

BIELLIPTIC SMOOTH PLANE CURVES AND QUADRATIC POINTS

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ABSTRACT. Let C_k be a smooth projective curve over a global field k , which is neither rational nor elliptic. Harris-Silverman [15], when $p = 0$, and Schweizer, when $p > 0$ together with an extra condition on the Jacobian variety $\text{Jac}(C_k)$ arising from Mordell's conjecture, showed that C has infinitely many quadratic points over some finite field extension L/k inside \bar{k} (a fixed algebraic closure of k) if and only if C is hyperelliptic or bielliptic.

Now, let C_k be a smooth plane curve of a fixed degree $d \geq 4$ with $p = 0$ or $p > (d-1)(d-2)+1$ (up to an extra condition on $\text{Jac}(C_k)$ in positive characteristic). Then, C_k admits always finitely many quadratic points unless $d = 4$; see Theorem 2.8. A so-called *geometrically complete families* for the different strata of smooth bielliptic plane quartic curves by their automorphism groups, are given; see Theorem 2.3. Interestingly, we show (in a very simple way) that there are only finitely many quadratic extensions $k(\sqrt{D})$ of a fixed number field k , in which we may have more solutions to the Fermat's and the Klein's equations of degree $d \geq 5$; $X^d + Y^d - Z^d = 0$ and $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$ respectively, than these over k (the same holds for any non-singular projective plane equation of degree $d \geq 5$ over k , and also in general when k is a global field after imposing an extra condition on $\text{Jac}(C_k)$); see Corollary 2.9.

Finally, given a stratum $\mathcal{M}(G)$ of smooth plane bielliptic quartic curves over a number field k associated to an automorphism group G , we conjecture in section §3 that there are subsets $\mathcal{E}, \mathcal{D} \subset \mathcal{M}(G)$ of infinite cardinality, such that all members of \mathcal{E} (resp. \mathcal{D}) have finitely (resp. infinitely) many quadratic points over k . We support our claim when $k = \mathbb{Q}$ and $G = \mathbb{Z}/6\mathbb{Z}$, or $\text{GAP}(16, 13)$; see Theorems 3.2, 3.4, 3.6 and 3.8.

1. THE INTERPLAY BETWEEN HYPERELLIPTIC (RESP. BIELLIPTIC) CURVES AND QUADRATIC POINTS

Let \bar{k} (resp. k^{sep}) be a fixed algebraic (resp. separable) closure of a field k of characteristic $p \neq 2$. By C_k we mean a smooth projective curve defined over k of geometric genus $g_C \geq 2$ (that is, the genus of the base extension $C_{\bar{k}} := C \otimes_k \bar{k}$ is at least two), and non-trivial automorphism group $\text{Aut}(C_{\bar{k}})$. The set of all k -points on C_k is denoted by $C(k)$.

An arithmetic geometer finds a lot of interest to investigate the cardinality of $C(k)$. When k is a global field; i.e. when k is a finite field extension of either \mathbb{Q} or $\mathbb{F}_p(T)$, or it is the function field of \mathbb{P}^1 over the finite field \mathbb{F}_p (in this case, we denote the finite field $k \cap \bar{\mathbb{F}}_p$ by \mathbb{F}_q , where q is a power of p , a priori).

In zero characteristic, we have the following result on *Mordell's Conjecture* due to Faltings [11, 12]:

Theorem 1.1 (Faltings). *Given a smooth projective curve C_k as above defined over a number field k , the set $C(k)$ is always finite.*

On the other hand, we obtain by Grauert [14] and Samuel [27, Theorem 4 and 5b] the next result in positive characteristic:

Theorem 1.2 (Grauert-Samuel). *Let C_k be a smooth projective curve over a global field k of characteristic $p > 0$. Assume also that C_k is conservative (see the definition in §1.1). Then, $C(k)$ is always finite except possibly when $C_k \otimes_k k^{\text{sep}}$ is isomorphic to a smooth projective curve C' over a finite field \mathbb{F}_{q^n} (in this situation, C_k or $C_k \otimes_k k^{\text{sep}}$ is called an isotrivial curve). More concretely, for $C(k)$ to be infinite, it suffices the existence of a finite Galois extension ℓ'/k , an ℓ' -isomorphism $\varphi : C_{\ell'} = C_k \otimes_k \ell' \rightarrow C' \otimes_{\mathbb{F}_{q^n}} \ell'$, an injection $\text{Gal}(\ell'/k) \hookrightarrow \text{Aut}(C' \otimes_{\mathbb{F}_{q^n}} \ell') : s \mapsto j_s := \varphi^s \circ \varphi^{-1}$, and a point $z \in C'(\ell') \setminus C'(\bar{\mathbb{F}}_p)$ satisfying $j_s(z) = z^s$ for all $s \in \text{Gal}(\ell'/k)$. Under these conditions, it exists a finite family $(x_i)_{i \in I}$ of points of $C'(\ell')$ with $x_i^s = j_s(x_i)$ for all $i \in I, s \in \text{Gal}(\ell'/k)$, and*

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moreover the infinite set $C(k)$ is given by:

$$C(k) = \left(\bigcup_{i \in I, m \geq 0} \varphi^{-1}(f^m(x_i)) \right) \cup \left(C(k) \cap (\varphi^{-1}(C'(\overline{\mathbb{F}}_{q^n}))) \right)$$

where $f : x \mapsto x^{q^{n\mathfrak{B}_0}}$ and \mathfrak{B}_0 is the strict least positive integer \mathfrak{B} satisfying $(x \mapsto x^{q^{n\mathfrak{B}}}) \circ j_s = j_s \circ (x \mapsto x^{q^{n\mathfrak{B}}})$ for all $s \in \text{Gal}(\ell'/k)$.¹

Remark 1.3. *Grauert-Samuel Theorem requires C_k to be geometrically non-singular instead of being conservative, but if C_k is conservative then it is also geometrically non-singular by [14].*

Remark 1.4. *An example of Theorem 1.2, where an infinite number of points occur, is the Fermat curve $C_k : X^d + Y^d = Z^d$ over the global field $k = \mathbb{F}_p(w, v)/(w^d + v^d - 1)$ with d and p coprime. It is isomorphic to the Fermat curve over \mathbb{F}_p , and we get an infinite set of points over k on it namely, $(f^n(w), f^n(v), 1)$ for each positive integer n by using the point $(w, v, 1) \in C(k) \setminus C(\overline{\mathbb{F}}_p)$ where f is the Frobenius $x \mapsto x^p$.*

For a finite field extension L/k inside \overline{k} , the set of quadratic points of C_k over L , denoted by $\Gamma_2(C, L)$, is given by

$$\Gamma_2(C, L) := \bigcup \{C(L') : L \subseteq L' \subseteq \overline{k} \text{ with } [L' : L] \leq 2\},$$

where (by an abuse of notation) $C(L')$ denotes the set of L' -points of $C_{L'} := C \otimes_k L'$.

It is a natural question to study whether $\Gamma_2(C, L)$ is finite or not.

Definition 1.5. A smooth projective curve C_k is called *hyperelliptic* (resp. *bielliptic*) over k if there exists a degree two k -morphism to a projective line \mathbb{P}_k^1 (resp. to an elliptic curve E_k) over k . We simply call it *hyperelliptic* (resp. *bielliptic*) if $C_{\overline{k}}$ is hyperelliptic (resp. bielliptic) over \overline{k} .

Remark 1.6. *Clearly, if C_k is hyperelliptic over k and k is not a finite field, then $\Gamma_2(C, k)$ is an infinite set. Also, if C_k is bielliptic over k and E_k has infinitely many k -points, then $\Gamma_2(C, k)$ is again an infinite set.*

The following result is well-known in the literature (cf. [28] for (ii)).

Proposition 1.7. *Let C_k be a smooth projective curve over k . Then,*

- (i) *C_k is hyperelliptic if and only if there exists a (hyperelliptic) involution $w \in \text{Aut}(C_{\overline{k}})$, having exactly $2g_C + 2$ fixed points. In particular, if C_k is hyperelliptic, then w is unique, defined over a finite purely inseparable extension ℓ/k of k , and it is called the hyperelliptic involution of C_k .*
- (ii) *C_k is bielliptic if and only if there exists a (bielliptic) involution $\tilde{w} \in \text{Aut}(C_{\overline{k}})$, having $2g_C - 2$ fixed points. If C_k is bielliptic and $g_C \geq 6$, then there is a unique bielliptic involution, which belongs to the center of $\text{Aut}(C_{\overline{k}})$ and defined over a finite purely inseparable extension ℓ of k .*

1.1. Conservative curves. Let C_k be a smooth projective curve over a global field k of characteristic $p > 0$. The genus of C_k relative to k is defined to be the integer $g_{C,k}$ that makes the Riemann-Roch formula hold, that is, for any k -divisor D of C , of sufficiently large degree; $\ell(D) = \deg(D) + 1 - g_{C,k}$, where $\ell(D)$ is the dimension of the k -(Riemann-Roch) vector space associated to D . The relative genus may change under inseparable extensions of k inside \overline{k} , see for example [32]. The absolute genus of C_k is defined to be the genus of $C_{\overline{k}}$ relative to \overline{k} , in particular it equals to the geometric genus g_C we have seen before.

