

Combinatorial Hopf algebras for interconnected nonlinear input-output systems with a view towards discretization

Luis A. Duffaut Espinosa, Kuruş Ebrahimi-Fard, and W. Steven Gray

Abstract A detailed expose of the Hopf algebra approach to interconnected input-output systems in nonlinear control theory is presented. The focus is on input-output systems that can be represented in terms of Chen–Fliess functional expansions or Fliess operators. This provides a starting point for a discrete-time version of this theory. In particular, the notion of a discrete-time Fliess operator is given and a class of parallel interconnections is described in terms of the quasi-shuffle algebra.

1 Introduction

A central problem in control theory is understanding how dynamical systems behave when they are interconnected. In a typical design problem, one is given a fixed system representing the *plant*, say a robotic manipulator or a spacecraft. The objective is to find a second system, usually called the *controller*, which when interconnected with the first will make the output track a pre-specified trajectory. When the component systems are nonlinear, these problems are difficult to address by purely analytical means. The prevailing methodologies are geometric in nature and based largely on state variable analysis [43, 47, 53, 60]. A complementary approach, however, has begun to emerge where the input-output map of each component system

Luis A. Duffaut Espinosa
Department of Electrical and Biomedical Engineering, University of Vermont, Burlington, Vermont 05405, USA, e-mail: l.duffautespinosa@gmail.com

Kuruş Ebrahimi-Fard
Department of Mathematical Sciences, Norwegian University of Science and Technology – NTNU, NO-7491 Trondheim, Norway, e-mail: kurusch.ebrahimi-fard@ntnu.no, <https://folk.ntnu.no/kurusche/>

W. Steven Gray
Department of Electrical and Computer Engineering, Old Dominion University, Norfolk, Virginia 23529, USA, e-mail: sgray@odu.edu

is represented in terms of a Chen–Fliess functional expansion or Fliess operator. In this setting, concepts from combinatorics and algebra are employed to produce an explicit description of the interconnected system. The genesis of this method can be found in the work of Fliess [24, 25] and Ferfera [19, 20], who described key elements of the underlying algebras in terms of noncommutative formal power series. In particular, they identified the central role played by the shuffle algebra in this theory. The method was further developed by Gray et al. [38, 39, 40, 65] and Wang [66] who addressed basic analytical questions such as what inputs guarantee convergence of the component series, when are the interconnections well defined, and what is the nature of the output functions? A fundamental open problem on the algebraic side until 2011 was how to explicitly compute the generating series of two feedback interconnected Fliess operators. Largely inspired by interactions at the 2010 Trimester in Combinatorics and Control (COCO2010) in Madrid with researchers in the area of quantum field theory (see, for example, [21]), the problem was eventually solved by identifying a new combinatorial Hopf algebra underlying the calculation. The evolution of this structure took several distinct steps. The single-input, single-output (SISO) case was first addressed by Gray and Duffaut Espinosa in [28] via a certain graded Hopf algebra of combinatorial type. Foissy then introduced a new grading in [26] which rendered a connected version of this combinatorial Hopf algebra. This naturally provided a fully recursive formula for the antipode, which is central to the feedback calculation [31]. The multivariable, i.e., multi-input, multi-output (MIMO) case, was then treated in [32]. Next, a full combinatorial treatment, including a Zimmermann type forest formula for the antipode [2], was presented in [14]. This last result, based on an equivalent combinatorial Hopf algebra of decorated rooted circle trees, greatly reduces the number of computations involved by eliminating the inter-term cancelations that are intrinsic in the usual antipode calculation. Practical problems would be largely intractable without this innovation. The final and most recent development method is a description of this Hopf algebra based entirely on (co)derivation(-type) maps applied to the (co)product. This method was first observed to be implicit in the work of Devlin on the classical Poincaré center problem [12, 16]. In a state space setting, it can be related to computing iterated Lie derivatives to determine series coefficients [43]. Control applications ranging from guidance and chemical engineering to systems biology can be found in [15, 30, 32, 33, 34, 37].

This article has two general goals. First, an introduction to the method of combinatorial Hopf algebra in the context of feedback control theory is given in its most complete and up-to-date form. The idea is to integrate all of the advances described above into a single uniform treatment. This results in a new and distinct presentation of these ideas. In particular, the full solution to the problem of computing the generating series for a multivariable continuous-time dynamic output feedback system will be described. The origins of the forest formula related to this calculation will also be outlined. Then the focus is shifted to the second objective, which is largely an open problem in this field, namely how to recast this theory in a discrete-time setting. One approach suggested by Fliess in [23] is to describe the generating series of a discrete-time input-output map in terms of a complete tensor algebra defined

on an algebra of polynomials over a noncommutative alphabet. While this is a very general approach and can be related to the notion of a Volterra series as described by Sontag in [61], to date it has not lead to any particularly useful algebraic structures in the context of system interconnections. Another approach developed by the authors in the context of numerical approximation is to define the notion of a discrete-time Fliess operator in terms of a series of iterated sums over a noncommutative alphabet [35, 36]. While not the most general set up, it has been shown to be related to a class of state affine rational input discrete-time systems in the case where the generating series is rational. Furthermore, this class of so called rational discrete-time Fliess operators is guaranteed to always converge. So this will be the approach taken here. As the main interest in this paper is interconnection theory, the analysis begins with the simplest type of interconnections, the parallel sum and parallel product connections. The former is completely trivial, but the later induces the quasi-shuffle algebra of Hoffman (see [41]) on the vector space of generating series. Of particular interest is whether rationality is preserved under the quasi-shuffle product. It is well known to be the case for the shuffle product [24].

The paper is organized as follows. In Section 2, some preliminaries on Fliess operators and graded connected Hopf algebras are given to set the notation and provide some background. The subsequent section is devoted to describing the combinatorial algebras that are naturally induced by the interconnection of Fliess operators. In Section 4, the Hopf algebra of coordinate functions for the output feedback Hopf algebra is described in detail. The final section addresses elements of the discrete-time version of this theory.

2 Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \dots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X , $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X . The *length* of the word η , denoted $|\eta|$, is given by the number of letters it contains. The set of all words with length k is denoted by X^k . The set of all words including the empty word, \emptyset , is designated by X^* , while $X^+ := X^* - \{\emptyset\}$. The set X^* forms a monoid under catenation. The set ηX^* is comprised of all words with the prefix $\eta \in X^*$. For any fixed integer $\ell \geq 1$, a mapping $c : X^* \rightarrow \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as $(c, \eta) \in \mathbb{R}^\ell$ and called the *coefficient* of the word η in c . Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta) \eta$. If the *constant term* $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c , $\text{supp}(c)$, is the set of all words having nonzero coefficients in c . The *order* of c , $\text{ord}(c)$, is the length of the shortest word in its support ($\text{ord}(0) := \infty$).¹ The \mathbb{R} -vector space of all formal power series over X^* with coefficients in \mathbb{R}^ℓ is denoted by $\mathbb{R}^\ell \langle\langle X \rangle\rangle$.² It forms a unital associative \mathbb{R} -

¹ For notational convenience, $p = (p, \emptyset) \emptyset \in \mathbb{R} \langle X \rangle$ is often abbreviated as $p = (p, \emptyset)$.

² The superscript ℓ will be dropped when $\ell = 1$.

algebra under the catenation product. Specifically, for $c, d \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ the catenation product is given by $cd = \sum_{\eta \in X^*} (cd, \eta) \eta$, where

$$(cd, \eta) = \sum_{\eta = \xi \nu} (c, \xi)(d, \nu), \quad \forall \eta \in X^*,$$

and the product on \mathbb{R}^ℓ is defined componentwise. The unit in this case is $\mathbf{1} := 1\emptyset$. $\mathbb{R}^\ell \langle\langle X \rangle\rangle$ is also a unital, commutative and associative \mathbb{R} -algebra under the shuffle product, denoted here by the shuffle symbol \sqcup . The shuffle product of two words is defined inductively by

$$(x_i \eta) \sqcup (x_j \xi) = x_i (\eta \sqcup (x_j \xi)) + x_j ((x_i \eta) \sqcup \xi) \quad (1)$$

with $\eta \sqcup \emptyset = \emptyset \sqcup \eta = \eta$ given any words $\eta, \xi \in X^*$ and letters $x_i, x_j \in X$ [24, 56, 57]. For instance, $x_i \sqcup x_j = x_i x_j + x_j x_i$ and

$$x_{i_1} x_{i_2} \sqcup x_{i_3} x_{i_4} = x_{i_1} x_{i_2} x_{i_3} x_{i_4} + x_{i_3} x_{i_4} x_{i_1} x_{i_2} + x_{i_1} x_{i_3} (x_{i_2} \sqcup x_{i_4}) + x_{i_3} x_{i_1} (x_{i_2} \sqcup x_{i_4}).$$

The definition of the shuffle product is extended linearly to any two series $c, d \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ by letting

$$c \sqcup d = \sum_{\eta, \xi \in X^*} (c, \eta)(d, \xi) \eta \sqcup \xi, \quad (2)$$

where again the product on \mathbb{R}^ℓ is defined componentwise. For a fixed word $\nu \in X^*$, the coefficient

$$(\eta \sqcup \xi, \nu) = 0 \text{ if } |\eta| + |\xi| \neq |\nu|.$$

Hence, the infinite sum in (2) is always well defined since the family of polynomials $\{\eta \sqcup \xi\}_{\eta, \xi \in X^*}$ is locally finite. The unit for this product is $\mathbf{1}$.

Some standard concepts regarding rational formal power series, which are used in Section 5, are provided next [3]. A series $c \in \mathbb{R} \langle\langle X \rangle\rangle$ is called *invertible* if there exists a series $c^{-1} \in \mathbb{R} \langle\langle X \rangle\rangle$ such that $cc^{-1} = c^{-1}c = \mathbf{1}$. In the event that c is not proper, it is always possible to write

$$c = (c, \emptyset)(\mathbf{1} - c'),$$

where (c, \emptyset) is nonzero, and $c' \in \mathbb{R} \langle\langle X \rangle\rangle$ is proper. It then follows that

$$c^{-1} = \frac{1}{(c, \emptyset)} (\mathbf{1} - c')^{-1} = \frac{1}{(c, \emptyset)} (c')^*, \quad (3)$$

where

$$(c')^* := \sum_{i=0}^{\infty} (c')^i.$$

In fact, c is invertible if and *only if* c is not proper. Now let S be a subalgebra of the \mathbb{R} -algebra $\mathbb{R} \langle\langle X \rangle\rangle$ with the catenation product. S is said to be *rationally closed* when every invertible $c \in S$ has $c^{-1} \in S$ (or equivalently, every proper $c' \in S$ has

$(c')^* \in S$). The *rational closure* of any subset $E \subset \mathbb{R}\langle\langle X \rangle\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle X \rangle\rangle$ containing E .

Definition 1. [3] A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is *rational* if it belongs to the rational closure of $\mathbb{R}\langle X \rangle$. The subset of all rational series in $\mathbb{R}\langle\langle X \rangle\rangle$ is denoted by $\mathbb{R}_{rat}\langle\langle X \rangle\rangle$.

From Definition 1 it is clear that every series in $\mathbb{R}_{rat}\langle\langle X \rangle\rangle$ is generated by a finite number of rational operations (scalar multiplication, addition, catenation, and inversion) applied to a finite set of polynomials over X . In the case where the alphabet X has an infinite number of letters, such a series can only involve a finite subset of X . Therefore, it is rational in exactly the sense described above when restricted to this sub-alphabet. This will be the notion of rationality employed in this manuscript whenever X is infinite. But the reader is cautioned that other notions of rationality for infinite alphabets appear in the literature, see, for example, [54].

It turns out that an entirely different characterization of a rational series is possible using a monoid structure on the set of $n \times n$ matrices over \mathbb{R} , denoted $\mathbb{R}^{n \times n}$, where the product is conventional matrix multiplication and the unit is the $n \times n$ identity matrix I .

Definition 2. [3] A *linear representation* of a series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is any triple (μ, γ, λ) , where

$$\mu : X^* \rightarrow \mathbb{R}^{n \times n}$$

is a monoid morphism, and $\gamma, \lambda^T \in \mathbb{R}^{n \times 1}$ are such that

$$(c, \eta) = \lambda \mu(\eta) \gamma, \quad \forall \eta \in X^*. \quad (4)$$

The integer n is the dimension of the representation.

Definition 3. [3] A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is called *recognizable* if it has a linear representation.

Theorem 1. [59] *A formal power series is rational if and only if it is recognizable.*

A third characterization of rationality is given by the notion of *stability*. Define for any letter $x_i \in X$ and word $\eta = x_j \eta' \in X^*$ the left-shift operator

$$x_i^{-1}(\eta) = \delta_{ij} \eta', \quad (5)$$

where δ_{ij} is the standard Kronecker delta. Higher order shifts are defined inductively via $(x_i \xi)^{-1}(\cdot) = \xi^{-1} x_i^{-1}(\cdot)$, where $\xi \in X^*$. The left-shift operator is assumed to act linearly on $\mathbb{R}\langle\langle X \rangle\rangle$.

Definition 4. [3] A subset $V \subset \mathbb{R}\langle\langle X \rangle\rangle$ is called *stable* when $\xi^{-1}(c) \in V$ for all $c \in V$ and $\xi \in X^*$.

Theorem 2. [3] *A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is rational if and only if there exists a stable finite dimensional \mathbb{R} -vector subspace of $\mathbb{R}\langle\langle X \rangle\rangle$ containing c .*

2.1 Chen–Fliess series and Fliess operators

One can associate with any formal power series $c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle$ a functional series, F_c , known as a *Chen–Fliess series*. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \rightarrow \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$ -norm for a measurable real-valued function, u_i , defined on $[t_0, t_1]$. Let $L_{\mathfrak{p}}^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B_{\mathfrak{p}}^m(R)[t_0, t_1] := \{u \in L_{\mathfrak{p}}^m[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define inductively for each word $\eta \in X^*$ the map $E_{\eta} : L_1^m[t_0, t_1] \rightarrow C[t_0, t_1]$ by setting $E_{\emptyset}[u] = 1$ and letting

$$\begin{aligned} E_{x_{i_1} \bar{\eta}}[u](t, t_0) &:= \int_{t_0}^t u_{i_1}(\tau_1) E_{\bar{\eta}}[u](\tau_1, t_0) d\tau_1 \\ &= \int_{\Delta_{[t_0, t]}^n} u_{i_1}(\tau_1) u_{i_2}(\tau_2) \cdots u_{i_n}(\tau_n) d\tau_n \cdots d\tau_2 d\tau_1, \end{aligned}$$

where $\Delta_{[t_0, t]}^n := \{(\tau_1, \dots, \tau_n), t \geq \tau_1 \geq \dots \geq \tau_n \geq t_0\}$, $\eta = x_{i_1} \cdots x_{i_n} = x_{i_1} \bar{\eta} \in X^*$, and $u_0 := 1$. For instance, the words x_i and $x_{i_1} x_{i_2}$ correspond to the integrals

$$E_{x_i}[u](t, t_0) = \int_{t_0}^t u_i(\tau) d\tau, \quad E_{x_{i_1} x_{i_2}}[u](t, t_0) = \int_{t_0}^t u_{i_1}(\tau_1) \int_{t_0}^{\tau_1} u_{i_2}(\tau_2) d\tau_2 d\tau_1.$$

The *Chen–Fliess series* corresponding to $c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle$ is defined to be

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_{\eta}[u](t, t_0). \quad (6)$$

In the event that there exist real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \forall \eta \in X^*, \quad (7)$$

then F_c constitutes a well defined causal operator from $B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$ into $B_q^\ell(S)[t_0, t_0 + T]$ for some $S > 0$ provided $\bar{R} := \max\{R, T\} < 1/M_c(m+1)$, and the numbers $\mathfrak{p}, q \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/q = 1$ [40]. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) In this case, F_c is called a *Fliess operator* and said to be *locally convergent* (LC). The set of all series satisfying (7) is denoted by $\mathbb{R}_{LC}^\ell \langle \langle X \rangle \rangle$. When c satisfies the more stringent growth condition

$$|(c, \eta)| \leq K_c M_c^{|\eta|}, \quad \forall \eta \in X^*, \quad (8)$$

the series (6) defines a Fliess operator from the extended space $L_{\mathfrak{p}, e}^m(t_0)$ into $C[t_0, \infty)$, where

$$L_{\mathfrak{p}, e}^m(t_0) := \{u : [t_0, \infty) \rightarrow \mathbb{R}^m : u|_{[t_0, t_1]} \in L_{\mathfrak{p}}^m[t_0, t_1], \quad \forall t_1 \in (t_0, \infty)\},$$

and $u|_{[t_0, t_1]}$ denotes the restriction of u to the interval $[t_0, t_1]$ [40]. In this case, the operator is said to be *globally convergent* (GC), and the set of all series satisfying (8) is designated by $\mathbb{R}_{GC}^\ell \langle\langle X \rangle\rangle$.

