proof of (42), we have

$$(65) \quad \ln R(n+1, k)^2 \leq C(k, n_0, a) - \frac{a}{\sin \pi k} \left( \frac{2}{\pi} - \varepsilon \right) \ln n.$$

Suppose  $E$  and  $a$  satisfy the assumption for  $q = 0$  in Theorems 2.3 and 2.4. Then we have

$$(66) \quad \frac{a}{\sin \pi k} \left( \frac{2}{\pi} - \varepsilon \right) > 1$$

for small  $\varepsilon > 0$ . By (65), we obtain that  $R(n, k)$  is in  $\ell^2(\mathbb{N})$ . By changing the initial condition  $\theta(0)$ , we can make the  $\ell^2(\mathbb{N})$  solution  $u$  satisfy the given boundary condition (3). We finish the proof for  $q = 0$ . □

In the following arguments, we will continue to use the idea “making the  $\ell^2(\mathbb{N})$  solution  $u$  satisfy the given boundary condition (3) by changing the initial condition  $\theta(0)$ ”. In order to avoid the repetition, sometimes we omit the details.

**Proof of Theorems 2.3 and 2.4 for  $q \geq 2$ .** Fix  $E \in S_q$  with even  $q \geq 2$ .

By Theorem 4.1 and induction, we can prove that there exist  $a_{j,m}^\pm$ ,  $j = 1, 2, \dots, \frac{q}{2}$ ,  $m \geq 0$ , and  $\tilde{\delta}$  such that the following statements hold,

- for all  $j = 1, 2, \dots, \frac{q}{2}$ ,

$$(67) \quad |a_{j,m}^{\pm} - a| \leq \tilde{\delta}.$$

- 

$$(68) \quad V(n_0 + mq + p_j^{\pm}) = \pm \frac{a_{j,m}^{\pm}}{1 + n_0 + mq}.$$

- 

$$(69) \quad \begin{aligned} \theta(n_0 + mq, k) &= mqk + \theta(n_0, k) + O(1) \sum_{j=1}^m \frac{1}{1 + n_0 + j^2} \\ &= \frac{1}{2q} + O(1) \sum_{j=1}^m \frac{1}{1 + n_0 + j^2} \pmod{\mathbb{Z}} \\ &= \frac{1}{2q} + \frac{O(1)}{1 + n_0} \pmod{\mathbb{Z}}. \end{aligned}$$

By (69) and (17), one has for any  $0 \leq j \leq q-1$ ,

$$(70) \quad \sin \pi(\theta(n_0 + mq + j, k) - \frac{1}{2q} - kj) = \frac{O(1)}{1 + n_0}.$$

By (69), (70) and (26), one has

$$(71) \quad \frac{1}{q} \sum_{j=0}^{q-1} |\sin \pi \theta(n_0 + mq + j, k)| = A_q + \frac{O(1)}{1 + n_0}.$$

By (45), (46), (68), and (70), we have for any  $0 \leq j \leq q-1$ ,

$$\frac{V(n_0 + mq + j)}{\sin \pi k} \sin 2\pi \theta(n_0 + mq + j, k) > 0.$$

By (67) and (71) (letting  $n_0$  be large), we have

$$(72) \quad \frac{1}{q} \sum_{j=0}^{q-1} V(n_0 + mq + j) \sin 2\pi \theta(n_0 + mq + j, k) \geq \frac{(a - \tilde{\delta})}{1 + n_0 + mq} (A_q - \varepsilon).$$

By (15) and (72), we have for  $m \geq 1$ ,

$$(73) \quad \begin{aligned} &\ln R(n_0 + mq, k)^2 \\ &\leq \ln R(n_0, k)^2 - \frac{1}{\sin \pi k} \sum_{j=0}^{m-1} \sum_{i=0}^{q-1} V(n_0 + jq + i) \sin(2\pi \theta(n_0 + jq + i, k)) + \sum_{j=1}^m \frac{O(1)}{n_0 + j^2} \\ &\leq C(k, n_0, a) - \frac{1}{\sin \pi k} \sum_{j=0}^{m-1} q \frac{a - \tilde{\delta}}{n_0 + jq} (A_q - \varepsilon) \\ &\leq C(k, n_0, a) - \frac{(a - \tilde{\delta})(A_q - \varepsilon)}{\sin \pi k} \ln m \\ &\leq C(k, n_0, a) - \frac{(a - \tilde{\delta})(A_q - \varepsilon)}{\sin \pi k} \ln(n_0 + mq). \end{aligned}$$