The relative genus $g_{C,k}$ to k is an upper bound for the absolute genus g_C . We call C_k *conservative over k* , if $g_C = g_{C,k}$ (in particular, it is not genus-changing under inseparable extensions between k and \overline{k}).

First, we prove:

Proposition 1.8. *Let C_k be a smooth projective curve defined over a global field k of characteristic $p > 0$, that is conservative over k . Assume also that C_k is hyperelliptic with hyperelliptic involution w defined over a finite purely inseparable extension ℓ/k in \overline{k} . Then, there is a (unique) degree two ℓ -morphism φ to a conic Q over ℓ . Moreover, if C_ℓ (or more generally if $C_\ell/\langle w \rangle$) has an ℓ -point, then we reduce to that Q is ℓ -isomorphic to \mathbb{P}_ℓ^1 .*

¹It is assumed in [27, Theorem 5b] that all automorphisms of C' are also defined over \mathbb{F}_{q^n} , in particular, $f \circ j_s = j_s \circ f$ for some power of the Frobenius. Accordingly, we will impose the latter condition directly.

Proof. By assumption $C_\ell/\langle w \rangle$ is a genus 0 curve defined over ℓ . Thus it corresponds to a conic over ℓ , since by definition it is a twist for \mathbb{P}_ℓ^1 . Next, the covering $\pi : C_\ell \rightarrow C_\ell/\langle w \rangle$ is Galois (being cyclic of degree 2), hence π is defined over a separable extension of ℓ inside \bar{k} , a priori. Fix a separable closure $\ell^{\text{sep}} \subseteq \bar{k}$ of ℓ and let $\text{Gal}(\ell^{\text{sep}}/\ell)$ denotes the absolute Galois group. By the uniqueness of the hyperelliptic involution w , π and $\sigma\pi$ only differs by an automorphism ξ_σ of $\mathbb{P}_{\ell^{\text{sep}}}^1$, for any $\sigma \in \text{Gal}(\ell^{\text{sep}}/\ell)$. In other words, for any $\sigma \in \text{Gal}(\ell^{\text{sep}}/\ell)$, we obtain $\xi_\sigma \in \text{PGL}_2(\ell^{\text{sep}})$; the projective general linear group of 2×2 matrices over ℓ^{sep} , where $\sigma\pi = \xi_\sigma \circ \pi$. It can be easily checked that $\xi_{\sigma\tau} = {}^\sigma\xi_\tau \circ \xi_\sigma$ for all $\sigma, \tau \in \text{Gal}(\ell^{\text{sep}}/\ell)$. Therefore,

$$\xi : \text{Gal}(\ell^{\text{sep}}/\ell) \rightarrow \text{PGL}_2(\ell^{\text{sep}}) : \sigma \mapsto \xi_\sigma$$

defines a 1-cocycle, in particular, an element of the first Galois cohomology set $H^1(\text{Gal}(\ell^{\text{sep}}/\ell), \text{PGL}_2(\ell^{\text{sep}}))$. Using the Twisting Theory for Varieties (cf. [29, III.1]), it exists a conic Q (a twist for \mathbb{P}_ℓ^1) over ℓ and an isomorphism $\varphi_0 : Q \rightarrow C_\ell/\langle w \rangle$ given by the rule $\xi_\sigma = {}^\sigma\varphi_0 \circ \varphi_0^{-1}$, for all $\sigma \in \text{Gal}(\ell^{\text{sep}}/\ell)$. Consequently, $\varphi := \varphi_0^{-1} \circ \pi : C_\ell \rightarrow Q$ is an ℓ -morphism from C_ℓ to Q .

The rest is direct, since a conic over ℓ that has an ℓ -point is ℓ -isomorphic to \mathbb{P}_ℓ^1 . This obviously happens if C_ℓ or $C_\ell/\langle w \rangle$ has an ℓ -point via the morphism φ . \square

Corollary 1.9. *Let C_k be a smooth projective curve defined over a global field k of characteristic $p > 0$, that is conservative over k . Then, C_k is hyperelliptic if and only if there exists a finite extension L/k inside \bar{k} where $C_k \otimes_k L$ is hyperelliptic over L . In this situation, $\Gamma_2(C, L)$ is an infinite set.*

Also, we show:

Proposition 1.10. *Let C_k be a smooth projective curve defined over a global field k of characteristic $p > 0$, that is conservative over k . Then, C_k is bielliptic if and only if there exists a finite extension L/k inside \bar{k} where $C_k \otimes_k L$ is bielliptic over L (hence, $\Gamma_2(C, L')$ is an infinite set for some finite extension L'/L inside \bar{k}).*

Proof. Assume that C_k is bielliptic and consider a bielliptic involution $\tilde{w} \in \text{Aut}(C_{\bar{k}})$ as in Proposition 1.7. Since $g_C \geq 2$, $\text{Aut}(C_{\bar{k}})$ is a finite group, and so we only have finitely many possibilities for the Galois group $\text{Gal}(\bar{k}/k)$ -action on \tilde{w} . Accordingly, \tilde{w} must be defined over a finite field extension L_0/k inside \bar{k} . Because C_k is also conservative over k , then by making a finite extension L/L_0 with $L \subseteq \bar{k}$, we get a degree two L -morphism from $C_k \otimes_k L$ to a genus one curve that has L -points, hence to an elliptic curve E over L , and hence C_k is bielliptic over L . Finally, it suffices to apply base extension to some $L \subseteq L' \subseteq \bar{k}$ of finite index so that $E \otimes_L L'$ has positive rank. Consequently, $\Gamma_2(C, L')$ is infinite and we conclude. \square

Furthermore, inspired by the case of number fields (Theorems 1.16 and 1.17 below), we have [28, Theorem 5.1]:

Theorem 1.11 (Schweizer, version I). *Let C_k be a conservative smooth projective curve over a finite field extension k of $\mathbb{F}_q(T)$; i.e. k is a global field of characteristic $p > 0$. Under the conditions that; $g_C \geq 3$, $C(k) \neq \emptyset$, C_k is not hyperelliptic over k , and that the Jacobian variety $\text{Jac}(C_k)$ over \bar{k} has no non-zero homomorphic images defined over $\bar{\mathbb{F}}_q$, then $\Gamma_2(C, k)$ is an infinite set only if $\text{Jac}(C_k)$ over k contains an elliptic curve E_k of positive rank, moreover there is a degree two morphism from C_k to E_k .*

Remark 1.12. *The conditions on $\text{Jac}(C_k)$ imply that E_k is not isotrivial, (i.e. its j -invariant of E_k does not belong to $\bar{\mathbb{F}}_q$) and also that any factor of the Jacobian is a Jacobian of an isotrivial curve (as it happens, for example, by the Jacobian of the Fermat curve $X^d + Y^d = Z^d$ over a global field of positive characteristic, relatively prime with d , which is isotrivial).*

Remark 1.13. *To ensure that the degree two morphism from C_k to E_k is also defined over k , it suffices to assume that $g_C \geq 6$ and then to follow the argument of [15, Lemma 5], provided that Castelnuovo's inequality holds over the function field extensions over a global field k that are involved in Castelnuovo's inequality (we mention that by the proof of [15, Lemma 5] all of these extensions are of degree 2 in our situation). By [31, Theorem 3.11.3], the inequality is true when k is perfect, which is not the case. However, in the proof of [31, Theorem 3.11.3] under the assumption that C_k is conservative, one only needs to take the characteristic*

p big enough so that inseparable extensions do not appear (for example, to be greater than the degrees of the function field extensions involved in Castelnuovo's inequality). Consequently, under the hypothesis of Theorem 1.11 together with the assumptions $g_C \geq 6$ and $p > 2$, we have C_k is bielliptic over k .

The next result is a weaker version of Theorem 1.11.

Theorem 1.14 (Schweizer, version II). *Let C_k be a smooth projective curve defined over a global field k of characteristic $p > 0$, that is conservative over k . Under the conditions that; $g_C \geq 3$, and that the Jacobian variety $\text{Jac}(C_k)$ over \bar{k} has no non-zero homomorphic images defined over \mathbb{F}_q , then, there exists a finite field extension L/k inside \bar{k} such that $\Gamma_2(C, L)$ is infinite if and only if C_k is bielliptic or hyperelliptic.*

1.2. Number fields. Let us now assume that k is a global field of characteristic zero, that is, k is a number field. If we follow the same line of discussion in the proofs of Propositions 1.7, 1.8 and 1.10, we deduce, since inseparable extensions between k and \bar{k} do not exist, that:

Proposition 1.15. *A smooth projective curve C_k defined over a number field k is hyperelliptic with hyperelliptic involution w such that $C/\langle w \rangle(k) \neq \emptyset$ if and only if it is hyperelliptic over k . Also, for $g_C \geq 6$, C_k is bielliptic if and only if it is bielliptic over k .*

The next results are quite known to the specialists (they follow from the arguments of Abramovich-Harris in [1], or from Harris-Silverman in [15]). One also can read a proof in [7].

