The analytic integrability problem for perturbations of homogeneous quadratic Lotka-Volterra systems

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August 3, 2021

Abstract

We solve the analytic integrability problem for differential systems in the plane whose origin is an isolated singularity and the first homogeneous component is a quadratic Lotka-Volterra type. As an application, we give the analytically integrable systems of a class of systems $\dot{x} = x(P_1+P_2)$, $\dot{y} = y(Q_1+Q_2)$, being P_i, Q_i homogeneous polynomials of degree i.

1 Introduction and statement of the main result

We focus on the study of the analytic integrability of a planar differential system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{1.1}$$

where $\mathbf{F}$ is analytic in a neighborhood of the origin.

Writing the Taylor expansion $\mathbf{F} = \mathbf{F}_n + \mathbf{F}_{n+1} + \cdots$, $\mathbf{F}_n \not\equiv 0$, we notice that the condition of polynomial integrability of $\mathbf{F}_n$, lowest degree homogeneous term of the vector field, is a necessary condition in order to be $\mathbf{F}$ analytically integrable.

The analytically integrable differential systems with non-null linear part, i.e. $n = 1$, and $\mathbf{F}_1$ polynomially integrable, are orbitally linearizable. Indeed, it has the following cases in function the eigenvalues of $D(\mathbf{F}_1)(\mathbf{0})$: if $\lambda_1\lambda_2 \neq 0$, the origin is either a saddle, or node or a non-degenerate monodromic singular point (with complex eigenvalues). If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, the origin is a saddle-node. Finally if $\lambda_1 = \lambda_2 = 0$, it has a nilpotent singular point. The nodes and saddle-nodes are not analytically integrable. A non-degenerate monodromic point is analytically integrable if, and only if, it is orbitally equivalent to $(-y, x)^T$, and a resonant saddle has an analytic first integral around the singular point if, and only if, it is orbitally equivalent to $(px, -qy)^T$ with $p, q \in \mathbb{N}$, see [12, 16]. The most studied systems whose origin is a resonant saddle are the Lotka-Volterra systems, see [7, 8, 9, 10, 11, 13, 14, 17] and references therein.

Recently, it is proved that a nilpotent singular point is analytically integrable if, and only if, the vector field is orbital equivalent to its lowest degree quasi-homogeneous term, see [5].

For vector fields with null linear part (a degenerate singular point) some partial results are known. Not any analytically integrable vector field with null linear part is orbitally equivalent to its first quasi-homogeneous component, see [3, 4]. The analytic integrability problem when the first quasi-homogeneous component of $\mathbf{F}$ is conservative whose Hamiltonian function h has only simple factors is completely solved in [2]. In [1] it is studied a particular case with h having multiple factors.

In this work, we deal with perturbations of homogeneous quadratic Lotka-Volterra systems,

$$\mathbf{F} = \mathbf{F}_2 + \cdots, \quad \mathbf{F}_2(x, y) = (xP_1(x, y), yQ_1(x, y))^T$$

with P_1 and Q_1 homogeneous polynomials of degree one (vector field with null linear part) and the origin is an isolated singular point of $\dot{\mathbf{x}} = \mathbf{F}_2(\mathbf{x})$.

Here, we solve the analytic integrability problem for these systems. More specifically, we prove that, under the condition of polynomial integrability of $\mathbf{F}_2$, the vector field $\mathbf{F}$ is analytically integrable if, and only if, it is orbitally equivalent to its lowest degree component. (Theorem 3.13).

As consequence, we characterize its analytic integrability through the existence of a Lie symmetry (Theorem 3.14) and of an inverse integrating factor (Theorem 3.15).

We emphasize that for the vector fields $\mathbf{F} = \mathbf{F}_3 + \cdots$, whose first homogeneous component is $\mathbf{F}_3(x, y) = (xP_2(x, y), yQ_2(x, y))^T$, with P_2 and Q_2 homogeneous polynomials of degree two, the existence of an analytic first integral is not equivalent to the orbital equivalence of its lowest degree component. In fact, the vector field $(x(-3y^2 - x^2), y(y^2 + 3x^2))^T + (y^4, 0)^T$ is a perturbation of a cubic Lotka-Volterra type, analytically integrable since it is Hamiltonian. Nevertheless, it is not possible to transforms it into its lowest degree component. So, for $n \geq 3$, the problem is still open.

Finally, in Section 4, we calculate the systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(-x + 3y) \\ y(3x - y) \end{pmatrix} + \begin{pmatrix} x(a_{20}x^2 + a_{11}xy + a_{02}y^2) \\ y(b_{20}x^2 + b_{11}xy + b_{02}y^2) \end{pmatrix}$$

with an analytic first integral at the origin.

1.1 Invariant curves and first integrals of vector fields

First we give the definition of invariant curve and its associated cofactor.

We deal with a vector field $\mathbf{F} = (P, Q)^T$ with P, Q analytic at the origin and $P(\mathbf{0}) = Q(\mathbf{0}) = 0$. Throughout the paper, we will denote by F the operator associated to the vector field $\mathbf{F}$, that is, $F := P\partial_x + Q\partial_y$. We recall the concept of invariant curve and its associated cofactor.

Definition 1.1 *It is said that $C \in \mathbf{C}[[x, y]]$ (algebra of formal power series in x, y over $\mathbf{C}$), with $C(\mathbf{0}) = 0$, is an invariant curve of the vector field $\mathbf{F}$, if there exists $K \in \mathbf{C}[[x, y]]$, named cofactor of C, such that $F(C) = KC$.*
Moreover, if $K \equiv 0$, it is said that $\mathbf{F}$ is formally integrable and C is a first integral of $\mathbf{F}$.

Let note that any formal function C with $C(\mathbf{0}) \neq 0$, satisfies $F(C) = KC$ with $K = F(C)/C \in \mathbf{C}[[x, y]]$.

We will denote by $\mathcal{P}_k$ the vector space of homogeneous scalar polynomials of degree k, and by $\mathcal{Q}_k$ the vector space of polynomial homogeneous vector fields of degree k. We will use Taylor expansion of functions and vector fields without to consider questions of convergence. We note that analytic integrability is equivalent to formal integrability, see Mattei & Moussu [15].