Most of the work regarding nonlinear input-output systems in control theory prior to the work of Fliess was based on Volterra series, see, for example, [4, 48, 58]. In many ways this earlier work set the stage for the introduction of the noncommutative algebraic framework championed by Fliess. As Fliess operators are series of weighted iterated integrals of control functions, they are also related to the work of K. T. Chen, who revealed that iterated integrals come with a natural algebraic structure [9, 11, 10]. Indeed, products of iterated integrals can again be written as linear combinations of iterated integrals. This is implied by the classical integration by parts rule for indefinite Riemann integrals, which yields for instance that

$$E_{x_{i_1}}[u](t, t_0)E_{x_{i_2}}[u](t, t_0) = E_{x_{i_1}x_{i_2}}[u](t, t_0) + E_{x_{i_2}x_{i_1}}[u](t, t_0).$$

Linearity allows one to relate this to the shuffle product (1)

$$E_{x_{i_1}}[u](t, t_0)E_{x_{i_2}}[u](t, t_0) = F_{x_{i_1}x_{i_2}+x_{i_2}x_{i_1}}[u](t) = F_{x_{i_1} \sqcup x_{i_2}}[u](t).$$

This generalizes naturally to the shuffle product for iterated integrals with respects to words $\eta, \nu \in X^*$

$$E_\eta[u](t, t_0)E_\nu[u](t, t_0) = F_{\eta \sqcup \nu}[u](t), \quad (9)$$

which in turn implies for Fliess operators corresponding to $c, d \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ that

$$F_c[u](t)F_d[u](t) = F_{c \sqcup d}[u](t). \quad (10)$$

A Fliess operator F_c defined on $B_{\mathbb{p}}^m(\mathcal{R})[t_0, t_0 + T]$ is said to be *realizable* when there exists a state space model consisting of n ordinary differential equations and ℓ output functions

$$\dot{z}(t) = g_0(z(t)) + \sum_{i=1}^m g_i(z(t))u_i(t), \quad z(t_0) = z_0 \quad (11a)$$

$$y_j(t) = h_j(z(t)), \quad j = 1, 2, \dots, \ell, \quad (11b)$$

where each g_i is an analytic vector field expressed in local coordinates on some neighborhood \mathcal{W} of z_0 , and each output function h_j is an analytic function on \mathcal{W} such that (11a) has a well defined solution $z(t)$, $t \in [t_0, t_0 + T]$ for any given input $u \in B_{\mathbb{p}}^m(\mathcal{R})[t_0, t_0 + T]$, and $y_j(t) = F_{c_j}[u](t) = h_j(z(t))$, $t \in [t_0, t_0 + T]$, $j = 1, 2, \dots, \ell$. It can be shown that for any word $\eta = x_{i_k} \cdots x_{i_1} \in X^*$

$$(c_j, \eta) = L_{g_\eta} h_j(z_0) := L_{g_{i_1}} \cdots L_{g_{i_k}} h_j(z_0), \quad (12)$$

where $L_{g_i} h_j$ is the *Lie derivative* of h_j with respect to g_i . For any $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$, the \mathbb{R} -linear mapping $\mathcal{H}_c : \mathbb{R}\langle X \rangle \rightarrow \mathbb{R}^\ell \langle\langle X \rangle\rangle$ uniquely specified by $(\mathcal{H}_c(\eta), \xi) =$

$(c, \xi \eta)$, $\xi, \eta \in X^*$ is called the *Hankel mapping* of c . The series c is said to have finite *Lie rank* $\rho_L(c)$ when the range of \mathcal{H}_c restricted to the \mathbb{R} -vector space of Lie polynomials over X , i.e., the free Lie algebra $\mathcal{L}(X) \subset \mathbb{R}\langle X \rangle$, has dimension $\rho_L(c)$. It is well known that F_c is realizable if and only if $c \in \mathbb{R}_{LC}^\ell \langle \langle X \rangle \rangle$ has finite Lie rank [24, 25, 43, 44, 45, 46, 62, 63]. In which case, all minimal realization have dimension $\rho_L(c)$ and are unique up to a diffeomorphism. In the event that \mathcal{H}_c has finite rank on the entire vector space $\mathbb{R}\langle X \rangle$, usually referred to as the *Hankel rank* $\rho_H(c)$ of c , and $c \in \mathbb{R}_{GC}^\ell \langle \langle X \rangle \rangle$ then F_c has a minimal bilinear state space realization

$$\begin{aligned} \dot{z}(t) &= A_0 z(t) + \sum_{i=1}^m A_i z(t) u_i(t), \quad z(t_0) = z_0 \\ y_j(t) &= C_j z(t), \quad j = 1, 2, \dots, \ell \end{aligned}$$

of dimension $\rho_H(c) \geq \rho_L(c)$, where A_j and C_j are real matrices of appropriate dimensions [24, 25, 43]. Here the state $z(t)$ is well defined on any interval $[t_0, t_0 + T]$, $T > 0$, when $u \in L_{1,e}^m(t_0)$, and the operator F_c always converges globally. In addition, (12) simplifies to

$$(c_j, x_{i_k} \cdots x_{i_1}) = C_j A_{i_k} \cdots A_{i_1} z_0. \quad (13)$$

In light of (4), it is not hard to see that c is recognizable in SISO case if and only if F_c has a bilinear realization with $A_i = \mu(x_i)$ for $i = 0, 1$ and $z_0 = \gamma$, and $C = \lambda$.

2.2 Graded connected Hopf algebras

All algebraic structures here are considered over the base field \mathbb{K} of characteristic zero, for instance, \mathbb{C} or \mathbb{R} . Multiplication in \mathbb{K} is denoted by $m_{\mathbb{K}} : \mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K}$.

2.2.1 Algebra

A \mathbb{K} -algebra is denoted by the triple (A, m_A, η_A) where A is a \mathbb{K} -vector space carrying an associative product $m_A : A \otimes A \rightarrow A$, i.e., $m_A \circ (m_A \otimes \text{id}_A) = m_A \circ (\text{id}_A \otimes m_A) : A \otimes A \otimes A \rightarrow A$, and a unit map $\eta_A : \mathbb{K} \rightarrow A$. The algebra unit corresponding to the latter is denoted by 1_A . A \mathbb{K} -subalgebra of the \mathbb{K} -algebra A is a \mathbb{K} -vector subspace $B \subseteq A$ such that $m_A(b \otimes b') \in B$ for all $b, b' \in B$. A \mathbb{K} -subalgebra $I \subseteq A$ is called a (*right-*) *left-ideal* if for any elements $i \in I$ and $a \in A$ the product $(m_A(i \otimes a)) m_A(a \otimes i)$ is in I . An *ideal* $I \subseteq A$ is both a left- and right-ideal.

In order to motivate the concept of a \mathbb{K} -coalgebra, the definition of a \mathbb{K} -algebra A is rephrased in terms of commutative diagrams. Associativity of the \mathbb{K} -vector space morphism $m_A : A \otimes A \rightarrow A$ translates into commutativity of the diagram

$$\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m_A \otimes \text{id}_A} & A \otimes A \\
\text{id}_A \otimes m_A \downarrow & & \downarrow m_A \\
A \otimes A & \xrightarrow{m_A} & A
\end{array} \quad (14)$$

The \mathbb{K} -algebra A is unital if the \mathbb{K} -vector space map $\eta_A : \mathbb{K} \rightarrow A$ satisfies the commutative diagram

$$\begin{array}{ccccc}
\mathbb{K} \otimes A & \xrightarrow{\eta_A \otimes \text{id}_A} & A \otimes A & \xleftarrow{\text{id}_A \otimes \eta_A} & A \otimes \mathbb{K} \\
& \searrow \alpha_l & \downarrow m_A & \swarrow \alpha_r & \\
& & A & &
\end{array} \quad (15)$$

Here α_l and α_r are the isomorphisms sending $k \otimes a$ respectively $a \otimes k$ to ka for $k \in \mathbb{K}$, $a \in A$. Let $\tau := \tau_{A,A} : A \otimes A \rightarrow A \otimes A$ be the flip map, $\tau_{A,A}(x \otimes y) := y \otimes x$. The \mathbb{K} -algebra A is *commutative* if the next diagram commutes

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\tau} & A \otimes A \\
m_A \downarrow & \swarrow m_A & \\
A & &
\end{array} \quad (16)$$

An important observation is that nonassociative \mathbb{K} -algebras play a key role in the context of Hopf algebras, in particular, for those of combinatorial type. Recall that the Lie algebra $\mathcal{L}(A)$ associated with a \mathbb{K} -algebra A follows from antisymmetrization of the algebra product m_A . Algebras that give Lie algebras in this way are called Lie admissible. Another class of Lie admissible algebras are pre-Lie \mathbb{K} -algebras. A *left pre-Lie algebra* [5, 8, 51] is a vector space V equipped with a bilinear product $\triangleright : V \otimes V \rightarrow V$ such that the *(left) pre-Lie identity*,

$$a \triangleright (b \triangleright c) - (a \triangleright b) \triangleright c = b \triangleright (a \triangleright c) - (b \triangleright a) \triangleright c, \quad (17)$$

holds for $a, b, c \in V$. This identity rewrites as $L_{[a,b]} = [L_a, L_b]$, where $L_a : V \rightarrow V$ is defined by $L_a b := a \triangleright b$. The bracket on the left-hand side is defined by $[a, b] := a \triangleright b - b \triangleright a$ and satisfies the Jacobi identity. Right pre-Lie algebras are defined analogously. Note that the (left) pre-Lie identity (17) can be understood as a relation between associators, i.e., let $\alpha_{\triangleright} : V \otimes V \otimes V \rightarrow V$ be defined by

$$\alpha_{\triangleright}(a, b, c) := a \triangleright (b \triangleright c) - (a \triangleright b) \triangleright c,$$

then (17) simply says that $\alpha_{\triangleright}(a, b, c) = \alpha_{\triangleright}(b, a, c)$. From this it is easy to see that any associative algebra is pre-Lie.

Example 1. [5] Let A be a commutative \mathbb{K} -algebra endowed with commuting derivatives $D := \{\partial_1, \dots, \partial_n\}$. For $a \in A$ define $a\partial_i : A \rightarrow A$ by $(a\partial_i)(b) := a\partial_i b$ and $V(n) := \{\sum_{i=1}^n a_i \partial_i \mid a_i \in A, \partial_i \in D\}$. The algebra $(V(n), \triangleleft)$, where $\sum_{i=1}^n a_i \partial_i \triangleleft \sum_{j=1}^n a_j \partial_j := \sum_{j,i=1}^n a_j (\partial_j a_i) \partial_i$, is a right pre-Lie algebra called the pre-Lie Witt algebra.

Example 2. [5, 8, 51] The next example of a pre-Lie algebra is of a geometric nature and similar to the one above. Let M be a differentiable manifold endowed with a flat and torsion-free connection. The corresponding covariant derivation operator ∇ on the space $\chi(M)$ of vector fields on M provides it with a left pre-Lie algebra structure, which is defined via $a \triangleright b := \nabla_a b$ by virtue of the two equalities $\nabla_a b - \nabla_b a = [a, b]$, $\nabla_{[a, b]} = [\nabla_a, \nabla_b]$. They express the vanishing of torsion and curvature respectively. Let $M = \mathbb{R}^n$ with its standard flat connection. For $a = \sum_{i=1}^n a_i \partial_i$ and $b = \sum_{i=1}^n b_i \partial_i$ it follows that

$$a \triangleright b = \sum_{i=1}^n \left(\sum_{j=1}^n a_j (\partial_j b_i) \right) \partial_i.$$

Example 3. [5] Denote by W the space of all words over the alphabet $\{a, b\}$. Define for words v and $w = w_1 \cdots w_n$ in W the product $w \circ v := \sum_{i=0}^n \varepsilon(i) w \triangleleft_i v$, where $w \triangleleft_i v$ denotes inserting the word v between letters w_i and w_{i+1} of w , i.e., $w \triangleleft_i v = w_1 \cdots w_i v w_{i+1} \cdots w_n$ and

$$\varepsilon(i) := \begin{cases} -1, & w_i = a, w_{i+1} = b \\ +1, & w_i = b, w_{i+1} = a \text{ or } \emptyset \\ +1, & w_i = \emptyset, w_{i+1} = a \\ 0, & \text{otherwise.} \end{cases}$$

The algebra (W, \circ) is right pre-Lie. For example,

$$a \circ a = aa, \quad a \circ ab = aba, \quad ab \circ a = aab - aab + aba = aba, \quad ba \circ ab = baba.$$

2.2.2 Coalgebra

The definition of a \mathbb{K} -coalgebra is most easily obtained by reversing the arrows in diagrams (14) and (15). Thus, a \mathbb{K} -coalgebra is a triple $(C, \Delta_C, \varepsilon_C)$, where C is a \mathbb{K} -vector space carrying a *coassociative coproduct* map $\Delta_C : C \rightarrow C \otimes C$, i.e., $(\Delta_C \otimes \text{id}_C) \circ \Delta_C = (\text{id}_C \otimes \Delta_C) \circ \Delta_C : C \rightarrow C \otimes C \otimes C$, and $\varepsilon_C : C \rightarrow \mathbb{K}$ is the counit map which satisfies $(\varepsilon_C \otimes \text{id}_C) \circ \Delta_C = \text{id}_C = (\text{id}_C \otimes \varepsilon_C) \circ \Delta_C$. Its kernel $\ker(\varepsilon_C) \subset C$ is called the *augmentation ideal*.

A simple example of a coalgebra is the field \mathbb{K} itself with the coproduct $\Delta_{\mathbb{K}} : \mathbb{K} \rightarrow \mathbb{K} \otimes \mathbb{K}$, $c \mapsto c \otimes 1$ and $\varepsilon_{\mathbb{K}} := \text{id}_{\mathbb{K}} : \mathbb{K} \rightarrow \mathbb{K}$.

Using Sweedler's notation for the coproduct of an element $x \in C$, $\Delta_C(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$, provides a simple description of coassociativity

$$\sum_{(x)} \left(\sum_{(x^{(1)})} x^{(1)(1)} \otimes x^{(1)(2)} \right) \otimes x^{(2)} = \sum_{(x)} x^{(1)} \otimes \left(\sum_{(x^{(2)})} x^{(2)(1)} \otimes x^{(2)(2)} \right).$$

It permits the use of a transparent notation for iterated coproducts: $\Delta_C^{(n)} : C \rightarrow C^{\otimes n+1}$, where $\Delta_C^{(0)} := \text{id}$, $\Delta_C^{(1)} := \Delta_C$, and

$$\Delta_C^{(n)} := (\text{id} \otimes \Delta_C^{(n-1)}) \circ \Delta_C = (\Delta_C^{(n-1)} \otimes \text{id}_C) \circ \Delta_C.$$

For example,

$$\Delta_C^{(2)}(x) := (\text{id}_C \otimes \Delta_C) \circ \Delta_C(x) = (\Delta_C \otimes \text{id}_C) \circ \Delta_C(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}.$$

A *cocommutative* coalgebra satisfies $\tau \circ \Delta_C = \Delta_C$, which amounts to reversing the arrows in diagram (16). An element x in a coalgebra $(C, \Delta_C, \epsilon_C)$ is called *primitive* if $\Delta_C(x) = x \otimes 1_C + 1_C \otimes x$. It is called *group-like* if $\Delta_C(x) = x \otimes x$. The set of primitive elements in C is denoted by $P(C)$. A subspace $I \subseteq C$ of a \mathbb{K} -coalgebra $(C, \Delta_C, \epsilon_C)$ is a *subcoalgebra* if $\Delta_C(I) \subseteq I \otimes I$. A subspace $I \subseteq C$ is called a (left-, right-) *coideal* if $(\Delta_C(I) \subseteq I \otimes C, \Delta_C(I) \subseteq C \otimes I)$ $\Delta_C(I) \subseteq I \otimes C + C \otimes I$.

2.2.3 Bialgebra

A \mathbb{K} -*bialgebra* consists of a \mathbb{K} -algebra and a \mathbb{K} -coalgebra which are compatible [1, 21, 50, 55, 64]. More precisely, a \mathbb{K} -bialgebra is a quintuple $(B, m_B, \eta_B, \Delta_B, \epsilon_B)$, where (B, m_B, η_B) is a \mathbb{K} -algebra, and $(B, \Delta_B, \epsilon_B)$ is a \mathbb{K} -coalgebra, such that m_B and η_B are morphisms of \mathbb{K} -coalgebras with the natural coalgebra structure on the space $B \otimes B$. Commutativity of the following diagrams encodes the compatibilities

$$\begin{array}{ccc} B \otimes B & \xrightarrow{m_B} & B \\ \tau_2(\Delta_B \otimes \Delta_B) \downarrow & & \downarrow \Delta_B \\ B \otimes B \otimes B \otimes B & \xrightarrow{m_B \otimes m_B} & B \otimes B \end{array} \quad \begin{array}{ccc} B \otimes B & \xrightarrow{\epsilon_B \otimes \epsilon_B} & \mathbb{K} \otimes \mathbb{K} \\ m_B \downarrow & & \downarrow m_{\mathbb{K}} \\ B & \xrightarrow{\epsilon_B} & \mathbb{K} \end{array} \quad (18)$$

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta_B} & B \\ \Delta_{\mathbb{K}} \downarrow & & \downarrow \Delta_B \\ \mathbb{K} \otimes \mathbb{K} & \xrightarrow{\eta_B \otimes \eta_B} & B \otimes B \end{array} \quad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta_B} & B \\ \text{id}_{\mathbb{K}} \searrow & & \swarrow \epsilon_B \\ & \mathbb{K} & \end{array} \quad (19)$$

where $\tau_2 := (\text{id}_B \otimes \tau \otimes \text{id}_B)$. Equivalently, Δ_B and ϵ_B are morphisms of \mathbb{K} -algebras, with the natural algebra structure on the space $B \otimes B$. By a slight abuse of notation one writes $\Delta_B(m_B(b \otimes b')) = \Delta_B(b)\Delta_B(b')$ for $b, b' \in B$, saying that the *coproduct of the product is the product of the coproduct*. The identity element in B will be denoted by $\mathbf{1}_B$, and all algebra morphisms are required to be unital. Note that if x_1, x_2 are primitive in B , then $[x_1, x_2] := m_B(x_1 \otimes x_2) - m_B(x_2 \otimes x_1)$ is primitive as well, i.e., the set $P(B)$ of primitive elements of a bialgebra B is a Lie subalgebra of the Lie algebra $\mathcal{L}(B)$.

A bialgebra B is called *graded* if there are \mathbb{K} -vector spaces $B_n, n \geq 0$, such that

1. $B = \bigoplus_{n \geq 0} B_n$,
2. $m_B(B_n \otimes B_m) \subseteq B_{n+m}$,

$$3. \Delta_B(B_n) \subseteq \bigoplus_{p+q=n} B_p \otimes B_q.$$

Elements $x \in B_n$ are given a degree $\deg(x) = n$. For a *connected* graded bialgebra B , the degree zero part is $B_0 = \mathbb{K}\mathbf{1}_B$. Note that $\mathbf{1}_B$ is group-like. A graded bialgebra $B = \bigoplus_{n \geq 0} B_n$ is said to be of *finite type* if its homogeneous components B_n are \mathbb{K} -vector spaces of finite dimension.

Let B be a connected graded \mathbb{K} -bialgebra. One can show [50] that the coproduct of any element $x \in B$ is given by

$$\Delta_B(x) = x \otimes \mathbf{1}_B + \mathbf{1}_B \otimes x + \sum_{(x)}' x' \otimes x'',$$

where

$$\Delta'(x) := \sum_{(x)}' x' \otimes x'' \in \bigoplus_{\substack{p+q=n \\ p>0, q>0}} B_p \otimes B_q$$

is the *reduced coproduct*, which is coassociative on the augmentation ideal $\ker(\varepsilon_B) := B^+ := \bigoplus_{n>0} B_n$. Elements in the kernel of Δ'_B are primitive elements of B .

