Affirmative Solutions On Local Antimagic Chromatic Number

Gee-Choon Lau[a,1], Ho-Kuen Ng[b], Wai-Chee Shiu[c]

[a]*Faculty of Computer & Mathematical Sciences,*
Universiti Teknologi MARA (Segamat Campus),
85000, Johor, Malaysia.
geeclau@yahoo.com

[b]*Department of Mathematics, San José State University,*
San José CA 95192 USA.
ho-kuen.ng@sjsu.edu

[c,2]*Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, P.R. China,*
College of Global Talents, Beijing Institute of Technology, Zhuhai, P.R. China.
wcshiu@associate.hkbu.edu.hk

Abstract

An edge labeling of a connected graph $G = (V, E)$ is said to be local antimagic if it is a bijection $f : E \to \{1, \dots, |E|\}$ such that for any pair of adjacent vertices x and y, $f^+(x) \neq f^+(y)$, where the induced vertex label $f^+(x) = \sum f(e)$, with e ranging over all the edges incident to x. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of distinct induced vertex labels over all local antimagic labelings of G. In this paper, we give counterexamples to the lower bound of $\chi_{la}(G \vee O_2)$ that was obtained in [Local antimagic vertex coloring of a graph, Graphs and Combin., 33 : 275 - 285 (2017)]. A sharp lower bound of $\chi_{la}(G \vee O_n)$ and sufficient conditions for the given lower bound to be attained are obtained. Moreover, we settled Theorem 2.15 and solved Problem 3.3 in the affirmative. We also completely determined the local antimagic chromatic number of complete bipartite graphs.

Keywords: Local antimagic labeling, Local antimagic chromatic number
2010 AMS Subject Classifications: 05C78; 05C69.

1 Introduction

A connected graph $G = (V, E)$ is said to be local antimagic if it admits a local antimagic edge labeling, i.e., a bijection $f : E \to \{1, \dots, |E|\}$ such that the induced vertex labeling $f^+ : V \to \mathbb{Z}$ given by $f^+(x) = \sum f(e)$ (with e ranging over all the edges incident to x) has the property that any two adjacent vertices have distinct induced vertex labels. The number of distinct induced vertex labels under f is denoted by $c(f)$, and is called the *color number* of f. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is $\min\{c(f) : f$ is a local antimagic labeling of $G\}$. In [3], Haslegrave proved that the local antimagic chromatic number is well-defined for every connected graph other than K_2. Thus, for every connected graph $G \neq K_2$, $\chi_{la}(G) \geq \chi(G)$, the chromatic number of G.

For any graph G, the graph $H = G \vee O_n$, $n \geq 1$, is defined by $V(H) = V(G) \cup \{v_i : 1 \leq i \leq n\}$ and $E(H) = E(G) \cup \{uv_i : u \in V(G)\}$. In [1, Theorem 2.16], it was claimed that for any G with

[1]Corresponding author.

[2]The third author was supported by Tianjin Research Program of Application Foundation and Advanced Technology (No.14JCYBJC43100) for visiting Tianjin University of Technology and Education.

order $m \geq 4$,

$$\chi_{la}(G)+1 \leq \chi_{la}(G \vee O_2) \leq \begin{cases} \chi_{la}(G)+1 & \text{if } m \text{ is even,} \\ \chi_{la}(G)+2 & \text{if } m \text{ is odd.} \end{cases}$$

In Section 2, we give counterexamples to the above lower bound for each $m \geq 3$. A sharp lower bound is then given. Moreover, sufficient conditions for the above lower bound to be attained are also presented. In Section 3, we settled [1, Theorem 2.15] and solved [1, Problem 3.3] in the affirmative. In Section 4, we completely determined the local antimagic chromatic number of complete bipartite graphs.

2 Counterexamples and sharp bound

In this section, we will make use of the existence of magic rectangles. From [2,8], we know that a $h \times k$ magic rectangle exists when $h, k \geq 2$, $h \equiv k \pmod 2$ and $(h,k) \neq (2,2)$. For $a, b \in \mathbb{Z}$ and $a \leq b$, we use $[a,b]$ to denote the set of integers from a to b. We first introduce some notation about matrices.

Let m, n be two positive integers. For convenience, we use $M_{m,n}$ to denote the set of $m \times n$ matrices over $\mathbb{Z}$. For any matrix $M \in M_{m,n}$, let $r_i(M)$ and $c_j(M)$ denote the i-th row sum and the j-th column sum of M, respectively.

We shall assign the integers in $[1, q+r+qr]$ to matrices $PR \in M_{1,r}$, $QR \in M_{q,r}$ and $QP = (PQ)^T \in M_{q,1}$ such that the matrix

$$M = \begin{pmatrix} * & PR \\ QP & QR \end{pmatrix}$$

has the following properties:

P.1 Each integer in $[1, q+r+qr]$ appears once.

P.2 $r_{i+1}(M)$ is a constant not equal to $r_1(M) + c_1(M)$, $1 \leq i \leq q$.

P.3 $c_{j+1}(M)$ is a constant not equal to $r_{i+1}(M)$ or $r_1(M) + c_1(M)$, $1 \leq j \leq r$.

Let $\{u\}$, $\{v_1, v_2, \ldots, v_q\}$ and $\{w_1, w_2, \ldots, w_r\}$ be the three independent vertex set of $K(1,q,r)$, $r \geq q \geq 2$. For $1 \leq i \leq q, 1 \leq j \leq r$, let the i-entry of QP be the edge label of uv_i, the j-entry of PR be the edge label of uw_j, the (i,j)-entry of QR be the edge label of v_iw_j. It follows that $r_1(M) + c_1(M)$ is the sum of all the incident edge labels of u, $r_{i+1}(M)$ is the sum of all the incident edge labels of v_i, and $c_{j+1}(M)$ is the sum of all the incident edge labels of w_j. Thus, M corresponds to a local antimagic labeling of $K(1,q,r)$ with $\chi_{la}(K(1,q,r)) = 3$ if such M exists.

Theorem 2.1. *For $r \geq 2$, $\chi_{la}(K(1,2,r)) = 3$.*

Proof. Suppose r is even. If $r = 2$, a required labeling is given by

$$M = \begin{pmatrix} * & 1 & 5 \\ 6 & 7 & 2 \\ 8 & 3 & 4 \end{pmatrix}$$

Consider $r \geq 4$. Let A be a $3 \times (r+1)$ magic rectangle. Exchanging columns and exchanging rows if necessary so that $3(r+1)$ is put at the $(1,1)$-entry of A. Now, M is obtained by letting PR be the $1 \times r$ matrix obtained from the first row of A by deleting the $(1,1)$-entry; letting QP be the 3×1 matrix obtained from the first column of A by deleting the $(1,1)$-entry; and letting QR be the $2 \times r$ matrix obtained from A by deleting the first row and the first column.

It is easy to check that $c_1(M)+r_1(M) = \frac{(r+4)(3r+4)}{2} - 6(r+1) \neq r_2(M) = r_3(M) = \frac{(r+1)(3r+4)}{2} \neq c_{j+1} = \frac{3(3r+4)}{2}$, $1 \leq j \leq r$.

