Remarks on Liouville type theorems for the 3D steady axially symmetric Navier-Stokes equations

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Abstract

In this note, we investigate the 3D steady axially symmetric Navier-Stokes equations, and obtained Liouville type theorems if the velocity or the vorticity satisfies some a priori decay assumptions.

Keywords: Liouville type theorem, Navier-Stokes equations, axially symmetric Navier-Stokes equations

1 Introduction

An interesting question about Liouville type theorem of the 3D stationary Navier-Stokes equations in R^3 is as follows: whether the solution of

$$\begin{cases} -\Delta u + u\cdot\nabla u = -\nabla p, \\ \nabla\cdot u = 0, \end{cases} \tag{1}$$

satisfying the vanishing property at infinity

$$\lim_{|x|\to\infty} u(x) = 0, \tag{2}$$

and the bounded Dirichlet energy

$$D(u) = \int_{R^3} |\nabla u|^2 dx < \infty \tag{3}$$

implies $u \equiv 0$ is still an open problem, which is related to J. Leray (see also P12. Galdi [7]).

Many conditional criteria have been obtained for this issue. For example, Galdi proved the above Liouville type theorem by assuming $u \in L^{\frac{9}{2}}(R^3)$ in [7]. Chae in [2] showed the condition $\triangle u \in L^{\frac{6}{5}}(R^3)$ is sufficient for the vanishing property of u. Also, Chae-Wolf gave an improvement of logarithmic form for Galdi's result in [4] by assuming that $\int_{R^3} |u|^{\frac{9}{2}}\{\ln(2+\frac{1}{|u|})\}^{-1}dx < \infty$. Seregin obtained the conditional criterion $u \in BMO^{-1}(R^3)$ in [12]. Moreover, Kozonoa-Terasawab-Wakasugib proved $u \equiv 0$ if the vorticity $w = o(|x|^{-\frac{5}{3}})$ or $\|u\|_{L^{\frac{9}{2},\infty}(R^3)} \leq \delta D(u)^{1/3}$ for a small constant δ in [10]. It is shown that all the above norms $u \in L^{\frac{9}{2}}(R^3)$, the log form of $u \in L^{\frac{9}{2}}(R^3)$ or $u \in L^{\frac{9}{2},\infty}(R^3)$ can be replaced by the norms in the annular domain $B_R \setminus B_{R/2}$ in [16] by Seregin and the author, where the following energy description was stated:

$$\int_{B_{R/2}} |\nabla u|^2 dx \quad \leq \quad CR^{-2}\left(\int_{B_R \setminus B_{R/2}} |u|^2 dx\right) + C(q)R^{2-\frac{9}{q}}\|u\|^3_{L^{q,\infty}(B_R\setminus B_{R/2})}$$

where $B_R = B_R(0)$ is a ball centered at 0 and $q > 3$. Note that the conditions (2) and (3) are not used in [16] as in [4]. More references, we refer to [3, 13, 14] and the references therein.

Moreover, the problem is not solved, even for the case of axially symmetric Navier-Stokes equations, to the best of the author's knowledge. Motivated by the result Seregin in [14], where he proved that the condition $|u| \lesssim \frac{1}{|x'|^{\mu}}$ with $x' = (x_1, x_2)$ and $\mu \approx 0.77$ implies $u \equiv 0$, we are aimed to improve the decay assumption. At first, let us introduce the axially symmetric Navier-Stokes equations. Let $u(x) = u_r(t,r,z)e_r + u_\theta(t,r,z)e_\theta + u_z(t,r,z)e_z$, where

$$\begin{aligned} e_r &= (\frac{x_1}{r}, \frac{x_2}{r}, 0) = (\cos\theta, \sin\theta, 0),\\ e_\theta &= (-\frac{x_2}{r}, \frac{x_1}{r}, 0) = (-\sin\theta, \cos\theta, 0),\\ e_z &= (0,0,1) \end{aligned}$$

and (1) becomes

$$\begin{cases} b\cdot\nabla u_r - \Delta_0 u_r + \dfrac{u_r}{r^2} - \dfrac{u_\theta^2}{r} + \partial_r p = 0,\\ b\cdot\nabla u_\theta - \Delta_0 u_\theta + \dfrac{u_\theta}{r^2} + \dfrac{u_r u_\theta}{r} = 0,\\ b\cdot\nabla u_z - \Delta_0 u_z + \partial_z p = 0,\\ \partial_r(ru_r) + \partial_z(ru_z) = 0, \end{cases} \tag{4}$$

where

$$b = u_r e_r + u_z e_z, \quad \Delta_0 = \partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz}.$$

The vorticity is represented as

$$w = w_r e_r + w_\theta e_\theta + w_z e_z = (-\partial_z u_\theta) e_r + (\partial_z u_r - \partial_r u_z) e_\theta + \frac{\partial_r (r u_\theta)}{r} e_z.$$

There are also many developments on the Liouville type theorems of axi-symmetric case. For example, Liouville type theorem was proved by assuming no swirl (i.e. $u_\theta = 0$), see Koch-Nadirashvili-Seregin-Sverak [9] or Korobkov-Pileckas-Russo[11]. The condition $ru_\theta \in L^q$ with some $q \geq 1$ or $b \in L^3$ is enough, see Chae-Weng in [5]. Specially, for the axially symmetric case, the decay of the velocity or the vorticity can be obtained: Choe-Jin [6], Weng [17] proved that

$$|u_r(r,z)| + |u_z(r,z)| + |u_\theta(r,z)| \lesssim \sqrt{\frac{\ln r}{r}},$$
$$|w_\theta(r,z)| \lesssim r^{-(\frac{19}{16})^-}, \quad |w_r(r,z)| + |w_z(r,z)| \lesssim r^{-(\frac{17}{16})^-}$$

Recently, Carrillo-Pan-Zhang in [1] gave an alternative method for the decay of u and an improvement for the decay bound of the vorticity

$$|w_\theta(r,z)| \lesssim r^{-\frac{5}{4}} (\ln r)^{\frac{3}{4}}, \quad |w_r(r,z)| + |w_z(r,z)| \lesssim r^{-\frac{9}{8}} (\ln r)^{\frac{11}{8}}$$

by using Brezis-Gallouet inequality.