Theorem 1.16. *Let C_k be a smooth projective curve over a number field k with $g_C \geq 2$. Hence, $\Gamma_2(C, k)$ is an infinite set if and only if C_k is hyperelliptic over k or bielliptic over k , such that it exists a degree two k -morphism from C_k to an elliptic curve E_k of positive rank over k .*

The next result in [15] is a weaker version of Theorem 1.16, not controlling the base field for quadratic points:

Theorem 1.17 (Harris-Silverman). *Let C_k be a smooth projective curve over a number field k with $g_C \geq 2$. Hence, there exists a finite field extension L/k inside \bar{k} such that $\Gamma_2(C, L)$ is an infinite set if and only if C_k is hyperelliptic or bielliptic.*

2. BIELLIPTIC SMOOTH PLANE CURVES AND THEIR QUADRATIC POINTS

Let k be a field and $k \subseteq L \subseteq \bar{k}$ be a field extension. A curve C is said to be a *smooth (k, L) -plane curve of degree d* or equivalently, L is a *plane model field of definition for C* , if C as a smooth projective curve is defined over k such that $C_L := C \otimes_k L$ admits a non-singular plane model over L , that is, C_L is L -isomorphic to a non-singular homogenous projective polynomial equation $F(X, Y, Z) = 0$ of degree d with coefficients in L . When \bar{k} is a plane model field of definition for C , then $C_{\bar{k}}$ has a unique g_d^2 -linear series modulo $\text{PGL}_3(\bar{k})$ -conjugation, which allows us to embed $C_{\bar{k}}$ into $\mathbb{P}_{\bar{k}}^2$ as a non-singular plane model over \bar{k} of degree d for some d (in this case, $g := g_C = (d-1)(d-2)/2$). Also, we call C a *smooth plane curve of degree d over k* when k itself is plane model field of definition for C .

The present authors and et al. addressed in [5, §2] the problem where non-singular plane models of smooth projective curves (also of their twists) are defined. For instance, we showed:

Theorem 2.1 (Badr-Bars-García). *Given a smooth (k, k^{sep}) -plane curve C of degree $d \geq 4$, it does not necessarily have a non-singular plane model defined over the field k . However, it does in any of the following cases; if d is coprime with 3, if $C(k) \neq \emptyset$, or if the 3-torsion $\text{Br}(k)[3]$ of the Brauer group $\text{Br}(k)$ is trivial. In general, C is a smooth (k, L) -plane curve of degree d for some cubic Galois field extension L/k .*

It is a basic fact in algebraic geometry that a smooth (k, \bar{k}) -plane curve of degree $d \geq 4$ is non-hyperelliptic. On the other hand, the (geometric) gonality of a smooth projective curve C is defined to be the minimum degree of a \bar{k} -morphism from $C_{\bar{k}}$ to the projective line $\mathbb{P}_{\bar{k}}^1$. For the special case of smooth (k, \bar{k}) -plane curves C of degree $d \geq 4$, the gonality equals to $d-1$ (cf. Namba [25] for zero characteristic and Homma [17] for positive characteristic, also one can read the assertion in [35, p. 341]). Consequently, when $d \geq 6$, the gonality of C is at least 5, and C can not be bielliptic. When $d = 5$, we use the fact that $C_{\bar{k}}$ has an automorphism \tilde{w} of order 2

(a bielliptic involution) that also fixes $2g - 2 = 10$ points on $C_{\bar{k}}$. Following the techniques in [3], we may assume (up to $\mathrm{PGL}_3(\bar{k})$ -conjugation) that $\tilde{w} : X \mapsto X, Y \mapsto Y, Z \mapsto -Z$ and $C_{\bar{k}} : Z^4 L_{1,Z} + Z^2 L_{2,Z} + L_{5,Z}$, where $L_{i,Z}$ denotes a homogenous degree i , binary form in X, Y . Thus, \tilde{w} fixes exactly $d + 1 = 6 < 10$ points on $C_{\bar{k}}$, a contradiction, and C can not be bielliptic. This proves the result:

Proposition 2.2. *A smooth (k, \bar{k}) -plane curve C of degree $d \geq 5$ is neither hyperelliptic nor bielliptic. Also, C is never hyperelliptic when $d = 4$.*

2.1. Smooth plane quartic curves, which are bielliptic. Next, we characterize smooth (k, \bar{k}) -plane quartic curves that might be bielliptic, hence, $\Gamma_2(C, L)$ is infinite for some finite field extension L/k inside \bar{k} .

Let \mathcal{M}_g be the (coarse) moduli space, representing \bar{k} -isomorphism classes of smooth curves of genus g , and define the substratum $\widetilde{\mathcal{M}}_g^{\mathrm{P}1}(G) \subset \mathcal{M}_g$, where G is a finite non-trivial group, whose \bar{k} -points are \bar{k} -isomorphism classes of smooth plane curves $[C]$, such that $\mathrm{Aut}(C_{\bar{k}})$ is $\mathrm{PGL}_3(\bar{k})$ -conjugated to $\varrho(G)$, for some injective representation $\varrho : G \hookrightarrow \mathrm{PGL}_3(\bar{k})$.

Lercier-Ritzenthaler-Rovetta-Sijsling in [20, §2] introduced the notions of complete, finite and representative families for strata of the moduli space \mathcal{M}_g when k is any field of characteristic $p = 0$ or $p > 2g + 1$. In the case of plane curves, a family \mathcal{C} of smooth plane curves over k is said to be *complete over k for $\widetilde{\mathcal{M}}_g^{\mathrm{P}1}(G)$* if, for any algebraic extension k'/k inside \bar{k} and any k' -point $[C]/k'$ in the stratum $\widetilde{\mathcal{M}}_g^{\mathrm{P}1}(G)$, there exists a non-singular plane model for C defined over k' in the family \mathcal{C} , moreover if such a model is always unique, then the family \mathcal{C} is *representative over k for $\widetilde{\mathcal{M}}_g^{\mathrm{P}1}(G)$* . A family \mathcal{C} is *geometrically complete* (resp. *geometrically representative*) if $\mathcal{C} \otimes_k \bar{k}$ is complete (resp. representative) over \bar{k} .

Theorem 2.3 (Bielliptic quartic curves). *Let C be a smooth (k, \bar{k}) -plane quartic curve over a field k of characteristic $p = 0$ or $p > 7$ ². Then, C is bielliptic if and only if $C \otimes_k \bar{k}$ is isomorphic to a non-singular plane model of the form $Z^4 + Z^2 L_{2,Z} + L_{4,Z} = 0$, where $L_{i,Z}$ is a homogenous binary form in $\bar{k}[X, Y]$ of degree i (in this case, $\tilde{w} : X \mapsto X, Y \mapsto Y, Z \mapsto -Z$ is a bielliptic involution).*

Table 1 below gives a geometrically complete families for each substrata (in GAP library [13] notation) of \mathcal{M}_3 of smooth plane quartic curves over \bar{k} , which are bielliptic.

TABLE 1. Bielliptic geometrically complete classification

| $\mathrm{Aut}(C_{\bar{k}})$ | Families | Restrictions |
|--|--|--|
| $\mathbb{Z}/2\mathbb{Z}$ | $Z^4 + Z^2 L_{2,Z}(X, Y) + L_{4,Z}(X, Y)$ | $L_{2,Z}(X, Y) \neq 0$, not below |
| $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $Z^4 + Z^2(bY^2 + cX^2) + (X^4 + Y^4 + aX^2Y^2)$ | $a \neq \pm b \neq c \neq \pm a$ |
| $\mathbb{Z}/6\mathbb{Z}$ | $Z^4 + aZ^2Y^2 + (X^3Y + Y^4)$ | $a \neq 0$ |
| S_3 | $(X^3 + Y^3)Z + X^2Y^2 + aXYZ^2 + bZ^4$ | $a \neq b, ab \neq 0$ |
| D_4 | $Z^4 + bXYZ^2 + (X^4 + Y^4 + aX^2Y^2)$ | $b \neq 0, \pm \frac{2a}{\sqrt{1-a}}$ |
| GAP(16, 13) | $Z^4 + (X^4 + Y^4 + aX^2Y^2)$ | $\pm a \neq 0, 2, 6, 2\sqrt{-3}$ |
| S_4 | $Z^4 + aZ^2(Y^2 + X^2) + (X^4 + Y^4 + aX^2Y^2)$ | $a \neq 0, \frac{-1 \pm \sqrt{-7}}{2}$ |
| GAP(48, 33) | $Z^4 + (X^4 + Y^4 + (4\zeta_3 + 2)X^2Y^2)$ | – |
| GAP(96, 64) | $Z^4 + (X^4 + Y^4)$ | – |
| $\mathrm{PSL}_2(\mathbb{F}_7)$ | $X^3Y + Y^3Z + Z^3X$ | – |

The algebraic restrictions for the parameters over \bar{k} , in the last column, are taken so that the defining equation is non-singular and has no bigger automorphism group. For example, the term “not below” means to assume more restrictions for no larger automorphism group to occur.