Suppose r is odd. If $r = 3$, a required labeling is given by

$$M = \begin{pmatrix} * & 2 & 4 & 6 \\ 8 & 5 & 11 & 3 \\ 10 & 9 & 1 & 7 \end{pmatrix}$$

Consider $r \geq 5$. Now for $r \equiv 1 \pmod 4$, let $r = 4s+1, s \geq 1$. The entries of a required labeling matrix M is given in tabular form as follows.

$PR =$

1	3	$\cdots$	$4s-5$	$4s-3$	$4s-1$	$10s+3$	2	4	$\cdots$	$4s-4$	$4s-2$	$4s$

$QP + QR =$

$12s+5$	$12s+3$	$6s+1$	$\cdots$	$4s+5$	$10s+5$	$4s+3$	$4s+1$
$12s+4$	$6s+2$	$12s+2$	$\cdots$	$10s+6$	$4s+4$	$10s+4$	$4s+2$

$10s+2$	$8s+1$	$\cdots$	$6s+5$	$8s+4$	$6s+3$
$8s+2$	$10s+1$	$\cdots$	$8s+5$	$6s+4$	$8s+3$

Clearly, we get $c_1(M) + r_1(M) = 8s^2 + 36s + 12 \neq r_2(M) = r_3(M) = 32s^2 + 27s + 6 \neq c_{j+1}(M) = 18s + 6$, $1 \leq j \leq r$.

Finally for $r \equiv 3 \pmod 4$, let $r = 4s+3, s \geq 1$. To get PR, we assign $2k-1$ to column k if $1 \leq k \leq 2s+2$, and assign $2k-4s-4$ to column k if $2s+3 \leq k \leq 4s+3$. For row 1 of QR, we assign $6s+6-k$ to column k if $k = 1,3,5,\ldots,2s+1$; assign $12s+10-k$ to column k if $k = 2,4,6,\ldots,2s+2$; assign $10s+9-k$ to column k if $k = 2s+3, 2s+5, 2s+7, \ldots, 4s+3$; and assign $12s+10-k$ to column k if $k = 2s+4, 2s+6, 2s+8, \ldots, 4s+2$. For row 2 of QR, we assign $12s+10-k$ to column k if $k = 1,3,5,\ldots,2s+1$; assign $6s+6-k$ to column k if $k = 2,4,6,\ldots,2s+2$; assign $12s+10-k$ to column k if $k = 2s+3, 2s+5, 2s+7, \ldots, 4s+3$; and assign $10s+9-k$ to column k if $k = 2s+4, 2s+6, 2s+8, \ldots, 4s+2$. For QP, the two entries are $12s+10$ and $12s+11$. Lastly, we exchange the labels $4s-1$ and $4s+6$; the labels $4s-2$ and $6s+8$; and the labels $4s+2$ and $8s+7$. The resulting matrices are given by the following tables.

$PR =$

1	3	5	$\cdots$	$4s-3$	$\mathbf{4s+6}$	$4s+1$	$4s+3$	2	4	$\cdots$	$4s-4$	$\mathbf{6s+8}$	$4s$	$\mathbf{8s+7}$

$QP + QR =$

$12s+10$	$6s+5$	$12s+8$	$6s+3$	$\cdots$	$4s+7$	$10s+10$	$4s+5$	$10s+8$
$12s+11$	$12s+9$	$6s+4$	$12s+7$	$\cdots$	$10+11$	$\mathbf{4s-1}$	$10s+9$	$4s+4$

$8s+6$	$10s+6$	$\cdots$	$8s+10$	$\mathbf{4s-2}$	$8s+8$	$6s+6$
$10s+7$	$8s+5$	$\cdots$	$6s+9$	$8s+9$	$6s+7$	$\mathbf{4s+2}$

Thus, we now have a required M with $c_1(M)+r_1(M) = 8s^2+44s+49 \neq r_2(M) = r_3(M) = 32s^2+59s+19 \neq c_{j+1}(M) = 18s+15, 1 \leq j \leq r$. □

Observe that $K(1,2,r) = K(1,r) \vee O_2$. Obviously, $\chi_{la}(K(1,r)) = r+1, r \geq 2$.

Corollary 2.2. *For each $m \geq 3$, there exists a graph G of order m such that $\chi_{la}(G \vee O_2) - \chi_{la}(G) = 3 - m \leq 0$.*

Corollary 2.2 serves as counterexamples to the lower bound of [1, Theorem 2.16]. Interested readers may refer to [4] for more general results on $\chi_{la}(K(p,q,r)), r \geq q \geq p \geq 1$. The next theorem gives a sharp lower bound of $\chi_{la}(G \vee O_n)$ for $n \geq 1$.

Theorem 2.3. *For $n \geq 1$, $\chi_{la}(G \vee O_n) \geq \chi(G) + 1$ and the bound is sharp.*

Proof. It is obvious that for $n \geq 1$, we have $\chi_{la}(G \vee O_n) \geq \chi(G \vee O_n) = \chi(G) + 1$. In [5], the authors obtained that for $h \geq 2, k \geq 1$, $\chi_{la}(C_{2h} \vee O_{2k}) = 3$ and $\chi_{la}(C_{2h-1} \vee O_{2k-1}) = 4$. Since $\chi(C_{2h}) = 2$ and $\chi(C_{2h-1}) = 3$, the bound is sharp. □

Observe that if $\chi_{la}(G) = \chi(G)$, then $\chi_{la}(G \vee O_n) \geq \chi(G \vee O_n) = \chi(G) + 1 = \chi_{la}(G) + 1$. Thus we have proved the sufficiency of the following conjecture.

Conjecture 2.1. *For $n \geq 1$, $\chi_{la}(G \vee O_n) \geq \chi_{la}(G) + 1$ if and only if $\chi_{la}(G) = \chi(G)$.*

In [1], we have for $m \geq 2$, $\chi(C_{2m-1}) = \chi_{la}(C_{2m-1}) = 3 = \chi_{la}(C_{2m})$ and $\chi(C_{2m}) = 2$. This provides a supporting evidence that the conjecture holds.

Let $a_{i,j}$ be the (i,j)-entry of a magic (m,n)-rectangle with row constant $n(mn+1)/2$ and column constant $m(mn+1)/2$. The following theorems partially answer Conjecture 2.1.

Theorem 2.4. *Suppose G is of order $m \geq 3$ with $m \equiv n \pmod 2$ and $\chi(G) = \chi_{la}(G)$. If (i) $n \geq m$, or (ii) $m \geq n^2/2$ and $n \geq 4$, then $\chi_{la}(G \vee O_n) = \chi_{la}(G) + 1$.*

Proof. Let G has size e such that $V(G) = \{u_i : 1 \leq i \leq m\}$ and $V(O_n) = \{v_j : 1 \leq j \leq n\}$. Suppose f is a local antimagic labeling of G that induces a t-coloring of G.