It's a natural question: whether there exist the sharp constants μ_1, μ_2 such that $|(u_r(r,z), u_z(r,z), u_\theta(r,z))| \lesssim \frac{1}{r^{\mu_1}}$ or $|(w_r(r,z), w_z(r,z), w_\theta(r,z))| \lesssim \frac{1}{r^{\mu_2}}$ implies that $u \equiv 0$ for the axially symmetric case?

With the help of energy estimates in [16] we can improve the result in [14] to $\mu > \frac{2}{3}$, which is almost a equivalent form of $u \in L^{\frac{9}{2},\infty}$.

Theorem 1.1. *Suppose that u is axially symmetric smooth solution of the equation (4) and for some $\mu > \frac{2}{3}$,*

$$|u| \leq \frac{C}{(1+r)^\mu}.$$

Then $u \equiv 0$.

Note that $\Gamma = r u_\theta$ satisfies the special structure

$$b \cdot \nabla \Gamma - \triangle_0 \Gamma + \frac{2}{r} \partial_r \Gamma = 0$$

and Maximum principle can be applied, thus the condition $u_\theta = o(\frac{1}{r})$ as $|x| \to \infty$ implies u is trivial. However, it's still known that whether $u_\theta = o(\frac{1}{r})$ can be replaced by $u_\theta = O(\frac{1}{r})$. But we show that the condition $|b| = O(\frac{1}{r})$ or $b \in BMO^{-1}(R^3)$ is sufficient, which improved the assumption $b \in L^3(R^3)$ in [5].

Here we say a function $f \in BMO^{-1}(R^3)$ if there exists a vector-value function $d \in R^3$ and $d_j \in BMO(R^3)$ such that $f = div\ d = d_{j,j}$. It's well-known that for the BMO space, we have

$$\Gamma(s) = \sup_{x_0 \in R^3, R>0} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |d - d_{x_0,R}|^s dx \right)^{\frac{1}{s}} < \infty.$$

for any $s \in [1, \infty)$.

In details, we obtained the following result.

Theorem 1.2. *Suppose that u is axially symmetric smooth solution of the equation (4) satisfying (2) and (3). Then $u \equiv 0$ if one of the following conditions is satisfied*

$$(i) \quad b = (u_r, u_z) \in BMO^{-1}(R^3);$$
$$(ii) \quad |b| \leq \frac{C}{r}$$

For the decay of the vorticity, we also state the following corresponding result.

Theorem 1.3. *Suppose that u is axially symmetric smooth solution of the equation (4) satisfying (2) and (3). Moreover,*

$$|(w_r, w_\theta, w_z)| \leq \frac{C}{r^\beta}, \quad \beta > \frac{5}{3}.$$

Then $u \equiv 0$.

Remark 1. *This conclusion generalized the result of [10] to the axially symmetric case, where the condition $|w| = o(|x|^{-\frac{5}{3}})$ was put.*

Throughout this article, C denotes a constant, which may be different from line to line.

2 Proof of Theorem 1.1

Recall a Caccioppoli inequality in [16], which is stated as follows.

Proposition 2.1. *Let (u, p) be the smooth solution of (1). Then for $0 < \delta \leq 1$ and $\frac{6(3-\delta)}{6-\delta} < q < 3$, we have*

$$\int_{B_{R/2}} |\nabla u|^2 dx \leq \frac{C}{R^2} \left(\int_{B_R \setminus B_{R/2}} |u|^2 dx \right) + C(\delta) \left(\|u\|^{3-\delta}_{L^{q,\infty}(B_R \setminus B_{R/2})} R^{2 - \frac{9-3\delta}{q} - \frac{\delta}{2}} \right)^{\frac{2}{2-\delta}}$$

Proof of Theorem 1.1. Let C_R denote the cylindrical region $\{x; |x'| \le R, |z| \le R\}$, then it's easy to check that

$$B_R \subset C_R \subset B_{\sqrt{2}R}.$$

Hence, it follows from Proposition 2.1 that

$$\begin{aligned}
\int_{C_{\frac{\sqrt{2}}{4}R}} |\nabla u|^2 dx &\le \frac{C}{R^2}\left(\int_{C_R \backslash C_{\frac{\sqrt{2}}{4}R}} |u|^2 dx\right) \\
&\quad + C(\delta)\left(\|u\|^{3-\delta}_{L^{q,\infty}(C_R \backslash C_{\frac{\sqrt{2}}{4}R})} R^{2-\frac{9-3\delta}{q}-\frac{\delta}{2}}\right)^{\frac{2}{2-\delta}} \\
&\le C\|u\|^2_{L^q(C_R)} R^{1-\frac{6}{q}} + C(\delta, q)\left(\|u\|^{3-\delta}_{L^q(C_R)} R^{2-\frac{9-3\delta}{q}-\frac{\delta}{2}}\right)^{\frac{2}{2-\delta}} \qquad (5)
\end{aligned}$$

for $q > 2$, where we used the property of Lorentz space

$$\|u\|_{L^{q,\infty}(\Omega)} \le C(q, \ell)\|u\|_{L^{q,\ell}(\Omega)}$$

(for example, see Proposition 1.4.10 in [8]).