Example 4. Divided powers $(D, m_D, \eta_D, \Delta_D, \varepsilon_D)$ are a graded bialgebra, where $D = \bigoplus_{n=0}^{\infty} D_n$, $D_n := \mathbb{K}d_n$. The product is given by $m_D(d_m \otimes d_n) = \binom{m+n}{m} d_{m+n}$, and the unital map is $\eta_D : \mathbb{K} \rightarrow D$, $1_{\mathbb{K}} \mapsto d_0 := \mathbf{1}_D$. The coproduct $\Delta_D : D \rightarrow D \otimes D$ maps $d_n \mapsto \sum_{k=0}^n d_k \otimes d_{n-k}$, and $\varepsilon_D : D \rightarrow \mathbb{K}$, $d_n \mapsto \delta_{0,n} 1_{\mathbb{K}}$, where $\delta_{0,n}$ is the usual Kronecker delta.

For a \mathbb{K} -algebra A and a \mathbb{K} -coalgebra C , the *convolution product* of two linear maps $f, g \in L(C, A) := \text{Hom}_{\mathbb{K}}(C, A)$ is defined to be the linear map $f \star g \in L(C, A)$ given for $a \in C$ by

$$(f \star g)(a) := m_A \circ (f \otimes g) \circ \Delta_C(a) = \sum_{(a)} f(a^{(1)}) g(a^{(2)}). \quad (20)$$

In other words

$$C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m_A} A.$$

It is easy to see that associativity of A and coassociativity of C imply the following.

Theorem 3. [1, 21, 50, 55, 64] *$L(C, A)$ with the convolution product (20) is an unital associative \mathbb{K} -algebra with unit $\eta := \eta_A \circ \varepsilon_C$.*

The algebra A can be replaced by the base field \mathbb{K} . For a bialgebra B the theorem describes the convolution algebra structure on $L(B, B)$ with unit $\eta := \eta_B \circ \varepsilon_B$.

For the maps $f_i \in L(C, A)$, $i = 1, \dots, n$, $n > 1$, multiple convolution products are defined by

$$f_1 \star f_2 \star \dots \star f_n := m_A \circ (f_1 \otimes f_2 \otimes \dots \otimes f_n) \circ \Delta_C^{(n-1)}. \quad (21)$$

Recall that $\Delta_C^{(0)} := \text{id}_C$, and for $n > 0$, $\Delta_C^{(n)} := (\Delta_C^{(n-1)} \otimes \text{id}_C) \circ \Delta_C$.

2.2.4 Hopf algebra

Definition 5. [1, 55, 64] A *Hopf algebra* is a \mathbb{K} -bialgebra $(H, m_H, \eta_H, \Delta_H, \varepsilon_H, S)$ together with a particular \mathbb{K} -linear map $S : H \rightarrow H$ called the *antipode*, which satisfies the Hopf algebra axioms [1, 55, 64]. The algebra unit in H is denoted by $\mathbf{1}_H$.

The antipode map has the property of being an antihomomorphism for both the algebra and the coalgebra structures, i.e., $S(m_H(x \otimes y)) = m_H(S(y) \otimes S(x))$ and $\Delta_H \circ S = (S \otimes S) \circ \tau \circ \Delta_H$. The necessarily unique antipode $S \in L(H, H)$ is the inverse of the identity map $\text{id}_H : H \rightarrow H$ with respect to the convolution product

$$S \star \text{id}_H = m_H \circ (S \otimes \text{id}_H) \circ \Delta_H = \eta_H \circ \varepsilon_H = m_H \circ (\text{id}_H \otimes S) \circ \Delta_H = \text{id}_H \star S. \quad (22)$$

If the Hopf algebra H is commutative or cocommutative, then $S \circ S = \text{id}_H$.

Recall that the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ of a Lie algebra \mathcal{L} has the structure of a Hopf algebra [57] and provides a natural example. An important observation is contained in the next result.

Proposition 1. [21] Any connected graded bialgebra $H = \bigoplus_{n \geq 0} H_n$ is a connected graded Hopf algebra. The antipode is defined by the geometric series $S := \text{id}_H^{\star(-1)} = (\eta_H \circ \varepsilon_H - (\eta_H \circ \varepsilon_H - \text{id}_H))^{\star(-1)}$.

See [21] for a proof and more details. Hence, for any $x \in H_n$ the antipode is computed by

$$S(x) = \sum_{k \geq 0} (\eta_H \circ \varepsilon_H - \text{id}_H)^{\star k}(x). \quad (23)$$

It is well-defined as the sum on the right-hand side terminates at $\deg(x) = n$ due to the fact that the projector $P := \text{id}_H - \eta_H \circ \varepsilon_H$ maps H to its augmentation ideal $\ker(\varepsilon_H)$. Note that the antipode preserves the grading, i.e., $S(H_n) \subseteq H_n$.

Corollary 1. [21] The antipode S for a connected graded Hopf algebra $H = \bigoplus_{n \geq 0} H_n$ may be defined recursively in terms of either of the two formulae

$$S(x) = -S \star P(x) = -x - \sum_{(x)}' S(x')x'', \quad (24a)$$

$$S(x) = -P \star S(x) = -x - \sum_{(x)}' x'S(x''), \quad (24b)$$

for $x \in \ker(\varepsilon_H) = \bigoplus_{n > 0} H_n$, which follow readily from (22) and $S(\mathbf{1}_H) = \mathbf{1}_H$.

These recursions make sense due to the fact that on the righthand side the antipode is calculated on elements x' or x'' , which are of strictly smaller degree than the element x . Let A be a \mathbb{K} -algebra and H a Hopf algebra. An element $\phi \in L(H, A)$ is called a *character* if ϕ is a unital algebra morphism, that is, $\phi(\mathbf{1}_H) = \mathbf{1}_A$ and

$$\phi(m_H(x \otimes y)) = m_A(\phi(x) \otimes \phi(y)). \quad (25)$$

An *infinitesimal character* with values in A is a linear map $\xi \in L(H, A)$ such that for $x, y \in \ker(\varepsilon_H)$, $\xi(m_H(x \otimes y)) = 0$, which implies $\xi(\mathbf{1}_H) = 0$. An equivalent way to characterize infinitesimal characters is as *derivations*, i.e.,

$$\xi(m_H(x \otimes y)) = m_A(\eta_A \circ \varepsilon_H(x) \otimes \xi(y)) + m_A(\xi(x) \otimes \eta_A \circ \varepsilon_H(y)) \quad (26)$$

for any $x, y \in H$. The set of characters (respectively infinitesimal characters) is denoted by $G_A \subset L(H, A)$ (respectively $g_A \subset L(H, A)$).

Let A be a commutative \mathbb{K} -algebra. The linear space of infinitesimal characters, g_A , forms a Lie algebra with respect to the Lie bracket defined on $L(H, A)$ in terms of the convolution product

$$[\alpha, \beta] := \alpha \star \beta - \beta \star \alpha.$$

Moreover, A -valued characters, G_A , form a group. The inverse is given by composition with the antipode S of H . Since $\alpha(\mathbf{1}_H) = 0$ for $\alpha \in g_A$, the exponential defined by its power series with respect to convolution, $\exp^*(\alpha)(x) := \sum_{j \geq 0} \frac{1}{j!} \alpha^{\star j}(x)$, is a finite sum terminating at $j = n$ for any $x \in H_n$.

Proposition 2. [21, 50] \exp^* restricts to a bijection from g_A onto G_A .

The compositional inverse of \exp^* is given by the logarithm defined with respect to the convolution product, $\log^*(\eta_A \circ \varepsilon_H + \gamma)(x) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \gamma^{\star k}(x)$, where $\gamma \in g_A$. Again the sum terminates at $k = n$ for any $x \in H_n$ as $\gamma(\mathbf{1}_H) = 0$. For details and proofs of these well-known facts the reader is referred to [21, 50].

Remark 1. An important result which is due to Milnor and Moore [52] concerns the structure of cocommutative connected graded Hopf algebras of finite type. It states that any such Hopf algebra H is isomorphic to the universal enveloping algebra of its primitive elements, i.e., $H \cong \mathcal{U}(P(H))$. See also [22].

Remark 2. An observation concerning the relationship between the group $G_A \subset L(H, A)$ and the Hopf algebra H will be important in the analysis which follows. The reader is referred to the paper of Manchon and Frabetti [27] for details and additional references. By definition, elements in G_A map all of H into the commutative unital algebra A . However, an element $x \in H$ can also be seen as an A -valued function on G_A . Indeed, let $\Phi \in G_A$, then $x(\Phi) := \Phi(x) \in A$ and the usual pointwise product of functions $(xy)(\Phi) = x(\Phi)y(\Phi)$ follows from (25) since $\Phi \in G_A$. The definition of the convolution product (20) in terms of the coproduct of H implies a natural coproduct on functions $x \in H$, that is, $\Delta(x)(\Phi, \Psi) := (\Phi \star \Psi)(x) \in A$. Similarly, the inverse of G_A as well as its unit correspond naturally to the antipode and counit map on H , respectively. This *reversed* perspective on the relationship between H and its group of characters G_A allows one to interpret H as the (Hopf) algebra of *coordinate functions* of the group G_A . More precisely, H contains the *representative functions* over G_A . The reader is directed to Cartier's work [7] for a comprehensive review of this topic. In the context of input-output systems in nonlinear control theory, a particular group of unital Fliess operators is central. Its product, unit and inverse are used to identify its Hopf algebra of coordinate functions.

Example 5. Three examples of Hopf algebra are presented: the unshuffle, shuffle [3] and quasi-shuffle Hopf algebras [41].

1. Let $X := \{x_1, x_2, x_3, \dots, x_m\}$ be an alphabet with m letters. As before, X^* is the set of words with letters in X . The length of a word $\eta = x_{i_1} \cdots x_{i_n}$ in X^* is defined by the number of letters it contains, i.e., $|\eta| := n$. The empty word $\mathbf{1} \in X^*$ has length zero. The vector space $\mathbb{K}\langle X \rangle$, which is freely generated by X^* and graded by length, becomes a noncommutative, unital, connected, graded algebra by concatenating words, i.e., for $\eta = x_{i_1} \cdots x_{i_n}$ and $\eta' = x_{j_1} \cdots x_{j_l}$, $\eta \cdot \eta' := x_{i_1} \cdots x_{i_n} x_{j_1} \cdots x_{j_l}$, and $|\eta \cdot \eta'| = n + l$. The *unshuffle coproduct* is defined by declaring the elements in X to be primitive, i.e., $\Delta^{\sqcup}(x_i) := x_i \otimes \mathbf{1} + \mathbf{1} \otimes x_i$ for all $x_i \in X$, and by extending it multiplicatively. In this case, $\mathbb{K}\langle X \rangle$ is turned into the cocommutative *unshuffle Hopf algebra* H_{conc} . For instance, for the letters $x_{i_1}, x_{i_2} \in X$ the coproduct of the length two word $\eta = x_{i_1} x_{i_2}$ is

$$\Delta^{\sqcup}(\eta) = \Delta^{\sqcup}(x_{i_1})\Delta^{\sqcup}(x_{i_2}) = x_{i_1}x_{i_2} \otimes \mathbf{1} + \mathbf{1} \otimes x_{i_1}x_{i_2} + x_{i_1} \otimes x_{i_2} + x_{i_2} \otimes x_{i_1}.$$

The general form of Δ^{\sqcup} for an arbitrary word $\eta = x_{i_1} \cdots x_{i_l} \in X^*$ is given by

$$\Delta^{\sqcup}(\eta) = \prod_{j=1}^l \Delta^{\sqcup}(x_{i_j}) = \sum_{\alpha, \beta \in X^*} \langle \alpha \sqcup \beta, \eta \rangle \alpha \otimes \beta. \quad (27)$$

The coefficient in the sum over words $\alpha, \beta \in X^*$ on the right-hand side is defined through the linearly extended bracket $\langle \alpha, \nu \rangle := 1$ if $\alpha = \nu$, and zero otherwise. The product \sqcup displayed in (27) is the shuffle product of words (1) introduced above. The antipode of H_{conc} turns out to be

$$S(x_{i_1} \cdots x_{i_l}) = (-1)^l x_{i_l} \cdots x_{i_1}. \quad (28)$$

It is interesting to check that (28) satisfies the recursions (24).

2. The same space $\mathbb{K}\langle X \rangle$ can be turned into a unital, connected, graded, commutative, noncocommutative Hopf algebra, H_{\sqcup} , known as *shuffle Hopf algebra*, by defining its algebra structure in terms of the shuffle product on words (1), and its coproduct by *deconcatenation*. The latter is defined on words $\eta = x_{i_1} \cdots x_{i_l} \in X^*$ as follows

$$\Delta(\eta) = \eta \otimes \mathbf{1} + \mathbf{1} \otimes \eta + \sum_{k=1}^{l-1} x_{i_1} \cdots x_{i_k} \otimes x_{i_{k+1}} \cdots x_{i_l}. \quad (29)$$

It is easy to show – and left to the reader, – that this gives a coassociative coproduct which is compatible with the shuffle product. The antipode of H_{\sqcup} is the same as that of H_{conc} , i.e., $S(x_{i_1} \cdots x_{i_l}) := (-1)^l x_{i_l} \cdots x_{i_1}$. Again, it is interesting to verify that it satisfies both recursions (24).

3. The last example of a connected graded Hopf algebra is defined on the countable alphabet A . Here A^* denotes the monoid of words $w = a_{i_1} \cdots a_{i_l}$ generated by the letters from A with concatenation as product. The degree of an element $a_i \in A$

is defined to be $\deg(a_i)$. It is extended to words additively, i.e., $\deg(a_{i_1} \cdots a_{i_l}) = \deg(a_{i_1}) + \cdots + \deg(a_{i_l})$. The empty word $\mathbf{1} \in A^*$ is of degree zero. Moreover, it is assumed that A itself is a graded commutative semigroup with bilinear product $[- -]: A \times A \rightarrow A$. The degree $\deg([a_i a_j]) := \deg(a_i) + \deg(a_j)$. Commutativity and associativity of $[- -]$ allow for a notational simplification, i.e., $[a_{i_1} \cdots a_{i_n}] := [a_{i_1} [\cdots [a_{i_{n-1}} a_{i_n}] \cdots]]$. The free noncommutative algebra of words $w = a_{i_1} \cdots a_{i_l}$ over the alphabet A is denoted $\mathbb{K}\langle A \rangle$. The commutative and associative quasi-shuffle product on words $w, v \in \mathbb{K}\langle A \rangle$ is defined by

- i) $\mathbf{1} \star v := v \star \mathbf{1} := v$,
- ii) $a_i v \star a_j w := a_i(v \star a_j w) + a_j(a_i v \star w) + [a_i a_j](v \star w)$,

where a_i, a_j are letters in A . For instance,

$$\begin{aligned} a_{i_1} \star a_{i_2} &= a_{i_1} a_{i_2} + a_{i_2} a_{i_1} + [a_{i_1} a_{i_2}] \\ a_{i_1} \star a_{i_2} a_{i_3} &= a_{i_1} a_{i_2} a_{i_3} + a_{i_2} a_{i_1} a_{i_3} + a_{i_2} a_{i_3} a_{i_1} + [a_{i_1} a_{i_2}] a_{i_3} + a_{i_2} [a_{i_1} a_{i_3}]. \end{aligned}$$

Hoffman [41] showed that the quasi-shuffle algebra H_\star is a Hopf algebra with respect to the deconcatenation coproduct (29). The antipode $S: H_\star \rightarrow H_\star$ is deduced from the recursion (24a), e.g.,

$$\begin{aligned} S(a_{i_1} \cdots a_{i_n}) &= -(S \star P)(a_{i_1} \cdots a_{i_n}) \\ &= (-\text{id}_{H_\star} - m_\star \circ (S \otimes \text{id}_{H_\star}) \circ \Delta')(a_{i_1} \cdots a_{i_n}) \end{aligned} \quad (30)$$

$$= -a_{i_1} \cdots a_{i_n} - \sum_{l=1}^{n-1} S(a_{i_1} \cdots a_{i_l}) \star a_{i_{l+1}} \cdots a_{i_n}, \quad (31)$$

where the projector $P := \text{id}_{H_\star} - \eta_{H_\star} \circ \varepsilon_{H_\star}$ maps H_\star to its augmentation ideal $\ker(\varepsilon_{H_\star})$. For example, the antipode for the letter a_i and the word $a_i a_j$ are respectively

$$S(a_i) = -a_i, \quad S(a_i a_j) = -a_i a_j + a_i \star a_j = a_j a_i + [a_i a_j].$$

If $[a_i a_j] = 0$ for any letters $a_i, a_j \in A$, then the quasi-shuffle product reduces to the ordinary shuffle product (1) on words, and H_\star reduces to H_{\sqcup} .

In Section 4 another example of a connected graded Hopf algebra is presented, one which plays a key role in the context of the output feedback interconnection. In Section 5 the quasi-shuffle product is employed in the context of products of discrete-time Fliess operators.

3 Algebras Induced by the Interconnection of Fliess Operators

In engineering applications, where input-output systems are represented in terms of Fliess operators, it is natural to interconnect systems to create models of more complex systems. It is known that all the basic interconnection types, such as the

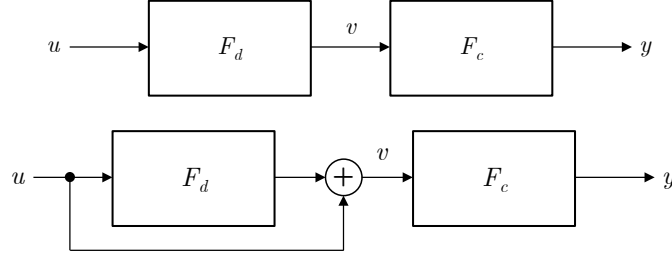


Fig. 1 Fliess operator cascades yielding the composition product (top) and modified composition product (bottom)

parallel, cascade and feedback connections, are well-posed. For example, under suitable assumptions, the output of a given Fliess operator generates an admissible input for another Fliess operator in the cascade connection. In each case it is also known that the aggregate system has a Fliess operator representation. Therefore, a family of formal power series products is naturally induced whereby the generating series of an aggregate system can be computed in terms of the generating series of its subsystems. Of particular interest here are the cascade and feedback interconnections, as their respective products define a semi-group and group that are central in control theory. In this section, these algebraic structures are described in detail.