Suppose  $E$  and  $a$  satisfy the assumption for  $q \geq 2$  in Theorems 2.3 and 2.4. Then we have

$$(74) \quad \frac{(a - \tilde{\delta})(A_q - \varepsilon)}{\sin \pi k} > 1$$

for small  $\tilde{\delta}, \varepsilon > 0$ . By (73) and (74), one has

$$\sum_m R(n_0 + mq, k)^2 < \infty.$$

This implies (using (15)),

$$\sum_n R(n, k)^2 < \infty.$$

By (13) and (14), we have  $u(n, k)$  is in  $\ell^2(\mathbb{N})$ . We finish the proof for  $q \geq 2$ .  $\square$

**Proof of Theorems 2.7 and 2.8.** Solve the following equation,

$$(75) \quad \cot(\pi\theta(n+1, k) - \pi k) = \cot \pi\theta(n, k) - \frac{V(n)}{\sin \pi k},$$

with

$$V(n) = \frac{a}{1+n} \operatorname{sgn}(\sin 2\pi\theta(n, k)),$$

where  $\operatorname{sgn}(\cdot)$  is the sign function. Thus equation (15) becomes

$$(76) \quad \frac{R(n+1, k)^2}{R(n, k)^2} = 1 - \frac{a}{\sin \pi k} \frac{|\sin 2\pi\theta(n, k)|}{1+n} + \frac{O(1)}{1+n^2}.$$

Applying (37) and (15), we have

$$\ln R(n+q, k)^2 \leq \ln R(n, k)^2 - q(B_q - \varepsilon) \frac{a}{\sin \pi k} \frac{1}{1+n}.$$

This implies

$$(77) \quad \begin{aligned} \ln R(n_0 + mq, k)^2 &\leq C(n_0, k, a) - (B_q - \varepsilon) \frac{a}{\sin \pi k} \sum_{j=0}^{m-1} \frac{q}{1+n_0 + jq} \\ &= C(n_0, k, a) - (B_q - \varepsilon) \frac{a}{\sin \pi k} \ln(n_0 + mq). \end{aligned}$$

Suppose  $E$  and  $a$  satisfy the assumption for odd  $q$  in Theorems 2.7 and 2.8. Then we have

$$(78) \quad (B_q - \varepsilon) \frac{a}{\sin \pi k} > 1$$

for small  $\varepsilon > 0$ . By (77), we obtain  $R(n, k)$  is in  $\ell^2(\mathbb{N})$ . We finish the proof for odd  $q \geq 3$ .  $\square$

**Proof of Corollary 2.6.** Let us consider  $q = 2$ . Then  $A_2 = 1$  and  $S_2 = \{0\}$ . Now the Corollary follows Theorem 2.3.  $\square$

**Proof of Corollary 2.9.** By (10) and the fact that  $a < \frac{\pi}{2}$ , we have for large  $q$ ,

$$\frac{1}{A_q} > a,$$

and

$$\left(-2\sqrt{1 - \frac{4a^2}{\pi^2}} + \varepsilon, 2\sqrt{1 - \frac{4a^2}{\pi^2}} - \varepsilon\right) \subset \left(-2\sqrt{1 - a^2 A_q^2}, 2\sqrt{1 - a^2 A_q^2}\right).$$

Theorem 2.1 implies that there are no eigenvalues in  $\left(-2\sqrt{1 - \frac{4a^2}{\pi^2}} + \varepsilon, 2\sqrt{1 - \frac{4a^2}{\pi^2}} - \varepsilon\right) \cap S_q$  for  $q = 0$  and large  $q$ . Now Corollary 2.9 follows.  $\square$

## 6. PROOF OF THEOREM 2.11

**Lemma 6.1.** [14, Lemma 4.4] Let  $\{e_i\}_{i=1}^N$  be a set of unit vector in a Hilbert space  $\mathcal{H}$  so that

$$\alpha = N \sup_{i \neq j} |\langle e_i, e_j \rangle| < 1.$$

Then

$$(79) \quad \sum_{i=1}^N |\langle g, e_i \rangle|^2 \leq (1 + \alpha) \|g\|^2.$$