For $\mu q > 2$, we have

$$\|u\|_{L^q(C_R)} \le C\left(R\int_0^R (1+r)^{1-\mu q} dr\right)^{\frac{1}{q}} \le C(\mu, q) R^{\frac{1}{q}}$$

Then the terms of the right hand side of (5) is controlled by

$$\int_{C_{\frac{\sqrt{2}}{4}R}} |\nabla u|^2 dx \le C(\mu, q) R^{1-\frac{4}{q}} + C(\delta, \mu, q)\left(R^{2-\frac{\delta}{2}-\frac{6-2\delta}{q}}\right)^{\frac{2}{2-\delta}} \qquad (6)$$

Claim that: for fixed $\mu > \frac{2}{3}$, there exist constants $\delta \in (0, 1)$ and q such that

$$\max\{6\frac{3-\delta}{6-\delta}, \frac{2}{\mu}\} < q < 3, \quad \text{and} \quad 2 - \frac{\delta}{2} - \frac{6-2\delta}{q} < 0 \qquad (7)$$

hence letting $R \to \infty$, by (6) we have

$$\int_{R^3} |\nabla u|^2 dx = 0,$$

which implies $u \equiv 0$.

Proof of (7). First for fixed $\mu > \frac{2}{3}$, we choose $\delta_0 \in (0, 1)$ such that

$$\frac{2}{\mu} < 4\frac{3-\delta_0}{4-\delta_0}$$

Since $0 < \delta_0 < 1$, we have

$$1 - \frac{\delta_0}{4} < 1 - \frac{\delta_0}{6},$$

and

$$6\frac{3-\delta_0}{6-\delta_0} < 4\frac{3-\delta_0}{4-\delta_0}$$

so we take

$$q = \frac{1}{2}\left(\max\{6\frac{3-\delta_0}{6-\delta_0}, \frac{2}{\mu}\} + 4\frac{3-\delta_0}{4-\delta_0}\right)$$

Then we have

$$\max\{6\frac{3-\delta_0}{6-\delta_0}, \frac{2}{\mu}\} < q < 4\frac{3-\delta_0}{4-\delta_0} < 3,$$

which implies (7).

Hence the proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

Let $\phi(x) = \phi(r, z) \in C_0^\infty(C_R)$ and $0 \leq \phi \leq 1$ satisfying

$$\phi(x) = \begin{cases} 1, & x \in C_{R/2}, \\ 0, & x \in C_R^c \end{cases}$$

and

$$|\nabla\phi| \leq \frac{C}{R}, \quad |\nabla^2\phi| \leq \frac{C}{R^2}.$$

Without loss of generality, by Theorem X.5.1 in [7] we can assume that

$$\lim_{|x|\to\infty} |p| + |u| = 0.$$

Note that $\triangle p = -\partial_i\partial_j(u_i u_j)$, then using Calderón-Zygmund estimates and gradient estimates of harmonic function, we have

$$\int_{R^3} |p|^3 + |u|^6 dx < CD(u)^3,$$

and

$$\|\nabla p\|_{L^{\frac{3}{2}}(R^3)} < CD(u),$$

since $\||\nabla u|u\|_{L^{\frac{3}{2}}(R^3)} \leq CD(u)$.

Multiplying $\phi u\cdot$ on both sides of (1), integration by parts yields that

$$\begin{aligned}
&\int_{C_R} \phi\left(|\nabla u_r|^2 + |\nabla u_\theta|^2 + |\nabla u_z|^2 + \frac{u_r^2}{r^2} + \frac{u_\theta^2}{r^2}\right) dx \\
\leq\ & \int_{C_R} \left(\frac{1}{2}|u|^2 + p\right)(u_r\partial_r + u_z\partial_z)\phi dx + C\|u\|^2_{L^6(C_R\setminus C_{R/2})} \\
\doteq\ & I + C\|u\|^2_{L^6(C_R\setminus C_{R/2})}
\end{aligned}$$

Case (i). Due to $u_r, u_z \in BMO^{-1}(R^3)$, we write

$$u_r = \partial_j d_{1,j}, \quad u_z = \partial_j d_{2,j}, \quad j = 1, 2, 3,$$

where $d_{1,j}, d_{2,j} \in BMO(R^3)$. Also, denote $\bar{f}$ as the mean value of f on the domain C_R. Then we have

$$\begin{aligned}
I &= \int_{C_R} \left(\frac{1}{2}|u|^2 + p\right)\left[\partial_j(d_{1,j} - \bar{d}_{1,j})\partial_r + \partial_j(d_{2,j} - \bar{d}_{2,j})\partial_z\right]\phi dx \\
&= -\int_{C_R} \partial_j\left(\frac{1}{2}|u|^2 + p\right)\left[(d_{1,j} - \bar{d}_{1,j})\partial_r\phi + (d_{2,j} - \bar{d}_{2,j})\partial_z\phi\right] dx \\
&\quad -\int_{C_R} \left(\frac{1}{2}|u|^2 + p\right)\left[(d_{1,j} - \bar{d}_{1,j})\partial_j(\partial_r\phi) + (d_{2,j} - \bar{d}_{2,j})\partial_j(\partial_z\phi)\right] dx
\end{aligned}$$