Throughout the paper, we will be denoted by $\mathbf{D} = (x, y)^T \in \mathcal{Q}_1$ (dissipative homogeneous vector field) and by $\mathbf{X}_h = (-\partial h/\partial y, \partial h/\partial x)^T$ (Hamiltonian vector field associated to the polynomial h).
The following splitting of a homogeneous vector field plays a main role in our study.

Proposition 1.2 *[2, Prop.2.7] Every $\mathbf{F}_k \in \mathcal{Q}_k$ can be uniquely written as $\mathbf{F}_k = \mathbf{X}_h + \mu\mathbf{D}$ with $h := \frac{1}{k+1}(\mathbf{D} \wedge \mathbf{F}_k) \in \mathcal{P}_{k+1}$ (product wedge of both vector fields) and $\mu := \frac{1}{k+1} div(\mathbf{F}_k) \in \mathcal{P}_{k-1}$ (divergence of $\mathbf{F}_k$).*

We give the Taylor expansion of a formal invariant curve of a formal vector field.

Proposition 1.3 *Consider* $\mathbf{F} = \sum_{j\geq n} \mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j$ *with* $\mathbf{F}_n \not\equiv \mathbf{0}$. *Let* C *a formal invariant cuve of* $\mathbf{F}$ *with cofactor* K. *Then,* $C = \sum_{j\geq s} C_j$, $C_j \in \mathcal{P}_j$ *and* $K = \sum_{j\geq n} K_j$, $K_j \in \mathcal{P}_j$, *being the polynomial* C_s *an invariant curve of the polynomial vector field* $\mathbf{F}_n$ *with cofactor* K_n.

Proof. It is enough to consider the lowest degree homogeneous term of the equality $F(C) - KC = 0$. ■

The following two results show the invariant curves of a homogeneous vector field.

Proposition 1.4 *Every homogeneous polynomial invariant curve of a homogeneous vector field* $\mathbf{F}_n$ *is given by* $g_1^{n_1} g_2^{n_2} \dots g_m^{n_m}$ *being each* g_j *a polynomial invariant curve of* $\mathbf{F}_n$.
Moreover, its cofactor is $n_1K_1 + \cdots + n_mK_m$, *being* K_j *the cofactor of* g_j.

Proof. We suppose that $g = g_1p$, (g_1 an irreducible homogeneous polynomial), is an invariant curve of $\mathbf{F}_n$ with K_n cofactor of g. It has that $F_n(g_1p) = g_1F_n(p) + pF_n(g_1) = K_ng_1p$, that is, $g_1(pK_n - F_n(p)) = pF_n(g_1)$. From the irreducibility of g_1, it has two situations: either g_1 is an irreducible invariant curve of $\mathbf{F}_n$, in such case, p is also an invariant curve of $\mathbf{F}_n$ and we repeat the process for p. Or $p = qg_1$, i.e. $g = g_1^2q$. We now have that $F_n(g_1^2q) = g_1^2F_n(q) + 2qg_1F_n(g_1) = K_ng_1^2q$. Thus, $g_1(qK_r - F_r(q)) = 2qF_r(g_1)$. Reasoning of similar way, it completes the proof.
The second part is obtained easily. ■

Proposition 1.5 *Given* $\mathbf{F}_n \in \mathcal{Q}_n$, *any factor of* h *is an invariant curve of* $\mathbf{F}_n$. *Conversely, any homogeneous polynomial invariant curve of* $\mathbf{F}_n$ *is a factor of* h.
Moreover, if I *is a polynomial first integral of* $\mathbf{F}_n$, *then* $I = g_1^{n_1} g_2^{n_2} \cdots g_m^{n_m}$ *where* $g_1, \dots, g_{n_m}$ *are all the irreducible factors of* h *and* $n_i > 0$.

Proof. We know that $\mathbf{F}_n = \mathbf{X}_h + \mu\mathbf{D}$ with $\mu = \frac{1}{n+1}\mathrm{div}(\mathbf{F}_n)$. Let $f \in \mathcal{P}_s$ a factor of h then $h = fg$ and $F_n(f) = X_{fg}(f) + \mu D(f) = fX_g(f) + s\mu f = (X_g(f) + s\mu)f$ Therefore, f is an invariant curve of $\mathbf{F}_n$.
If $f \in \mathcal{P}_s^{\mathbf{t}}$ is an irreducible invariant curve of $\mathbf{F}_n$ with cofactor K_n then $K_nf = F_n(f) = X_h(f) + \mu D(f) = X_h(f) + s\mu f$. Thus, $X_h(f) = (K_n - s\mu)f$ and f is an invariant curve of $\mathbf{X}_h$. So, f divides to h.
Last on, if I is a first integral of $\mathbf{F}_n$, it is an invariant curve of $\mathbf{F}_n$, that is, from Proposition 1.4, a factorization of I is formed by the irreducible factors of h. On the other hand, a first integral is zero on every invariant curve. So, $n_i > 0$. ■

1.2 Necessary condition of analytic integrability

Now we study the integrability problem for a vector field whose first homogeneous component is a quadratic type Lotka-Volterra. The following result determines the expression of the lowest degree component in the case of polynomial integrability of this class of vector fields.

Proposition 1.6 (Necessary condition of analytic integrability) *Let* $\mathbf{F} = \mathbf{F}_2 +$ *h.o.t. be with* $\mathbf{F}_2 = (x(a_1x + a_2y), y(b_1x + b_2y))^T$ *such that the origin of* $\dot{\mathbf{x}} = \mathbf{F}_2(\mathbf{x})$ *is isolated (*a_1, b_2, $a_2b_1 - a_1b_2$ *are different from zero). If* $\mathbf{F}$ *is formally integrable, then* $b_1 - a_1$, $b_2 - a_2$ *are different from zero, and there exists* $\Phi_1 \in \mathcal{Q}_1$, $\det(D\Phi_1(\mathbf{0})) \neq 0$ *such that* $\mathbf{G} := (\Phi_1)_* \mathbf{F} = \mathbf{G}_2 +$ *h.o.t., being*

$$\mathbf{G}_2 = (x(-qx + (q + r)y), y((p + r)x - py))^T, \ p, q, r \in \mathbb{N}, \ gcd(p, q, r) = 1$$

and $I_M = x^py^q(y - x)^r$ *is a polynomial first integral of* $\mathbf{G}_2$ *of degree* $M = p + q + r$.