**Lemma 6.2.** [18, Lemma 4.2] Suppose  $V(n) = \frac{O(1)}{1+n}$ . Let  $E_1, E_2 \in (-2, 2)$  be such that  $k(E_1) \neq k(E_2)$  and  $k(E_i) + k(E_j) \neq 1$  for  $i, j = 1, 2$ . Then for any  $\varepsilon > 0$ , there exist  $D(E_1, E_2, \varepsilon)$  and  $D(E_1, \varepsilon)$  such that

$$(80) \quad \left| \sum_{t=1}^n \frac{\cos 4\theta(t, E_1)}{1+t} \right| \leq D(E_1, \varepsilon) + \varepsilon \ln n,$$

and

$$(81) \quad \left| \sum_{t=1}^n \frac{\sin 2\theta(t, E_1) \sin 2\theta(t, E_2)}{1+t} \right| \leq D(E_1, E_2, \varepsilon) + \varepsilon \ln n.$$

**Proof of Theorem 2.11.** By the assumption of Theorem 2.11, for any  $M > a$ , we have

$$|V(n)| \leq \frac{M}{1+n}$$

for large  $n$ . By shifting the operator, we can assume

$$(82) \quad |V(n)| \leq \frac{M}{1+n}$$

for all  $n > 0$ .

Fix  $\varepsilon > 0$ , which is small.

Suppose  $E_1, E_2, \dots, E_N \in (0, 2)$  so that (1) has an  $\ell^2(\mathbb{N})$  solution for each  $E_i$ ,  $i = 1, 2, \dots, N$ . Let  $k_i = k(E_i)$  for  $i = 1, 2, \dots, N$ . It leads that

$$\sum_{i=1}^N R(n, k_i) \in \ell^2(\mathbb{N}),$$

and then there exists  $B_j \rightarrow \infty$  such that

$$(83) \quad R(B_j, k_i) \leq B_j^{-\frac{1}{2}},$$

for all  $i = 1, 2, \dots, N$ .

Notice that we have

$$R(n, k_i) = O(1)$$

for all  $i = 1, 2, \dots, N$ .

By (15), one has

$$(84) \quad \ln R(n+1, k)^2 - \ln R(n, k)^2 = -\frac{V(n)}{\sin \pi k} \sin 2\pi\theta(n, k) + \frac{O(1)}{1+n^2}.$$

By (83) and (84), we have

$$(85) \quad \sum_{n=1}^{B_j} V(n) \sin 2\pi\theta(n, k) \geq (\sin \pi k_i) B_j + O(1),$$



for all  $i = 1, 2, \dots, N$ .

We consider the Hilbert spaces

$$\mathcal{H}_j = \{u \in \mathbb{R}^{B_j} : \sum_{n=1}^{B_j} |u(n)|^2 (1+n) < \infty\}$$

with the inner product

$$\langle u, v \rangle = \sum_{n=1}^{B_j} u(n)v(n)(1+n).$$

In  $\mathcal{H}_j$ , by (82) we have

$$(86) \quad \|V\|_{\mathcal{H}_j}^2 \leq M^2 \log(1+B_j).$$

Let

$$e_i^j(n) = \frac{1}{\sqrt{A_i^j}} \frac{\sin 2\theta(n, k_i)}{1+n} \chi_{[0, B_j]}(n),$$

where  $A_i^j$  is chosen so that  $e_i^j$  is a unit vector in  $\mathcal{H}_j$ . We have the following estimate,

$$\begin{aligned} A_i^j &= \sum_{n=1}^{B_j} \frac{\sin^2 2\theta(n, k_i)}{1+n} \\ &= \sum_{n=1}^{B_j} \frac{1}{2(1+n)} - \sum_{n=1}^{B_j} \frac{\cos 4\theta(n, k_i)}{2(1+n)} \\ (87) \quad &= \frac{1}{2} \log B_j - \sum_{n=1}^{B_j} \frac{\cos 4\theta(n, k_i)}{2(1+n)} + O(1), \end{aligned}$$

Since  $E_i$  is positive ( $k_i \in (0, \frac{1}{2})$ ) for all  $i = 1, 2, \dots, N$ , one has

$$k_i + k_{\tilde{i}} \neq 1,$$

for all  $1 \leq i, \tilde{i} \leq N$ .

By (80), one has

$$(88) \quad \left| \sum_{n=1}^{B_j} \frac{\cos 4\theta(n, k_i)}{1+n} \right| \leq O(1) + \epsilon \ln B_j,$$

for all  $i = 1, 2, \dots, N$ .