Recall that $\phi(x) = \phi(r, z)$ and

$$\begin{aligned}
&\partial_j\partial_z\phi = \partial_z\partial_j\phi, \quad \text{for} \ \ j = 1, 2, 3, \\
&\partial_j\partial_r\phi = \partial_z\partial_j\phi, \quad \text{for} \ \ j = 3, \\
&\partial_1\partial_r\phi = \cos\theta\partial_r^2\phi, \quad \partial_2\partial_r\phi = \sin\theta\partial_r^2\phi,
\end{aligned}$$

which and the property of BMO function yield that

$$\begin{aligned}
I &\leq CR^{-1}\||\nabla(|u|^2)| + |\nabla p|\|_{L^{\frac{3}{2}}(C_R\setminus C_{R/2})}(\|d_{1,j} - \bar{d}_{1,j}\|_{L^3(C_R)} + \|d_{2,j} - \bar{d}_{2,j}\|_{L^3(C_R)}) \\
&\quad + CR^{-2}(\|u\|^2_{L^6(C_R\setminus C_{R/2})} + \|p\|_{L^3(C_R\setminus C_{R/2})})(\|d_{1,j} - \bar{d}_{1,j}\|_{L^{\frac{3}{2}}(C_R)} + \|d_{2,j} - \bar{d}_{2,j}\|_{L^{\frac{3}{2}}(C_R)}) \\
&\leq C\||\nabla(|u|^2)| + |\nabla p|\|_{L^{\frac{3}{2}}(C_R\setminus C_{R/2})} + C(\|u\|^2_{L^6(C_R\setminus C_{R/2})} + \|p\|_{L^3(C_R\setminus C_{R/2})}) \\
&\quad \to 0 \quad (\text{as} \ \ R \to \infty)
\end{aligned}$$

Hence, the proof of case (i) is complete.

Case (ii). When $|(u_r, u_z)| \leq \frac{C}{r}$ for $r > 0$,

$$I = \int_{C_R} \left(\frac{1}{2}|u|^2 + p\right)(u_r\partial_r + u_z\partial_z)\phi dx$$

$$\leq C\int_{C_R}\left(\frac{1}{2}|u|^2+|p|\right)(\partial_r\ln(r)|\partial_r\phi|+\partial_r\ln(r)|\partial_z\phi|)\,dx.$$

Let $g(r)=\ln(r)$ and $\bar{g}$ be the mean value of g on $\{x';|x'|\leq R\}$. Then we have

$$\begin{aligned}
I &\leq -C\int_{C_R}\partial_r(\frac{1}{2}|u|^2+|p|)(g-\bar{g})\left(|\partial_r\phi|+|\partial_z\phi|\right)dx\\
&\quad -C\int_{C_R}\left(\frac{1}{2}|u|^2+|p|\right)(g-\bar{g})\partial_r\left(|\partial_r\phi|+|\partial_z\phi|\right)dx\\
&\quad -C\int_{C_R}\left(\frac{1}{2}|u|^2+|p|\right)(g-\bar{g})\frac{1}{r}\left(|\partial_r\phi|+|\partial_z\phi|\right)dx\\
&\doteq I_1+I_2+I_3
\end{aligned}$$

Note that $g\in BMO(R^2)$ (see, for example, Chapter IV [15]), and we have

$$R^{-1}\left(\int_{C_R}|g-\bar{g}|^3dx\right)^{\frac{1}{3}}\leq C\left(R^{-2}\int_{|x'|\leq R}|g-\bar{g}|^3dx\right)^{\frac{1}{3}}\leq C$$

and

$$R^{-2}\left(\int_{C_R}|g-\bar{g}|^{\frac{2}{3}}dx\right)^{\frac{2}{3}}\leq C,\quad R^{-3}\left(\int_{C_R}|g-\bar{g}|^{12}dx\right)\leq C$$

Hence as the arguments of (i), we have

$$I_1+I_2\leq C\||\nabla(|u|^2)|+|\nabla p|\|_{L^{\frac{3}{2}}(C_R\backslash C_{R/2})}+C(\|u\|^2_{L^6(C_R\backslash C_{R/2})}+\|p\|_{L^3(C_R\backslash C_{R/2})})$$

For the term of I_3, we get

$$\begin{aligned}
I_3 &\leq CR^{-1}(\|u\|^2_{L^6(C_R\backslash C_{R/2})}+\|p\|_{L^3(C_R\backslash C_{R/2})})\|g-\bar{g}\|_{L^{12}(C_R)}\|\frac{1}{r}\|_{L^{\frac{12}{7}}(C_R)}\\
&\leq CR^{-\frac{1}{4}}(\|u\|^2_{L^6(C_R\backslash C_{R/2})}+\|p\|_{L^3(C_R\backslash C_{R/2})})\|g-\bar{g}\|_{L^{12}(C_R)}\\
&\leq C(\|u\|^2_{L^6(C_R\backslash C_{R/2})}+\|p\|_{L^3(C_R\backslash C_{R/2})})
\end{aligned}$$

Hence, we can conclude that

$$I\to 0\quad (\text{as R}\to\infty)$$

The proof of Theorem 1.2 is complete.

4 Proof of Theorem 1.3

We are going to prove that

Proposition 4.1. *Assume that the conditions of Theorem 1.3 hold. (1) Let* $w_\theta \le Cr^{-\beta}$ *with* $\beta > 1$*. Then we get for* $r > 1$

$$|u_r(r,z)| + |u_z(r,z)| \le C \begin{cases} (1+r)^{-\frac{3}{2}+\frac{1}{2(\beta-1)}}, & \beta > 2, \\ (1+r)^{1-\beta}, & 1 < \beta < 2, \\ (1+r)^{-1}\ln(r+1), & \beta = 2. \end{cases}$$

(2) Let $|w_r| + |w_z| \le Cr^{-\beta}$ *with* $\beta > 1$*. Then we get for* $r > 1$

$$|u_\theta(r,z)| \le C \begin{cases} (1+r)^{-\frac{3}{2}+\frac{1}{2(\beta-1)}}, & \beta > 2, \\ (1+r)^{1-\beta}, & 1 < \beta < 2, \\ (1+r)^{-1}\ln(r+1), & \beta = 2. \end{cases}$$

Proof of Theorem 1.3. It follows from Proposition 4.1 and Theorem 1.1 directly.