Proof. Let $I = I_N + \text{h.o.t.}$ be a formal first integral of $\mathbf{F}$. Equation $F(I) = 0$ for degree $N+1$ is $F_2(I_N) = 0$, i.e. $\mathbf{F}_2$ is polynomially integrable and I_N is a first integral of $\mathbf{F}_2$.
Polynomial h associated to $\mathbf{F}_2$ is $h = \frac{1}{3}xy((b_1 - a_1)x + (b_2 - a_2)y)$.
We suppose that $a_1 = b_1$ (analogously for $a_2 = b_2$). From Proposition 1.5, if there would exist a first integral of $\mathbf{F}_2$, it would have the expression $I_N = x^{n_1}y^{n_2}$ with $n_i > 0$. So, $F_2(I_N) = 0$ becomes

$$n_1a_1 + n_2b_1 = 0, \qquad n_2b_2 + n_1a_2 = 0.$$

Therefore, $n_1 = n_2 = 0$ since $a_2b_1 - a_1b_2 \neq 0$. This contradicts the existence of the first integral.
We assume that $b_1 \neq a_1$ and $b_2 \neq a_2$. From Proposition 1.5, the first integral is $I_{Ms} = x^{ps}y^{qs}((b_1-a_1)x+(b_2-a_2)y)^{rs}$ with p, q, r, s natural numbers, $\gcd(p, q, r) = 1$ and $M = p + q + r$. By imposing $F_2(I_{Ms}) = 0$, it has that

$$(p + r)a_1 + qb_1 = 0, \qquad (q + r)b_2 + pa_2 = 0,$$

i.e. $a_2 = -\frac{b_2(q+r)}{p}$ and $b_1 = -\frac{a_1(p+r)}{q}$.
The linear change $\Phi_1(x, y, t) = (-pa_1x, -qb_2y, st/pq)$ transforms $\mathbf{F}_2$ into $\mathbf{G}_2 = (x(-qx+(q + r)y), y((p + r)x - py))^T$ being $I_M = x^py^q(y - x)^r$ a first integral of $\mathbf{G}_2$.
∎

2 Normal Form for perturbations of homogeneous quadratic Lotka-Volterra systems

We will not consider questions of convergence in the normal forms because the formal integrability is equivalent to the analytical integrability for the vector fields analyzed, see [15].

In [5], the authors provide an orbital normal form of the vector field whose first quasi-homogeneous term is non-conservative. Here, we provide the expression of the normal form for the vector field $\mathbf{F} = \mathbf{F}_2 + \text{h.o.t.}$ with $\mathbf{F}_2 = \mathbf{X}_h + \mu\mathbf{D} \in \mathcal{Q}_2$.

For every $k \in \mathbb{N}$, we fix the subspaces Δ_{k+2} such that $\mathcal{P}_{k+2} = \Delta_{k+2} \bigoplus h\mathcal{P}_{k-1}$. We consider the linear operators:

$$\begin{aligned} \ell_k \quad : \quad & \mathcal{P}_{k-1} \longrightarrow \mathcal{P}_k^{\mathbf{t}} \\ & \eta_{k-1} \longrightarrow F_2(\eta_{k-1}), \end{aligned}$$

and

$$\begin{aligned} \ell_{k+3}^{c} \quad : \quad & \Delta_{k+2} \longrightarrow \Delta_{k+3} \\ & g_{k+2} \longrightarrow \text{Proy}_{\Delta_{k+3}}(F_2 - \tfrac{3}{k+3}\mu D)(g_{k+2}). \end{aligned}$$

Theorem 2.7 *Let* $\mathbf{F} = \sum_{j\geq 2}\mathbf{F}_j$, $\mathbf{F}_j \in \mathcal{Q}_j$*. If* $\text{Ker}\left(\ell_{k+3}^{c}\right) = \{0\}$ *for all* $k \in \mathbb{N}$ *then* $\mathbf{F}$ *is orbitally equivalent to*

$$\mathbf{G} = \mathbf{F}_2 + \sum_{j>2}\mathbf{G}_j, \ \textit{with } \mathbf{G}_j = \mathbf{X}_{g_{j+1}} + \eta_{j-1}\mathbf{D} \in \mathcal{Q}_j,$$

where $g_{j+1} \in \text{Cor}\left(\ell_{j+1}^{c}\right)$ *and* $\eta_{j-1} \in \text{Cor}\left(\ell_{j-1}\right)$*. (where Cor(·) is a complementary subspace to Range(·)).*

Next results are referred to vector fields whose first homogeneous component is polynomially integrable and quadratic Lotka-Volterra type.

Lemma 2.8 *Consider* $\mathbf{F}_2 = (x(-qx+(q+r)y), y((p+r)x-py))^T$ *with* p, q, r *natural numbers. It has that for all* $k \in \mathbb{N}$, $\mathrm{Ker}\left(\ell_{k+3}^{\mathrm{c}}\right) = \{0\}$. *Moreover,* $\mathrm{Cor}\left(\ell_{k+3}^{\mathrm{c}}\right) = \{0\}$.