²In case that k is a perfect field, Theorem 2.1 allows us to always assume that C is a smooth plane quartic curve over the base field k , since $d = 4$ is coprime with 3.

Remark 2.4. *The two highlighted cases in Table 1 are not in the prescribed form $Z^4 + Z^2L_{2,Z} + L_{4,Z} = 0$, that is $\text{diag}(1, 1, -1)$ is not a bielliptic involution. However, they do up to $\text{PGL}_3(\bar{k})$ -conjugation. The way to this is simple; any 3×3 projective linear transformation of order two of the plane fixes (pointwise) a line $\mathcal{L} \subset \mathbb{P}_k^2$, called its axis. So, we may ask for a change of variables ϕ such that the transformed \mathcal{L} becomes the projective line $Z = 0$. For example, for $G = S_3$, we may apply the change of variables*

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 0 \end{pmatrix},$$

moving the axis $\mathcal{L} : X - Y = 0$ of $\phi : X \leftrightarrow Y, Z \mapsto Z$ to $Z = 0$. In particular, we obtain the \bar{k} -equivalent model

$$Z^4 - 2Z^2(X^2 - 8XY + (2a + 7)Y^2) + (X^4 + 2(2a - 3)(XY)^2 - 8(a - 1)XY^3 + (4a + 16b - 3)Y^4).$$

Proof of Theorem 2.3. Given a smooth (k, \bar{k}) -plane quartic curve C defined by the form $Z^4 + Z^2L_{2,Z} + L_{4,Z} = 0$ over \bar{k} , the map $C_{\bar{k}} \rightarrow C_{\bar{k}}/\langle \text{diag}(1, 1, -1) \rangle$ is, by Riemann-Hurwitz formula, a two-to-one \bar{k} -morphism to a genus one curve over \bar{k} . Hence, $C_{\bar{k}}/\langle \text{diag}(1, 1, -1) \rangle$ is an elliptic over \bar{k} , and C is bielliptic.

Conversely, let C be a smooth plane quartic curve over k , which is bielliptic. In particular, $C_{\bar{k}}$ admits an order two automorphism that can be taken, up to \bar{k} -projective equivalence, as $\tilde{w} = \text{diag}(1, 1, -1)$ leaving invariant a non-singular plane model $F(X, Y, Z) = 0$ of $C_{\bar{k}}$. By non-singularity, $F(X, Y, Z)$ should be of degree at least 3 in each variable. Consequently, $F(X, Y, Z)$ reduces to the one of the forms $Z^4 + Z^2L_{2,Z}(X, Y) + L_{4,Z}(X, Y) = 0$ or $Z^3L_{1,Z} + ZL_{3,Z} = 0$ (cf. [3]). However, the latter case is absurd, since it decomposes into $Z \cdot G(X, Y, Z)$ and it becomes singular.

The stratification by automorphism groups follows from the work of Henn [16] (see also [6]), and for the families being geometrically complete, we refer to [20, 21]. \square

Be noted that any of the strata $\widetilde{\mathcal{M}}_3^{\text{Pl}}(G)$, whenever it is non-empty, corresponds to a unique representation $\varrho : G \hookrightarrow \text{PGL}_3(\bar{k})$. Consequently, any two smooth plane quartic curves with automorphism groups isomorphic to G are also \bar{k} -isomorphic (this is not the case for higher degrees, since there are smooth (k, \bar{k}) -plane curves C, C' of the same degree $d > 4$ with isomorphic but non-conjugated automorphism groups, in particular, $C_{\bar{k}}$ and $C'_{\bar{k}}$ are not \bar{k} -isomorphic (cf. [2, 4])). Accordingly, by the aid of GAP library [13], we can list down all bielliptic involutions that may happen by considering the fixed $\varrho(G)$ given by Henn [16] (cf. [6, 20, 21]):

Notations. We use ζ_n for a fixed primitive n th root of unity inside \bar{k} when the characteristic of k is coprime with n . A projective linear transformation $A = (a_{i,j})$ of the plane \mathbb{P}_k^2 is often written as

$$[a_{1,1}X + a_{1,2}Y + a_{1,3}Z : a_{2,1}X + a_{2,2}Y + a_{2,3}Z : a_{3,1}X + a_{3,2}Y + a_{3,3}Z],$$

where $\{X, Y, Z\}$ are the homogenous coordinates of \mathbb{P}_k^2 .

Corollary 2.5 (Bielliptic involutions). *Let C be a smooth (k, \bar{k}) -plane quartic curve over a field k of characteristic $p = 0$ or $p > 7$. Assume that C is bielliptic, then the set of bielliptic involutions, acting on a non-singular plane model for $C_{\bar{k}}$ in one of the families of Theorem 2.3, are classified as follows:*

- (i) when $G = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$; $\text{diag}(1, 1, -1)$,
- (ii) when $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; $\text{diag}(1, 1, -1)$, $\text{diag}(1, -1, 1)$, and $\text{diag}(-1, 1, 1)$,
- (iii) when $G = D_4$; $\text{diag}(1, 1, -1)$, $[Y : X : \pm Z]$, and $[Y : -X : \pm\zeta_4 Z]$,
- (iv) when $G = S_3$; $[Y : X : Z]$, $[\zeta_3 Y : \zeta_3^2 X : Z]$, and $[\zeta_3^2 Y : \zeta_3 X : Z]$,
- (v) when $G = \text{GAP}(16, 13)$; $\text{diag}(1, 1, -1)$, $\text{diag}(1, -1, 1)$, $\text{diag}(-1, 1, 1)$, $[Y : X : \pm Z]$, and $[Y : -X : \pm\zeta_4 Z]$,
- (vi) when $G = S_4$; $\text{diag}(1, 1, -1)$, $\text{diag}(1, -1, 1)$, $\text{diag}(-1, 1, 1)$, $[Y : X : \pm Z]$, $[Z : \pm Y : X]$, and $[\pm X : Z : Y]$,
- (vii) when $G = \text{GAP}(48, 33)$; $\text{diag}(1, 1, -1)$, $\text{diag}(1, -1, 1)$, $\text{diag}(-1, 1, 1)$, $[Y : X : \pm Z]$, and $[Y : -X : \pm\zeta_4 Z]$,
- (viii) when $G = \text{GAP}(96, 64)$; $\text{diag}(1, 1, -1)$, $\text{diag}(1, -1, 1)$, $\text{diag}(-1, 1, 1)$, $[Y : X : \pm Z]$, $[Y : -X : \pm\zeta_4 Z]$, $[Z : \pm Y : X]$, $[\pm X : Z : Y]$, $[Z : \pm\zeta_4 Y : -X]$, and $[\pm\zeta_4 X : -Z : Y]$,

(ix) when $G = \mathrm{PSL}_2(\mathbb{F}_7)$; Let $\alpha := \frac{-1+\sqrt{-7}}{2}$, and

$$g := \begin{pmatrix} -2 & \alpha & -1 \\ \alpha & -1 & 1-\alpha \\ -1 & 1-\alpha & -1-\alpha \end{pmatrix}, h := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, s := \begin{pmatrix} -3 & -6 & 2 \\ -6 & 2 & -3 \\ 2 & -3 & -6 \end{pmatrix}.$$

Then, we get the 21 involutions $\psi \circ (\phi_0 \circ s \circ \phi_0^{-1})$ for $\psi \in \langle g, h \rangle$, where

$$\phi_0 := \begin{pmatrix} 1 & 1 + \zeta_7 \alpha & \zeta_7^2 + \zeta_7^6 \\ 1 + \zeta_7 \alpha & \zeta_7^2 + \zeta_7^6 & 1 \\ \zeta_7^2 + \zeta_7^6 & 1 & 1 + \zeta_7 \alpha \end{pmatrix} \circ \begin{pmatrix} -\alpha & 1 & 2\alpha + 3 \\ 2\alpha + 3 & -\alpha & 1 \\ 1 & 2\alpha + 3 & -\alpha \end{pmatrix}.$$

Remark 2.6. One observes that for smooth plane quartic curves that are bielliptic, all bielliptic involutions are defined over a finite separable extension of the base field k , up to conjugation by a linear projective transformation of the plane.

2.2. Infinitude of quadratic points.

Lemma 2.7. Let C_k be a smooth plane curve of degree $d \geq 4$ over k , where k is a global field of characteristic $p > (d-1)(d-2) + 1$. Then, C_k is conservative over k .