Next we are aimed to prove Proposition 4.1. Firstly, we introduce a representation formula of u_r, u_z and u_θ with the help of the vorticity. Since $b = u_r e_r + u_z e_z$ and

$$\nabla \times b = w_\theta e_\theta, \quad \nabla \times (u_\theta e_\theta) = w_r e_r + w_z e_z$$

by Biot-Savart law, we can get the integral representation of the velocity as follows(for example, see Lemma 2.2 for a local version by Choe-Jin [6], also see Lemma 3.10 by Weng [17]).

Lemma 4.2. *Like the vorticity at the point* $(r\cos\theta, r\sin\theta, z)$ *denoted by* $(w_r, w_\theta, w_z)(r,z)$*, we write the vorticity at the point* $(\rho\cos\phi, \rho\sin\phi, k)$ *as* $(w_\rho, w_\phi, w_k)(\rho, k)$*. Then we have*

$$u_r(r,z) = \int_{-\infty}^{\infty}\int_0^{\infty} \Gamma_1(r,\rho,z-k) w_\phi(\rho,k)\rho d\rho dk, \tag{8}$$

$$u_z(r,z) = -\int_{-\infty}^{\infty}\int_0^{\infty} \Gamma_2(r,\rho,z-k) w_\phi(\rho,k)\rho d\rho dk \tag{9}$$

$$\begin{aligned} u_\theta(r,z) &= \int_{-\infty}^{\infty}\int_0^{\infty} \Gamma_3(r,\rho,z-k) w_k(\rho,k)\rho d\rho dk \\ &\quad - \int_{-\infty}^{\infty}\int_0^{\infty} \Gamma_1(r,\rho,z-k) w_\rho(\rho,k)\rho d\rho dk \end{aligned} \tag{10}$$

where

$$\Gamma_1(r,\rho,z-k) = \frac{1}{4\pi}\int_0^{2\pi} \frac{z-k}{[r^2+\rho^2-2r\rho\cos\phi+(z-k)^2]^{\frac{3}{2}}}\cos\phi d\phi$$

$$\Gamma_2(r,\rho,z-k) = -\frac{1}{4\pi}\int_0^{2\pi} \frac{\rho - r\cos\phi}{[r^2+\rho^2-2r\rho\cos\phi+(z-k)^2]^{\frac{3}{2}}} d\phi,$$

$$\Gamma_3(r,\rho,z-k) = -\frac{1}{4\pi}\int_0^{2\pi} \frac{\rho - r\cos\phi}{[r^2+\rho^2-2r\rho\cos\phi+(z-k)^2]^{\frac{3}{2}}}\cos\phi d\phi.$$

Secondly, we give the bounds of estimate of Γ_2, Γ_3 and Γ_1, which will be used in the proof. This is similar to that in [6], where $\rho \approx r$ was assumed. Here we consider all $\rho > 0$ and large $r > 0$. In details, we have the following estimates.

Lemma 4.3 (Estimate of Γ_2, Γ_3 and Γ_1)**.**

$$|\Gamma_2(r, \rho, z-k)| + |\Gamma_3(r, \rho, z-k)| \le \frac{C}{(\max\{\rho, r\})^\alpha [(r-\rho)^2 + (z-k)^2]^{\frac{2-\alpha}{2}}}, \tag{11}$$

for $r > 1$ and $0 \le \alpha \le 1$;

$$|\Gamma_1(r, \rho, z-k)| \le \frac{C|z-k|}{(\max\{\rho, r\})^\alpha [(r-\rho)^2 + (z-k)^2]^{\frac{3-\alpha}{2}}}, \tag{12}$$

where $r > 1$, $0 \le \alpha \le 1$ for $\frac{r}{4} \le \rho \le 4r$, and $0 \le \alpha \le 3$ for $\rho < \frac{r}{4}$ or $\rho \ge 4r$.

Thirdly, we assume Lemma 4.3 holds and complete the proof of Proposition 4.1 and Lemma 4.3 is proved later.

Proof of Proposition 4.1: At first, we estimate the term of $u_r(r, z)$. Let

$$\begin{aligned} I &= u_r(r,z) = \int_{-\infty}^{\infty}\int_0^{\infty} \Gamma_1 w_\phi \rho d\rho dk \\ &= \int_{-\infty}^{\infty}\int_0^{r^\gamma/8} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty}\int_{r^\gamma/8}^{r/4} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty}\int_{r/4}^{r-r^\delta/2} \Gamma_1 w_\phi \rho d\rho dk \\ &\quad + \int_{-\infty}^{\infty}\int_{r-r^\delta/2}^{r+r^\delta/2} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty}\int_{r+r^\delta/2}^{4r} \Gamma_1 w_\phi \rho d\rho dk + \int_{-\infty}^{\infty}\int_{4r}^{\infty} \Gamma_1 w_\phi \rho d\rho dk \\ &= I_1 + \cdots + I_6, \end{aligned}$$

where $0 \le \gamma, \delta \le 1$, to be decided.