Proof. Vector field $\mathbf{F}_2 = \mathbf{X}_h + \mu\mathbf{D}$ with $h = \frac{p+q+r}{3}xy(x-y)$ and $\mu = \frac{1}{3}((-2q+p+r)x+(q+r-2p)y)$. We choose the bases $\Delta_{k+2} = \langle x^{k+2},\, x^{k+1}y,\, y^{k+2}\rangle$ and $\Delta_{k+3} = \langle x^{k+3},\, x^{k+2}y,\, y^{k+3}\rangle$.
Consider

$$\mathbf{G}_2^{(k+3)} = \mathbf{F}_2 - \frac{3}{k+3}\mu\mathbf{D} = \begin{pmatrix} (-q-\frac{-2q+p+r}{3+k})x^2 + (q+r-\frac{q+r-2p}{3+k})xy \\ (p+r-\frac{-2q+p+r}{3+k})xy + (-p-\frac{q+r-2p}{3+k})y^2 \end{pmatrix}.$$

We have that

$$\begin{aligned} G_2^{(k+3)}(x^{k+2}) &= A_1x^{k+3} + B_1x^{k+2}y, \\ G_2^{(k+3)}(x^{k+1}y) &= A_2x^{k+2}y + B_2x^{k+1}y^2 = (A_2+B_2)x^{k+2}y - \tfrac{3B_2}{p+q+r}x^k h, \\ G_2^{(k+3)}(y^{k+2}) &= A_3xy^{k+2} + B_3y^{k+3} = A_3x^{k+2}y + A_3hp_k(x,y) + B_3y^{k+3}, \end{aligned}$$

with

$$\begin{aligned} A_1 &= -\tfrac{2+k}{3+k}(q+qk+p+r), \\ B_1 &= \tfrac{2+k}{3+k}(2q+qk+2r+rk+2p), \\ A_2+B_2 &= \tfrac{2+k}{3+k}(p+q+r+rk), \\ A_3 &= \tfrac{2+k}{3+k}(2p+pk+2r+rk+2q), \\ B_3 &= -\tfrac{2+k}{3+k}(p+pk+q+r), \end{aligned}$$

and p_k homogeneous polynomial of degree k. In this way, the determinant of the matrix of the operator ℓ_{k+3}^{c} is

$$\tfrac{(k+2)^3}{(k+3)^3}(q+qk+p+r)(q+p+rk+r)(p+pk+q+r),$$

which is different from zero. Therefore, both $\mathrm{Ker}\left(\ell_{k+3}^{\mathrm{c}}\right)$ and $\mathrm{Cor}\left(\ell_{k+3}^{\mathrm{c}}\right)$ are trivial subspaces. ∎

For computing $\mathrm{Cor}\left(\ell_k\right)$ with $k > n$, we need the following two technical lemmas.

Lemma 2.9 *Consider* $\mathbf{F}_n \in \mathcal{Q}_n$ *irreducible and* $f \in \mathbb{C}[x,y]$ *an irreducible invariant curve of* $\mathbf{F}_n$. *If* $F_n(p_k) \in \langle f\rangle$ *with* $p_k \in \mathcal{P}_k$, *then* $p_k \in \langle f\rangle$.

Proof. If $F_n(p_k) = 0$ then p_k is a first integral of $\dot{\mathbf{x}} = \mathbf{F}_n$. A first integral of $\mathbf{F}_n$ vanishes on any invariant curve of it, i.e., $p_k(\mathbf{x}) = 0$ when $f(\mathbf{x}) = 0$. Therefore, by Hilbert's Nullstellensatz $p_k \in rad\,\langle f\rangle$. Since $\langle f\rangle$ is a prime ideal, then $\langle f\rangle = rad\,\langle f\rangle$, in consequence $p \in \langle f\rangle$.

If $F_n(p_k) \neq 0$, let $\nu \in \mathbb{C}[x,y] \setminus \{0\}$ such that $f\nu = F_n(p_k)$. Consider $\gamma(t)$, real or complex, a solution curve of $\dot{\mathbf{x}} = \mathbf{F}_n(\mathbf{x})$ which is a parametrization of $f(\mathbf{x}) = 0$. We assume that $\lim_{t\to-\infty}\gamma(t) = 0$, (the other case $\lim_{t\to+\infty}\gamma(t) = 0$ is proved in a similar way). Taking into account that $p_k(\mathbf{0}) = 0$ then

$$\begin{aligned} p_k(\gamma(t)) &= p_k(\gamma(t)) - p_k(\mathbf{0}) = \int_{-\infty}^{t} \tfrac{dp_k(\gamma(s))ds}{ds} = \int_{-\infty}^{t} \nabla_{\mathbf{x}}p_k \cdot \mathbf{F}_n(\gamma(s))ds \\ &= \int_{-\infty}^{t} F_n(p_k)(\gamma(s))ds = \int_{-\infty}^{t} f(\gamma(s))\nu(\gamma(s))ds = 0. \end{aligned}$$

Recalling that $f(\mathbf{x}) = 0$ is the union of orbits, we have that $p_k(\mathbf{x}) = 0$ when $f(\mathbf{x}) = 0$. Therefore, by Hilbert's Nullstellensatz $p_k \in rad\,\langle f\rangle$. Since $\langle f\rangle$ is a prime ideal, then $\langle f\rangle = rad\,\langle f\rangle$, in consequence $p_k \in \langle f\rangle$. ∎

Remark. The hypothesis of the irreducibility of the invariant curve is fundamental. For instance, if we consider $\mathbf{F}_2 := (-2x^2, -3x^2 - 2xy + 3y^2)^T \in \mathcal{Q}_2$ irreducible and the invariant curve $(y-x)^2$, for $p_3 = x^2(y-x)$ we have that $F_2(p_3) = 3x^2(y-x)^2 \in \langle (y-x)^2 \rangle$ and nevertheless $p_3 \notin \langle (y-x)^2 \rangle$.

Lemma 2.10 *Consider* $\mathbf{F}_2 = (x(-qx+(q+r)y), y((p+r)x-py))^T$ *with* p, q, r *natural numbers. Let* k *and* m *natural numbers with* $p+q+r \neq p\frac{k}{j}$, $p+q+r \neq q\frac{k}{j}$, $p+q+r \neq r\frac{k}{j}$, $j = 1, \ldots, m-1$. *If* $p_k \in \mathcal{P}_k$ *such that* $F_2(p_k) \in \langle f_i^m \rangle$, *being* $f_1 = x$, $f_2 = y$, $f_3 = x - y$, *invariant curves of* $\mathbf{F}_2$, *then* $p_k \in \langle f_i^m \rangle$, $i = 1, 2, 3$.