Define $g : E(G \vee O_n) \to [1, e+mn]$ by

$$g(uv) = f(uv) \text{ for each } uv \in E(G),$$
$$g(u_iv_j) = e + a_{i,j} \text{ for } 1 \leq i \leq m, 1 \leq j \leq n.$$

It is clear that g is a bijection such that

$$g^+(u_i) = f^+(u_i) + ne + n(mn+1)/2 \text{ for } 1 \leq i \leq m,$$
$$g^+(v_j) = me + m(mn+1)/2 \text{ for } 1 \leq j \leq n.$$

(i) Suppose $n \geq m$, we have $g^+(u_i) > g^+(v_j)$ for all i and j.

(ii) Suppose $m \geq n^2/2$ and $n \geq 4$. We proceed to show that $g^+(v_j) - g^+(u_i) = (m-n)e + (m-n)(mn+1)/2 - f^+(u_i) > 0$ for all i and j. Note that $e \leq m(m-1)/2$, and $f^+(u_i) \leq e + (e-1) + \cdots + (e-m+2) = (2e-m+2)(m-1)/2$. Thus, $2(g^+(v_j) - g^+(u_i)) \geq 2(m-n)e + (m-n)(mn+1) - (2e-m+2)(m-1) = 2(1-n)e + (m-n)(mn+1) + (m-2)(m-1) \geq (1-n)m(m-1) + (m-n)(mn+1) + (m-2)(m-1) = 2m^2 + mn - 3m - mn^2 - n + 2 > m(2m-n^2) + m(n-4) + (m-n) \geq 0$.

In either case, g is a local antimagic labeling that induces a $(t+1)$-coloring of $G \vee O_n$. Hence, $\chi_{la}(G \vee O_n) \leq \chi_{la}(G) + 1$. Since $\chi_{la}(G \vee O_n) \geq \chi(G \vee O_n) = \chi(G) + 1 = \chi_{la}(G) + 1$, the theorem holds. □

Hence, we may assume that $m > n$.

Theorem 2.5. *Suppose G is an r-regular graph of order $m \geq 3$ with $m \equiv n \pmod 2$ and $\chi(G) = \chi_{la}(G)$. If $m > n$ and $r \geq \frac{(m-n)(mn+1)}{2mn}$, then $\chi_{la}(G \vee O_n) = \chi_{la}(G) + 1$.*

Proof. Let $V(G) = \{u_i : 1 \leq i \leq m\}$ and $V(O_n) = \{v_j : 1 \leq j \leq n\}$. Note that the size of G is $mr/2$. Suppose f is a local antimagic labeling of G that induces a t-coloring of G. Define $g : E(G \vee O_n) \to [1, mr/2 + mn]$ by

$$g(uv) = f(uv) + mn \text{ for each } uv \in E(G),$$
$$g(u_iv_j) = a_{i,j} \text{ for } 1 \leq i \leq m, 1 \leq j \leq n.$$

It is clear that g is a bijection such that

$$g^+(u_i) = f^+(u_i) + mnr + n(mn + 1)/2 \text{ for } 1 \leq i \leq m,$$
$$g^+(v_j) = m(mn + 1)/2, \text{ for } 1 \leq j \leq n.$$

Now, $2(g^+(u_i) - g^+(v_j)) = 2f^+(u_i) + 2mnr + (n - m)(mn + 1)$. Since $r \geq \frac{(m-n)(mn+1)}{2mn}$, we have $g^+(u_i) > g^+(v_j)$. This means g is a local antimagic labeling that induces a $(t+1)$-coloring of $G \vee O_n$. Hence, $\chi_{la}(G \vee O_n) \leq \chi_{la}(G)+1$. Since $\chi_{la}(G \vee O_n) \geq \chi(G \vee O_n) = \chi(G)+1 = \chi_{la}(G)+1$, the theorem holds. □

From [1, Theorem 2.14] and [5], for odd $m \geq 3, n \geq 1$, $\chi_{la}(C_m) = \chi(C_m) = 3$ and $\chi_{la}(C_m \vee O_n) = 4 = \chi(C_m) + 1$.

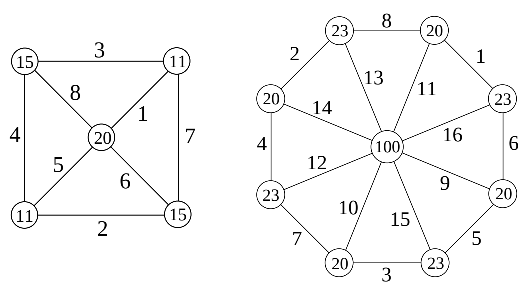
Problem 2.1. *Show that the condition $n \geq m$ in Theorem 2.4 or the condition $r \geq \frac{(m-n)(mn+1)}{2mn}$ in Theorem 2.5 can be omitted.*

3 Affirmative answers to [1, Theorem 2.15 and Problem 3.3]

In [1, Theorem 2.15], the authors show that $3 \leq \chi_{la}(W_n) \leq 5$ for $n \equiv 0 \pmod 4$. We now give the exact value of $\chi_{la}(W_n)$.

Theorem 3.1. *For $k \geq 1$, $\chi_{la}(W_{4k}) = 3$.*

Proof. Let $V(W_{4k}) = \{v\} \cup \{u_i : 1 \leq i \leq 4k\}$ and $E(W_{4k}) = \{vu_i : 1 \leq i \leq 4k\} \cup \{u_iu_{i+1} : 1 \leq i \leq 4k\}$, where $u_{4k+1} = u_1$. For $k = 1$ and 2, we have the labelings f in figures below for W_4 and W_8 showing that $c(f) = 3$.



For $k \geq 3$, we consider the following two tables.

Table 1.

C_1	C_2	C_3	$\ldots$	C_{k-2}	C_{k-1}	C_k	C_{k+1}	C_{k+2}	C_{k+3}	$\ldots$	C_{2k-2}	C_{2k-1}	C_{2k}
1	3	5	$\ldots$	$2k-5$	$2k-3$	$2k-1$	2	4	6	$\ldots$	$2k-4$	$2k-2$	$2k$
$3k$	$3k-1$	$3k-2$	$\ldots$	$2k+3$	$2k+2$	$2k+1$	$4k$	$4k-1$	$4k-2$	$\ldots$	$3k+3$	$3k+2$	$3k+1$
$8k$	$8k-1$	$8k-2$	$\ldots$	$7k+3$	$7k+2$	$7k+1$	$7k-1$	$7k-2$	$7k-3$	$\ldots$	$6k+2$	$6k+1$	$6k$

Table 2.