Proof. The result follows from [32, Corollary 2], since the relative genus $g_{C,k} = (d-1)(d-2)/2 < (p-1)/2$. \square

Theorem 2.8. Let C be a smooth plane curve of degree $d \geq 4$ over a global field k of characteristic $p = 0$ or $p > (d-1)(d-2) + 1$, and assume in positive characteristic that $\mathrm{Jac}(C_{\bar{k}})$ over \bar{k} has no non-zero homomorphic images defined over $\overline{\mathbb{F}_q}$. For any finite field extension L/k inside \bar{k} , the set of quadratic points $\Gamma_2(C, L)$ of C over L is always a finite set when $d \geq 5$, also it does when $d = 4$ and $\mathrm{Aut}(C_{\bar{k}}) \cong 1, \mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/9\mathbb{Z}$. Moreover,

- (i) if $d = 4$ and $p = 0$, then it exists a number field L/k inside \bar{k} for which $\Gamma_2(C, L)$ is an infinite set and only if $\mathrm{Aut}(C_{\bar{k}}) \cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, S_3, D_4, \mathrm{GAP}(16, 13), S_4, \mathrm{GAP}(48, 33), \mathrm{GAP}(96, 64)$, or $\mathrm{PSL}_2(\mathbb{F}_7)$.
- (ii) if $d = 4$ and $p > 7$, then $\mathrm{Jac}(C)$ over \bar{k} does not contain bielliptic quotients associated to isotrivial elliptic curves only if $C_{\bar{k}}$ belongs to an open set (of the same dimension) of one of the strata $\widetilde{\mathcal{M}}_3^{\mathrm{P}1}(G)$ with $G \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, S_3, D_4, \mathrm{GAP}(16, 13), S_4\}$ ³. Moreover, for such C 's (i.e. when no isotrivial elliptic curves appear), there is a finite field extension L/k inside \bar{k} such that $\Gamma_2(C, L)$ is an infinite set.

Proof. First, when $d \geq 5$ or $d = 4$ and $\mathrm{Aut}(C_{\bar{k}}) \cong 1, \mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/9\mathbb{Z}$, C is neither hyperelliptic nor bielliptic (Proposition 2.2 and Theorem 2.3). Therefore, $\Gamma_2(C, L)$ is a finite set (Theorems 1.14, 1.17). Second, if $d = 4$ and $p = 0$, then, by Theorem 2.3, C is bielliptic if and only if $\mathrm{Aut}(C_{\bar{k}}) \cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, S_3, D_4, \mathrm{GAP}(16, 13), S_4, \mathrm{GAP}(48, 33), \mathrm{GAP}(96, 64)$, or $\mathrm{PSL}_2(\mathbb{F}_7)$. Equivalently, $\Gamma_2(C, L)$ is an infinite set for some finite extension $k \subseteq L \subseteq \bar{k}$ (Theorem 1.17). Finally, assume that $d = 4$ and $p > 7$, then, by Lemma 2.7, C is conservative over k . If $\mathrm{Aut}(C_{\bar{k}}) = \mathbb{Z}/6\mathbb{Z}, \mathrm{GAP}(48, 33), \mathrm{GAP}(96, 64)$, or $\mathrm{PSL}_2(\mathbb{F}_7)$, then isotrivial elliptic curves appear in $\mathrm{Jac}(C)$ over \bar{k} (for more details, see Lemma 3.5 and its proof, in particular the MAGMA code to compute the Jacobian and j -invariants), hence we can not apply Theorem 1.14 in these situation. Furthermore, in some cases, the j -invariant depends on parameters where some particular specializations corresponds to isotrivial elliptic curves, but not the generic case (we again refer to Lemma 3.5), and this justifies $C_{\bar{k}}$ being a member of an open set of one of the prescribed strata $\widetilde{\mathcal{M}}_3^{\mathrm{P}1}(G)$. This shows the ‘‘only if’’ part. For the remaining situations of $\widetilde{\mathcal{M}}_3^{\mathrm{P}1}(G)$ where isotrivial elliptic curves do not occur, we may apply Proposition 1.10 to deduce that $\Gamma_2(C, L)$ is infinite for some finite field extension L/k inside \bar{k} . \square

The next result maybe is well-known to specialists, however we present a quite easy proof.

³By the virtue of the theorem of Grauert-Samuel in §1, the other non-empty strata $\widetilde{\mathcal{M}}_3^{\mathrm{P}1}(G)$ not satisfying the hypothesis of Theorem 2.8 (ii) may also have infinite number of point without need to extend to a degree 2 extension.

Corollary 2.9. *Let $d \geq 5$ be a fixed integer, and C be any smooth plane curve of degree d over a global field k of characteristic $p = 0$ or $p > (d-1)(d-2) + 1$ (and in positive characteristic we further assume that $\text{Jac}(C_k)$ over \bar{k} has no non-zero homomorphic images defined over \mathbb{F}_q). Then, the number of quadratic field extensions $k \subset k' \subseteq \bar{k}$ where $C(k') \neq C(k)$ are finitely many. In particular, if we consider the Fermat curve $C : X^d + Y^d - Z^d = 0$ of degree d over \mathbb{Q} , then $C(\mathbb{Q}) = C(\mathbb{Q}(\sqrt{D}))$ for all square-free integers D , except possibly finitely many values for D . That is, there are, in the worst case, only finitely many quadratic number fields in which we may have more solutions in $\mathbb{Q}(\sqrt{D})$ to $X^d + Y^d - Z^d = 0$ than these over \mathbb{Q} . The same is true for the Klein curve $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$, and any other smooth plane curves of degree d over \mathbb{Q} or number fields.*

Proof. By definition, we have $C(k) \subseteq C(k') \subseteq \Gamma(C, k)$ for any quadratic field extension $k \subset k' \subseteq \bar{k}$. We also know from Theorem 2.8 that $\Gamma_2(C, k)$ is a finite set. Hence, only finitely many k' could satisfy $C(k) \subsetneq C(k')$. \square

3. QUADRATIC POINTS ON SMOOTH PLANE CURVES FIXING THE BASE FIELD: CASE \mathbb{Q}

In this section, we restrict our attention to smooth $(\mathbb{Q}, \bar{\mathbb{Q}})$ -plane quartic curves C , which are bielliptic (see Theorem 2.3). Since, the degree $d = 4$ is relatively prime to 3, we have that C is a smooth plane curve over \mathbb{Q} , that is, \mathbb{Q} is not only a field of definition for C , but also a plane-model field of definition.

We find some interest to conjecture the following.

Conjecture 3.1. *Fix a stratum of the shape $\widetilde{\mathcal{M}}_3^{\text{P}1}(G)$, where all of its $\bar{\mathbb{Q}}$ -points are bielliptic, equivalently, take $G \neq 1, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}$. Then, there is an infinite family \mathcal{E} (resp. \mathcal{D}) $\subseteq \widetilde{\mathcal{M}}_3^{\text{P}1}(G)$ of \mathbb{Q} -isomorphism classes of smooth plane quartic curves over \mathbb{Q} , such that $\Gamma_2(C, \mathbb{Q})$ is a finite (resp. infinite) set, for all $[C] \in \mathcal{E}$.*

In what follows, we are going to support the above conjecture in two different situations. By the work of Lercier-Ritzenthaler-Rovetta-Sijsling in [20, §2], we have a parametrization of the (coarse) moduli space of smooth plane quartic curves in terms of complete and representative families over \mathbb{Q} for all the strata $\widetilde{\mathcal{M}}_3^{\text{P}1}(G)$, except when $G = \mathbb{Z}/2\mathbb{Z}$ (a representative family over \mathbb{R} does not exist). Moreover, the work of Lorenzo in [21] detailed the study of the twists of smooth plane quartic curves over \mathbb{Q} . These results helps us a lot in our study of quadratic points in the sense of the previous conjecture. The main idea is to start with a family of smooth plane quartic curves over \mathbb{Q} , having many parameters in the defining equation over \mathbb{Q} . This in turns allows us to construct a subfamily with infinite cardinality of non \mathbb{Q} -isomorphic smooth plane quartic curves with the same automorphism group G (up to group isomorphisms), where its members are mapped to concrete elliptic curves over \mathbb{Q} whose rank is zero (resp. positive) over \mathbb{Q} . We note that, in the first case, we may reduce up to change of variables to some twists over \mathbb{Q} that have a unique bielliptic involution defined over \mathbb{Q} (in particular, it suffices to deal with a specific elliptic curve of rank zero, not with family of a elliptic curves over \mathbb{Q}).

3.1. The conjecture 3.1 is true for $\widetilde{\mathcal{M}}_3^{\text{P}1}(\mathbb{Z}/6\mathbb{Z})$. Let k be a field of characteristic $p = 0$ or $p > 7$. The one parameter family \mathcal{C}_a defined by

$$\mathcal{C}_a : aZ^4 + Y^2(Y^2 + aZ^2) + X^3Y = 0,$$

where $a \neq 0, 4$, is a representative family over k for $\widetilde{\mathcal{M}}_3^{\text{P}1}(\mathbb{Z}/6\mathbb{Z})$, see [20, Theorem 3.3]. Thus, any smooth plane quartic curve C over k with automorphism group isomorphic to $\mathbb{Z}/6\mathbb{Z}$ has a non-singular plane model in \mathcal{C}_a for a unique $a \in k$, that is, there exists a finite extension L/k inside \bar{k} where $C \otimes_k L$ is L -isomorphic to a unique non-singular polynomial equation of the form $aZ^4 + Y^2(Y^2 + aZ^2) + X^3Y = 0$ for some $a \in k$. In particular, $\Gamma_2(C, L) = \Gamma_2(aZ^4 + Y^2(Y^2 + aZ^2) + X^3Y = 0, L)$.