Proof. We prove the case $i = 1$, $(f_1 = x)$, the cases $i = 2, 3$ are analogous.
Lemma 2.9 proves the statement for $m = 1$.

We first consider the case $m = 2$. We denote by $K_1 = -qx + (q+r)y$ the cofactor of x. If $F_2(p_k) \in \langle x^2 \rangle$ then $F_2(p_k) \in \langle x \rangle$ and by Lemma 2.9 we have that there exists $p_{k-1} \in \mathcal{P}_{k-1}$ such that $p_k = xp_{k-1}$, therefore

$$\begin{aligned} F_2(p_k) &= F_2(xp_{k-1}) = p_{k-1}F_2(x) + xF_2(p_{k-1}) = p_{k-1}K_1 x + xF_2(p_{k-1}) \\ &= x\left(\tfrac{K_1}{k-1}D(p_{k-1}) + F_2(p_{k-1})\right) = x(F_2 + \tfrac{K_1}{k-1}D)(p_{k-1}) \in \langle x^2 \rangle. \end{aligned}$$

Hence $(F_2 + \frac{K_1}{k-1}D)(p_{k-1}) \in \langle x \rangle$. Vector field

$$\mathbf{F}_2 + \tfrac{K_1}{k-1}\mathbf{D} = \frac{1}{k-1}\begin{pmatrix} xk(-qx + (q+r)y) \\ y((p+r)k - p - q - r)x + (-pk + p + q + r)y) \end{pmatrix}$$

is irreducible if, and only if, $p + q + r \neq pk$. Applying Lemma 2.9 we have that $p_{k-1} \in \langle x \rangle$ and consequently $p_k \in \langle x^2 \rangle$.

Consider now the case $m = 3$. If $F_2(p_k) \in \langle x^3 \rangle$ then $F_2(p_k) \in \langle x^2 \rangle$ and by the previous paragraph we have that there exists $p_{k-2} \in \mathcal{P}_{k-2}$ such that $p_k = x^2 p_{k-2}$, therefore

$$\begin{aligned} F_2(p_k) &= F_2(x^2 p_{k-2}) = p_{k-2}F_2(x^2) + x^2 F_2(p_{k-2}) = 2p_{k-2}K_1 x^2 + x^2 F_2(p_{k-2}) \\ &= x^2\left(\tfrac{2K_1}{k-2}D(p_{k-2}) + F_2(p_{k-2})\right) = x^2(F_2 + \tfrac{2K_1}{k-2}D)(p_{k-2}) \in \langle x^3 \rangle. \end{aligned}$$

Hence $(F_2 + \frac{2K_1}{k-2}D)(p_{k-2}) \in \langle x \rangle$ and as $\mathbf{F}_2 + \frac{2K_1}{k-2}\mathbf{D}$ is irreducible if, and only if, $p + q + r \neq p\frac{k}{2}$, applying Lemma 2.9 we have that $p_{k-2} \in \langle x \rangle$ and consequently $p_k \in \langle x^3 \rangle$. Reasoning by induction we get the result for $m \in \mathbb{N}$.
Reasoning as before, it is easy to prove that for $f_2 = y$ and $f_3 = x - y$, the conditions are $p + q + r \neq q\frac{k}{j}$ and $p + q + r \neq r\frac{k}{j}$, $j = 1, \ldots, m-1$, respectively. ∎

Next statement establishes a cyclicity relation between the co-ranges of the operators ℓ_k.

Lemma 2.11 *Consider* $\mathbf{F}_2 = (x(-qx+(q+r)y), y((p+r)x-py))^T$ *with* p, q, r *natural numbers and* $M = p + q + r$. *For* $k \geq 2$, *it is always possible to choose* $\mathrm{Cor}(\ell_{k+M})$, *a complementary subspace to* $\mathrm{Range}(\ell_{k+M})$, *such that* $\mathrm{Cor}(\ell_{k+M}) = I_M \mathrm{Cor}(\ell_k)$ *being* $I_M = x^p y^q (x-y)^r$.

Proof. We first see that both subspaces have the same dimension. Indeed, by Lemma 2.10, $Ker(\ell_k) = \langle I_M^l \rangle$ if $k-1 = lM$. Otherwise, $Ker(\ell_k) = \{0\}$. Thus, $\dim(\mathrm{Cor}(\ell_k)) = 2$ if $k = lM$ and $\dim(\mathrm{Cor}(\ell_k)) = 1$, otherwise; i.e., $\dim(\mathrm{Cor}(\ell_k)) = \dim(\mathrm{Cor}(\ell_{k+M}))$.

For completing the proof it is enough to prove that $I_M \mathrm{Cor}(\ell_k) \subset \mathrm{Cor}(\ell_{k+M})$ or equivalently that $I_M \mathrm{Cor}(\ell_k) \cap \mathrm{Range}(\ell_{k+M}) = \{0\}$ by *reductio ad absurdum*. Let $p_k \in \mathrm{Cor}(\ell_k) \setminus \{0\}$ such that $p_k I_M \in \mathrm{Range}(\ell_{k+M})$, then there exists $p_{k+M-1} \in \mathcal{P}^{\mathbf{t}}_{k+M-1} \setminus \{0\}$ such that $\ell_{k+M}(p_{k+M-1}) = p_k I_M$, that is, $\ell_{k+M}(p_{k+M-1})$ is multiple of I_M. As $\frac{p(k+M-1)}{j} > \frac{pM}{j} > M,\ j=1,\dots,p-1;\ \frac{q(k+M-1)}{j} > M,\ j=1,\dots,q-1;$ $\frac{r(k+M-1)}{j} > M,\ j=1,\dots,r-1$, by applying Lemma 2.10 we have that $p_{k+M-1} = p_{k-1}I_M$ with $p_{k-1} \in \mathcal{P}^{\mathbf{t}}_{k-1} \setminus \{0\}$ and consequently

$$p_k I_M = F_2(p_{k+M-1}) = F_2(p_{k-1}I_M) = I_M F_2(p_{k-1}).$$

Hence $p_k = F_2(p_{k-1})$, that is, $p_k \in \mathrm{Range}\,(\ell_k) \cap \mathrm{Cor}\,(\ell_k)$ which gives a contradiction. ■

Next result provides an orbital normal form of vector field whose first homogeneous component is integrable and quadratic Lotka-Volterra type. This normal form depends on the first integral of the first homogeneous component of the vector field and it is a suitable normal form for the applications.