For the term I_1, by (12) and $\|w_\phi\|^2_{L^2(R^3)} \le CD(u) < \infty$ we get

$$\begin{aligned} I_1 &\le C\left(\int_{-\infty}^{\infty}\int_0^{r^\gamma/8} |\Gamma_1(r,\rho,z-k)|^2 \rho d\rho dk\right)^{\frac12} \\ &\le C\left(\int_{-\infty}^{\infty}\int_0^{r^\gamma/8} \frac{|z-k|^2}{r^{2\alpha}[r^2+(z-k)^2]^{3-\alpha}} \rho d\rho dk\right)^{\frac12} \\ &\le Cr^{-\frac32}\left(\int_{-\infty}^{\infty}\int_0^{r^\gamma/8} \frac{r^{-2}|z-k|^2}{[1+r^{-2}(z-k)^2]^{3-\alpha}} r^{-1} dk\ \rho d\rho\right)^{\frac12} \le Cr^{-\frac32+\gamma} \end{aligned}$$

where $0 \le \alpha < \frac32$.

For the term I_2, using $r > 1$, (12) and $w_\theta \le Cr^{-\beta}$

$$I_2 \le C\int_{-\infty}^{\infty}\int_{r^\gamma/8}^{r/4} \Gamma_1 \rho^{1-\beta} d\rho dk$$

$$
\begin{aligned}
&\le C\left(\int_{-\infty}^{\infty}\int_{r^\gamma/8}^{r/4}\frac{|z-k|}{r^\alpha[r^2+(z-k)^2]^{\frac{3-\alpha}{2}}}\rho^{1-\beta}d\rho dk\right)\\
&\le C\begin{cases} r^{-1+\gamma(2-\beta)} \quad (\beta>2)\\ r^{-1}\ln r \quad (\beta=2)\\ r^{1-\beta} \quad (1<\beta<2)\end{cases}
\end{aligned}
$$

where $0\le\alpha<1$.

Moreover, for the term I_3, by (12) and $w_\theta\le Cr^{-\beta}$

$$
\begin{aligned}
I_3 &\le C\int_{-\infty}^{\infty}\int_{r/4}^{r-r^\delta/2}\Gamma_1\rho^{1-\beta}d\rho dk\\
&\le C\left(\int_{-\infty}^{\infty}\int_{r/4}^{r-r^\delta/2}\frac{|z-k|}{r^\alpha[(r-\rho)^2+(z-k)^2]^{\frac{3-\alpha}{2}}}\rho^{1-\beta}d\rho dk\right)\\
&\le Cr^{-\alpha-\delta+\alpha\delta}\left(\int_{-\infty}^{\infty}\int_{r/4}^{r-r^\delta/2}\frac{r^{-\delta}|z-k|}{[\frac{1}{4}+r^{-2\delta}(z-k)^2]^{\frac{3-\alpha}{2}}}r^{-\delta}dk\rho^{1-\beta}d\rho\right)\\
&\le C\begin{cases} r^{2-\beta-\alpha-\delta+\delta\alpha} \quad (\beta<2 \text{ or } \beta>2)\\ r^{-\alpha-\delta+\alpha\delta}\ln r \quad (\beta=2)\end{cases}
\end{aligned}
$$

where $0\le\alpha<1$.

Similarly, for I_5 we have

$$
I_5 \le C\begin{cases} r^{2-\beta-\alpha-\delta+\delta\alpha} \quad (\beta<2 \text{ or } \beta>2)\\ r^{-\alpha-\delta+\alpha\delta}\ln r \quad (\beta=2)\end{cases}
$$

where $0\le\alpha<1$.

Furthermore, for $0\le\alpha<1$ by (12) and $w_\theta\le Cr^{-\beta}$ we have

$$
\begin{aligned}
I_4 &\le C\int_{-\infty}^{\infty}\int_{r-r^\delta/2}^{r+r^\delta/2}\Gamma_1\rho^{1-\beta}d\rho dk\\
&\le C\left(\int_{-\infty}^{\infty}\int_{r-r^\delta/2}^{r+r^\delta/2}\frac{|z-k|}{r^\alpha[(r-\rho)^2+(z-k)^2]^{\frac{3-\alpha}{2}}}\rho^{1-\beta}d\rho dk\right)\\
&\le C\left(\int_{r-r^\delta/2}^{r+r^\delta/2}r^{-\alpha}(r-\rho)^{-1+\alpha}\rho^{1-\beta}d\rho\right)\\
&\le Cr^{1-\beta-\alpha}\left(\int_{r-r^\delta/2}^{r+r^\delta/2}(r-\rho)^{-1+\alpha}d\rho\right)\\
&\le Cr^{1-\beta-\alpha+\delta\alpha} \quad (\beta>1)
\end{aligned}
$$

Finally, (12) and $w_\theta\le Cr^{-\beta}$ yield that

$$
I_6 \le C\int_{-\infty}^{\infty}\int_{4r}^{\infty}\Gamma_1\rho^{1-\beta}d\rho dk
$$

$$\leq C\left(\int_{-\infty}^{\infty}\int_{4r}^{\infty}\frac{|z-k|}{\rho^\alpha[\rho^2+(z-k)^2]^{\frac{3-\alpha}{2}}}\rho^{1-\beta}d\rho dk\right)$$
$$\leq Cr^{1-\beta} \quad (\beta>1)$$

Hence, concluding the estimates of $I_1,\cdots,I_6$, we have the following arguments.

Case a. $\beta>2$. At this time, we have

$$I\leq C\left[r^{-\frac{3}{2}+\gamma}+r^{-1+\gamma(2-\beta)}+r^{2-\beta-\alpha-\delta+\delta\alpha}+r^{1-\beta-\alpha+\delta\alpha}+r^{1-\beta}\right]$$

where $0\leq\alpha<1$ and $0\leq\gamma,\delta\leq 1$.