By (87) and (88), we have

$$(89) \quad \left(\frac{1}{2} - \epsilon\right) \log B_j + O(1) \leq A_i^j \leq \left(\frac{1}{2} + \epsilon\right) \log B_j + O(1).$$

By (81), we have for  $i \neq \tilde{i}$ ,

$$(90) \quad -\epsilon \log B_j + O(1) \leq \sum_{n=1}^{B_j} \frac{\sin 2\theta(n, k_i) \sin 2\theta(n, k_{\tilde{i}})}{1+n} \leq \epsilon \log B_j + O(1).$$

By (89) and (90), one has

$$-4\epsilon + \frac{O(1)}{\log B_j} \leq \langle e_i^j, e_{\tilde{i}}^j \rangle \leq 4\epsilon + \frac{O(1)}{\log B_j},$$

for all  $1 \leq i, \tilde{i} \leq N$  and  $i \neq \tilde{i}$ . It implies for large  $j$ ,

$$(91) \quad -5\epsilon \leq \langle e_i^j, e_{\tilde{i}}^j \rangle \leq 5\epsilon,$$

for all  $1 \leq i, \tilde{i} \leq N$  and  $i \neq \tilde{i}$ .

By (89) and (85)

$$(92) \quad \langle V, e_i^j \rangle_{\mathcal{H}_j} \geq \sqrt{2}(1-2\epsilon) \sin \pi k_i \sqrt{\log B_j},$$

for large  $j$ . By (79) and (91), one has

$$(93) \quad \sum_{i=1}^N |\langle V, e_i^j \rangle_{\mathcal{H}_j}|^2 \leq (1+10N\epsilon) \|V\|_{\mathcal{H}_j}.$$

By (92), (93) and (86), we have

$$\sum_{i=1}^N 2(1-2\epsilon)^2 \sin^2(\pi k_i) \log B_j \leq (1+10N\epsilon) M^2 \log B_j + O(1).$$

Let  $j \rightarrow \infty$  and then  $\epsilon \rightarrow 0$ , we get

$$\sum_{i=1}^N 2 \sin^2(\pi k_i) \leq M^2,$$

for any  $M > a$ . This implies

$$\sum_{i=1}^N 2 \sin^2(\pi k_i) \leq a^2.$$

By the fact that  $E_i = 2 \cos \pi k_i$ , one has

$$(94) \quad \sum_{i=1}^N (4 - E_i^2) \leq 2a^2.$$

It yields that  $H_0 + V$  has at most countable eigen-solutions corresponding to positive energies. Let  $N \rightarrow \infty$  in (94), we have

$$(95) \quad \sum_{E_i \in (0,2) \cap P}^{\infty} (4 - E_i^2) \leq 2a^2.$$

Similarly, we can prove  $H_0 + V$  has at most countable negative eigenvalues in  $(-2, 0)$  with the same upper bound in (95). Thus

$$(96) \quad \sum_{E_i \in P, E_i \neq 0}^{\infty} (4 - E_i^2) \leq 4a^2.$$

We may add 4 in the bound of (96) if  $Hu = 0$  has an  $\ell^2(\mathbb{N})$  solution. However, if  $a < 1$ , by Theorem 2.1,  $Hu = 0$  can not have an  $\ell^2(\mathbb{N})$  solution. We finish the proof.  $\square$

## 7. PROOF OF THEOREMS 2.12 AND 2.13

Suppose  $u(n, E)$  is a solution of  $Hu = H_0u + Vu = Eu$ . Let

$$\tilde{R}(n, E) = \sqrt{u(n-1, E)^2 + u(n, E)^2}.$$

**Lemma 7.1.** ([32, Theorem 3.1]) *Let*

$$V(n) = \frac{M \sin(2\pi kn + \phi)}{n},$$

with  $k \neq \frac{1}{2}$  and  $H = H_0 + V$ . Then for every  $E \in (-2, 2)$  there exists a base  $u^+(n, E)$  and  $u^-(n, E)$  of  $Hu = Eu$  with the following asymptotics.