Theorem 2.12 *Vector field* $\mathbf{F} = \mathbf{F}_2 + h.o.t.$ *with* $\mathbf{F}_2 = (x(-qx+(q+r)y), y((p+r)x-py))^T,\ p,q,r$ *natural numbers and* $M = p+q+r$ *is orbitally equivalent to*

$$\dot{\mathbf{x}} = \mathbf{F}_2 + \sum_{j=2}^{M+1} \eta_j^{(0)}\mathbf{D} + \sum_{i=1}^{\infty}\sum_{j=2}^{M+1} \eta_j^{(i)} I_M^i \mathbf{D},$$

with $\eta_j^{(i)} \in \mathrm{Cor}(\ell_j)$ *and* $I_M = x^p y^q (x-y)^r$.

Proof. Applying Theorem 2.7 and Lemma 2.8, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{F}_2 + \sum_{j\ge 2}\eta_j\mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$. In order to finish the proof it is sufficient to apply Lemma 2.11 for the components of the normal form of degree greater than $M+1$. ■

3 Main results

Our purpose is to characterize the analytic integrable vector fields which are perturbations of quadratic Lotka-Volterra type. For that, we will assume that the lowest degree component of the vector field satisfies the necessary condition of analytic integrability given in Proposition 1.6, i.e. we deal with the vector field $\mathbf{F} = \mathbf{F}_2 + \text{h.o.t.}$ with $\mathbf{F}_2 = (x(-qx+(q+r)y), y((p+r)x-py))^T,\ p,q,r \in \mathbb{N},\ \gcd(p,q,r) = 1$.

We give the main result of our study. It solves the analytic integrability problem for vector fields which are perturbations of quadratic Lotka-Volterra vector fields whose first component is polynomially integrable. It also gives the expression of a first integral.

Theorem 3.13 *Let* $\mathbf{F} = \mathbf{F}_2 + h.o.t.$ *be with* $\mathbf{F}_2 = (x(-qx+(q+r)y), y((p+r)x-py))^T,\ p,q,r \in \mathbb{N},\ gcd(p,q,r) = 1$. *The vector field* $\mathbf{F}$ *is analytically integrable if, and only if, it is orbitally equivalent to* $\mathbf{F}_2$.
Moreover, in such a case, $\mathbf{F}$ *has an analytic first integral of the form* $I = I_M + h.o.t.$ *being* $I_M = x^p y^q (x-y)^r$ *a primitive first integral of* $\mathbf{F}_2$.

Proof. We see the sufficiency. The polynomial is $I_M = x^p y^q (y-x)^r$ is a first integral of $\mathbf{F}_2$ which it is transformed into a formal first integral $I = I_M + \text{h.o.t.}$ of $\mathbf{F}$ and from [Theorem A,[15]] $\mathbf{F}$ is analytically integrable.
We see the necessity of the condition. Applying Theorem 2.7 and Lemma 2.8, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_2 + \sum_{j\ge 2}\eta_j\mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$.
Let note that $\mathbf{F}$ has an analytic first integral equivalents to $\mathbf{G}$ has a formal first integral. Assume that $\mathbf{G}$ is formally integrable and not all the η_j are zero. Let N defined by $N = \min\{j > 1 : \eta_j \not\equiv 0\}$. A formal first integral of $\mathbf{G}$ is of the form $I = I_M^l + \sum_{j>Ml} I_j$ with $I_j \in \mathcal{P}_j$. Imposing the integrability condition we have

$$\begin{aligned} 0 &= (G(I))_{N+Ml} = (\eta_N D)(I_M^l) + F_2(I_{Ml+N-1}) \\ &= Ml\eta_N I_M^l + \ell_{Ml+N}\left(I_{Ml+N-1}\right). \end{aligned}$$

But this equation is incompatible since by Lemma 2.11 $Ml\eta_N I_M^l \in \mathrm{Cor}\,(\ell_{Ml+N})$ and $\ell_{Ml+N}\,(I_{Ml+N-1}) = -Ml\eta_N I_M^l \in \mathrm{Range}\,(\ell_{Ml+N})$ which is a contradiction. Consequently, $\mathbf{G} = \mathbf{F}_2$, i.e. $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_2$.

We now see the second part. First integrals of $\mathbf{F}_2$ are $\Psi(I_M)$ for any formal function Ψ. So, first integrals of $\mathbf{F}$ are $\Psi(I_M + \text{h.o.t.})$ since $\mathbf{F}$ is orbitally equivalent to $\mathbf{F}_2$. Thus, $I_M + \text{h.o.t.}$ is also a first integral of $\mathbf{F}$. ■

The following theorem characterizes the analytic integrability of a vector field whose first homogeneous component is quadratic Lotka-Volterra type through the existence of a Lie symmetry.

Theorem 3.14 *Let $\mathbf{F} = \mathbf{F}_2 + h.o.t.$ be with $\mathbf{F}_2 = (x(-qx + (q + r)y), y((p + r)x - py))^T$, $p, q, r \in \mathbb{N}$, $gcd(p, q, r) = 1$. Then $\mathbf{F}$ is analytically integrable if, and only if, there exist a formal vector field $\mathbf{G} = \sum_{j\geq 1} \mathbf{G}_j$, $\mathbf{G}_j \in \mathcal{Q}_j$, $\mathbf{G}_1 = (x, y)^T$ and a formal scalar function ν, $\nu(\mathbf{0}) = 1$ such that $[\mathbf{F}, \mathbf{G}] = \nu\mathbf{F}$, i.e. $\mathbf{F}$ has a Lie symmetry.*

The proof of Theorem 3.14 follows from Theorem 3.13 and applying [6, Theorem 1.3].