C_1	C_2	C_3	$\ldots$	C_{k-2}	C_{k-1}	C_k	C_{k+1}	C_{k+2}	C_{k+3}	C_{k+4}	$\ldots$	C_{2k-1}	C_{2k}
$3k+2$	$3k+1$	$3k$	$\ldots$	$2k+5$	$2k+4$	$2k+3$	$2k+2$	$2k$	$4k$	$4k-1$	$\ldots$	$3k+4$	$3k+3$
1	3	5	$\ldots$	$2k-5$	$2k-3$	$2k-1$	$2k+1$	2	4	6	$\ldots$	$2k-4$	$2k-2$
$6k-1$	$6k-2$	$6k-3$	$\ldots$	$5k+2$	$5k+1$	$5k$	$5k-1$	$7k$	$5k-2$	$5k-3$	$\ldots$	$4k+2$	$4k+1$

Observe that

(i) all integers in row 1 and row 2 of each table are in $[1, 4k]$;

(ii) the two rows 3 of both tables collectively give all integers in $[4k + 1, 8k]$;

(iii) Table 1 has constant column sum of $11k + 1$ and Table 2 has constant column sum of $9k + 2$;

(iv) in Table 1, all integers from column C_1 to C_k, and from C_{k+1} to C_{2k} of each row form an arithmetic progression;

(v) in Table 2, all integers from column C_1 to C_{k+1}, and from C_{k+2} to C_{2k} of each row (or from C_{k+3} to C_{2k} for row 1 and row 3) form an arithmetic progression.

Consider the following three sequences obtained by taking the first two entries of a particular column of Table 1 and the first two entries of a particular column of Table 2 alternately. Both entries taken are written in ordered pair respectively.

For even k, we have

(a) $(1, 3k)$, $(3k, 5)$, $(5, 3k-2)$, $(3k-2, 9)$, $(9, 3k-4)$, $(3k-4, 13)$, $\ldots$, $(2k-7, 2k+4)$, $(2k+4, 2k-3)$, $(2k-3, 2k+2)$, $(2k+2, 2k+1)$;

(b) $(2k+1, 2k-1)$, $(2k-1, 2k+3)$, $(2k+3, 2k-5)$, $(2k-5, 2k+5)$, $\ldots$, $(3k-3, 7)$, $(7, 3k-1)$, $(3k-1, 3)$, $(3, 3k+1)$;

(c) $(3k+1, 2k)$, $(2k, 2)$, $(2, 4k)$, $(4k, 4)$, $(4, 4k-1)$, $(4k-1, 6)$, $(6, 4k-2)$, $(4k-2, 8)$, $\ldots$, $(2k-6, 3k+4)$, $(3k+4, 2k-4)$, $(2k-4, 3k+3)$, $(3k+3, 2k-2)$, $(2k-2, 3k+2)$, $(3k+2, 1)$.

Here, sequences (a) and (b) are of length k and sequence (c) is of length $2k$.

Observe that $T = (a)+(b)+(c)$ is a sequence of $4k$ ordered pairs with every integers in $[1, 4k]$ appearing exactly twice, once as the left entry of an ordered pair and once as the right entry of another ordered pair. Therefore, taking the left entry of every ordered pair gives us a sequence S with $4k$ distinct integers in $[1, 4k]$. Define, $f : E(C_{4k}) \to S$ such that $f(e_i) = f(u_i u_{i+1})$ is the i-th entry of S, $1 \leq i \leq 4k$ and $u_{4k+1} = u_1$. Let $f(vu_{i+1})$ be the value in row 3 of the column that corresponds to the i-th entry of S. For $1 \leq j \leq 2k$, since all the $(2j-1)$-st ordered pairs of T are from Table 1 and all the $2j$-th ordered pairs are from Table 2, we now have $f^+(u_{2j}) = 11k + 1$

and $f^+(u_{2j-1}) = 9k+2$. Moreover, $f^+(v) = (4k+1) + \cdots + (8k) = 2k(12k+1)$. Thus, f is a local antimagic labeling of W_{4k} with $c(f) = 3$.

For odd k, we shall have different sequence (a) and sequence (b) as follows.

(a) $(1, 3k)$, $(3k, 5)$, $(5, 3k-2)$, $(3k-2, 9)$, $(9, 3k-4)$, $(3k-4, 13)$, …, $(2k-5, 2k+3)$, $(2k+3, 2k-1)$;

(b) $(2k-1, 2k+1)$, $(2k+1, 2k+2)$, $(2k+2, 2k-3)$, $(2k-3, 2k+4)$, $(2k+4, 2k-7)$, $(2k-7, 2k+6)$, …, $(3k-3, 7)$, $(7, 3k-1)$, $(3k-1, 3)$, $(3, 3k+1)$.

Here, sequences (a) and (b) are of length $k-1$ and $k+1$, respectively.

By an argument similar to that for even k, we also can obtain a local antimagic labeling f of W_{4k} with $c(f) = 3$ such that $f^+(u_{2j}) = 11k+1$ and $f^+(u_{2j-1}) = 9k+2$ for $1 \le j \le 2k$, and $f^+(v) = 2k(12k+1)$. Since $\chi_{la}(W_{4k}) \ge \chi(W_{4k}) = 3$, the theorem holds. □

Example 3.1. For $k = 3$, the tables defined in the proof of Theorem 3.1 are

Table 1.

C_1	C_2	C_3	C_4	C_5	C_6
1	3	5	2	4	6
9	8	7	12	11	10
24	23	22	20	19	18

Table 2.

C_1	C_2	C_3	C_4	C_5	C_6
11	10	9	8	6	12
1	3	5	7	2	4
17	16	15	14	21	13

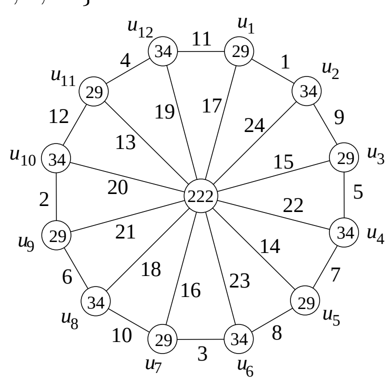
Thus, sequence T is given by:

(a) (1,9), (9,5);

(b) (5,7), (7,8), (8,3), (3,10);

(c) (10,6), (6,2), (2,12), (12,4), (4,11), (11,1)

and $S = \{1, 9, 5, 7, 8, 3, 10, 6, 2, 12, 4, 11\}$. Hence we have the following labeling:



In [1, Problem 3.3], the authors also asked:

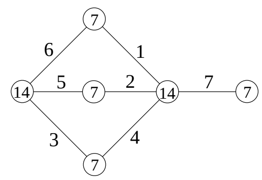
Problem 3.1. *Does there exist a graph G of order n with $\chi_{la}(G) = n - k$ for every $k = 0, 1, 2, \ldots, n-2$?*

The following theorem in [6] is needed to answer Problem 3.1. For completeness, the proof is also given.

Theorem 3.2. *Let G be a graph having k pendants. If G is not K_2, then $\chi_{la}(G) \ge k+1$ and the bound is sharp.*

Proof. Suppose G has size m. Let f be any local antimagic labeling of G. Consider the edge uv with $f(uv) = m$. We may assume u is not a pendant. Clearly, $f^+(u) > m \geq f^+(z)$ for every pendant z. Since all pendants have distinct induced colors, we have $\chi_{la}(G) \geq k + 1$.