Case 1. For  $E = 2 \cos \pi k$

$$u^+(n, E) = n^{\frac{M}{4 \sin \pi k}} (\cos(\pi kn + \phi/2) + o(1)),$$

and

$$(97) \quad u^-(n, E) = n^{-\frac{M}{4 \sin \pi k}} (\sin(\pi kn + \phi/2) + o(1)).$$

Case 2. For  $E = -2 \cos \pi k$

$$\begin{aligned} u^+(n, E) &= (-1)^n n^{\frac{M}{4 \sin \pi k}} (\sin(\pi kn + \phi/2) + o(1)), \\ u^-(n, E) &= (-1)^n n^{-\frac{M}{4 \sin \pi k}} (\cos(\pi n + \phi/2) + o(1)). \end{aligned}$$

Case 3. For  $E = 2 \cos \hat{\pi} k \in (-2; 2) \setminus \{\pm 2 \cos \pi k\}$

$$\begin{aligned} u^+(n, E) &= \exp(i\hat{k}n) + o(1), \\ u^-(n, E) &= \exp(-i\hat{k}n) + o(1). \end{aligned}$$

**Proposition 7.2.** *Let  $E \neq 0$  and  $A = \{\tilde{E}_j\}_{j=1}^m$  be in  $(-2, 2)$  such that  $E \notin (-A) \cup A$ . Suppose  $E$  and  $\{\tilde{E}_j\}_{j=1}^m$  are different. Suppose  $\theta_0 \in [0, \pi)$ . Let  $n_1 > n_0 > b$ . Then there exist constants  $K(E, A)$ ,  $C(E, A)$  (independent of  $b, n_0$  and  $n_1$ ), and potential  $\tilde{V}(n, E, A, n_0, n_1, b, \theta_0)$  such that for  $n_0 - b > K(E, A)$  the following holds:*

**Potential:** for  $n_0 \leq n \leq n_1$ ,  $\text{supp}(\tilde{V}) \subset (n_0, n_1)$ , and

$$(98) \quad |\tilde{V}(n, E, A, n_0, n_1, b, \theta_0)| \leq \frac{C(E, A)}{n - b}.$$

**Solution for  $E$ :** the solution of  $(H_0 + \tilde{V})u = Eu$  with boundary condition  $\frac{u(n_0, E)}{u(n_0 - 1, E)} = \tan \theta_0$  satisfies

$$(99) \quad \tilde{R}(n_1, E) \leq C(E, A) \left(\frac{n_1 - b}{n_0 - b}\right)^{-100} \tilde{R}(n_0, E)$$

and for  $n_0 < n < n_1$ ,

$$(100) \quad \tilde{R}(n, E) \leq C(E, A) \tilde{R}(n_0, E).$$

**Solution for  $\tilde{E}_j$ :** any solution of  $(H_0 + \tilde{V})u = \tilde{E}_j u$  satisfies for  $n_0 < n \leq n_1$ ,

$$(101) \quad \tilde{R}(n, \tilde{E}_j) \leq C(E, A) \tilde{R}(n_0, \tilde{E}_j).$$

*Proof.* For simplicity, denote by  $K = K(E, A)$ ,  $C = C(E, A)$  and  $k = k(E)$ . Let  $E = 2 \cos \pi k$ . By the assumption that  $E \neq 0$ , one has  $k \neq \frac{1}{2}$ . By the assumption, we have  $\tilde{E}_j \neq \pm E$ .

By shifting the operator  $b$  unit, we only need to consider  $b = 0$ . For  $n \geq n_0$ , define

$$(102) \quad \tilde{V}(n) = \frac{400}{\sin \pi k(E)} \frac{\sin(2\pi kn + \phi_E)}{n},$$

where  $\phi_E$  will be determined later.

By Case 1 of Lemma 7.1, one of the solution of  $Hu = H_0u + \tilde{V}u = Eu$  satisfies (97). By adapting  $\phi_E$  in (102), we can make sure that the solution of  $Hu = H_0u + \tilde{V}u = Eu$  with

boundary condition  $\frac{u(n_0, E)}{u(n_0-1, E)} = \tan \theta_0$  satisfies (97). Thus (choosing  $C$  large enough in (102)), one has

$$(103) \quad \tilde{R}(n_1, E) \leq C \left(\frac{n_1}{n_0}\right)^{-100} \tilde{R}(n_0, E)$$

and for  $n_0 < n < n_1$ ,

$$(104) \quad \tilde{R}(n, E) \leq C \tilde{R}(n_0, E).$$

Those prove (99) and (100).

(101) follows from Case 3 of Lemma 7.1. □

**Proof of Theorems 2.12 and 2.13.** Once we have Proposition 7.2 at hand, Theorems 2.12 and 2.13 can be proved by the piecewise functions gluing technics from [9, 19]. □

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