We solve the analytic integrability problem through the existence of a formal inverse integrating factor.

Theorem 3.15 *Let $\mathbf{F} = \mathbf{F}_2 + h.o.t.$ be with $\mathbf{F}_2 = (x(-qx + (q + r)y), y((p + r)x - py))^T$, $p, q, r \in \mathbb{N}$, $gcd(p, q, r) = 1$. Then $\mathbf{F}$ is analytically integrable if, and only if, it has a formal inverse integrating factor of the form $V = xy(x - y) + h.o.t.$.*

Proof. We prove that the condition is necessary. We assume that $\mathbf{F}$ is analytically integrable. From Theorem 3.13, it is orbitally equivalent to $\mathbf{F}_2 = \mathbf{X}_h + \mu\mathbf{D}$ being $h = \frac{p+q+r}{3}xy(x-y)$ and $\mu = \frac{1}{3}((-2q+p+r)x + (q+r-2p)y)$, which has the inverse integrating factor h. Undoing the change, it has that $\mathbf{F}$ has a formal inverse integrating $V = h + \text{h.o.t.}$.

Now we will see the sufficiency of the condition. Let $V = h + \text{h.o.t.}$ a formal inverse integrating factor of $\mathbf{F}$. Since Theorem 2.7 and Lemma 2.8, we can assert that $\mathbf{F}$ is orbital equivalent to $\mathbf{G} = \mathbf{F}_2 + \sum_{j\geq 2} \eta_j \mathbf{D}$ with $\eta_j \in \mathrm{Cor}\,(\ell_j)$. Therefore, $\mathbf{F}$ has a formal inverse integrating factor if, and only if, $\mathbf{G}$ has it too. Moreover, the formal inverse integrating factor W of $\mathbf{G}$ is also of the form $W = h + \text{h.o.t.}$. On the other hand, the unique invariant curves of $\mathbf{G}$ are $x, y, x - y$ and any u formal with $u(\mathbf{0}) = 1$, u is an unit element. So, we get $W = hu$ being u formal and $u(\mathbf{0}) = 1$. Equation $G(W) - W\mathrm{div}(\mathbf{G}) = 0$ is

$$0 = uG(h) + hG(u) - hu\mathrm{div}(\mathbf{G}).$$

As $G(h) = 3h\mu + \sum_{j>2} 3h\eta_j$ and $\mathrm{div}(\mathbf{G}) = 3\mu + \sum_{j>2}(j + 2)\eta_j$, it has that

$$0 = h(G(u) - u\sum_{j>2}(j - 1)\eta_j).$$

Expanding $u = 1 + \sum_{i\geq 1} u_i$, it is easy to prove that the equation to degree $i + 1$ becomes

$$0 = G_2(u_i) - i\eta_{i+1} + \sum_{k=1}^{i-1}(2k - i)\eta_{i-k+1}u_k \tag{3.2}$$

We see that $\eta_j = 0$ for all j. Indeed, otherwise, let $j_0 = \min\{j \in \mathbb{N} : \eta_{j+1} \not\equiv 0\}$. Equation (3.2) to degree $j_0 + 1$ is

$$G_2(u_{j_0}) = j_0\eta_{j_0+1} - \sum_{k=1}^{j_0-1}(2k - j_0)\eta_{j_0-k+1}u_k.$$

As $\eta_{j_0-k+1}=0$ for $1\le k\le j_0-1$, we get $G_2(u_{j_0})=j_0\eta_{j_0+1}$, i.e. $\eta_{j_0+1}\in \mathrm{Cor}\,(\ell_{j_0+1})$ and $\eta_{j_0+1}\in \mathrm{Range}\,(\ell_{j_0+1})$. We conclude that $\eta_{j_0+1}=0$. ∎

4 An application

Consider the analytic integrability problem of the following system

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\begin{pmatrix}x(3y-x)\\ y(3x-y)\end{pmatrix}+\begin{pmatrix}x(a_{20}x^2+a_{11}xy+a_{02}y^2)\\ y(b_{20}x^2+b_{11}xy+b_{02}y^2)\end{pmatrix}. \qquad (4.3)$$

The first homogeneous component of the vector field is $\mathbf{F}_2=(x(-x+3y),\ y(3x-y))^T$, where $\mathbf{F}_2=\mathbf{X}_h+\mu\mathbf{D}$ with $h=\frac{4}{3}xy(x-y)$ and $\mu=\frac{1}{3}(x+y)$. The vector field $\mathbf{F}_2$ is polynomially integrable and a primitive first integral is $I_4=xy(x-y)^2$.

The following result solves the integrability problem for this family.

Theorem 4.16 *System (4.3) is analytically integrable if, and only if, one of the following conditions holds:*
(1) $b_{11}+5b_{02}=b_{20}+2b_{02}=a_{11}+3b_{02}=a_{20}-b_{02}=a_{02}=0$,
(2) $b_{11}+3b_{02}=a_{02}+2b_{02}=a_{11}+5b_{02}=a_{20}-b_{02}=b_{20}=0$,
(3) $2a_{11}+2a_{02}-3b_{20}-3b_{11}-5b_{02}=a_{02}b_{20}+a_{02}b_{11}+3a_{02}b_{02}+2b_{20}b_{02}=2a_{20}+b_{20}+b_{11}+3b_{02}=0$,
(4) $a_{02}+5b_{02}=a_{11}+b_{11}=5a_{20}+b_{20}$,
(5) $a_{11}+b_{11}=a_{20}+b_{02}=a_{02}=b_{20}$,
(6) $b_{20}-b_{02}=a_{02}+b_{02}=a_{11}+b_{11}=a_{20}+b_{02}$.

Proof. To prove the necessary condition, it has computed the first coefficients of the normal form given in Theorem 2.12. By Theorem 3.13, the vanishing of the coefficients leads us to the integrability. In this case, it has been necessary the coefficients of the normal form up order 7,

$$(\dot x,\dot y)^T=\mathbf{F}_2+(\alpha_2x^2+\alpha_3x^3+\alpha_4x^4+\alpha_5xI_4+\beta_5yI_4+\alpha_6x^2I_4)(x,y)^T.$$

The coefficients $\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6$ and β_5 are polynomials too long, so we do not given them here. Their vanishing arrives to systems (4.3) for cases 1–6.