3.1 Cascade interconnections

Consider the cascade interconnections of two Fliess operators shown in Figure 1. The first is a simple cascade interconnection corresponding to a direct composition of the operators F_c and F_d , namely, $F_c \circ F_d$. The other involves a *direct feed term* passing from the input u to the input v so that $F_c \circ (I + F_d)$, where I denotes the identity operator. This direct feed term is a common feature found in some control systems and is particularly important in feedback systems as will be discussed shortly. The primary claim is that each cascade interconnection induces a locally finite product on the level of formal power series, which unambiguously describes the interconnected system as generating series of Fliess operators are known to be unique [25, 66]. Specifically, the *composition product* satisfies $F_c \circ F_d = F_{c \circ d}$, and the *modified composition product* satisfies $F_c \circ (I + F_d) = F_{c \circ d}$. Each product is defined in terms of a certain algebra homomorphism. The morphism for the modified composition product ultimately defines a pre-Lie product, which is at the root of all the underlying combinatorial structures at play.

For a fixed $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$ and alphabet $X = \{x_0, x_1, \dots, x_m\}$, let ψ_d be the continuous (in the ultrametric sense) algebra homomorphism mapping $\mathbb{R} \langle\langle X \rangle\rangle$ to the set of vector space endomorphisms $\text{End}(\mathbb{R} \langle\langle X \rangle\rangle)$ uniquely specified by $\psi_d(x_i \eta) = \psi_d(x_i) \circ \psi_d(\eta)$ for $x_i \in X$, $\eta \in X^*$ with

$$\psi_d(x_i)(e) = x_0(d_i \sqcup e),$$

$i = 0, 1, \dots, m$ and any $e \in \mathbb{R}\langle\langle X \rangle\rangle$, and where d_i is the i -th component series of $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ ($d_0 := \emptyset$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$. The following theorem describes the generating series for the direct cascade connection of two Fliess operators.

Theorem 4. [19, 20, 38, 65] *Given any $c \in \mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}_{LC}^m\langle\langle X \rangle\rangle$, the composition $F_c \circ F_d = F_{c \circ d}$, where the **composition product** of c and d is given by*

$$c \circ d = \sum_{\eta \in X^*} (c, \eta) \psi_d(\eta)(\mathbf{1}),$$

and $c \circ d \in \mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$.

It is not difficult to show that this composition product is locally finite (and therefore summable), associative and \mathbb{R} -linear in its left argument. As this product lacks an identity, it defines a semi-group on $\mathbb{R}^m\langle\langle X \rangle\rangle$. In addition, this product distributes to the left over the shuffle product, which reflects the fact that in general

$$F_{(c \sqcup d) \circ e} = (F_c F_d) \circ F_e = F_c[F_e]F_d[F_e] = F_{(c \circ e) \sqcup (d \circ e)}.$$

Finally, it is known that the composition product defines an ultrametric contraction on $\mathbb{R}^m\langle\langle X \rangle\rangle$.

Example 6. In the case of two linear time-invariant systems with analytic kernels $h_c(t) = \sum_{i \geq 0} (c, x_0^i x_1) t^i / i!$ and $h_d(t) = \sum_{i \geq 0} (d, x_0^i x_1) t^i / i!$, respectively, a direct calculation gives the kernel for the composition

$$\begin{aligned} (h_c * h_d)(t) &:= \int_0^t h_c(t - \tau) h_d(\tau) d\tau = \sum_{k=1}^{\infty} \left[\sum_{j=0}^{k-1} (c, x_0^{k-j-1} x_1) (d, x_0^j x_1) \right] \frac{t^k}{k!} \\ &= \sum_{k=1}^{\infty} (c \circ d, x_0^k x_1) \frac{t^k}{k!} =: h_{c \circ d}(t). \end{aligned}$$

The introduction of a direct feed term in the composition interconnection requires a modification to the set up, namely, the new algebra homomorphism ϕ_d from $\mathbb{R}\langle\langle X \rangle\rangle$ to $\text{End}(\mathbb{R}\langle\langle X \rangle\rangle)$ where $\phi_d(x_i \eta) = \phi_d(x_i) \circ \phi_d(\eta)$ for $x_i \in X$, $\eta \in X^*$ with

$$\phi_d(x_i)(e) = x_i e + x_0(d_i \sqcup e),$$

$i = 0, 1, \dots, m$ and any $e \in \mathbb{R}\langle\langle X \rangle\rangle$, and where $d_0 := 0$. Again, $\phi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$. The direct feed term is encoded in the new term $x_i e$ shown above. This yields the desired formal power series product as described next.

Theorem 5. [38, 49] *Given any $c \in \mathbb{R}_{LC}^\ell\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}_{LC}^m\langle\langle X \rangle\rangle$, the composition $F_c \circ (I + F_d) = F_{c \circ d}$, where the **modified composition product** of c and d is given by*

Table 1 Composition products involving $c, d, c_{\delta} = \delta + c, d_{\delta} = \delta + d$ when $X = \{x_0, x_1, \dots, x_m\}$

Name	Symbol	Map	Operator Identity	Remarks
composition	$c \circ d$	$\mathbb{R}^{\ell}\langle\langle X \rangle\rangle \times \mathbb{R}^m\langle\langle X \rangle\rangle \rightarrow \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$	$F_c \circ F_d = F_{c \circ d}$	associative
modified composition	$c \tilde{\circ} d$	$\mathbb{R}^{\ell}\langle\langle X \rangle\rangle \times \mathbb{R}^m\langle\langle X \rangle\rangle \rightarrow \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$	$F_c \circ (I + F_d)$	nonassociative
mixed composition	$c \circ d_{\delta}$	$\mathbb{R}^{\ell}\langle\langle X \rangle\rangle \times \mathbb{R}^m\langle\langle X_{\delta} \rangle\rangle \rightarrow \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$	$F_c \circ F_{d_{\delta}} = F_{c \circ d_{\delta}}$	$c \circ d_{\delta} = c \tilde{\circ} d$
group composition	$c \odot d$	$\mathbb{R}^m\langle\langle X \rangle\rangle \times \mathbb{R}^m\langle\langle X \rangle\rangle \rightarrow \mathbb{R}^m\langle\langle X \rangle\rangle$	$(I + F_c) \circ (I + F_d) = I + F_{c \odot d}$	$c \odot d = d + c \tilde{\circ} d$
group product	$c_{\delta} \circ d_{\delta}$	$\mathbb{R}^m\langle\langle X_{\delta} \rangle\rangle \times \mathbb{R}^m\langle\langle X_{\delta} \rangle\rangle \rightarrow \mathbb{R}^m\langle\langle X_{\delta} \rangle\rangle$	$F_{c_{\delta}} \circ F_{d_{\delta}} = F_{c_{\delta} \circ d_{\delta}}$	$c_{\delta} \circ d_{\delta} = \delta + c \odot d$

$$c \tilde{\circ} d = \sum_{\eta \in X^*} (c, \eta) \phi_d(\eta)(\mathbf{1}),$$

and $c \tilde{\circ} d \in \mathbb{R}_{LC}^{\ell}\langle\langle X \rangle\rangle$.

This product is also always summable, but it is not associative, in fact, in general

$$(c \tilde{\circ} d) \tilde{\circ} e = c \tilde{\circ} (d \tilde{\circ} e + e) \quad (32)$$

[49]. The following lemma describes some other elementary properties of the modified composition product.

Lemma 1. [32, 38] *The modified composition product*

- (1) *is left \mathbb{R} -linear;*
- (2) *satisfies $c \tilde{\circ} \mathbf{0} = c$;*
- (3) *satisfies $c \tilde{\circ} d = k \in \mathbb{R}^{\ell}$ for any fixed d if and only if $c = k$;*
- (4) *satisfies $(x_0 c) \tilde{\circ} d = x_0 (c \tilde{\circ} d)$ and $(x_i c) \tilde{\circ} d = x_i (c \tilde{\circ} d) + x_0 (d_i \sqcup (c \tilde{\circ} d))$;*
- (5) *distributes to the left over the shuffle product.*

It is also known that the modified composition product is an ultrametric contraction on $\mathbb{R}^m\langle\langle X \rangle\rangle$. A summary description of the composition and modified composition product is given in Table 1 along with other types of composition products which will be presented shortly.

3.2 Output feedback

A central object of study in control theory is the output feedback interconnection as shown in Figure 2. As with the cascade systems discussed above, this class of interconnections is also closed in the sense that the mapping $u \mapsto y$ always has a Fliess operator representation. The computation of the corresponding generating series, however, is much more involved. To see the source of the difficulty, consider the following calculation assuming $c, d \in \mathbb{R}_{LC}^m\langle\langle X \rangle\rangle$ (for the most general case, which requires two alphabets, see [32]). Clearly the function v in Figure 2 must satisfy the identity

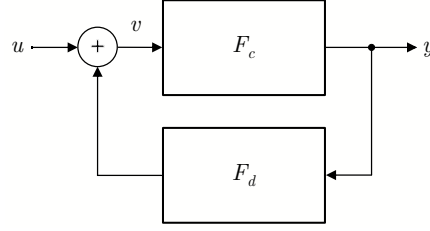


Fig. 2 Output feedback connection

$$v = u + F_{d \circ c}[v].$$

Therefore, from $(I + F_{-d \circ c})[v] = u$ one deduces that

$$v = (I + F_{(-d \circ c)})^{-1} [u] =: (I + F_{(-d \circ c)^{-1}}) [u],$$

where I is the identity operator, and the superscript “ -1 ” denotes the composition inverse in both the operator sense and in terms of formal power series. Thus, the generating series for the closed-loop system, denoted by the output feedback product $c@d$, is

$$F_{c@d}[u] = F_c[v] = F_{c \circ (-d \circ c)^{-1}} [u]. \quad (33)$$

The crux of the problem is how to compute the generating series $(-d \circ c)^{-1}$ of the inverse operator $(I + F_{(-d \circ c)})^{-1}$. The approach described next is based on identifying an underlying combinatorial Hopf algebra whose antipode acts on a certain character group in such a way as to explicitly produce this inverse generating series. This requires that one first identifies the relevant group structures.

Consider the set of *unital Fliess operators* $\mathcal{F}_\delta := \{I + F_c : c \in \mathbb{R}_{LC}^m \langle \langle X \rangle \rangle\}$. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for \mathcal{F}_δ will be denoted by $\mathbb{R}_{LC}^m \langle \langle X_\delta \rangle \rangle$. The central idea is that $(\mathcal{F}_\delta, \circ, I)$ forms a group under operator composition

$$\begin{aligned} F_{c_\delta} \circ F_{d_\delta} &= (I + F_c) \circ (I + F_d) = I + F_d + F_c \circ (I + F_d) \\ &= I + F_d + F_{c \circ d} =: F_{c_\delta \circ d_\delta}, \end{aligned}$$

where

$$c_\delta \circ d_\delta := \delta + d + c \circ d =: \delta + c \odot d.$$

(The series $c \odot d$ is clearly locally convergent since all the operations employed in its definition preserve local convergence.) This group will be referred to as the *output feedback group* of unital Fliess operators. It is natural to think of this group as acting on an arbitrary Fliess operator, say F_c , to produce another Fliess operator, as is evident in (33), by defining the right action $F_c \circ F_{d_\delta} = F_{c \odot d_\delta}$ with $c \odot d_\delta := c \circ d$. This product will be referred to as the *mixed composition product* on

$\mathbb{R}^\ell \langle \langle X \rangle \rangle \times \mathbb{R}^m \langle \langle X_\delta \rangle \rangle$.³ The following lemma gives the basic properties of the mixed composition product.

Lemma 2. [29] *The mixed composition product*

- (1) *is left \mathbb{R} -linear;*
- (2) *satisfies $(c \circ d_\delta) \circ e_\delta = c \circ (d_\delta \circ e_\delta)$ (mixed associativity);*
- (3) *is associative.*

Proof.

- (1) This claim follows from the left linearity of the modified composition product.
- (2) In light of the first item it is sufficient to prove the claim only for $c = \eta \in X^k$, $k \geq 0$. The cases $k = 0$ and $k = 1$ are trivial. Assume the claim holds up to some fixed $k \geq 0$. Then via item (4) in Lemma 1 and the induction hypothesis it follows that

$$\begin{aligned} ((x_0\eta) \circ d_\delta) \circ e_\delta &= (x_0(\eta \circ d_\delta)) \circ e_\delta = x_0((\eta \circ d_\delta) \circ e_\delta) = x_0(\eta \circ (d_\delta \circ e_\delta)) \\ &= (x_0\eta) \circ (d_\delta \circ e_\delta). \end{aligned}$$

In a similar fashion, apply the properties (1), (4), and (5) in Lemma 1 to get

$$\begin{aligned} ((x_i\eta) \circ d_\delta) \circ e_\delta &= [x_i(\eta \circ d_\delta) + x_0(d_i \sqcup (\eta \circ d_\delta))] \circ e_\delta \\ &= [x_i(\eta \circ d_\delta)] \circ e_\delta + [x_0(d_i \sqcup (\eta \circ d_\delta))] \circ e_\delta \\ &= x_i[(\eta \circ d_\delta) \circ e_\delta] + x_0[e_i \sqcup ((\eta \circ d_\delta) \circ e_\delta)] + \\ &\quad x_0[(d_i \circ e_\delta) \sqcup ((\eta \circ d_\delta) \circ e_\delta)]. \end{aligned}$$

Now employ the induction hypothesis so that

$$\begin{aligned} ((x_i\eta) \circ d_\delta) \circ e_\delta &= x_i[\eta \circ (d_\delta \circ e_\delta)] + x_0[e_i \sqcup (\eta \circ (d_\delta \circ e_\delta))] + \\ &\quad x_0[(d_i \circ e_\delta) \sqcup (\eta \circ (d_\delta \circ e_\delta))] \\ &= x_i[\eta \circ (d_\delta \circ e_\delta)] + x_0[(e_i + (d_i \circ e_\delta)) \sqcup (\eta \circ (d_\delta \circ e_\delta))] \\ &= x_i[\eta \circ (d_\delta \circ e_\delta)] + x_0[(d_\delta \circ e_\delta)_i \sqcup (\eta \circ (d_\delta \circ e_\delta))] \\ &= (x_i\eta) \circ (d_\delta \circ e_\delta). \end{aligned}$$

Therefore, the claim holds for all $\eta \in X^*$, and the identity is proved.

(3) Applying item (5) in Lemma 1 and the previous result it follows

$$\begin{aligned} (c_\delta \circ d_\delta) \circ e_\delta &= \delta + e + (c \odot d) \circ e_\delta \\ &= \delta + e + (d + c \circ d_\delta) \circ e_\delta \\ &= \delta + e + d \circ e_\delta + (c \circ d_\delta) \circ e_\delta \\ &= \delta + d \odot e + c \circ (d_\delta \circ e_\delta) \\ &= (c_\delta \circ (d_\delta \circ e_\delta)). \end{aligned}$$

³ The same symbol will be used for composition on $\mathbb{R}^m \langle \langle X \rangle \rangle$, $\mathbb{R}^m \langle \langle X_\delta \rangle \rangle$, and $\mathbb{R}^m \langle \langle X \rangle \rangle \times \mathbb{R}^m \langle \langle X_\delta \rangle \rangle$. It will always be clear which product is being used since the arguments of these products have a distinct notation, namely, c versus c_δ .

Given the uniqueness of generating series of Fliess operators, their set of generating series forms the group $(\mathbb{R}_{LC}^m \langle\langle X_\delta \rangle\rangle, \circ, \delta)$. In fact, it is a subgroup of the group described next since it can be shown that the composition inverse preserves local convergence [32].

Theorem 6. [32] *The triple $(\mathbb{R}^m \langle\langle X_\delta \rangle\rangle, \circ, \delta)$ is a group.*

Proof. By design, δ is the identity element of the group. Associativity of the group product was established above. For a fixed $c_\delta \in \mathbb{R}^m \langle\langle X_\delta \rangle\rangle$, the composition inverse, $c_\delta^{-1} = \delta + c^{-1}$, must satisfy $c_\delta \circ c_\delta^{-1} = \delta$ and $c_\delta^{-1} \circ c_\delta = \delta$, which reduce, respectively, to

$$c^{-1} = (-c) \tilde{\circ} c^{-1} \quad (34a)$$

$$c = (-c^{-1}) \tilde{\circ} c. \quad (34b)$$

It was shown in [38] that $e \mapsto (-c) \tilde{\circ} e$ is always a contraction on $\mathbb{R}^m \langle\langle X \rangle\rangle$ when viewed as an (complete) ultrametric space and thus has a unique fixed point, c^{-1} . So it follows directly that c_δ^{-1} is a right inverse of c_δ , i.e., satisfies (34a). To see that this same series is also a left inverse, first observe that (34a) is equivalent to

$$c^{-1} \tilde{\circ} 0 + c \tilde{\circ} c^{-1} = 0, \quad (35)$$

using Lemma 1, items (1) and (2). Substituting (35) back into itself where zero appears and applying (32) gives

$$\begin{aligned} c^{-1} \tilde{\circ} (c \tilde{\circ} c^{-1} + c^{-1}) + c \tilde{\circ} c^{-1} &= 0 \\ (c^{-1} \tilde{\circ} c) \tilde{\circ} c^{-1} + c \tilde{\circ} c^{-1} &= 0. \end{aligned}$$

Again from left linearity of the modified composition product it follows that

$$(c^{-1} \tilde{\circ} c + c) \tilde{\circ} c^{-1} = 0.$$

Finally, Lemma 1, item (3) implies that $c^{-1} \tilde{\circ} c + c = 0$, which is equivalent to (34b). This concludes the proof.

In light of the identities $c \circ \delta = c$ and Lemma 2, item (2), it is therefore established that $\mathbb{R}^m \langle\langle X_\delta \rangle\rangle$ acts as a right transformation group on $\mathbb{R}^m \langle\langle X \rangle\rangle$. In which case, the feedback product $c@d = c \tilde{\circ} (-d \circ c)^{-1} = c \circ (-d \circ c)_\delta^{-1}$ can be viewed as specific example of such a right action.