First, we choose $\gamma=\frac{1}{2(\beta-1)}$ such that $-\frac{3}{2}+\gamma=-1+\gamma(2-\beta)$. Furthermore, we take $\alpha\uparrow 1,\delta\uparrow 1$ such that

$$(1-\delta)(1-\alpha)\leq\beta-\frac{5}{2}+\frac{1}{2(\beta-1)}$$

which implies

$$-1+\gamma(2-\beta)\geq 2-\beta-\alpha-\delta+\delta\alpha$$

Moreover, note that

$$2-\beta-\alpha-\delta+\delta\alpha\geq 1-\beta\geq 1-\beta-\alpha+\delta\alpha$$

Then, we get for $r>1$

$$|u_r(r,z)|\leq Cr^{-\frac{3}{2}+\frac{1}{2(\beta-1)}}.$$

Case b. $\beta<2$. At this time, we have

$$I\leq C\left[r^{-\frac{3}{2}+\gamma}+r^{2-\beta-\alpha-\delta+\delta\alpha}+r^{1-\beta-\alpha+\delta\alpha}+r^{1-\beta}\right]$$

where $0\leq\alpha<1$ and $0\leq\gamma,\delta\leq 1$. We choose $\gamma=0$ and $\delta=1$, then we get

$$|u_r(r,z)|\leq Cr^{1-\beta}.$$

Case c. $\beta=2$. At this time, we have

$$I\leq C\left[r^{-\frac{3}{2}+\gamma}+r^{-1}\ln r+r^{-\alpha-\delta+\alpha\delta}\ln r+r^{1-\beta-\alpha+\delta\alpha}+r^{1-\beta}\right]$$

where $0\leq\alpha<1$ and $0\leq\gamma,\delta\leq 1$. We choose $\gamma=0$ and $\delta=1$, then we get

$$|u_r(r,z)|\leq Cr^{-1}\ln r.$$

Hence we complete the estimate of $u_r(r,z)$.

Note that the bound of Γ_1 used as above is similar to the estimates of Γ_2 and Γ_3. Hence similar arguments hold for u_z and u_θ. The proof of Proposition 4.1 is complete.

Proof of Lemma 4.3. The remaining part is devoted to proving Lemma 4.3, which is similar to that of [6], where the case $\frac{r}{4} < \rho < 4r$ is discussed. Here we consider all the value $\rho > 0$ and sketch the proof. First, for $k > 0$ and $\beta \geq 1$ we find

$$I = \int_0^{\frac{\pi}{2}} \frac{d\phi}{(\sqrt{1 + k\sin^2\phi})^\beta} \leq \begin{cases} C(\delta)\min\{1, k^{-\frac{\delta}{2}}\}, & \beta = 1 \\ C(\beta)\min\{1, k^{-\frac{1}{2}}\}, & \beta > 1 \end{cases} \tag{13}$$

for any $0 \leq \delta < 1$. Obviously, $k \leq C$ holds, and next we assume that k is large enough. Then for $0 < \ell < 1$

$$I \leq \ell + \int_\ell^{\frac{\pi}{2}} \frac{d\phi}{(k\sin^2\phi)^{\beta/2}}$$

Due to $\phi \leq 2\sin\phi$ for $\phi \in (0, \frac{\pi}{2})$, we have

$$I \leq \ell + 2k^{-\beta/2}(\ln(\frac{\pi}{2}) - \ln\ell), \quad \beta = 1,$$

and

$$I \leq \ell + 2^\beta k^{-\beta/2}\frac{(\frac{\pi}{2})^{1-\beta} - \ell^{1-\beta}}{1-\beta}, \quad \beta > 1,$$

which yield the required bound (13) by choosing a suitable ℓ.

Obviously, from the formulas of Γ_2, Γ_3 and Γ_1, we have

$$|\Gamma_i(r, \rho, z-k)| \leq \frac{\rho + r}{[(r-\rho)^2 + (z-k)^2]^{\frac{3}{2}}}, \quad i = 2, 3; \tag{14}$$

$$|\Gamma_1(r, \rho, z-k)| \leq \frac{|z-k|}{[(r-\rho)^2 + (z-k)^2]^{\frac{3}{2}}} \tag{15}$$

for all $\rho > 0$ and $r > 0$.

Next we go on estimating Γ_2, Γ_3, and Γ_1 carefully, respectively.

Step I. Noting the periodic and even property and variable transform for ϕ, we also have

$$\begin{aligned} \Gamma_2 &= -\int_0^{2\pi} \frac{1}{4\pi} \frac{\rho - r\cos\phi}{[r^2 + \rho^2 - 2r\rho\cos\phi + (z-k)^2]^{\frac{3}{2}}} d\phi \\ &= -\int_0^{\frac{\pi}{2}} \frac{1}{\pi} \frac{\rho - r\cos 2\phi}{[r^2 + \rho^2 - 2r\rho\cos 2\phi + (z-k)^2]^{\frac{3}{2}}} d\phi \end{aligned}$$

and

$$\Gamma_2 = -\int_0^{\frac{\pi}{2}} \frac{1}{2\pi} \frac{\rho^2 - 2r\rho\cos 2\phi + r^2 + \rho^2 - r^2}{\rho[(r-\rho)^2 + 4r\rho\sin^2\phi + (z-k)^2]^{\frac{3}{2}}} d\phi$$

$$\begin{aligned} &\leq C\frac{1}{\rho\sqrt{(r-\rho)^2+(z-k)^2}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1+K\sin^2\phi}} \\ &\quad -\frac{1}{2\pi}\frac{1}{\rho[(r-\rho)^2+(z-k)^2]^{\frac{3}{2}}}\int_0^{\pi/2}\frac{\rho^2-r^2}{(\sqrt{1+K\sin^2\phi})^3}d\phi \\ &\doteq I_1+I_2 \end{aligned}$$

where

$$K=\frac{4r\rho}{(r-\rho)^2+(z-k)^2}$$

When $K\leq 1$, that is $4r\rho\leq(r-\rho)^2+(z-k)^2$, we have $(r-\rho)^2+(z-k)^2\geq\frac{1}{2}r^2$ for $\rho\leq\frac{r}{2}$ and $(r-\rho)^2+(z-k)^2\geq 2r^2$ for $\frac{r}{2}\leq\rho\leq 4r$. Moreover, for $\rho\geq 4r$ we have

$$(r-\rho)^2+(z-k)^2\geq(\frac{3}{4}\rho)^2\geq(\frac{3}{5}(\rho+r))^2\geq\frac{9}{25}(\rho+r)^2$$