We prove the sufficiency. System (4.3) for case 1 has an analytic first integral $xy(x-y-b_{02}x^2+\frac{1}{3}b_{02}xy)^2(1-b_{02}x-b_{02}y)^{-3}$.
System (4.3) for case 2 is transformed into system (4.3) for case 1 by using the involution $(x,y)\leftrightarrow(y,x)$.
System (4.3) for case 3 has an inverse integrating factor $xy(x-y)(2+b_{20}x+b_{11}x+3b_{02}x-2b_{02}y)$ whose first component is h. Applying Theorem 3.15, the vector field is analytically integrable.
System (4.3) for case 4 has a polynomial first integral $xy(3x-3y-3a_{20}x^2+b_{11}xy+3b_{02}y^2)^2$.
System (4.3) for case 5 has an analytic first integral

$$xy(3x-3y+(b_{02}+b_{11})xy)^2(2+2b_{02}x-2b_{02}y+(b_{02}b_{11}+b_{02}^2)xy)^{-3}.$$

System (4.3) in the case 6, for $b_{11}=-2b_{02}$, has an inverse integrating factor $xy(x-y)(1+b_{02}x-b_{02}y)$. Otherwise, we have not found the expression of an inverse integrating factor starting by h. In this case, we center on proving its existence in order to apply Theorem 3.15.
Consider $V=xyC(x,y)$ with C the invariant curve given by Lemma 4.17. It has that $F(V)=xyF(C)+xCF(y)+yCF(x)=V(K^{(1)}+K^{(2)}+K^{(3)})$ with $K^{(1)},K^{(2)}$ and $K^{(3)}$ the cofactors of x,y and C, respectively, $K^{(1)}=-x+3y-b_{02}x^2-b_{11}xy-b_{02}y^2$, $K^{(2)}=3x-y+b_{02}x^2+b_{11}xy+b_{02}y^2$ and $K^{(3)}=-(x+y)(1+2b_{02}x-2b_{02}y)$ and as $K^{(1)}+K^{(2)}+K^{(3)}=x+y-2b_{02}x^2+2b_{02}y^2=\mathrm{div}(\mathbf{F})$, V is an inverse integrating factor of $\mathbf{F}$. This concludes the proof. ∎

Lemma 4.17 *System (4.3) for case 6 has an invariant curve $C = x - y + h.o.t.$ with cofactor $K = -(x+y)(1 + 2b_{02}x - 2b_{02}y)$.*

Proof. System (4.3) for case 6 is $\dot{\mathbf{x}} = \mathbf{F}_2 + \mathbf{F}_3$ with $\mathbf{F}_2 = (x(-x+3y),\ y(3x-y))^T$ and $\mathbf{F}_3 = (x(-b_{02}x^2 - b_{11}xy - b_{02}y^2),\ y(b_{02}x^2 + b_{11}xy + b_{02}y^2))^T$.
We claim that there exists a formal invariant curve of $\mathbf{F}$ of the form $C = \sum_{j\geq 1} C_j$ with

$$C_{2j-1} = A_{2j-1}x^{j-1}y^{j-1}(x-y), \quad C_{2j} = x^{j-1}y^{j-1}(A_{2j}x^2 + B_{2j}xy - A_{2j}y^2), \quad (4.4)$$

for any $j \geq 1$, with cofactor $K_1 + K_2$ being $K_1 = -x - y$ and $K_2 = -2b_{02}x^2 + 2b_{02}y^2$.
We are going to verify that C satisfies $F(C) - KC = 0$ degree to degree.
For the degree 2, $F_2(C_1) - K_1C_1 = 0$ arrives to $C_1 = x - y$, and for the degree 3, $F_2(C_2) + F_3(C_1) - C_2K_1 - C_1K_2 = 0$, we get $C_2 = b_{02}x^2 + \frac{1}{3}(b_{11} - 4b_{02})xy + b_{02}y^2$.
Thus, C_1 and C_2 have the form given by (4.4). Assume that (4.4) is true for $2j_0 - 1$ and $2j_0$ and we prove that also it holds for $2j_0 + 1$ and $2j_0 + 2$.
Equation $F(C) - KC = 0$ for degree $2j_0 + 2$ is

$$F_2(C_{2j_0+1}) - C_{2j_0+1}K_1 = -F_3(C_{2j_0}) + C_{2j_0}K_2 = 2x^{j_0}y^{j_0}(x-y)(x+y)(A_{2j_0}b_{11} - B_{2j_0}b_{02}).$$

A solution of this equation is $C_{2j_0+1} = (A_{2j_0}b_{11} - B_{2j_0}b_{02})x^{j_0-1}y^{j_0-1}(x-y)$, i.e. C_{2j_0+1} is of the form given by (4.4) with $A_{2j_0+1} = A_{2j_0}b_{11} - B_{2j_0}b_{02}$.
Analogously, equation $F(C) - KC = 0$ for degree $2j_0 + 3$ is

$$F_2(C_{2j_0+2}) - C_{2j_0+2}K_1 = -A_{2j_0+1}x^{j_0}y^{j_0}(x+y)(b_{02}x^2 - (b_{11} + 4b_{02})xy + b_{02}y^2).$$

A solution of this equation is

$$C_{2j_0+2} = A_{2j_0+1}x^{j_0}y^{j_0}\left(-\frac{b_{02}}{2j_0-1}x^2 + (\frac{b_{11}}{2j_0+3} + \frac{4(2j_0+1)b_{02}}{(2j_0-1)(2j_0+3)})xy + \frac{b_{02}}{2j_0-1}y^2\right),$$

i.e. C_{2j_0+2} is of the form given by (4.4). Therefore, the result is proved. ∎

Acknowledgments. The authors are partially supported by a MINECO/FEDER grant number MTM2014-56272-C2-2 and by the *Consejería de Educación y Ciencia de la Junta de Andalucía* (projects P12-FQM-1658, FQM-276).

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