For $k \geq 2$, since $\chi_{la}(S_k) = k + 1$, where S_k is a star with maximum degree k, the lower bound is sharp. The labeling in figure below shows that the lower bound is sharp for $k = 1$.



□

For $m \geq 2, t \geq 1$, let $CT(m,t)$ be the coconut tree obtained by identifying the central vertex of a $K(1,t)$ with an end-vertex of a path P_m. Note that $CT(2,t) = K(1,t+1)$ with $\chi_{la}(K(1,t+1)) = t + 2$. Moreover, $CT(m,1) = P_{m+1}$.

Theorem 3.3. *For $m \geq 2, t \geq 1$, $\chi_{la}(CT(m,t)) = t + 2$.*

Proof. By [1, Theorem 2.7], the theorem holds for $t = 1$. Consider $t \geq 2$. Let $P_m = v_1 v_2 \cdots v_m$ and $E(K(1,t)) = \{v_m x_j : 1 \leq j \leq t\}$. Denote by e_i the edge $v_i v_{i+1}$ for $1 \leq i \leq m-1$. Define $f : E(CT(m,t)) \to [1, m+t-1]$ by

(i) $f(e_i) = (i+1)/2$ for odd i,

(ii) $f(e_i) = m - i/2$ for even i,

(iii) $f(v_m x_j) = m + j - 1$ for $1 \leq j \leq t$.

It is easy to verify that f is a bijection with $f^+(x_j) = m+j-1$ for $1 \leq j \leq t$, $f^+(v_m) \geq 2m+1$, $f^+(v_1) = 1$, $f^+(v_i) = m + 1$ for even $1 < i < m$ and $f^+(v_i) = m + 2$ for odd $1 < i < m$. Thus, f is a local antimagic labeling that induces a $(t+2)$-coloring so that $\chi_{la}(CT(m,t)) \leq t+2$. By Theorem 3.2, we know that $\chi_{la}(CT(m,t)) \geq t + 2$. Hence, $\chi_{la}(CT(m,t)) = t + 2$. □

Theorem 3.4. *For each possible n, k, there exists a graph G of order n such that $\chi_{la}(G) = n-k$ if and only if $n \geq k + 3 \geq 3$.*

Proof. By definition, $k \geq 0$ and $n \geq 3$. Suppose $n \geq k + 3 \geq 3$. Let $G = CT(m,t)$ of order $n = m+t \geq 3$. For $t \geq 1$, we have that $\chi_{la}(CT(m,t)) = t+2 = n-(m-2)$. Letting $m-2 = k$, we have $\chi_{la}(CT(k+2, n-k-2)) = n - k$. Thus, for every possible $0 \leq k \leq n-3$, there is a graph G of order n such that $\chi_{la}(G) = n - k$. This proves the sufficiency.

We prove the necessity by contrapositive. Suffice to assume $n = k + 2$. It is easy to check that there is no graph G of order $n = 3, 4$ such that $\chi_{la}(G) = n - k = 2$. □

Theorem 3.4 and the following theorem in [6] solve Problem 3.1 completely. For completeness, the proof is stated as well.

Theorem 3.5. *Suppose $n \geq 3$. There is a graph G of order n with $\chi_{la}(G) = 2$ if and only if $n \neq 3, 4, 5, 7$.*

Proof. Suppose $n = 3, 4, 5, 7$, it is routine to check that all graphs G of order n has $\chi_{la}(G) \geq 3$. This proves the necessity by contrapositive.

We now prove the sufficiency. Suppose n is odd and $n \geq 9$. Since $n = 6s+1$ $(s \geq 2)$, $n = 6s+3$ $(s \geq 1)$ or $6s+5$ $(s \geq 1)$, we consider the following three cases.

Case (a). $n = 6s+1$. Suppose $s \geq 3$. We shall construct a bipartite graph G with bipartition (A, B), where $|A| = 3$ and $|B| = 6s-2$, such that all vertices in B are of degree 2. If G exists, then G is of order $6s+1$ and size $12s-4$. Suppose there is a local antimagic labeling f of G such that $c(f) = 2$, then f corresponds to a labeling matrix M of size $3 \times (6s-2)$ such that each of its entry is either an integer in $[1, 12s-4]$ or $*$. Moreover, each integer in $[1, 12s-4]$ appears as entry of A once. Note that the total sum of integers in $[1, 12s-4]$ is $3(6s-2)(4s-1)$. We now arrange integers in $[1, 12s-4]$ to form matrix M as follows:

(1). In row 1, assign k to column k if $k = 2, 4, 6, \ldots, 4s-2, 6s-2$; assign $12s-3-k$ to column k if $k = 3, 5, 7, \ldots, 4s-1$.

(2). In row 2, assign k to column k if $k = 1, 2s-1, 2s+1, 2s+3, \ldots, 6s-3$; assign $12s-3-k$ to column k if $k = 2s, 2s+2, 2s+4, \ldots, 6s-4$.

(3). In row 3, assign $12s-4$ to column 1; assign k to column k if $k = 3, 5, 7, \ldots, 2s-3, 4s, 4s+2, 4s+4, \ldots, 6s-4$; assign $12s-3-k$ to column k if $k = 2, 4, 6, \ldots, 2s-2, 4s+1, 4s+3, 4s+5, \ldots, 6s-3, 6s-2$.

(4). All the remaining columns of each row is assigned with $*$.

The resulting matrix is given by the following table:

$*$	2	$12s-6$	$\cdots$	$2s-2$	$10s-2$	$2s$	$10s-4$	$\cdots$
1	$*$	$*$	$\cdots$	$*$	$2s-1$	$10s-3$	$2s+1$	$\cdots$
$12s-4$	$12s-5$	3	$\cdots$	$10s-1$	$*$	$*$	$*$	$*$

$8s$	$4s-2$	$8s-2$	$*$	$\cdots$	$*$	$*$	$*$	$6s-2$
$4s-3$	$8s-1$	$4s-1$	$8s-3$	$\cdots$	$6s-5$	$6s+1$	$6s-3$	$*$
$*$	$*$	$*$	$4s$	$\cdots$	$6s+2$	$6s-4$	$6s$	$6s-1$

It is easy to check that the first row contains $4s-1$ numbers, the second row contains $4s$ numbers and the third row contains $4s-3$ numbers. Moreover, each column sum is $12s-3$ and each row sum is $(6s-2)(4s-1)$. Thus, G exists and $\chi_{la}(G) = 2$.

When $s = 2$ $(n = 13)$, a required labeling matrix is as follow:

$M =$

$*$	2	18	4	16	6	14	$*$	$*$	10
1	$*$	3	17	5	15	7	13	9	$*$
20	19	$*$	$*$	$*$	$*$	$*$	8	12	11

Case (b). $n = 6s+3$. Similar to Case (a), a labeling matrix M of size $3 \times 6s$ such that each of its entry is either an integer in $[1, 12s]$ or $*$ can be obtained.