4 The Hopf Algebra of Coordinate Functions

In this section the Hopf algebra of coordinate functions for the group $\mathbb{R}^m \langle\langle X_\delta \rangle\rangle$ is described. This provides an explicit computational framework for computing group inverses, and thus, to calculate the feedback product as described in the previous

section. The strategy is to first introduce a connected graded commutative algebra and then a compatible noncocommutative coalgebra, resulting in a connected graded commutative noncocommutative bialgebra. The connectedness property ensures then that this bialgebra is a connected graded Hopf algebra. The section ends by providing a purely inductive formula for the antipode of this Hopf algebra.

4.1 Multivariable Hopf algebra of output feedback

Let $X = \{x_0, x_1, x_2, \dots, x_m\}$ be a finite alphabet with $m + 1$ letters. As usual the monoid of words is denoted by X^* and includes the empty word $e = \emptyset$. The degree of a word $\eta = x_{i_1} \cdots x_{i_n} \in X^*$ of length $|\eta| := n$, where $x_{i_i} \in X$, is defined by

$$\|\eta\| := 2|\eta|_0 + |\eta|_1. \quad (36)$$

Here $|\eta|_0$ denotes the number of times the letter $x_0 \in X$ appears in η , and $|\eta|_1$ is the number letters $x_{j \neq 0} \in X$ appearing in the word η . Note that $|e| = 0 = \|e\|$.

Recall Remark 2 above. For any word $\eta \in X^*$ and $i = 1, \dots, m$, the *coordinate function* a_η^i is defined to be the element of the dual space $\mathbb{R}^* \langle \langle X_\delta \rangle \rangle$ giving the coefficient of the i -th component series for the word $\eta \in X^*$, namely,

$$a_\eta^i(c) := (c_i, \eta).$$

In this context, a_δ^i denotes the coordinate function with respect to δ , where $a_\delta^i(\delta) = 1$ and zero otherwise. Consider the vector space V generated by the coordinate functions a_η^i , where $\eta \in X^*$ and $1 \leq i \leq m$. It is turned into a polynomial algebra H with unit denoted by $\mathbf{1}$. By defining the degree of elements in H as $\deg(\mathbf{1}) := 0$ and for $k > 0$, $\eta \in X^*$

$$\deg(a_\eta^k) := 1 + \|\eta\|, \quad (37)$$

$\deg(a_\eta^k a_\kappa^l) := \deg(a_\eta^k) + \deg(a_\kappa^l)$, H becomes a graded connected algebra, $H := \bigoplus_{n \geq 0} H_n$. Note, in particular, that $\deg(a_\epsilon^k) := 1$.

The left- and right-shift maps, $\theta_j : H \rightarrow H$ respectively $\tilde{\theta}_j : H \rightarrow H$ are defined by

$$\theta_j a_\eta^k := a_{x_j \eta}^k, \quad \tilde{\theta}_j a_\eta^k := a_{\eta x_j}^k$$

for $x_j \in X$, and $\theta_j \mathbf{1} = \tilde{\theta}_j \mathbf{1} = 0$. On products these maps act by definition as derivations

$$\theta_j a_\eta^k a_\mu^l := (\theta_j a_\eta^k) a_\mu^l + a_\eta^k (\theta_j a_\mu^l),$$

and analogously for $\tilde{\theta}_j$. For a word $\eta = x_{i_1} \cdots x_{i_n} \in X^*$

$$\theta_\eta := \theta_{i_1} \circ \cdots \circ \theta_{i_n}, \quad \tilde{\theta}_\eta := \tilde{\theta}_{i_n} \circ \cdots \circ \tilde{\theta}_{i_1}.$$

Hence, any element a_η^i , $\eta \in X^*$ can be written

$$a_\eta^i = \theta_\eta a_e^i = \tilde{\theta}_\eta a_e^i.$$

Both maps can be used to define a particular coproduct $\Delta : H \rightarrow H \otimes H$. In this approach, the right-shift map is considered first. Later it will be shown that the left-shift map gives rise to the same coproduct. The coordinate function with respect to the empty word, a_e^l , $1 \leq l \leq m$, is defined to be primitive

$$\Delta a_e^l := a_e^l \otimes \mathbf{1} + \mathbf{1} \otimes a_e^l. \quad (38)$$

The next step is to define Δ inductively on any a_η^i , $|\eta| > 0$, by specifying intertwining relations between the map $\tilde{\theta}_\eta$ and the coproduct

$$\Delta \circ \tilde{\theta}_i := \left(\tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i + \delta_{0i} \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)} \right) \circ \Delta, \quad (39)$$

where δ_{0i} is the usual Kronecker delta. The map $A_e^{(j)}$ for $0 < j \leq m$ is defined by

$$A_e^{(j)} a_\eta^i := a_\eta^i a_e^j. \quad (40)$$

The following notation is used, $\Delta \circ \tilde{\theta}_i = \tilde{\Theta}_i \circ \Delta$, where

$$\tilde{\Theta}_i := \tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i + \delta_{0i} \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)}, \quad (41)$$

and $\tilde{\Theta}_\eta := \tilde{\Theta}_{i_n} \circ \dots \circ \tilde{\Theta}_{i_1}$ for $\eta = x_{i_1} \dots x_{i_n} \in X^*$. The functions $a_{x_j}^l$ for $0 < l, j \leq m$ are primitive since

$$\begin{aligned} \Delta a_{x_j}^l &= (\tilde{\theta}_j \otimes \text{id} + \text{id} \otimes \tilde{\theta}_j) \circ \Delta a_e^l = (\tilde{\theta}_j \otimes \text{id} + \text{id} \otimes \tilde{\theta}_j)(a_e^l \otimes \mathbf{1} + \mathbf{1} \otimes a_e^l) \\ &= a_{x_j}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_j}^l, \end{aligned}$$

which follows from $\tilde{\theta}_j \mathbf{1} = 0$. However, for $a_{x_0}^l$ the coproduct is

$$\begin{aligned} \Delta a_{x_0}^l &= \tilde{\Theta}_0 \circ \Delta a_e^l = \left(\tilde{\theta}_0 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_0 + \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)} \right) \circ \Delta a_e^l \\ &= a_{x_0}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_0}^l + \sum_{j=1}^m a_{x_j}^l \otimes a_e^j. \end{aligned} \quad (42)$$

Observe that the coproduct is compatible with the grading. Indeed, $\deg(a_{x_0}^l) = 1 + 2$ and $\deg(a_{x_j}^l \otimes a_e^j) = \deg(a_{x_j}^l) + \deg(a_e^j) = 1 + 1 + 1$ for any $j > 0$. For the element $a_{x_i x_j}^l$, $i, j > 0$, one finds the following coproduct

$$\begin{aligned} \Delta a_{x_i x_j}^l &= \tilde{\Theta}_j \circ \tilde{\Theta}_i \circ \Delta a_e^l = (\tilde{\theta}_j \otimes \text{id} + \text{id} \otimes \tilde{\theta}_j) \circ (\tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i) \circ \Delta a_e^l \\ &= (\tilde{\theta}_j \tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_j \tilde{\theta}_i + \tilde{\theta}_j \otimes \tilde{\theta}_i + \tilde{\theta}_i \otimes \tilde{\theta}_j) \circ \Delta a_e^l \end{aligned}$$

$$= a_{x_i x_j}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_i x_j}^l. \quad (43)$$

Again, this follows from $\tilde{\theta}_k \mathbf{1} = 0$ and generalizes to any word η in the monoid \tilde{X}^* made from the reduced alphabet $\tilde{X} := X - \{x_0\} = \{x_1, \dots, x_m\}$. For the coproduct of $a_{x_i x_0}^l$, $i > 0$, it follows that

$$\begin{aligned} \Delta a_{x_i x_0}^l &= \tilde{\Theta}_0 \circ \tilde{\Theta}_i \circ \Delta a_e^l \\ &= \left(\tilde{\theta}_0 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_0 + \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)} \right) \circ \left(\tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i \right) \circ \Delta a_e^l \\ &= \left(\tilde{\theta}_0 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_0 + \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)} \right) (a_{x_i}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_i}^l) \\ &= a_{x_i x_0}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_i x_0}^l + \sum_{j=1}^m a_{x_i x_j}^l \otimes a_e^j. \end{aligned} \quad (44)$$

This should be compared with the coproduct of $a_{x_0 x_i}^l$, $i > 0$

$$\begin{aligned} \Delta a_{x_0 x_i}^l &= \tilde{\Theta}_i \circ \tilde{\Theta}_0 \circ \Delta a_e^l \\ &= \left(\tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i \right) \circ \left(\tilde{\theta}_0 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_0 + \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)} \right) \circ \Delta a_e^l \\ &= \left(\tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i \right) \left(a_{x_0}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_0}^l + \sum_{j=1}^m a_{x_j}^l \otimes a_e^j \right) \\ &= a_{x_0 x_i}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_0 x_i}^l + \sum_{j=1}^m a_{x_j x_i}^l \otimes a_e^j + \sum_{j=1}^m a_{x_j}^l \otimes a_{x_i}^j. \end{aligned} \quad (45)$$

Finally, the coproduct of $a_{x_0 x_0}^l$ is calculated

$$\begin{aligned} \Delta a_{x_0 x_0}^l &= \tilde{\Theta}_0 \circ \tilde{\Theta}_0 \circ \Delta a_e^l \\ &= \left(\tilde{\theta}_0 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_0 + \sum_{n=1}^m \tilde{\theta}_n \otimes A_e^{(n)} \right) \circ \left(\tilde{\theta}_0 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_0 + \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)} \right) \circ \Delta a_e^l \\ &= \left(\tilde{\theta}_0 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_0 + \sum_{n=1}^m \tilde{\theta}_n \otimes A_e^{(n)} \right) \left(a_{x_0}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_0}^l + \sum_{j=1}^m a_{x_j}^l \otimes a_e^j \right) \\ &= a_{x_0 x_0}^l \otimes \mathbf{1} + \mathbf{1} \otimes a_{x_0 x_0}^l + \sum_{j=1}^m a_{x_j x_0}^l \otimes a_e^j + \sum_{n=1}^m a_{x_0 x_n}^l \otimes a_e^n \\ &\quad + \sum_{j=1}^m a_{x_j}^l \otimes a_{x_0}^j + \sum_{n,j=1}^m a_{x_j x_n}^l \otimes a_e^j a_e^n. \end{aligned} \quad (46)$$

The coproduct Δ is extended multiplicatively to all of H , and the unit $\mathbf{1}$ is defined to be $\Delta(\mathbf{1}) := \mathbf{1} \otimes \mathbf{1}$. In particular, $\Delta a_\eta^l \in V \otimes H$ as opposed to simply $H \otimes H$ due to the left linearity of the mixed composition product as shown in Lemma 2. This has

interesting practical consequences for the antipode calculation as described in [14]. Namely, there is a significant difference between the computational efficiencies of the two antipode recursions in (24).

Theorem 7. *The algebra H with the multiplicatively extended coproduct*

$$\Delta a_\eta^l := \tilde{\Theta}_\eta(a_e^l \otimes \mathbf{1} + \mathbf{1} \otimes a_e^l) \in V \otimes H \quad (47)$$

is a connected graded commutative noncocommutative Hopf algebra.

Proof. Since H is connected, graded and commutative by construction, the antipode can be calculated recursively via one of the recursions in (24). It is clear as well that the coproduct (47) is noncocommutative. Hence, coassociativity remains to be checked. This is done inductively with respect to the length of the word $\eta \in X^*$. First observe that

$$\Delta(a_{\eta x_i}^k) = \Delta \circ \tilde{\theta}_i(a_\eta^k) = \tilde{\Theta}_i \circ \Delta(a_\eta^k).$$

This implies that

$$\begin{aligned} (\Delta \otimes \text{id}) \circ \Delta(a_{\eta x_i}^k) &= (\Delta \otimes \text{id}) \circ \tilde{\Theta}_i \circ \Delta(a_\eta^k) \\ &= \left(\Delta \circ \tilde{\theta}_i \otimes \text{id} + (\text{id} \otimes \text{id}) \circ \Delta \circ \tilde{\theta}_i + \delta_{0i} \sum_{j=1}^m \Delta \circ \tilde{\theta}_j \otimes A_e^{(j)} \right) \circ \Delta(a_\eta^k) \\ &= \left(\tilde{\Theta}_i \otimes \text{id} + \text{id} \otimes \text{id} \otimes \tilde{\theta}_i + \delta_{0i} \sum_{j=1}^m \tilde{\Theta}_j \otimes A_e^{(j)} \right) (\Delta \otimes \text{id}) \circ \Delta(a_\eta^k) \\ &= \left(\tilde{\theta}_i \otimes \text{id} \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \text{id} \otimes \tilde{\theta}_i \right. \\ &\quad \left. + \delta_{0i} \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)} \otimes \text{id} + \delta_{0i} \sum_{j=1}^m \tilde{\theta}_j \otimes \text{id} \otimes A_e^{(j)} \right. \\ &\quad \left. + \delta_{0i} \sum_{j=1}^m \text{id} \otimes \tilde{\theta}_j \otimes A_e^{(j)} \right) (\text{id} \otimes \Delta) \circ \Delta(a_\eta^k) \\ &= \left(\tilde{\theta}_i \otimes \text{id} \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i + \delta_{0i} \sum_{j=1}^m \tilde{\theta}_j \otimes (A_e^{(j)} \otimes \text{id} + \text{id} \otimes A_e^{(j)}) \right) (\text{id} \otimes \Delta) \circ \Delta(a_\eta^k) \\ &= (\text{id} \otimes \Delta) \circ \left(\tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i + \delta_{0i} \sum_{j=1}^m \tilde{\theta}_j \otimes A_e^{(j)} \right) \circ \Delta(a_\eta^k) \\ &= (\text{id} \otimes \Delta) \circ \Delta(a_{\eta x_i}^k). \end{aligned} \quad (49)$$

The identity

$$\Delta \circ A_e^{(i)} = (A_e^{(i)} \otimes \text{id} + \text{id} \otimes A_e^{(i)}) \circ \Delta$$

was used above, which follows from $A_e^{(l)} a_\eta^k = a_e^l a_\eta^k$ and a_e^l being primitive for all $0 < l \leq m$.

Remark 3. Note that the coproduct (47) can be simplified

$$\Delta a_\eta^l := \tilde{\Theta}_\eta(a_e^l \otimes \mathbf{1}) + \mathbf{1} \otimes a_\eta^l, \quad (50)$$

which follows from $\tilde{\theta}_k \mathbf{1} = 0$ and the form of $\tilde{\Theta}_i$.

A variant of Sweedler's notation is used for the reduced coproduct, i.e., $\Delta'(a_\eta^l) = \sum' a_{\eta'}^l \otimes a_{\eta''}^l$, as well as for the full coproduct

$$\Delta(a_\eta^l) = \sum a_{\eta(1)}^l \otimes a_{\eta(2)}^l = a_\eta^l \otimes \mathbf{1} + \mathbf{1} \otimes a_\eta^l + \Delta'(a_\eta^l).$$

Connectedness of H implies that its antipode $S : H \rightarrow H$ can be calculated using (24), namely,

$$S a_\eta^l = -a_\eta^l - \sum' S(a_{\eta'}^l) a_{\eta''}^l = -a_\eta^l - \sum' a_{\eta'}^l S(a_{\eta''}^l). \quad (51)$$

A few examples are given next. The coproduct (38) implies for the elements a_e^k that $S a_e^k = -a_e^k$. For $0 < j, k, l \leq m$

$$S a_{x_j}^k = -a_{x_j}^k, \quad S a_{x_0}^l = -a_{x_0}^l + \sum_{i=1}^m a_{x_i}^l a_e^i. \quad (52)$$

The next theorem uses the coproduct formula (47) to provide an alternative formula for the antipode of H .