We also note that the above family is \bar{k} -isomorphic to the geometrically complete family in Theorem 2.3 via a diagonal change of variables [20], hence by Corollary 2.5, $\tilde{w} = \text{diag}(1, 1, -1)$ is again a unique bielliptic involution for any smooth curve in the family \mathcal{C}_a .

Theorem 3.2. *Consider a smooth bielliptic quartic plane curve C_a of the form $aZ^4 + Y^2(Y^2 + aZ^2) + X^3Y = 0$ for some $a \in \mathbb{Q} \setminus \{0, 4\}$. Then, the quotient $C_a / \langle \tilde{w} \rangle$ is an elliptic curve of positive rank over \mathbb{Q} . In particular, \mathcal{C}_a with $a \in \mathbb{Q} \setminus \{0, 4\}$ is an infinite family of bielliptic smooth plane quartic curves over \mathbb{Q} whose full automorphism group isomorphic to $\mathbb{Z}/6\mathbb{Z}$, and such that $\Gamma_2(\mathcal{C}_a, \mathbb{Q})$ is an infinite set.*

Proof. We work affine with $az^4 + az^2 + x^3 + 1 = 0$ by taking $Y = 1$ (observe that, we have a unique point on C_a with $Y = 0$ that is also fixed by \tilde{w}). In particular, $C_a/\langle\tilde{w}\rangle : az^2 + az + x^3 + 1 = 0$. We apply the \mathbb{Q} -change of variables $x \mapsto -x$ and $z \mapsto z - \frac{1}{2}$ to obtain $az^2 = x^3 + a/4 - 1$. Next, change $z \mapsto (1/a^2)z$ and $x \mapsto (1/a)x$ to finally get the elliptic curve $E/\mathbb{Q} : z^2 = x^3 - a^3(1 - a/4)$ whose j -invariant equals to zero. Furthermore, the point $P_a := (x, z) = (a, a^2/2)$ is a non-torsion point on E/\mathbb{Q} (if it is not, then it has order $m \leq 10$ or $m = 12$ (cf. [24, Theorem 7]), and one can check by MAGMA that the order of P_a is distinct from these values). Consequently, $\text{rank}_{\mathbb{Q}}(C_a/\langle\tilde{w}\rangle) \geq 1$ for any $a \in \mathbb{Q} \setminus \{0, 4\}$, and thus $\Gamma_2(C_a, \mathbb{Q})$ is an infinite set (Theorem 1.16). Finally, the family \mathcal{C}_a is representative for $\widetilde{\mathcal{M}}_3^{\text{Pl}}(\mathbb{Z}/6\mathbb{Z})$ over \mathbb{Q} , that is C_a and $C_{a'}$ in the family with $a \neq a' \in \mathbb{Q}$ can not be $\overline{\mathbb{Q}}$ -isomorphic. Therefore, we get infinitely many smooth plane quartic curves that have infinitely many quadratic points over \mathbb{Q} . \square

Now, if we are interested to parameterize the set of k -isomorphism classes of smooth plane quartic curves over k with automorphism group isomorphic to $\mathbb{Z}/6\mathbb{Z}$, then we may use [21, Proposition 3.2.8] to have that the family with $A, n, m \in k^*$;

$$\mathcal{C}_{A,n,m} : Am^2Z^4 + mY^2Z^2 + nX^3Y + Y^4 = 0,$$

where $(A, n, m) \in k^* \times k^*/k^{*3} \times k^*/k^{*2}$ does⁴. That is, any smooth plane quartic curve over k with automorphism group $\mathbb{Z}/6\mathbb{Z}$ is k -isomorphic to a non-singular plane model in the family $\mathcal{C}_{A,n,m}$ for some triple $(A, n, m) \in k^* \times k^*/k^{*3} \times k^*/k^{*2}$, in particular, it is bielliptic with unique bielliptic involution $\tilde{w} = \text{diag}(1, 1, -1)$, by Corollary 2.5. Moreover, two such curves are \overline{k} -isomorphic if and only if they have the same parameter $A \in k^*$. Therefore, it suffices to assume $\Gamma_2(\mathcal{C}_{A,n,m}, k)$ for $(A, n, m) \in k^* \times k^*/k^{*3} \times k^*/k^{*2}$ in order to investigate the infinitude of quadratic points over k of smooth plane quartic curves inside $\widetilde{\mathcal{M}}_3^{\text{Pl}}(\mathbb{Z}/6\mathbb{Z})$ over k .

Lemma 3.3. *For an arbitrary but a fixed $(A, n) \in k^* \times k^*/k^{*3}$, the set $\Gamma_2(\mathcal{C}_{A,n,m}, k)$ being finite or infinite is independent of the choice of m .*

Proof. One finds that $\mathcal{C}_{A,n,m}/\langle\tilde{w}\rangle$ is k -isomorphic to the elliptic curve $\mathcal{D}_{A,n}/k : z^2 = x^3 + \frac{n^2A^2(1-4A)}{4}$ over k . Indeed, one works affine $Y = 1$ to easily obtain that $\mathcal{C}_{A,n,m}/\langle\tilde{w}\rangle$ is k -isomorphic to the elliptic curve $Az^2 + z + nx^3 + 1 = 0$ over k , in particular, its defining equation and its rank is independent from m . To reach the Weierstrass form $\mathcal{D}_{A,n}$, we follow the usual way (cf. [30]); first, by the change of variables $z \mapsto z - \frac{1}{2A}$ and $x \mapsto -x$, we get $z^2 = (n/A)x^3 + \frac{1-4A}{4A^2}$, and after by $z \mapsto (1/nA^2)z$ and $x \mapsto (1/An)x$. \square

Theorem 3.4. *There are infinitely many smooth plane quartic curves C over \mathbb{Q} in the family $\mathcal{C}_{A,n,m}$, with $(A, n, m) \in \mathbb{Q}^* \times \mathbb{Q}^*/\mathbb{Q}^{*3} \times \mathbb{Q}^*/\mathbb{Q}^{*2}$, such that $\Gamma_2(C, \mathbb{Q})$ is a finite set.*

Proof. For example, we may consider the subfamily $\mathcal{C}_{A(t),n(t),m} : A(t)m^2Z^4 + mY^2Z^2 + n(t)X^3Y + Y^4 = 0$ where $A(t) := (108t^2 + 1)/4$ and $n(t) := 4/t(108t^2 + 1)$, for $t = a/b \in \mathbb{Q}^*$ with a, b odd relatively prime integers (note that $n(t) \in \mathbb{Q}^{*3}$ only if either a or b is even). In this situation, $\mathcal{C}_{A(t),n(t),m}/\langle\tilde{w}\rangle$ is always \mathbb{Q} -isomorphic to $\mathcal{D}_{A(t),n(t)} : z^2 = x^3 - 27$, which has rank 0 over \mathbb{Q} . Since, two curves in the family associated to the triples $(A(t), n(t), m)$ and $(A(t'), n(t'), m')$ with $A(t) \neq A(t')$ are not $\overline{\mathbb{Q}}$ -isomorphic, we get infinitely many smooth plane quartic curves over \mathbb{Q} that have infinitely many quadratic points over \mathbb{Q} , which was to be shown. \square

3.2. The conjecture 3.1 is true for $\widetilde{\mathcal{M}}_3^{\text{Pl}}(\text{GAP}(16, 13))$. The family defined by

$$\mathcal{C}_a : Z^4 + (X^4 + Y^4 + aX^2Y^2) = 0,$$

where $\pm a \neq 0, 2, 6, 2\sqrt{-3}$, is a geometrically complete family for the stratum $\widetilde{\mathcal{M}}_3^{\text{Pl}}(\text{GAP}(16, 13))$ over $\overline{\mathbb{Q}}$. As already mentioned in Corollary 2.5, we have exactly seven bielliptic involutions, namely $\iota_1 := \text{diag}(1, 1, -1)$, $\iota_2 := \text{diag}(-1, 1, 1)$, $\iota_3 := \text{diag}(1, -1, 1)$, $\iota_4 := [Y : X : Z]$, $\iota_5 := [Y : X : -Z]$, $\iota_6 := [Y : -X : \zeta_4 Z]$ and $\iota_7 := [Y : -X : -\zeta_4 Z]$, where ζ_4 is a fixed primitive 4th root of unity in $\overline{\mathbb{Q}}$.