For $s \geq 1$, we let the matrix $M = \begin{pmatrix} M_1 & M_2 & M_3 \end{pmatrix}$, where each M_i is a $3 \times 2s$ matrix. For the first matrix, we assign $1, 2, \ldots, 2s$ at the first row; $*$ at each entry of the second row; $12s, 12s-1, \ldots, 10s+1$ at the third row. We then swap the $(1, j)$-entry with $(3, j)$-entry of this matrix when $j \equiv 2, 3 \pmod 4$ and $2 \leq j \leq 2s$. The resulting matrix is M_1. Similarly, for the second matrix, we assign $10s, 10s-1, \ldots, 8s+1$ at the first row; $2s+1, 2s+2, \ldots, 4s$ at the second row; $*$ at each entry of the third row. We then swap the $(1, j)$-entry with $(2, j)$-entry of this matrix when $j \equiv 2, 3 \pmod 4$ and $2 \le j \le 2s$. The resulting matrix is M_2. For the third matrix, we assign $*$ at each entry of the first row; $8s, 8s-1, \dots, 6s+1$ at the second row; $4s+1, 4s+2, \dots, 6s$ at the third row. We then swap the $(2, j)$-entry with $(3, j)$-entry of this matrix when $j \equiv 2, 3 \pmod 4$ and $2 \le j \le 2s$. The resulting matrix is M_3.

So when s is odd, we have

$M_1 =$

1	$12s-1$	$12s-2$	4	5	$12s-5$	$\cdots$	$10s+4$	$2s-2$	$2s-1$	$10s+1$
$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$
$12s$	2	3	$12s-3$	$12s-4$	6	$\cdots$	$2s-3$	$10s+3$	$10s+2$	$2s$

$M_2 =$

$10s$	$2s+2$	$2s+3$	$10s-3$	$10s-4$	$2s+6$	$\cdots$	$4s-3$	$8s+3$	$8s+2$	$4s$
$2s+1$	$10s-1$	$10s-2$	$2s+4$	$2s+5$	$10s-5$	$\cdots$	$8s+4$	$4s-2$	$4s-1$	$8s+1$
$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$

$M_3 =$

$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$
$8s$	$4s+2$	$4s+3$	$8s-3$	$8s-4$	$4s+6$	$\cdots$	$6s-3$	$6s+3$	$6s+2$	$6s$
$4s+1$	$8s-1$	$8s-2$	$4s+4$	$4s+5$	$8s-5$	$\cdots$	$6s+4$	$6s-2$	$6s-1$	$6s+1$

One may check that the first row sum of M_1, the second row sum of M_2 and the third row sum of M_3 are the same which is $12s + \frac{1}{4}(2s-2)(24s+2) = 12s^2 + s - 1$. It is easy to see that the third row sum of M_1, the first row sum of M_2 and the second row sum of M_3 also are the same and equals $(12s+1)(2s) - (12s^2 + s - 1) = 12s^2 + s + 1$. Hence each row sum of M is $24s^2 + 2s$ and each column sum is $12s + 1$.

When s is even, we have

$M_1 =$

1	$12s-1$	$12s-2$	4	5	$\cdots$	$2s-4$	$2s-3$	$10s+3$	$10s+2$	$2s$
$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$	$*$
$12s$	2	3	$12s-3$	$12s-4$	$\cdots$	$10s+5$	$10s+4$	$2s-2$	$2s-1$	$10s+1$

$M_2 =$

$10s$	$2s+2$	$2s+3$	$10s-3$	$10s-4$	$\cdots$	$8s+5$	$8s+4$	$4s-2$	$4s-1$	$8s+1$
$2s+1$	$10s-1$	$10s-2$	$2s+4$	$2s+5$	$\cdots$	$4s-4$	$4s-3$	$8s+3$	$8s+2$	$4s$
$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$	$*$

$M_3 =$

$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$	$*$
$8s$	$4s+2$	$4s+3$	$8s-3$	$8s-4$	$\cdots$	$6s+5$	$6s+4$	$6s-2$	$6s-1$	$6s+1$
$4s+1$	$8s-1$	$8s-2$	$4s+4$	$4s+5$	$\cdots$	$6s-4$	$6s-3$	$6s+3$	$6s+2$	$6s$

One may check that all the numerical row sum of M_i are the same which is $\frac{1}{4}(2s)(24s+2) + 10s + 2 = 12s^2 + s$. Hence each row sum of M is $24s^2 + 2s$ and each column sum is $12s + 1$.

Thus, G exists and $\chi_{la}(G) = 2$.

Case (c). $n = 6s + 5$. Suppose $s \ge 2$. We shall construct a bipartite graph G with bipartition (A, B), where $|A| = 3$ and $|B| = 6s + 2$, such that B has a vertex of degree 1 and the remaining $6s + 1$ vertices are of degree 2. If G exists, then G is of order $6s + 5$ and size $12s + 3$. Note that the total sum of integers in $[1, 12s + 3]$ is $3(6s + 2)(4s + 1)$. Similar to the above construction, we want to arrange integers in $[1, 12s + 3]$ to form a $3 \times (6s + 2)$ matrix M as follows:

(1). In row 1, assign k to column k if $k = 2, 4, 6, \dots, 4s + 2, 6s$; assign $12s + 3 - k$ to column k if $k = 3, 5, 7, \dots, 4s + 1$.

(2). In row 2, assign k to column k if $k = 1, 2s + 1, 2s + 3, 2s + 5, \dots, 6s + 1$; assign $12s + 3 - k$ to column k if $k = 2s, 2s + 2, 2s + 4, \dots, 6s - 2$.

(3). In row 3, assign $12s+2$ to column 1 and $12s+3$ to column $6s+2$; assign k to column k if $k = 3, 5, 7, \ldots, 2s-1, 4s, 4s+2, 4s+4, \ldots, 6s-2$; assign $12s+3-k$ to column k if $k = 2, 4, 6, \ldots, 2s-2, 4s+3, 4s+5, 4s+7, \ldots, 6s+1$.

(4). All the remaining columns of each row is assigned with $*$.

The resulting matrix is given by the following table:

$*$	2	$12s$	4	$\cdots$	$10s+4$	$2s$	$10s+2$	$\cdots$	$4s$
1	$*$	$*$	$*$	$\cdots$	$*$	$10s+3$	$2s+1$	$\cdots$	$8s+3$
$12s+2$	$12s+1$	3	$12s-1$	$\cdots$	$2s-1$	$*$	$*$	$\cdots$	$*$

$8s+2$	$4s+2$	$*$	$\cdots$	$*$	$*$	$6s$	$*$	$*$
$4s+1$	$8s+1$	$4s+3$	$\cdots$	$6s+5$	$6s-1$	$*$	$6s+1$	$*$
$*$	$*$	$8s$	$\cdots$	$6s-2$	$6s+4$	$6s+3$	$6s+2$	$12s+3$

It is easy to check that the first row contains $4s+1$ numbers, the second row contains $4s+2$ numbers and the third row contains $4s$ numbers. Moreover, each column sum is $12s+3$ and each row sum is $(6s+2)(4s+1)$.