Theorem 8. *For any nonempty word $\eta = x_{i_1} \cdots x_{i_l}$, the antipode $S : H \rightarrow H$ satisfies*

$$S a_\eta^k = (-1)^{|\eta|+1} \tilde{\Theta}'_\eta(a_e^k), \quad (53)$$

where

$$\tilde{\Theta}'_\eta := \tilde{\theta}'_{i_l} \circ \cdots \circ \tilde{\theta}'_{i_1} \quad (54)$$

and

$$\tilde{\theta}'_l := -\tilde{\theta}_l + \delta_{0l} \sum_{j=1}^m a_e^j \tilde{\theta}_j. \quad (55)$$

Proof. The claim is equivalent to saying that

$$S \circ \tilde{\theta}_\eta = -\tilde{\theta}'_{i_l} \circ S \circ \tilde{\theta}_{i_{l-1}} \circ \cdots \circ \tilde{\theta}_{i_1}$$

for the word $\eta = x_{i_1} \cdots x_{i_l} \in X^*$. The proof is via induction on the degree of a_η^k . The degree one case is excluded by assumption as it corresponds to $S a_e^k = -a_e^k$. For degree two, three, four and five it is quickly verified for $i > 0$ that

$$S a_{x_i}^k = S \circ \tilde{\theta}_i a_e^k = -\tilde{\theta}'_i \circ S a_e^k = \tilde{\theta}'_i a_e^k = -\tilde{\theta}_i a_e^k = -a_{x_i}^k$$

and

$$S a_{x_0}^k = S \circ \tilde{\theta}_0 a_e^k = \tilde{\theta}'_0 a_e^k = -a_{x_0}^k + \sum_{i=1}^m a_{x_i}^k a_e^i,$$

which coincide with (52). For $j > 0$

$$\begin{aligned} Sa_{x_j x_0}^k &= S \circ \tilde{\theta}_0 \circ \tilde{\theta}_j a_e^k = (-1) \tilde{\theta}'_0 \circ \tilde{\theta}'_j(a_e^k) = -a_{x_j x_0}^k + \sum_{i=1}^m a_{x_j x_i}^k a_e^i \\ &= \tilde{\theta}'_0 \circ \tilde{\theta}'_j S(a_e^k) \\ &= -\tilde{\theta}'_0 \circ S \circ \tilde{\theta}_j(a_e^k). \end{aligned}$$

For degree five

$$\begin{aligned} Sa_{x_0 x_0}^k &= S \circ \tilde{\theta}_0 \circ \tilde{\theta}_0 a_e^k = -\tilde{\theta}'_0 \circ \tilde{\theta}'_0(a_e^k) \\ &= -\left(-\tilde{\theta}_0 + \sum_{j=1}^m a_e^j \tilde{\theta}_j\right) \left(-\tilde{\theta}_0 + \sum_{n=1}^m a_e^n \tilde{\theta}_n\right)(a_e^k) \\ &= \left(-\tilde{\theta}_0 \tilde{\theta}_0 + \sum_{n=1}^m a_{x_0}^n \tilde{\theta}_n + \sum_{n=1}^m a_e^n \tilde{\theta}_0 \tilde{\theta}_n + \sum_{j=1}^m a_e^j \tilde{\theta}_j \tilde{\theta}_0 - \sum_{j=1}^m \sum_{n=1}^m a_e^j \tilde{\theta}_j a_e^n \tilde{\theta}_n\right)(a_e^k) \\ &= \left(-\tilde{\theta}_0 \tilde{\theta}_0 + \sum_{n=1}^m a_{x_0}^n \tilde{\theta}_n + \sum_{n=1}^m a_e^n \tilde{\theta}_0 \tilde{\theta}_n + \sum_{j=1}^m a_e^j \tilde{\theta}_j \tilde{\theta}_0 \right. \\ &\quad \left. - \sum_{j=1}^m \sum_{n=1}^m a_e^j a_{x_j}^n \tilde{\theta}_n - \sum_{j=1}^m \sum_{n=1}^m a_e^j a_e^n \tilde{\theta}_j \tilde{\theta}_n\right)(a_e^k) \\ &= -a_{x_0 x_0}^k + \sum_{n=1}^m a_{x_n}^k a_{x_0}^n + \sum_{n=1}^m a_{x_n x_0}^k a_e^n + \sum_{n=1}^m a_{x_0 x_n}^k a_e^n \\ &\quad - \sum_{j=1}^m \sum_{n=1}^m a_{x_n}^k a_{x_j}^n a_e^j - \sum_{j=1}^m \sum_{n=1}^m a_{x_n x_j}^k a_e^n a_e^j. \end{aligned}$$

Recall that Sweedler's notation for the reduced coproduct is in use. It is assumed that the theorem holds up to degree $n \geq 2$. Recall that $\tilde{\theta}'_i$ are derivations on H and that for the augmentation ideal projector P it holds that $P\mathbf{1} = 0$. Working with the second recursion in (51) one finds for $\deg(a_\eta^k) = n + 1$, $\eta = x_{i_1} \cdots x_{i_l} = \bar{\eta} x_{i_l} \in X^*$ that

$$\begin{aligned} Sa_\eta^k &= m_H \circ (P \otimes S) \circ \Delta a_\eta^k \\ &= m_H \circ \left(P \circ \tilde{\theta}_{i_l} \otimes S + P \otimes S \circ \tilde{\theta}_{i_l} + \delta_{0i_l} \sum_{n=1}^m P \circ \tilde{\theta}_n \otimes S \circ A_e^{(n)}\right) \circ \Delta a_\eta^k \\ &= m_H \circ \left(P \circ \tilde{\theta}_{i_l} \otimes S\right) \circ \left(a_\eta^k \otimes \mathbf{1} + \Delta' a_\eta^k\right) + m_H \circ \left(P \otimes S \circ \tilde{\theta}_{i_l}\right) \circ \Delta' a_\eta^k \\ &\quad + m_H \circ \left(\delta_{0i_l} \sum_{n=1}^m P \circ \tilde{\theta}_n \otimes S \circ A_e^{(n)}\right) \circ \left(a_\eta^k \otimes \mathbf{1} + \Delta' a_\eta^k\right). \end{aligned}$$

The critical term is

$$m_H \circ (P \otimes S \circ \tilde{\theta}_{i_l}) \circ \Delta' a_\eta^k = m_H \circ (P \otimes S \circ \tilde{\theta}_{i_l}) \circ \sum' a_{\eta'}^l \otimes a_{\eta''}^l.$$

Since $\deg(\tilde{\theta}_i a_{\eta''}^l) < n + 1$, it can be written as

$$m_H \circ (P \otimes S \circ \tilde{\theta}_i) \circ \Delta' a_{\eta}^k = -m_H \circ (P \otimes \tilde{\theta}'_i \circ S) \circ \Delta' a_{\eta}^k.$$

This yields

$$\begin{aligned} Sa_{\eta}^k &= m_H \circ (P \otimes S) \circ \Delta a_{\eta}^k \\ &= m_H \circ (P \otimes \tilde{\theta}_i \otimes S - P \otimes \tilde{\theta}'_i \circ S + \delta_{0i} \sum_{n=1}^m P \otimes \tilde{\theta}_n \otimes S \circ A_e^{(n)}) \circ \Delta a_{\eta}^k \\ &= m_H \circ (\tilde{\theta}_i \otimes \text{id} + \text{id} \otimes \tilde{\theta}_i - \text{id} \otimes \delta_{0i} \sum_{n=1}^m A_e^{(n)} \tilde{\theta}_n - \delta_{0i} \sum_{n=1}^m \tilde{\theta}_n \otimes A_e^{(n)}) \circ (P \otimes S) \circ \Delta a_{\eta}^k \\ &= -(-\tilde{\theta}_i + \delta_{0i} \sum_{n=1}^m A_e^{(n)} \tilde{\theta}_n) m_H \circ (P \otimes S) \circ \Delta a_{\eta}^k \\ &= -\tilde{\theta}'_i Sa_{\eta}^k, \end{aligned}$$

which proves the theorem. Note that the next to the last equality used the fact that the $\tilde{\theta}_i$ are derivations on H .

Remark 4. Consider the case where $m = 1$ in Theorem 7. That is, the alphabet $X := \{x_0, x_1\}$, and the Hopf algebra H is generated by the coordinate functions a_{η} , $\eta \in X^*$. Note that the upper index on the coordinate functions can be dismissed as $m = 1$. The element $a_{\eta} \in H$ has the coproduct defined in terms of $\Delta \circ \tilde{\theta}_i = \tilde{\Theta}_i \circ \Delta$, $i = 0, 1$, where $\tilde{\Theta}_0 = \tilde{\theta}_0 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_0$ and $\tilde{\Theta}_1 = \tilde{\theta}_1 \otimes \text{id} + \text{id} \otimes \tilde{\theta}_1 + \tilde{\theta}_1 \otimes A_e$. The antipode $S: H \rightarrow H$ for any nonempty word $\eta = x_{i_1} \cdots x_{i_l}$ is given by $Sa_{\eta} = (-1)^{|\eta|+1} \tilde{\Theta}'_{\eta}(a_e)$, where $\tilde{\Theta}'_{\eta} := \tilde{\theta}'_{i_l} \circ \cdots \circ \tilde{\theta}'_{i_1}$ and $\tilde{\theta}'_l := -\tilde{\theta}_l + \delta_{0l} a_e \tilde{\theta}_l$, $l = 0, 1$. Here $|\eta| := 2|\eta|_0 + |\eta|_1$, where $|\eta|_0$ denotes the number of times the letter x_0 appears in η , and $|\eta|_1$ is the number of times the letter x_1 is appearing in the word η . One can verify directly that this Hopf algebra coincides with the single-input/single-output (SISO) feedback Hopf algebra described in [28]. The reader is also referred to [14, 16] for more details. This connection between Hopf algebras will be studied further in future work regarding the multivariable (MIMO) case as described in [32].

Finally, returning to the antipode recursions in (51) one realizes quickly the intricacies that result from the signs of the different terms. Surprisingly, the computational aspects of the two formulas are rather different. It turns out that the rightmost recursion is optimal in the sense that its expansion is free of cancellations [14]. This triggers immediately the question whether the antipode formula (53) shares similar properties, which is answered by the next result.

Proposition 3. *The antipode formula (53) is free of cancellations.*

Proof. First recall that the algebra of coordinate functions is polynomially free by construction. Then the absence of cancellations follows from looking at (54) and (55) and noting that $\tilde{\theta}_i \tilde{\theta}_j \neq \tilde{\theta}_j \tilde{\theta}_i$.

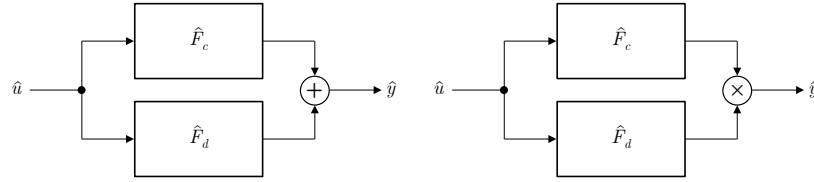


Fig. 3 Parallel sum (left) and parallel product (right) interconnections of two discrete-time Fliess operators.

5 Towards Discretization

This section lays the foundation for a discrete-time analogue of the continuous-time Fliess operator theory described in the previous sections. The starting point is the introduction of a *discrete-time* Fliess operator, where the basic idea is to replace the iterated integrals in (6) with iterated sums. This concept was originally exploited in [35, 36] to provide numerical approximations of continuous-time Fliess operators. The main results were developed without any a priori assumption regarding the existence of a state space realization. It was shown, however, that discrete-time Fliess operators are realizable by the class of state space realizations which are rational in the input and affine in the state whenever the generating series is rational. Some specific examples of this will be given here.

The main focus of this section is on parallel interconnections of discrete-time Fliess operators as shown in Figure 3. In the continuous-time theory presented earlier, it was evident that virtually all the results about interconnections flow from the shuffle algebra, which is induced by the parallel product interconnection. The hypothesis here is that an analogous situation holds in the discrete-time case. So it will be shown that the parallel product of discrete-time Fliess operators induces a *quasi*-shuffle algebra on the set of generating series. Given the natural suitability of rational generating series for discrete-time realization theory, a natural question to pursue is whether rationality is preserved under the quasi-shuffle product. The question was affirmatively answered in [42] but without proof. So here a complete proof will be given.

5.1 Discrete-time Fliess operators

The set of admissible inputs for discrete-time Fliess operators will be drawn from the real sequence space

$$l_{\infty}^{m+1}[N_0] := \{\hat{u} = (\hat{u}(N_0), \hat{u}(N_0 + 1), \dots) : \exists \hat{R}_u \text{ with } 0 \leq |\hat{u}(N)| < \hat{R}_u < \infty, \forall N \geq N_0\},$$

where $\hat{u} = [\hat{u}_0, \hat{u}_1, \dots, \hat{u}_m]^T$ and $|\hat{u}(N)| := \max_{i=0,1,\dots,m} |\hat{u}_i(N)|$. In which case, $\|\hat{u}\|_\infty := \sup_{N \geq N_0} |\hat{u}(N)|$ is always finite. Define a ball of radius \hat{R} in $l_\infty^{m+1}[N_0]$ as

$$B_\infty^{m+1}[N_0](\hat{R}) = \{\hat{u} \in l_\infty^{m+1}[N_0] : \|\hat{u}\|_\infty \leq \hat{R}\}.$$

The subset of finite sequences over $[N_0, N_f]$ is denoted by $B_\infty^{m+1}[N_0, N_f](\hat{R})$. That is, $\hat{u} \in B_\infty^{m+1}[N_0, N_f](\hat{R})$ if $\max_{N \in [N_0, N_f]} |\hat{u}(N)| \leq \hat{R}$. The following definition is of central importance.

Definition 6. [35, 36] For any $c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle$, the corresponding **discrete-time Fliess operator** is

$$\hat{y}(N) = \hat{F}_c[\hat{u}](N) = \sum_{\eta \in X^*} (c, \eta) S_\eta[\hat{u}](N), \quad (56)$$

where $\hat{u} \in l_\infty^{m+1}[1]$, $N \geq 1$, and the iterated sum for any $x_i \in X$ and $\eta \in X^*$ is defined inductively by

$$S_{x_i \eta}[\hat{u}](N) = \sum_{k=1}^N \hat{u}_i(k) S_\eta[\hat{u}](k) \quad (57)$$

with $S_\emptyset[\hat{u}](N) := 1$.

The following lemma will be used for providing sufficient conditions for the convergence of such operators.

Lemma 3. [35, 36] If $\hat{u} \in B_\infty^{m+1}[1](\hat{R})$ then for any $\eta \in X^*$ and $N \geq 1$

$$|S_\eta[\hat{u}](N)| \leq \hat{R}^{|\eta|} \binom{N-1+|\eta|}{|\eta|} \leq 2^{N-1} (2\hat{R})^{|\eta|}.$$

Proof. If $\eta = x_{i_j} \cdots x_{i_1} \in X^*$ then for any $N \geq 1$

$$\begin{aligned} |S_\eta[\hat{u}](N)| &= \left| \sum_{k_j=1}^N \hat{u}_{i_j}(k_j) \sum_{k_{j-1}=1}^{k_j} \hat{u}_{i_{j-1}}(k_{j-1}) \cdots \sum_{k_1=1}^{k_2} \hat{u}_{i_1}(k_1) \right| \\ &\leq \sum_{k_j=1}^N |\hat{u}_{i_j}(k_j)| \sum_{k_{j-1}=1}^{k_j} |\hat{u}_{i_{j-1}}(k_{j-1})| \cdots \sum_{k_1=1}^{k_2} |\hat{u}_{i_1}(k_1)| \\ &\leq \hat{R}^{|\eta|} \sum_{k_j=1}^N \sum_{k_{j-1}=1}^{k_j} \cdots \sum_{k_1=1}^{k_2} 1 = \hat{R}^{|\eta|} \binom{N-1+|\eta|}{|\eta|}, \end{aligned}$$

using the fact that the final nested sum above has $\binom{N-1+|\eta|}{|\eta|}$ terms [6]. The remaining inequality is standard.

Since the upper bound on $|S_\eta[\hat{u}](N)|$ in this lemma is achievable, it is not difficult to see that when the generating series c satisfies the growth bound (7), the series

(56) defining \hat{F}_c can diverge. For example, if $(c, \eta) = K_c M_c^{|\eta|} |\eta|!$ for all $\eta \in X^*$, and $\hat{u}_i(N) = \hat{R}$, $N \geq 1$, $i = 0, 1, \dots, m$ then

$$\begin{aligned} F[\hat{u}](N) &= \sum_{\eta \in X^*} K_c M_c^{|\eta|} |\eta|! \hat{R}^{|\eta|} \binom{N-1+|\eta|}{|\eta|} \\ &= K_c \sum_{j=0}^{\infty} (M_c(m+1)\hat{R})^j j! \binom{N-1+j}{j}. \end{aligned}$$

Since $\lim_{j \rightarrow \infty} \binom{N-1+j}{j} = 1$, this series diverges even when $\hat{R} < 1/M_c(m+1)$. The next theorem shows that this problem is averted when c satisfies the stronger growth condition (8).

Theorem 9. [35, 36] Suppose $c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle$ has coefficients which satisfy

$$|(c, \eta)| \leq K_c M_c^{|\eta|}, \quad \forall \eta \in X^*.$$

Then there exists a real number $\hat{R} > 0$ such that for each $\hat{u} \in B_\infty^{m+1}[1](\hat{R})$, the series (56) converges absolutely for any $N \geq 1$.

Proof. Fix $N \geq 1$. From the assumed coefficient bound and Lemma 3, it follows that

$$\begin{aligned} |\hat{F}_c(\hat{u})(N)| &\leq \sum_{j=0}^{\infty} \sum_{\eta \in X^j} |(c, \eta)| |S_\eta[\hat{u}](N)| \leq \sum_{j=0}^{\infty} K_c (M_c(m+1))^j 2^{N-1} (2\hat{R})^j \\ &= \frac{K_c 2^{N-1}}{1 - 2M_c(m+1)\hat{R}}, \end{aligned}$$

provided $\hat{R} < 1/2M_c(m+1)$.

The final convergence theorem shows that the restriction on the norm of \hat{u} can be removed if an even more stringent growth condition is imposed on c .

Theorem 10. [35, 36] Suppose $c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle$ has coefficients which satisfy

$$|(c, \eta)| \leq K_c M_c^{|\eta|} \frac{1}{|\eta|!}, \quad \forall \eta \in X^*$$

for some real numbers $K_c, M_c > 0$. Then for every $\hat{u} \in l_\infty^{m+1}[1]$, the series (56) converges absolutely for any $N \geq 1$.

Proof. Following the same argument as in the proof of the previous theorem, it is clear for any $\hat{u} \in l_\infty^{m+1}[1]$ and $N \geq 1$ that

$$|\hat{F}_c(\hat{u})(N)| \leq \sum_{j=0}^{\infty} K_c (M_c(m+1))^j \frac{1}{j!} 2^{N-1} (2\|\hat{u}\|_\infty)^j = K_c 2^{N-1} e^{2M_c(m+1)\|\hat{u}\|_\infty}.$$

Assuming the analogous definitions for local convergence (LC) and global convergence (GC) of the operator \hat{F}_c , note the incongruence between the convergence

Table 2 Summary of convergence conditions for F_c and \hat{F}_c

Growth Rate	F_c	\hat{F}_c
$ (c, \eta) \leq K_c M_c^{ \eta } \eta !$	LC	divergent
$ (c, \eta) \leq K_c M_c^{ \eta }$	GC	LC
$ (c, \eta) \leq K_c M_c^{ \eta } \frac{1}{ \eta !}$	GC (at least)	GC

conditions for continuous-time and discrete-time Fliess operators as summarized in Table 2. In each case, for a fixed series c , the sense in which the discrete-time Fliess operator \hat{F}_c converges is *weaker* than that for F_c . The source of this dichotomy is the observation in Lemma 3 that iterated sums of \hat{u} do not grow as a function of word length like $\hat{R}^{|\eta|} / |\eta|!$, which is the case for iterated integrals, but at a potentially faster rate. On the other hand, it is well known that rational series have coefficients that grow as in (8). Thus, as indicated in Table 2, their corresponding discrete-time Fliess operators always converge locally. Therefore, in the following sections this case will be considered in further detail.