Lemma 3.5. *For $a \in \mathbb{Q} \setminus \{0, \pm 2, \pm 6\}$ and assuming that $\mathcal{C}_a/\langle\iota_i\rangle$ has a \mathbb{Q} -point, then it is \mathbb{Q} -isomorphic to an elliptic curve E/\mathbb{Q} of one of the forms:*

⁴It remains to determine the algebraic restrictions on the parameters to ensure non-singularity and no larger automorphism group. For example, $A \neq 1/4$, since we get the singular points $(0 : \pm\sqrt{m/2} : 1)$ on $\mathcal{C}_{1/4,n,m}$, otherwise.

| <i>Involution</i> | <i>E</i> | <i>j-invariant</i> |
|--------------------------------------|------------------------------|--|
| ι_1 | $z^2 = x^3 - ax^2 - 4x + 4a$ | $(16a^6 + 576a^4 + 6912a^2 + 27648)/(a^4 - 8a^2 + 16)$ |
| ι_2, ι_3 | $z^2 = x^3 + (a^2 - 4)x$ | 1728 |
| $\iota_4, \iota_5, \iota_6, \iota_7$ | $z^2 = x^3 + (1 - a^2/4)x$ | 1728 |

Proof. The next MAGMA code applied for $\mathcal{C}_a/\langle\iota_6\rangle$ can also be adapted elsewhere. It assures that $\mathcal{C}_a/\langle\iota_6\rangle$ is a smooth curve of genus 1 over \mathbb{Q} . Because $\mathcal{C}_a/\langle\iota_i\rangle$ has rational points, it is \mathbb{Q} -isomorphic to its Jacobian variety over \mathbb{Q} , which is an elliptic curve over \mathbb{Q} , a priori.

```

> R<x>:=PolynomialRing(Integers());
> K<k>:=NumberField(x^2+1);
> L<a>:=FunctionField(K,1);
> P2<X,Y,Z>:=ProjectiveSpace(L,2);
> g:=X^4+Y^4+Z^4+aX^2Y^2;
> C:=Curve(P2,g);
> phi1:=iso<C->C|[Y,-X,k*Z],[Y,-X,k*Z]>;
> G1:=AutomorphismGroup(C,[phi1]);
> CG1,prj:=CurveQuotient(G1);
> Genus(CG1);
1
> Jacobian(CG1); E:=Jacobian(CG1);
Elliptic Curve defined by y^2 = x^3+(-1/4a^2+1)x over Multivariate rational function field
of rank 1 over K
> jInvariant(E);
1728

```

The condition that $\mathcal{C}_a/\langle\iota_i\rangle$ has rational points is verified in many cases. For example, we can see that $\mathcal{C}_a/\langle\iota_1\rangle$ is defined inside $\mathbb{P}_{\mathbb{Q}}^3$ by the two quadrics $-X_2X_3 + X_4^2 = 0$, and $X_1^2 + X_2^2 + X_3^2 + aX_2X_3 = 0$ over \mathbb{Q} . Hence, if we impose $-(a+2) \in \mathbb{Q}^* \setminus \{4\}$, then $(\sqrt{-(a+2)} : 1 : 1 : 1)$ is an obvious \mathbb{Q} -point. \square

Consider the family $\mathcal{C}'_A : AX^4 + Y^4 + Z^4 + X^2Y^2 = 0$, where $\pm A \neq \{1/4, 1/36, 1/12\}$. This is a representative family over \mathbb{Q} (cf. [21, p. 36,37]), in particular, any smooth plane quartic curve over \mathbb{Q} with automorphism group isomorphic to $\text{GAP}(16, 13)$ is isomorphic (not necessarily over \mathbb{Q}) to a smooth curve $\mathcal{C}_A : AX^4 + Y^4 + Z^4 + X^2Y^2 = 0$ in the family \mathcal{C}'_A for an unique $A \in \mathbb{Q}^* \setminus \{\pm 1/4, \pm 1/36, \pm 1/12\}$.

The transformed seven bielliptic involutions of any smooth plane quartic curve in the family \mathcal{C}'_A over \mathbb{Q} are $\iota'_i := P^{-1}\iota_i P$, for $i = 1, \dots, 7$, where

$$P := \begin{pmatrix} \frac{1}{\sqrt[4]{A}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We also impose, once and for all, that $A \in \mathbb{Q}^* \setminus \mathbb{Q}^{*4}$. In particular, $\iota'_1 = \iota_1$, $\iota'_2 = \iota_2$, $\iota'_3 = \iota_3$ are the only bielliptic involutions defined over \mathbb{Q} .

By the work of E. Lorenzo García in [21, Chp. 3], we know that any *diagonal twist* of \mathcal{C}_A , for a fixed A , in the family \mathcal{C}'_A is \mathbb{Q} -isomorphic to

$$\mathcal{C}_{A,m,q} : mAX^4 + q^2mY^4 + Z^4 + qmX^2Y^2 = 0,$$

for some $A, m, q \in \mathbb{Q}^*$, where two of twists $\{A, m, q\}$ and $\{A', m', q'\}$ are \mathbb{Q} -isomorphic if $A = A'$, $m \equiv m' \pmod{\mathbb{Q}^{*4}}$ and $q \equiv q' \pmod{\mathbb{Q}^{*2}}$. First, we consider smooth curves of the form $\mathcal{C}_{A,m,q}$ with $(A, m, q) \in \mathbb{Q}^* \times \mathbb{Q}^*/\mathbb{Q}^{*4} \times \mathbb{Q}^*/\mathbb{Q}^{*2}$. Next, the quotient curve $\mathcal{C}_{A,m,q}/\langle\iota'_3\rangle$ is a genus one curve \mathbb{Q} -isomorphic to $y^2 + (1/m)z^4 - (1/4 - A) = 0$, in particular it is independent of the parameter $q \in \mathbb{Q}^*/\mathbb{Q}^{*2}$. Let us assume that $f := 1/4 - A$ is a square (hence, $\mathcal{C}_{A,m,q}/\langle\iota'_3\rangle$ has a \mathbb{Q} -point), and after one may use Maple's **Weierstrassform** function

```

> algcurves:-Weierstrassform(y^2 - (-1/m)z^4 - f, x, y, u, v);

```

which returns the normal form in variables u, ν :

$$\nu^2 = u^3 + ((1 - 4A)/m)u.$$

Theorem 3.6. *Consider a smooth bielliptic quartic plane curve in the family $\mathcal{C}_{A,m,q} : mAX^4 + q^2mY^4 + Z^4 + qmX^2Y^2 = 0$, for some $q \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$ and $A, m \in \mathbb{Q}^* \setminus \mathbb{Q}^{*4}$ such that $1/4 - A$ is a square. Then, the quotient $\mathcal{C}_{A,m,q}/\langle \tilde{w} \rangle$, where $\tilde{w} = \text{diag}(1, -1, 1)$, is an elliptic curve of positive rank over \mathbb{Q} . In particular, $\mathcal{C}_{A,m,q}$ gives an infinite family of smooth plane quartic curves C over \mathbb{Q} whose automorphism group is isomorphic to $\text{GAP}(16, 13)$, and $\Gamma_2(C, \mathbb{Q})$ is an infinite set.*

Proof. We have seen above that $\mathcal{C}_{A,m,q}/\langle \tilde{w} \rangle$ is \mathbb{Q} -isomorphic to $\mathcal{D}_{A,m} : \nu^2 = u^3 + Du$ with $D := (1 - 4A)/m$ independently from the parameter $q \in \mathbb{Q}^*/\mathbb{Q}^{*2}$. It now suffices to specialize A, m accordingly so that $\text{rank}_{\mathbb{Q}}(\mathcal{D}_{A,m})$ is positive, hence $\Gamma_2(C, \mathbb{Q})$ is infinite. For example, take $D := (1 - 4A)/m = p$, where p is a prime integer < 1000 and congruent to 5 modulo 8, so the rank is always 1 in accordance with the conjecture of Selmer and Mordell (see [8]). Take the case $D := (1 - 4A)/m = -p$, where p is a Fermat or a Mersenne prime, then the ranks 0, 1 and 2 were found (see [19]). Take $D := (1 - 4A)/m = -n$, where n is related to the positive integer solutions of the diophantine equation $n = \alpha^4 + \beta^4 = \gamma^4 + \mu^4$, then the rank is at least 3 (see [18]). Take $D := (1 - 4A)/m = -pq$, where p and q are two different odd primes, then, up to an extra condition, a family of rank 4 was found (see [22]), etc... Finally, by [21, Proposition 3.2.9], two twists $\mathcal{C}_{A,q,m}$ and $\mathcal{C}_{A,q',m'}$, with $(A, m, q), (A, m', q') \in \mathbb{Q}^* \times \mathbb{Q}^*/\mathbb{Q}^{*2} \times \mathbb{Q}^*/\mathbb{Q}^{*4}$ are \mathbb{Q} -isomorphic only if $m = m'$ in $\mathbb{Q}^*/\mathbb{Q}^{*4}$. Consequently, for any fixed A , we can run m and q as before to obtain infinitely many non- \mathbb{Q} -isomorphic smooth plane quartic curves with infinitely many quadratic points over \mathbb{Q} . \square

Now, we ask for an infinite family of smooth quartic curves over \mathbb{Q} in the stratum $\widetilde{\mathcal{M}}_3^{\text{P1}}(\text{GAP}(16, 13))$ such that the quotient by any bielliptic involutions may only provide elliptic curves of rank 0 over \mathbb{Q} . In particular, the set quadratic points over \mathbb{Q} is finite. To do so, we turn out to *non-diagonal* twists of C_A , for a fixed $A \in \mathbb{Q}^*$, that are parameterized (see [21, Proposition 3.2.9]) by

$$\mathcal{C}_{\underline{a}, \underline{b}, m, q, A} : 2\underline{a}X^4 + 8\underline{b}mX^3Y + 12\underline{a}mX^2Y^2 + 8\underline{b}m^2XY^3 + 2\underline{a}m^2Y^4 + q(X^2 - mY^2)^2 + Z^4 = 0$$

where $m \in \mathbb{Q}^*$, $\underline{a}, \underline{b}, q \in \mathbb{Q}$ satisfy $\underline{a}^2 - \underline{b}^2m = q^4A$. Two such twists $\{\underline{a}, \underline{b}, m\}$ and $\{\underline{a}', \underline{b}', m'\}$ for C_A are equivalent if and only if $m \equiv m' \pmod{\mathbb{Q}^{*2}}$ and that there exist $c, d \in \mathbb{Q}$ such that $\underline{a} + \underline{b}\sqrt{m} = (c + d\sqrt{m})^4(\underline{a}' + \underline{b}'\sqrt{m})$.