When $s = 1$ ($n = 11$), a required labeling matrix is as follow:

$M =$

1	3	4	5	6	8	13	$*$
$*$	12	$*$	10	9	7	2	$*$
14	$*$	11	$*$	$*$	$*$	$*$	15

Thus, G exists and $\chi_{la}(G) = 2$.

Suppose $n \geq 6$ is even. In [1, Theorem 2.11] (see Theorem 4.1), we have $\chi_{la}(K_{p,q}) = 2$, where $p \neq q$ and $n = p + q \geq 6$ is an even integer. Therefore, there exists a graph G of order n such that $\chi_{la}(G) = 2$ for every even $n \geq 6$. □

Example 3.2. *Let* $n = 19$, *we have*

$M =$

$*$	2	30	4	28	6	26	8	24	10	22	$*$	$*$	$*$	$*$	16
1	$*$	$*$	$*$	5	27	7	25	9	23	11	21	13	19	15	$*$
32	31	3	29	$*$	$*$	$*$	$*$	$*$	$*$	$*$	12	20	14	18	17

Let $n = 21$, *we have*

$M =$

1	35	34	4	5	31	30	8	9	27	26	12	$*$	$*$	$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$*$	7	29	28	10	11	25	24	14	15	21	20	18
36	2	3	33	32	6	$*$	$*$	$*$	$*$	$*$	$*$	13	23	22	16	17	19

Let $n = 15$, *we have*

$M =$

1	23	22	4	20	6	7	17	$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$	5	19	18	8	16	10	11	13
24	2	3	21	$*$	$*$	$*$	$*$	9	15	14	12

Let $n = 17$, *we have*

$M =$

$*$	2	24	4	22	6	20	8	18	10	$*$	12	$*$	$*$
1	$*$	$*$	23	5	21	7	19	9	17	11	$*$	13	$*$
26	25	3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	16	15	14	27

4 Complete Bipartite Graphs

Theorem 4.1. *[1, Theorems 2.11-12] For $p, q \geq 1$ and $(q, p) \neq (1, 1)$,*

$$\chi_{la}(K_{p,q}) = \begin{cases} q+1 & \text{if } q > p = 1, \\ 3 & \text{if } p = 2, q = 2 \text{ or } q \text{ is odd}, \\ 2 & \text{if } p \geq 2, p \neq q \text{ and } p \equiv q \pmod 2. \end{cases}$$

We next determine $\chi_{la}(K_{p,q})$ for all p, q not considered in Theorem 4.1. Suppose f is a local antimagic labeling of $K_{p,q}$ and M is a $p \times q$ matrix with row sums and column sums correspond to the vertex labels under f accordingly.

The following lemma [5] is needed.

Lemma 4.2. *Let G be a graph of size q. Suppose there is a local antimagic labeling of G inducing a 2-coloring of G with colors x and y, where $x < y$. Let X and Y be the numbers of vertices of colors x and y, respectively. Then G is a bipartite graph whose sizes of parts are X and Y with $X > Y$, and $xX = yY = \frac{q(q+1)}{2}$.*

Theorem 4.3. *For $p \geq 3$, $\chi_{la}(K_{p,p}) = 3$.*

Proof. By Lemma 4.2, $\chi_{la}(K_{p,p}) \geq 3$. Suppose $p \geq 4$ is even. Let A be a magic rectangle of size $p \times 2$ by using integers in $[1, 2p]$ and B be a magic rectangle of size $p \times (p-2)$ by using integers in $[2p+1, p^2]$. Note that, for the construction of magic rectangles, one may find from [2, 8]. Now, for $M = \begin{pmatrix} A & B \end{pmatrix}$, each row sum is $\frac{1}{2}p(p^2+1)$, each of the first two column sums is $\frac{1}{2}p(2p+1)$, and each other column sum is $\frac{1}{2}p(p+1)^2$. Thus, $\chi_{la}(K_{p,p}) = 3$ for even $p \geq 2$.

Suppose $p = 2n+1 \geq 3$ is odd. Consider the $(2n+1) \times (2n+1)$ magic square A constructed by Siamese method:
Starting from the $(1, n+1)$-entry (i.e, $A_{1,n+1}$) with the number 1, the fundamental movement for filling the entries is diagonally up and right, one step at a time. When a move would leave the matrix, it is wrapped around to the last row or first column, respectively. If a filled entry is encountered, one moves vertically down one box instead, then continuing as before. One may find the detail in [7].

Note that each of the ranges $[1, p]$, $[p+1, 2p]$, $\ldots$, $[p^2 - p + 1, p^2]$ occupies a diagonal of the matrix, wrapping at the edges. Namely, the range $[1, p]$ starts at $A_{1,n+1}$ and ends at $A_{2,n}$; the range $[p+1, 2p]$ starts at $A_{3,n}$ ends at $A_{4,n-1}$; the range $[2p+1, 3p]$ starts at $A_{5,n-1}$ and ends at $A_{6,n-2}$, etc. In general, the range $[ip+1, (i+1)p]$ starts at $A_{2i+1,n+1-i}$ and ends at $A_{2i+2,n-i}$, where $0 \leq i \leq p-1$ and the indices are taken modulo p. It is easy to see that the $(n+1)$-st column of A is $(1, p+2, \ldots, p^2)$ which is an arithmetic sequence with common difference $p+1$.

Now let M be the matrix obtained from A by shifting up the $(n+1)$-st column by one entry (the top entry moves to the bottom). Hence each column sum of M is still the magic number $\frac{1}{2}p(p^2+1)$. Each row sum of M is $\frac{1}{2}p(p^2+1) + p + 1$ except the last row sum which is $\frac{1}{2}p(p^2+1) - p^2 + 1$. Thus, we conclude that $\chi_{la}(K_{p,p}) = 3$. □

Example 4.1. *Suppose $p = 5$. We have the following magic square of order 5:*

$$A = \begin{pmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{pmatrix}$$

Now

$$M = \left|\begin{array}{ccccc|c} 17 & 24 & 7 & 8 & 15 & 71 \\ 23 & 5 & 13 & 14 & 16 & 71 \\ 4 & 6 & 19 & 20 & 22 & 71 \\ 10 & 12 & 25 & 21 & 3 & 71 \\ 11 & 18 & 1 & 2 & 9 & 41 \\ \hline 65 & 65 & 65 & 65 & 65 & sum \end{array}\right.$$

Theorem 4.4. *For $p, q \geq 3$ and $p \not\equiv q \pmod 2$, $\chi_{la}(K_{p,q}) = 3$.*

Proof. If $\chi_{la}(K_{p,q}) = 2$, then the corresponding matrix M is a magic rectangle. Since there is no magic rectangle of size $p \times q$ for $p \not\equiv q \pmod 2$, we know $\chi_{la}(K_{p,q}) \geq 3$. Without loss of generality, assume $q \geq 3$ is odd and $p \geq 4$ is even.