5.2 Rational discrete-time Fliess operators

An example of a system that can be described by a discrete-time Fliess operator is

$$\begin{aligned}
\begin{bmatrix} z_1(N+1) \\ z_2(N+1) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(N) \\ z_2(N) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{u}_1(N+1) + \begin{bmatrix} 0 \\ z_1(N) \end{bmatrix} \hat{u}_2(N+1) \\
&\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}_1(N+1) \hat{u}_2(N+1), \\
y(N) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(N) \\ z_2(N) \end{bmatrix},
\end{aligned} \tag{58}$$

where \hat{u}_1 and \hat{u}_2 are suitable input sequences. Letting $z_1(0) = z_2(0) = 0$. Observe that

$$z_1(N+1) = z_1(N) + \hat{u}_1(N+1)$$

implies that $z_1(N) = \sum_{k=1}^N \hat{u}_1(k)$. Thus, it follows that

$$\begin{aligned}
z_2(N+1) &= z_2(N) + z_1(N) \hat{u}_2(N+1) + \hat{u}_2(N+1) \hat{u}_1(N+1) \\
&= z_2(N) + \hat{u}_2(N+1) z_1(N+1) \\
&= \sum_{k_2=1}^{N+1} \hat{u}_2(k_2) \sum_{k_1=1}^{k_2} \hat{u}_1(k_1).
\end{aligned} \tag{59}$$

The corresponding output is then

$$y(N) = \sum_{k_2=1}^{N+1} \hat{u}_2(k_2) \sum_{k_1=1}^{k_2} \hat{u}_1(k_1) = S_{x_2x_1}[\hat{u}](N),$$

which has the form of (56). System (58) falls into the category of *polynomial input and state affine* systems [61]. A simple discretization procedure can also yield discrete-time systems that are rational functions of the inputs. Consider, for instance, the following continuous-time system

$$\dot{z}(t) = z(t)u(t), \quad z(0) = 0. \quad (60)$$

For small $\Delta > 0$, an Euler type approximation gives

$$\begin{aligned} \tilde{z}((N+1)\Delta) &= \tilde{z}(N\Delta) + \int_{N\Delta}^{(N+1)\Delta} \tilde{z}(t)u(t) dt \\ &\approx \tilde{z}(N\Delta) + \int_{N\Delta}^{(N+1)\Delta} u(t) dt \tilde{z}((N+1)\Delta) \\ &= \tilde{z}(N\Delta) + \hat{u}(N+1) \tilde{z}((N+1)\Delta), \end{aligned}$$

and therefore, letting $\hat{z}(N) = \tilde{z}(N\Delta)$, observe that

$$\hat{z}(N+1) = (1 - \hat{u}(N+1))^{-1} \hat{z}(N) \quad (61)$$

In this case, $(1 - \hat{u}(N+1))^{-1}$ is a rational function and fall into the following class of systems.

Definition 7. [36] A discrete-time state space realization is *rational input* and *state affine* if its transition map has the form

$$\hat{z}_i(N+1) = \sum_{j=1}^n r_{ij}(\hat{u}(N+1)) \hat{z}_j(N) + s_i(\hat{u}(N+1)),$$

$i = 1, 2, \dots, n$, where $\hat{z}(N) \in \mathbb{R}^n$, $\hat{u} = [\hat{u}_0, \hat{u}_1, \dots, \hat{u}_m]^T$, r_{ij} and s_i are rational functions, and the output map $h : \hat{z} \mapsto \hat{y}$ is linear.

The general situation is described by the following realization theorem.

Theorem 11. [36] Let $c \in \mathbb{R}\langle\langle X \rangle\rangle$ be a rational series over $X = \{x_0, x_1, \dots, x_m\}$ with linear representation (μ, γ, λ) . Then $\hat{y} = \hat{F}_c[\hat{u}]$ has a finite dimensional rational input and state affine realization on $B_\infty^{m+1}[0, N_f](\hat{R})$ for any $N_f > 0$ provided $\hat{R} < \left(\sum_{j=0}^m \|\mu(x_j)\|\right)^{-1}$, where $\|\cdot\|$ is any matrix norm.

5.3 Parallel interconnections and the quasi-shuffle algebra

Given two continuous-time Fliess operators F_c and F_d with $c, d \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$, the parallel interconnections as shown in Figure 3 satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup d}$

[24]. In the discrete-time case, the parallel sum interconnection is characterized trivially by the addition of generating series, i.e., $\hat{F}_c + \hat{F}_d = \hat{F}_{c+d}$ due to the vector space nature of $\mathbb{R}_{LC}\langle\langle X \rangle\rangle$. But the parallel product connection in this case is characterized by the so-called *quasi-shuffle* product introduced in Example 5. The main objective of this section is to give a description of the quasi-shuffle algebra $H_{qsh} = (\mathbb{R}\langle X \rangle, \otimes)$ in the context of discrete-time Fliess operators and show that rationality is preserved under the quasi-shuffle product.

5.3.1 Quasi-shuffle algebra

The shuffle product (1) describes the product of iterated integrals. However, it cannot account for products of iterated sums. For instance, observe that the product

$$\sum_{i=1}^N \hat{u}_1(i) \sum_{j=1}^N \hat{u}_2(j) = \sum_{i=1}^N \sum_{j=1}^i \hat{u}_1(i) \hat{u}_2(j) + \sum_{i=1}^N \sum_{j=1}^i \hat{u}_1(j) \hat{u}_2(i) - \sum_{i=1}^N \hat{u}_1(i) \hat{u}_2(i), \quad (62)$$

where $\hat{u} \in B_\infty^{m+1}[0, N_f](R)$ for suitable R and N_f . If $X = \{x_0, x_1, x_2\}$, then using (57) it follows that (62) can be written as

$$S_{x_1}[\hat{u}](N) S_{x_2}[\hat{u}](N) = S_{x_1 x_2}[\hat{u}](N) + S_{x_2 x_1}[\hat{u}](N) - \sum_{i=1}^N \hat{u}_1(i) \hat{u}_2(i).$$

Note that the last term $\sum_{i=1}^N \hat{u}_1(i) \hat{u}_2(i)$ does not correspond to a letter in X nor to a word in X^* . Therefore, the alphabet X needs to be augmented to account for this fact. Associating the input $\hat{u}_1 \hat{u}_2$ with the new letter $x_{1,2}$, one can now write

$$S_{x_1}[\hat{u}](N) S_{x_2}[\hat{u}](N) = S_{x_1 x_2}[\hat{u}](N) + S_{x_2 x_1}[\hat{u}](N) + S_{x_{1,2}}[\hat{u}](N).$$

Therefore, the general setting in which products of iterated sums are considered requires a countable alphabet. The extra letters, in addition to those in $X = \{x_0, x_1, \dots, x_m\}$, account for all possible finite products of inputs. Recall item 3 in Example 5, where the quasi-shuffle Hopf algebra is defined. Here the alphabet X is extended to a graded commutative semigroup by defining the commutative bracket operation of letters in X to be $[x_i x_j] = x_{i,j}$, which is assumed to be associative, i.e., $[[x_i x_j] x_l] = [x_i [x_j x_l]]$ for letters $x_i, x_j, x_l \in X$. Iterated brackets may therefore be denoted by $x_{i_1, \dots, i_n} := [[[x_{i_1} x_{i_2}] \dots] x_{i_n}]$. The augmented alphabet \bar{X} contains X as well as all finitely iterated brackets x_{i_1, \dots, i_n} . The monoid of words with letters from \bar{X} is denoted \bar{X}^* . The definition (57) of iterated sums has to be extended to include the additional words in \bar{X}^* , for instance,

$$S_{x_k x_{i_1, i_2, \dots, i_n}}[\hat{u}](N) := \sum_{i=1}^N \hat{u}_k(i) \sum_{j=1}^i \hat{u}_{i_1}(j) \hat{u}_{i_2}(j) \cdots \hat{u}_{i_n}(j).$$

It follows now that the product $S_{x_1}[\hat{u}](N)S_{x_2}[\hat{u}](N)$ is encoded symbolically in terms of a quasi-shuffle product on \bar{X}^*

$$x_1 \otimes x_2 = x_1x_2 + x_2x_1 - x_{1,2} \in \mathbb{R}\langle\bar{X}\rangle. \quad (63)$$

The foundation of discrete-time Fliess operator theory is the summation operator, which is used inductively in the construction of the iterated sums in (57). In general, the summation operator Z is defined as

$$Z(f)(x) := \sum_{k=1}^{\lfloor x/\theta \rfloor} \theta f(\theta k) \quad (64)$$

for a suitable class of functions f . It is known to satisfy the so-called Rota–Baxter relation of weight θ [18]

$$Z(f)(x)Z(g)(x) = Z(Z(f)g + fZ(g) - \theta fg)(x). \quad (65)$$

This relation generalizes the integration by parts rule for indefinite Riemann integrals and provides the corresponding formula for iterated sums. Specifically, (62) corresponds to (65) where $\theta = 1$, $f = \hat{u}_1$ and $g = \hat{u}_2$. The quasi-shuffle product, introduced in item 3 of Example 5, defined on \bar{X}^* provides an extension of (63) and (65). For words $\eta = \eta_1 \cdots \eta_n$ and $\xi = \xi_1 \cdots \xi_m$, where $\eta_i, \xi_j \in \bar{X}$, the recursive definition of the quasi-shuffle product on \bar{X}^* is given by

$$\eta \otimes \xi = \eta_1(\eta_1^{-1}(\eta) \otimes \xi) + \xi_1(\eta \otimes \xi_1^{-1}(\xi)) - [\eta_1 \xi_1](\eta_1^{-1}(\eta) \otimes \xi_1^{-1}(\xi)) \quad (66)$$

with $\emptyset \otimes \eta = \eta \otimes \emptyset = \eta$ for $\eta \in \bar{X}^*$, and $\eta_1^{-1}(\cdot)$ is the left-shift operator defined in (5). This implies that

$$S_\eta[\hat{u}](N) \cdot S_\xi[\hat{u}](N) = S_{\eta \otimes \xi}[\hat{u}](N) \quad (67)$$

with $\eta \otimes \xi \in \mathbb{R}\langle\bar{X}\rangle$. Observe that since $|\eta|, |\xi| < \infty$, then $\text{supp}\{\eta \otimes \xi\}$ is generated by a finite subset of \bar{X} . The quasi-shuffle product \otimes is linearly extended to series $c, d \in \mathbb{R}\langle\langle\bar{X}\rangle\rangle$ so that

$$c \otimes d = \sum_{\eta, \xi \in \bar{X}^*} (c, \eta)(d, \xi) \eta \otimes \xi = \sum_{v \in \bar{X}^*} \underbrace{\sum_{\eta, \xi \in \bar{X}^*} (c, \eta)(d, \xi)(\eta \otimes \xi, v)}_{(c \otimes d, v)} v.$$

Note that the coefficient $(\eta \otimes \xi, v) \neq 0$ only when $v \in \bar{X}^*$ is such that $|\eta| + |\xi| - \min(|\eta|, |\xi|) \leq |v| \leq |\eta| + |\xi|$. Therefore, $(c \otimes d, v)$ is finite since the set $I_\otimes(v) \triangleq \{(\eta, \xi) \in \bar{X}^* \times \bar{X}^* : (\eta \otimes \xi, v) \neq 0\}$ is finite. Hence, the summation defining $c \otimes d$ is locally finite, and therefore summable. It can be shown that the quasi-shuffle product is commutative, associative and distributes over addition [17, 41]. Thus, the vector space $\mathbb{R}\langle\langle\bar{X}\rangle\rangle$ endowed with the quasi-shuffle product forms a commutative \mathbb{R} -algebra, the so-called *quasi-shuffle algebra* with multiplicative identity element $\mathbf{1}$.

5.3.2 Rationality of the quasi-shuffle product

In this section the question of whether the quasi-shuffle product of two rational series is again rational is addressed. In light of Definition 1 and the remark thereafter, it is clear that a rational series c over \bar{X} is also rational over a finite sub-alphabet $X_c \subset \bar{X}$. In which case, Theorems 1 and 2 still apply in the present setting. Also note that in the context of the parallel product connection the underlying alphabets for the generating series of \hat{F}_c and \hat{F}_d are always the same since the inputs are identical. But there is no additional complexity introduced if the alphabets are allowed to be distinct. So let $X_c, X_d \subset \bar{X}$ be finite sub-alphabets of \bar{X} corresponding to the generating series c and d and with cardinalities N_c and N_d , respectively. Define $[X_c X_d] = \{[x_i^c x_j^d] : x_i^c \in X_c, x_j^d \in X_d, i = 1, \dots, N_c, j = 1, \dots, N_d\}$. The main theorem of the section is given first.

Theorem 12. *Let $c, d \in \mathbb{R}\langle\langle\bar{X}\rangle\rangle$ be rational series with underlying finite alphabets $X_c, X_d \subset \bar{X}$, then $e = c \otimes d \in \mathbb{R}\langle\langle\bar{X}\rangle\rangle$ is rational with underlying alphabet $X_e = X_c \cup X_d \cup [X_c X_d] \subset \bar{X}$.*

Proof. In light of (66), the series $e = c \otimes d$ is clearly defined over the finite alphabet X_e . Therefore, a stable finite dimensional vector space V_e is constructed which contains e in order to apply Theorem 2. Since c and d are both rational, let V_c and V_d be stable finite dimensional vector subspaces of $\mathbb{R}\langle\langle X_c \rangle\rangle$ and $\mathbb{R}\langle\langle X_d \rangle\rangle$ containing c and d , respectively. Let $\{\bar{c}_i\}_{i=1}^{n_c}$ and $\{\bar{d}_j\}_{j=1}^{n_d}$ denote their corresponding bases. Define

$$V_e = \text{span}_{\mathbb{R}}\{\bar{c}_i \otimes \bar{d}_j : i = 1, \dots, n_c, j = 1, \dots, n_d\}.$$

Clearly, $V_e \subset \mathbb{R}\langle\langle X_e \rangle\rangle$ is finite dimensional. If one writes

$$c = \sum_{i=1}^{n_c} \alpha_i \bar{c}_i, \quad d = \sum_{j=1}^{n_d} \beta_j \bar{d}_j,$$

it then follows directly that

$$e = c \otimes d = \sum_{i,j=1}^{n_c, n_d} \alpha_i \beta_j \bar{c}_i \otimes \bar{d}_j \in V_e.$$

So it only remains to be shown that V_e is stable. Observe from (66) that for any $x \in X_e$ the left-shift operator acts on the quasi-shuffle product as

$$x^{-1}(\eta \otimes \xi) = x^{-1}(\eta) \otimes \xi + \eta \otimes x^{-1}(\xi) + \delta_{x, [x_i x_j]}(x_i^{-1}(\eta) \otimes x_j^{-1}(\xi)), \quad (68)$$

where $\eta = x_i \eta'$, $\xi = x_j \xi' \in \bar{X}^*$ and $\delta_{x, [x_i x_j]} = 1$ if $x = [x_i x_j]$ and 0 otherwise. Writing $c = (c, \emptyset) + \sum_{i=0}^{N_c} x_i^c (x_i^c)^{-1}(c)$ and $d = (d, \emptyset) + \sum_{i=0}^{N_d} x_i^d (x_i^d)^{-1}(d)$ and using the bilinearity of the quasi-shuffle, it follows that

$$x^{-1}(e) = x^{-1}(c) \otimes d + c \otimes x^{-1}(d) + \sum_{i,j=0}^{N_c N_d} \delta_{x_i^c, x_j^d} ((x_i^c)^{-1}(c) \otimes (x_j^d)^{-1}(d)).$$

But since $V_c, V_d \subset \mathbb{R}\langle\langle X_e \rangle\rangle$ are stable vector spaces by assumption, it is immediate that $(x_i^c)^{-1}(c) \in V_c$ and $(x_j^d)^{-1}(d) \in V_d$, and therefore $x^{-1}(e) \in V_e$ as well. It then follows that V_e is a stable vector space, and hence e is rational.

The following corollary describes the generating series for the parallel product connection in the context of rational series.

Corollary 2. *If $c, d \in \mathbb{R}\langle\langle \bar{X} \rangle\rangle$ are rational series with underlying finite alphabets $X_c, X_d \subset \bar{X}$, then $\hat{F}_c \hat{F}_d = \hat{F}_{c \otimes d}$ with $e = c \otimes d \in \mathbb{R}_{\text{rat}}\langle\langle X_e \rangle\rangle$, where $X_e = X_c \cup X_d \cup [X_c X_d]$.*

Proof. From (67) the product connection of two operators as in (56) is

$$\begin{aligned} \hat{F}_c[\hat{u}](N) \hat{F}_d[\hat{u}](N) &= \sum_{\eta \in X_c^*} (c, \eta) S_\eta[\hat{u}](N) \cdot \sum_{\xi \in X_d^*} (d, \xi) S_\xi[\hat{u}](N) \\ &= \sum_{\eta \in X_c^*, \xi \in X_d^*} (c, \eta)(d, \xi) S_{\eta \otimes \xi}[\hat{u}](N) \\ &= F_{c \otimes d}[\hat{u}](N) =: F_e[\hat{u}](N). \end{aligned}$$

Here $e \in \mathbb{R}_{\text{rat}}\langle\langle X_e \rangle\rangle$ since by Theorem 12 the quasi-shuffle preserves rationality.

The following lemma will be used in the final example of this section.

Lemma 4. *For any $i, j \geq 0$*

$$x_1^i \otimes x_1^j = \sum_{k=0}^{\min\{i,j\}} \binom{i+j-2k}{\min\{i,j\}-k} x_{1,1}^k \sqcup x_1^{i+j-2k}. \quad (69)$$

Proof. Without loss of generality assume $i \geq j$. The identity is proved by induction over $i+j$. The cases for $i+j = 0, 1$ are trivial. Assume (69) holds up to some fixed $i+j$. Using (66) compute

$$x_1^i \otimes x_1^{j+1} = x_1 (x_1^{i-1} \otimes x_1^{j+1}) + x_{1,1} (x_1^i \otimes x_1^j) + x_{1,1} (x_1^{i-1} \otimes x_1^j).$$

By the induction hypothesis and since $i \leq j$,

$$\begin{aligned} x_1^i \otimes x_1^{j+1} &= \sum_{k=0}^{i-1} \binom{i+j-2k}{i-1-k} x_1 (x_{1,1}^k \sqcup x_1^{i+j-2k}) \\ &\quad + \sum_{k=0}^i \binom{i+j-2k}{i-k} x_{1,1} (x_{1,1}^k \sqcup x_1^{i+j-2k}) + \sum_{k=0}^{i-1} \binom{i+j-1-2k}{i-1-k} x_{1,1} (x_{1,1}^k \sqcup x_1^{i+j-1-2k}) \\ &= \sum_{k=0}^{i-1} \binom{i+j-2k}{i-1-k} x_1 (x_{1,1}^k \sqcup x_1^{i+j-2k}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=0}^{i-1} \binom{i+j-2k}{i-k} x_1(x_{1,1}^k \sqcup x_1^{i+j-2k}) + \binom{j-i}{0} x_1(x_{1,1}^k \sqcup x^{j-i}) \\
& + \sum_{k=1}^{i-1} \binom{i+j-2k+1}{i-k} x_{1,1}(x_{1,1}^k \sqcup x_1^{i+j-2k+1}) + \binom{j-i+1}{0} x_1(x_{1,1}^{k-1} \sqcup x^{j-i+1}) \\
& = (x_{1,1}^k \sqcup x_1^{i+j-2k}) + \binom{i+j+1}{i} x_1^{i+j+1} + \sum_{k=0}^{i-1} \binom{i+j-2k+1}{i-k} x_1(x_{1,1}^k \sqcup x_1^{i+j-2k}) \\
& + \sum_{k=1}^{i-1} \binom{i+j-2k+1}{i-k} x_{1,1}(x_{1,1}^k \sqcup x_1^{i+j-2k+1}) \\
& = \sum_{k=0}^i \binom{i+j-2k+1}{i-k} x_{1,1}^k \sqcup x_1^{i+j-2k+1}.
\end{aligned}$$

This complete the proof since it was assumed that $\min\{i, j\} = i$.