Hence for $K\leq 1$ we have

$$\Gamma_2\leq C\frac{1}{\rho\sqrt{(r-\rho)^2+(z-k)^2}} \tag{16}$$

When $K>1$, by (13) we have

$$\begin{aligned} \Gamma_2 &\leq C(\delta)\frac{1}{\rho\sqrt{(r-\rho)^2+(z-k)^2}} \\ &\quad\cdot\left[\left(\frac{(r-\rho)^2+(z-k)^2}{4r\rho}\right)^{\frac{\delta}{2}}+\frac{|\rho^2-r^2|}{(r-\rho)^2+(z-k)^2}\left(\frac{(r-\rho)^2+(z-k)^2}{4r\rho}\right)^{\frac{1}{2}}\right] \end{aligned} \tag{17}$$

where $0\leq\delta<1$.

Case a. For $r>1$ and $\rho\leq\frac{r}{4}$ or $\rho>4r$, by (14) we know the estimate (11) holds.

Case b. For $r>1$ and $\frac{r}{4}\leq\rho\leq 4r$ with $K\leq 1$, by (14) and (16) we know the estimate (11) holds.

Case c. For $r>1$ and $\frac{r}{4}\leq\rho\leq 4r$ with $K>>1$, by (14) and (17) we know the estimate (11) holds by noting that $(r-\rho)^2+(z-k)^2\leq 16r^2$ and

$$\frac{|\rho^2-r^2|}{(r-\rho)^2+(z-k)^2}\left(\frac{(r-\rho)^2+(z-k)^2}{4r\rho}\right)^{\frac{1}{2}}\leq\frac{\rho+r}{\sqrt{4r\rho}}\leq 5.$$

Hence the proof of Γ_2 is complete.

Step II. The term Γ_2 is similar and we omitted the details.

Step III. The term Γ_1 is estimated as follows.

$$\Gamma_1(r,\rho,z-k) = \frac{1}{2\pi}\int_0^{\pi}\frac{z-k}{[r^2+\rho^2-2r\rho\cos\phi+(z-k)^2]^{\frac{3}{2}}}\cos\phi d\phi$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{z-k}{[(r-\rho)^2+4r\rho\sin^2\phi+(z-k)^2]^{\frac{3}{2}}}\cos 2\phi d\phi \\
&\le C\frac{|z-k|}{[(r-\rho)^2+(z-k)^2]^{\frac{3}{2}}}\int_0^{\pi/2}\frac{1}{(\sqrt{1+K\sin^2\phi})^3}d\phi \\
&\doteq I'
\end{aligned}
$$

where

$$
K = \frac{4r\rho}{(r-\rho)^2+(z-k)^2}
$$

When $K \le 1$, i.e. $4r\rho \le (r-\rho)^2+(z-k)^2$, we have $(r-\rho)^2+(z-k)^2 \ge \frac{1}{2}r^2$ for $\rho \le \frac{r}{2}$ and $(r-\rho)^2+(z-k)^2 \ge 2r^2$ for $\frac{r}{2} \le \rho \le 4r$. Moreover, for $\rho \ge 4r$ we have

$$
(r-\rho)^2+(z-k)^2 \ge (\frac{3}{4}\rho)^2
$$

Hence for $K \le 1$ we have

$$
(r-\rho)^2+(z-k)^2 \ge \frac{1}{2}(\max\{r,\rho\})^2
$$

Using (15), for $K \le 1$ we get

$$
|\Gamma_1(r,\rho,z-k)| \le \frac{C|z-k|}{(\max\{\rho,r\})^\alpha[(r-\rho)^2+(z-k)^2]^{\frac{3-\alpha}{2}}}, \tag{18}
$$

where $0 \le \alpha \le 3$.

When $K > 1$, i.e. $4r\rho \ge (r-\rho)^2+(z-k)^2$, which implies $\rho > \frac{1}{8}r$, by (13) we have

$$
\begin{aligned}
|\Gamma_1(r,\rho,z-k)| &\le \frac{C|z-k|}{[(r-\rho)^2+(z-k)^2]^{\frac{3}{2}}}\left(\frac{(r-\rho)^2+(z-k)^2}{4r\rho}\right)^{\frac{1}{2}} \\
&\le \frac{C|z-k|}{\sqrt{r\rho}[(r-\rho)^2+(z-k)^2]}
\end{aligned}
$$

Thus for $\frac{1}{8}r < \rho < 4r$, we have

$$
|\Gamma_1(r,\rho,z-k)| \le \frac{C|z-k|}{(\max\{\rho,r\})^\alpha[(r-\rho)^2+(z-k)^2]^{\frac{3-\alpha}{2}}} \tag{19}
$$

where $0 \le \alpha \le 1$. For $\rho \ge 4r$, by (15) we also derive that

$$
|\Gamma_1(r,\rho,z-k)| \le \frac{C|z-k|}{(\max\{\rho,r\})^\alpha[(r-\rho)^2+(z-k)^2]^{\frac{3-\alpha}{2}}} \tag{20}
$$

where $0 \le \alpha \le 3$.

Concluding the estimates (18), (19) and (20), we complete the proof of the inequality (12).

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