Consider $p \geq 6$. Let A be a $3 \times q$ magic rectangle using integer in $[1, 3q]$ and B be a $(p-3) \times q$ magic rectangle using integers in $[3q+1, qp]$. Let $M = \begin{pmatrix} A \\ B \end{pmatrix}$. Thus, each column sum of M is $y = \frac{1}{2}(qp+1)p$, each of the first three row sums of M is $x = \frac{1}{2}(3q+1)q$, and each other row sum of M is $z = \frac{1}{2}(qp + 3q + 1)q$. Clearly $x < z$.

Suppose $p > q$. It is easy to see that $x < y$. Consider $2(y - z) = (qp + 1)(p - q) - 3q^2$. If $p \geq q + 3$, then $y - z > 0$. Suffice to consider $p = q + 1$. In this case, $2(y - z) = -2q^2 + q + 1 \neq 0$ when $q \geq 3$. So M corresponds a local antimagic labeling of $K_{p,q}$ for this case.

Suppose $q > p$. $2(x - y) = 3q^2 + (1 - p^2)q - p \equiv -p \pmod q$. So $x - y \neq 0$ since $q > p \geq 6$. Now $2(y - z) = (qp + 1)(p - q) - 3q^2 < 0$. So M corresponds a local antimagic labeling of $K_{p,q}$ for this case.

The remaining case is when $p = 4$. Let A be a $4 \times q$ matrix whose first row is the sequence of odd integers in $[1, 2q]$ in natural order; second row is the sequence of even integers in $[2q + 1, 4q]$ in reverse natural order; third row is the sequence of even integers in $[1, 2q]$ in natural order; last row is the sequence of odd integers in $[2q + 1, 4q]$ in reverse natural order. It is clear that each column sum is $2(4q + 1)$, the first row sum is q^2, the second row sum is $3q^2 + q$, the third row sum is $q^2 + q$, and the last row sum is $3q^2$.

Suppose $q \equiv 3 \pmod 4$. Now $A_{1,(3q+3)/4}$ (the $(1, (3q + 3)/4)$-entry of A) and $A_{2,(3q+3)/4}$ are $\begin{pmatrix} (3q+1)/2 \\ (5q+1)/2 \end{pmatrix}$, respectively. Swap these two entries to obtain a matrix M. Thus, the first row sum of M is $q^2 + q$, the second row sum is $3q^2$, the third row sum is $q^2 + q$, and the last row sum is $3q^2$.

Suppose $q \equiv 1 \pmod 4$. Now the $A_{1,(3q+5)/4}$ and $A_{2,(3q+5)/2}$ are $\begin{pmatrix} (3q+3)/2 \\ (5q-1)/2 \end{pmatrix}$, respectively. To obtain M we swap these two entries first, and then swap $A_{1,1}$ with $A_{3,1}$ and swap $A_{2,1}$ with $A_{4,1}$. Now, the first row sum of M is $q^2 + q - 1$, the second row sum is $3q^2 + 1$, the third row sum is $q^2 + q - 1$, and the last row sum is $3q^2 + 1$.

It is easy to see that the column sum $8q + 2$ cannot be equal to each row sum. Hence this completes the proof. □

Example 4.2. *Consider the graph $K_{4,7}$. Let*

$$A = \left|\begin{array}{ccccccc|c} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 49 \\ 28 & 26 & 24 & 22 & 20 & 18 & 16 & 154 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 56 \\ 27 & 25 & 23 & 21 & 19 & 17 & 15 & 147 \\ \hline 58 & 58 & 58 & 58 & 58 & 58 & 58 & Sum \end{array}\right.$$

After swapping $A_{1,6}$ with $A_{2,6}$ we have

$$M = \left|\begin{array}{ccccccc|c} 1 & 3 & 5 & 7 & 9 & 18 & 13 & 56 \\ 28 & 26 & 24 & 22 & 20 & 11 & 16 & 147 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 56 \\ 27 & 25 & 23 & 21 & 19 & 17 & 15 & 147 \\ \hline 58 & 58 & 58 & 58 & 58 & 58 & 58 & Sum \end{array}\right.$$

Next we consider the graph $K_{4,5}$. Let

$$A = \left|\begin{array}{ccccc|c} 1 & 3 & 5 & 7 & 9 & 25 \\ 20 & 18 & 16 & 14 & 12 & 80 \\ 2 & 4 & 6 & 8 & 10 & 30 \\ 19 & 17 & 15 & 13 & 11 & 75 \\ \hline 42 & 42 & 42 & 42 & 42 & Sum \end{array}\right.$$

After swapping the $A_{1,5}$ with $A_{2,5}$, $A_{1,1}$ with $A_{3,1}$, and $A_{2,1}$ with $A_{4,1}$ we have

$$M = \left|\begin{array}{ccccc|c} 2 & 3 & 5 & 7 & 12 & 29 \\ 19 & 18 & 16 & 14 & 9 & 76 \\ 1 & 4 & 6 & 8 & 10 & 29 \\ 20 & 17 & 15 & 13 & 11 & 76 \\ \hline 42 & 42 & 42 & 42 & 42 & Sum \end{array}\right.$$

Corollary 4.5. *For $q \geq p \geq 1$ and $q \geq 2$,*

$$\chi_{la}(K_{p,q}) = \begin{cases} q+1 & \text{if } q > p = 1, \\ 2 & \text{if } q > p \geq 2 \text{ and } p \equiv q \pmod 2, \\ 3 & \text{otherwise.} \end{cases}$$

Acknowledgements This paper was initiated during the first author's visit to Harbin Engineering University. He is grateful to the university for providing full financial support and to Universiti Technologi MARA for granting the leaves. He would also like to dedicate this paper to Prof Yee-Hock Peng on the occasion of his 65th birthday.

References

[1] Arumugam S., Premalatha K., Bacă M., Semaničová-Feňovčíková, A.: Local antimagic vertex coloring of a graph. Graphs and Combin. 33, 275–285 (2017) .

[2] Chai, E.S., Das, A. and Midha, C.: Construction of magic rectangles of odd order. Australas. J. Combin. 55, 131–144 (2013).

[3] Haslegrave, J.: Proof of a local antimagic conjecture. Discrete Maths. & Theor. Comp. Sc., **Vol. 20:1** (2018) #18.

[4] Lau, G.C., Ng, H.K., Shiu, W.C.: Cartesian magicness of 3-dimensional boards. (2018) arXiv:1805.04890.

[5] Lau, G.C., Ng, H.K., Shiu, W.C.: On local antimagic chromatic number of cycle related join graphs. Discuss. Math. Graph Theory, (2018) doi:10.7151/dmgt.2177.

[6] Lau, G.C., Shiu, W.C., Ng, H.K.: On local antimagic chromatic number of graphs with cut-vertices. (2018) arXiv:1805.04801.

[7] Kraitchik, M.: Magic Squares. Mathematical Recreations, New York: Norton, pp. 142–192, 1942.

[8] De Los Reyes J.P.: A matrix approach to construct magic rectangles of even order. Australas. J. Combin. 40, 293–300 (2008).