Example 7. Let $X = \{x_1\}$ and consider the rational series $c = x_1^* := \sum_{k \geq 0} x_1^k$. It can be shown directly that

$$x_1^* \sqcup x_1^* = \sum_{n=0}^{\infty} \sum_{i=1}^n \binom{n}{i} x_1^n = \sum_{\eta \in X^*} 2^{|\eta|} \eta, \quad (70)$$

using the identity $x_1^i \sqcup x_1^j = \binom{i+j}{i} x_1^{i+j}$ [66]. Since the shuffle product is known to preserve rationality, it follows from Theorem 1 that $x_1^* \sqcup x_1^*$ must have a linear representation (μ, γ, λ) , in this case $\mu(\eta) = 2^{|\eta|}$ and $\gamma = \lambda = 1$. This is easily verified by setting $z_i = F_c[u]$, which gives the bilinear state space realization $\dot{z}_i = z_i u$, $y_i = z_i$. Then the parallel product connection $y = y_1 y_2 = F_c^2[u] = z$ has the realization $\dot{z} = 2zu$, $y = z$. One can confirm using iterated Lie derivatives that the generating series for this system is exactly (70). \square

Example 8. The goal now is to produce the quasi-shuffle analogue of (70). Note here that $X = \{x_1, x_{1,1}\}$. Using Lemma 4 it follows that

$$x_1^* \otimes x_1^* = \sum_{i,j=0}^{\infty} x_1^i \otimes x_1^j = \sum_{n=0}^{\infty} \sum_{i+j=n} \sum_{k=0}^{\min\{i,j\}} \binom{i+j-2k}{\min\{i,j\}-k} x_{1,1}^k \sqcup x_1^{i+j-2k}.$$

For fixed $k', n' \in \mathbb{N} \cup \{0\}$, let $\eta \in X^{n'}$ be such that $|\eta|_{x_{1,1}} = k'$, where $|\eta|_{x_i}$ denotes the number of times the letter $x_i \in X$ appears in $\eta \in X^*$. It follows that the coefficient $(x_1^* \otimes x_1^*, \eta)$ is written as

$$(x_1^* \otimes x_1^*, \eta) = \sum_{n=0}^{\infty} \sum_{i+j=n} \sum_{k=0}^{\min\{i,j\}} \binom{i+j-2k}{\min\{i,j\}-k} (x_{1,1}^k \sqcup x_1^{i+j-2k}, \eta). \quad (71)$$

Notice first that the number of elements in $\text{supp}(x_{1,1}^k \sqcup x_1^{i+j-2k})$ is $\binom{i+j-k}{k}$, and secondly that it implies that $(x_{1,1}^k \sqcup x_1^{i+j-2k}, \eta) = 1$ when $k = k'$ and $i + j = n' + k'$, otherwise $(x_{1,1}^k \sqcup x_1^{i+j-2k}, \eta) = 0$. The former statement is related to the fact that $Q^N = \sum_{i_1+\dots+i_m=N} q_1^{i_1} \sqcup \dots \sqcup q_m^{i_m}$ for an arbitrary alphabet $Q = \{q_1, \dots, q_m\}$ [13]. The coefficient $(x_1^* \otimes x_1^*, \eta)$ can be further developed as

$$\begin{aligned} (x_1^* \otimes x_1^*, \eta) &= \sum_{i+j=n'+k'} \binom{n'-k'}{\min\{i, j\} - k'} = \sum_{i+j=n'+k'} \binom{n'-k'}{\min\{i-k', j-k'\}} \\ &= \sum_{i=k'}^{n'} \binom{n'-k'}{\min\{i-k', n'-i\}} = \sum_{i=k'}^{n'} \binom{n'-k'}{i-k'} = \sum_{i=0}^{n'-k'} \binom{n'-k'}{i} = 2^{|\eta|_{x_1}}. \end{aligned}$$

Thus, one can write

$$x_1^* \otimes x_1^* = \sum_{\eta \in X^*} (x_1^* \otimes x_1^*, \eta) \eta = \sum_{\eta \in X^*} 2^{|\eta|_{x_1}} \eta. \quad (72)$$

In light of Theorem 12, the series $x_1^* \otimes x_1^*$ must be rational. In particular, a straightforward linear representation for $x_1^* \otimes x_1^*$ can be obtained due to (72). That is, (μ, γ, λ) is identified as $\mu(\eta) = 2^{|\eta|_{x_1}}$ and $\lambda = \gamma = 1$. Finally, a direct computation confirms that

$$\begin{aligned} x_1^* \otimes x_1^* &= \emptyset + 2x_1 + x_{1,1} + 4x_1^2 + 2x_1x_{1,1} + 2x_{1,1}x_1 + x_{1,1}^2 + 8x_1^3 + 4x_{1,1}^2x_{1,1} \\ &\quad + 4x_1x_{1,1}x_1 + 4x_{1,1}x_1^2 + 2x_1x_{1,1}^2 + 2x_{1,1}x_1x_{1,1} + 2x_{1,1}^2x_1 + x_{1,1}^3 + 16x_1^4 \\ &\quad + 8x_{1,1}^3x_{1,1} + 8x_{1,1}^2x_{1,1}x_1 + 8x_1x_{1,1}x_{1,1}^2 + 8x_{1,1}x_1^3 + 4x_{1,1}^2x_{1,1}^2 + 4x_1x_{1,1}x_1x_{1,1} \\ &\quad + 4x_1x_{1,1}^2x_1 + 4x_{1,1}x_{1,1}^2x_{1,1} + 4x_{1,1}x_1x_{1,1}x_1 + 4x_{1,1}^2x_1^2 + 2x_1x_{1,1}^3 + 2x_{1,1}x_1x_{1,1}^2 \\ &\quad + 2x_{1,1}^2x_1x_{1,1} + 2x_{1,1}^3x_1 + x_{1,1}^4 + \dots, \end{aligned}$$

where it is clear that (72) holds. \square

Acknowledgements The third author was supported by grant SEV-2011-0087 from the Severo Ochoa Excellence Program at the Instituto de Ciencias Matemáticas in Madrid, Spain. This research was also supported by a grant from the BBVA Foundation.

References

1. E. Abe, *Hopf Algebras*, Cambridge University Press, Cambridge, 1980.
2. M. Anshelevich, E. G. Effros and M. Popa, Zimmerman type cancellation in the free Faà di Bruno algebra, *J. Funct. Anal.*, **237** (2006) 76–104.
3. J. Berstel and C. Reutenauer, *Rational Series and Their Languages*, Springer-Verlag, Berlin, 1988.

4. R. Brockett, Volterra series and geometric control theory, *Automatica*, **12** (1976) 167–176, (addendum with E. Gilbert, **12** (1976), p. 635).
5. D. Burde, Left-symmetric algebras, or pre-Lie algebras in geometry and physics, *Central European Journal of Mathematics*, **4**, number 3 (2006) 323–357.
6. S. Butler and P. Karasik, A note on nested sums, *J. Integer Seq.*, **13** (2010) article 10.4.4.
7. P. Cartier, A primer of Hopf algebras, in *Frontiers in Number Theory, Physics and Geometry II*, P. Cartier, P. Moussa, B. Julia, and P. Vanhove, Eds., Berlin Heidelberg, Springer, 2007, pp. 537–615.
8. —, Vinberg algebras, Lie groups and combinatorics, *Clay Math. Proc.*, **11** (2011) 107–126.
9. K. T. Chen, Iterated integrals and exponential homomorphisms, *Proc. London Math. Soc.*, **4** (1954) 502–512.
10. —, Algebraization of iterated integration along paths, *Bull. AMS*, **73** (1967) 975–978.
11. —, Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula, *Ann. Math.*, 2nd Ser., **65** (1975) 163–178.
12. J. Devlin, Word problems related to periodic solutions of a non-autonomous system, *Math. Proc. Camb. Phil. Soc.*, **108** (1990) 127–151.
13. L. A. Duffaut Espinosa, Interconnections of nonlinear systems driven by L_2 -Itô stochastic processes, Diss. Old Dominion University, Norfolk, Virginia 2009.
14. L. A. Duffaut Espinosa, K. Ebrahimi-Fard, and W. S. Gray, A combinatorial Hopf algebra for nonlinear output feedback control systems, *J. Algebra*, **453** (2016) 609–643. arXiv:1406.5396
15. L. A. Duffaut Espinosa and W. S. Gray, Integration of output tracking and trajectory generation via analytic left inversion, *Proc. 21st Inter. Conf. on System Theory, Control and Computing*, Sinaia, Romania, 2017, pp. 802–807.
16. K. Ebrahimi-Fard and W. S. Gray, Center Problem, Abel Equation and the Faà di Bruno Hopf Algebra for Output Feedback, *Int. Math. Res. Not.*, **2017**, Issue 17, (2017) 5415–5450. arXiv:1507.06939
17. K. Ebrahimi-Fard and L. Guo, Mixable shuffles, quasi-shuffles and Hopf algebras, *J. of Algebraic Combin.*, **24** (2006) 83–101.
18. K. Ebrahimi-Fard and F. Patras, La structure combinatoire du calcul intégral, *Gaz. Math.*, **138** (2013).
19. A. Ferfera, Combinatoire du monoïde libre appliquée à la composition et aux variations de certaines fonctionnelles issues de la théorie des systèmes, Diss. University of Bordeaux I, Talence 1979.
20. —, Combinatoire du monoïde libre et composition de certains systèmes non linéaires, *Astérisque*, **75-76** (1980) 87–93.
21. H. Figueroa and J. M. Gracia-Bondía, Combinatorial Hopf algebras in quantum field theory I, *Rev. Math. Phys.*, **17** (2005) 881–976.
22. H. Figueroa, J. M. Gracia-Bondía and J. C. Várilly, *Elements of Noncommutative Geometry*, Birkhäuser, (2001).
23. M. Fliess, Generating series for discrete-time nonlinear systems, *IEEE Trans. Automat. Control*, **AC-25** (1980) 984–985.
24. —, Fonctionnelles causales non linéaires et indéterminées non commutatives, *Bull. Soc. Math. France*, **109** (1981) 3–40.
25. —, Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives, *Invent. Math.*, **71** (1983) 521–537.
26. L. Foissy, The Hopf algebra of Fliess operators and its dual pre-Lie algebra, *Comm. Algebra*, **43** (2015) 4528–4552. arXiv:0805.4385v2, 2009.
27. A. Frabetti and D. Manchon, Five interpretations of Faà di Bruno’s formula, in *Faà di Bruno Hopf Algebras, Dyson-Schwinger Equations, and Lie-Butcher Series*, K. Ebrahimi-Fard and F. Fauvet, Eds., IRMA Lect. Math. Theor. Phys., Vol. 21, Eur. Math. Soc., Zürich, Switzerland, 2015, pp. 91–147.
28. W. S. Gray and L. A. Duffaut Espinosa, A Faà di Bruno Hopf algebra for a group of Fliess operators with applications to feedback, *Systems Control Lett.*, **60** (2011) 441–449.

29. —, Feedback transformation groups for nonlinear input-output systems, *Proc. 52nd IEEE Conf. Decision and Control*, Florence, Italy, 2013, pp. 2570–2575.
30. —, A Faà di Bruno Hopf algebra for analytic nonlinear feedback control systems, in *Faà di Bruno Hopf Algebras, Dyson-Schwinger Equations, and Lie-Butcher Series*, K. Ebrahimi-Fard and F. Fauvet, Eds., IRMA Lectures in Mathematics and Theoretical Physics, Vol. 21, European Mathematical Society, Zürich, Switzerland, 2015, pp. 149–217.
31. W. S. Gray, L. A. Duffaut Espinosa, and K. Ebrahimi-Fard, Recursive algorithm for the antipode in the SISO feedback product, *Proc. 21st Inter. Symp. Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, 2014, pp. 1088–1093.
32. —, Faà di Bruno Hopf algebra of the output feedback group for multivariable Fliess operators, *Systems Control Lett.*, **74** (2014) 64–73.
33. —, Analytic left inversion of SISO Lotka-Volterra models, *Proc. 49th Conference on Information Sciences and Systems*, Baltimore, Maryland, 2015.
34. —, Analytic left inversion of multivariable Lotka-Volterra models, *Proc. 54th IEEE Conference on Decision and Control*, Osaka, Japan, 2015, pp. 6472–6477.
35. —, Discrete-time approximations of Fliess operators, *Proc. 2016 American Control Conference*, Boston, Massachusetts, 2016, pp. 2433–2439.
36. —, Discrete-Time Approximations of Fliess Operators, *Numerische Mathematik*, **137** (2017) 35–62. arXiv:1510.07901
37. W. S. Gray, L. A. Duffaut Espinosa, and M. Thitsa, Left inversion of analytic nonlinear SISO systems via formal power series methods, *Automatica*, **50** (2014) 2381–2388.
38. W. S. Gray and Y. Li, Generating series for interconnected analytic nonlinear systems, *SIAM J. Control Optim.*, **44** (2005), 646–672.
39. W. S. Gray and M. Thitsa, A unified approach to generating series for cascaded nonlinear input-output systems, *Inter. J. Control*, **85** (2012) 1737–1754.
40. W. S. Gray and Y. Wang, Fliess operators on L_p spaces: convergence and continuity, *Systems Control Lett.*, **46** (2002) 67–74.
41. M. E. Hoffman, Quasi-shuffle products, *J. Algebraic Combin.*, **11** (2000) 49–68.
42. V. Houseaux, G. Jacob, N. E. Oussous and, M. Petitot, A complete Maple package for non-commutative rational power series, in *Computer Mathematics: Proceeding of the Sixth Asian Symposium*, Lecture Notes Series on Computing vol. 10, Z. Li, W. Y. Sit, Eds., World Scientific, Singapore, 2003, pp. 174–188.
43. A. Isidori, *Nonlinear Control Systems*, 3rd Ed., Springer-Verlag, London, 1995.
44. B. Jakubczyk, Existence and uniqueness of realizations of nonlinear systems, *SIAM J. Control Optim.* **18** (1980), 445–471.
45. —, Local realization of nonlinear causal operators, *SIAM J. Control Optim.*, **24** (1986) 230–242.
46. —, Realization theory for nonlinear systems: Three approaches. In *Algebraic and Geometric Methods in Nonlinear Control Theory* (ed. by M. Fliess and M. Hazewinkel). D. Reidel Publishing Company, Dordrecht 1986, 3–31.
47. H. K. Khalil, *Nonlinear Systems*, 3rd Ed., Pearson India, 2014.
48. C. Lesiak and A. J. Krener, The existence and uniqueness of Volterra series for nonlinear systems, *IEEE Trans. Automat. Contr.* **AC-23** (1978), 1090–1095.
49. Y. Li, Generating series of interconnected nonlinear systems and the formal Laplace-Borel Transform, Diss. Old Dominion University, Norfolk, Virginia 2004.
50. D. Manchon, *Hopf algebras, from basics to applications to renormalization*, Comptes-rendus des Rencontres mathématiques de Glanon 2001, arXiv:0408405.
51. D. Manchon, A short survey on pre-Lie algebras, in *Noncommutative Geometry and Physics: Renormalisation, Motives, Index Theory*, A. Carey, Ed., Eur. Math. Soc., Zürich, Switzerland, 2011, pp. 89–102.
52. J. W. Milnor and J. C. Moore, On the structure of Hopf algebras, *Ann. Math.*, **81** (1965) 211–264.
53. H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, New York, 1990.

54. M. Pittou and G. Rahonis, Weighted recognizability over infinite alphabets, *Acta Cybern.*, **23**(1) (2017) 283–317.
55. D. E. Radford, *Hopf Algebras*, World Scientific Publishing, Hackensack, NJ, 2012.
56. R. Ree, Lie elements and an algebra associated with shuffles, *Ann. Math., 2nd Ser.*, **68** (1958) 210–220.
57. C. Reutenauer, *Free Lie Algebras*, Oxford University Press, New York, 1993.
58. W. J. Rugh, *Nonlinear System Theory, The Volterra/Wiener Approach*, The Johns Hopkins University Press, Baltimore, Maryland, 1981.
59. M. P. Schützenberger, On the definition of a family of automata, *Inform. Control*, **4** (1961) 245–270.
60. J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*, Pearson, 1991.
61. E. D. Sontag, *Polynomial Response Maps*, Springer-Verlag, Berlin, 1979.
62. H. J. Sussmann, Existence and uniqueness of minimal realizations of nonlinear systems, *Math. Syst. Theory*, **10** (1977) 263–284.
63. —, A proof of the realization theorem for convergent generating series of finite Lie rank, internal report SYCON-90-02, Rutgers Center for Systems and Control, 1990.
64. M. E. Sweedler, *Hopf Algebras*, W. A. Benjamin, Inc., New York, 1969.
65. M. Thitsa and W. S. Gray, On the radius of convergence of interconnected analytic nonlinear input-output systems, *SIAM J. Control Optim.* **50** (2012), 2786–2813.
66. Y. Wang, Differential equations and nonlinear control systems, Diss. Rutgers University, New Brunswick, New Jersey, 1990.