The transformed seven bielliptic involutions of any smooth plane quartic curve $\mathcal{C}_{\underline{a}, \underline{b}, m, q, A}$ in the family $\mathcal{C}_{\underline{a}, \underline{b}, m, q, A}$ over \mathbb{Q} are $\nu'_i := P'^{-1}\nu'_i P'$, for $i = 1, \dots, 7$, where

$$P' := \begin{pmatrix} \sqrt[4]{\underline{a} + \underline{b}\sqrt{m}} & \sqrt{m}\sqrt[4]{\underline{a} + \underline{b}\sqrt{m}} & 0 \\ \sqrt[4]{\underline{a} - \underline{b}\sqrt{m}} & -\sqrt{m}\sqrt[4]{\underline{a} - \underline{b}\sqrt{m}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can check that $\text{diag}(1, 1, -1)$ is the unique bielliptic involution defined over \mathbb{Q} for $\mathcal{C}_{\underline{a}, \underline{b}, m, q, A}$, when $A \notin \mathbb{Q}^{*4}$ and $m \notin \mathbb{Q}^{*2}$. Thus, the only way to obtain a degree two \mathbb{Q} -morphisms to elliptic curves over \mathbb{Q} is to quotient by $\tilde{w} := \text{diag}(1, 1, -1)$. Assume moreover that the quotient family $\mathcal{C}_{\underline{a}, \underline{b}, m, q, A}/\langle \tilde{w} \rangle$ has \mathbb{Q} -points (for example, we fix $q = -2\underline{a}$ and we always get the point $(0 : 1 : 0)$), hence $\mathcal{C}_{\underline{a}, \underline{b}, m, q = -2\underline{a}, A}/\langle \tilde{w} \rangle$ is \mathbb{Q} -isomorphic to its Jacobian variety, which (by MAGMA) is given by (for simplicity, we impose $\underline{a} = -\underline{b}$, which is enough for our proposes):

$$\mathcal{E}_{m,q} : y^2 = x^3 + 8qm x^2 + 16q^2 m^3 x.$$

With respect to the specialization $q = -2\underline{a} = 2\underline{b}$, we should have $A = \frac{(1-m)}{4q^2}$ and we always impose $A \notin \mathbb{Q}^{*4}$.

Lemma 3.7. *The two parameter family $\mathcal{E}_{m,q} : y^2 = x^3 + 8qm x^2 + 16q^2 m^3 x$, for $q \in \mathbb{Q}$ and $m \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$ such that $\frac{1-m}{4q^2} \notin \mathbb{Q}^{*4}$, contains infinitely many elliptic curves $E_{m,q}$ of rank 0 over \mathbb{Q} . More precisely, for any fixed $m \in \mathbb{Z} \setminus \mathbb{Z}^2$ with $m \notin \{1 - h^2 \mid h \in \mathbb{Z}\}$, there exist infinitely many $q \in \mathbb{Q}^* \pmod{\mathbb{Q}^{*2}}$ with qm square-free integer, and for which $\text{rank}_{\mathbb{Q}}(E_{m,q}) = 0$.*

Proof. If we specialize $m, mq \in \mathbb{Z}$ so that $m \notin \mathbb{Z}^2$ and mq is square-free, then the elliptic curve $E_{m,q}$ is \mathbb{Q} -isomorphic to the quadratic twist $E_D : Dy^2 = x^3 + 8x^2 + 16mx$ for $E : y^2 = x^3 + 8x^2 + 16mx$ over \mathbb{Q} , associated

to $D = mq$; indeed, if we multiply the defining equation for E_D by D^3 and then apply the change of variables $y \mapsto (1/D^2)y$ and $x \mapsto (1/D)x$ we obtain the equation of $E_{m,q}$. We know that E/\mathbb{Q} is modular by the work of so many people around; mainly C. Breuil, B. Conrad, F. Diamond, R. Taylor and A. Wiles (cf. [9, 10, 33, 34]). Therefore, we deduce from by L. Main-R. Murty [23] (cf. see also the first line of the abstract [26]) that for a fixed m there are infinitely many square-free integers $D = mq$ so that E_D has rank 0 over \mathbb{Q} (each D is congruent to 1 mod $4n$, where n is the conductor of the elliptic curve E/\mathbb{Q}). In particular, we get infinitely many elliptic curves $E_{m,q}$ (corresponding to infinitely many such q 's mod \mathbb{Q}^{*2}) of rank 0 over \mathbb{Q} . The condition that $m \notin \mathbb{Z}^2$ with $m \notin \{1 - h^2 \mid h \in \mathbb{Z}\}$ is to ensure that A is not a fourth power. \square

Theorem 3.8. *The two parameter family*

$$C_{m,q} : -qX^4 + 4qmX^3Y - 6qmX^2Y^2 + 4qm^2XY^3 - qm^2Y^4 + q(X^2 - mY^2)^2 + Z^4 = 0,$$

for $q \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$ and $m \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ such that $A = \frac{1-m}{4q^2} \notin \mathbb{Q}^{*4}$, contains infinitely many, non \mathbb{Q} -isomorphic, smooth plane curves C over \mathbb{Q} that have only finitely many quadratic points over \mathbb{Q} .

Proof. If we specialize $q = -2\underline{a} = 2\underline{b}$ in the family $C_{\underline{a},\underline{b},m,q,A}$ mentioned above, then we get the subfamily $C_{m,q}$. In particular, any smooth plane curve in $C_{m,q}$ is bielliptic and has only one bielliptic involution \tilde{w} over \mathbb{Q} . Furthermore, by Lemma 3.7, for any $m \in \mathbb{Z} \setminus \mathbb{Z}^2$ with $m \notin \{1 - h^2 \mid h \in \mathbb{Z}\}$, there exists an infinite number of $q \in \mathbb{Q}^* \text{ mod } \mathbb{Q}^{*2}$ with mq a square-free integer, such that $C_{m,q}/\langle \tilde{w} \rangle$ is an elliptic curve over \mathbb{Q} and whose rank is zero. Therefore, the number of quadratic points over \mathbb{Q} is finite by Theorem 1.16. Moreover, if any two curves $C_{m,q}$ and $C_{m,q'}$, which gives rank 0, are \mathbb{Q} -isomorphic, then $C_{(1-m)/4q^2} : \frac{1-m}{4q^2}X^4 + Y^4 + Z^4 + X^2Y^2 = 0$ and $C_{(1-m)/4q'^2} : \frac{1-m}{4q'^2}X^4 + Y^4 + Z^4 + X^2Y^2 = 0$ are $\overline{\mathbb{Q}}$ -isomorphic (recall that $C_{m,q}$ and $C_{m,q'}$ are twists for $C_{(1-m)/4q^2}$ and $C_{(1-m)/4q'^2}$, respectively), but this contradicts the fact that the family $C'_A : AX^4 + Y^4 + Z^4 + X^2Y^2 = 0$ is a representative family for over \mathbb{Q} for the stratum $\widetilde{\mathcal{M}}_3^{\text{P}1}(\text{GAP}(16, 13))$. \square

Remark 3.9. *The previous discussion tends to be applicable for any zero-dimensional stratum $\widetilde{\mathcal{M}}_3^{\text{P}1}(G)$, that is, when $G = \{\text{GAP}(48, 33), \text{GAP}(96, 64), \text{PSL}_2(\mathbb{F}_7)\}$. However, once we start with families that parameterize the twists over \mathbb{Q} , we need to precise the algebraic restrictions on the parameters that appear in the defining equations, which characterize when two twists are \mathbb{Q} -equivalent. This is a key point in order to construct an infinite family of non \mathbb{Q} -isomorphic smooth plane quartic curves with infinitely (resp. finitely) many quadratic points over \mathbb{Q} . In particular, one may check Conjecture 3.1 for Fermat and Klein quartic curves considering all the details of constructing their twists in [21].*

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REFERENCES

- [1] D. Abramovich and J. Harris, *Abelian varieties and curves in $W_d(C)$* . *Compositio Math.* **78** (1991), 227-238.
- [2] E. Badr, *On the stratification of smooth plane curves by automorphism groups*. PhD thesis, September 2017, Universitat Autònoma de Barcelona.
- [3] E. Badr and F. Bars, *Non-singular plane curves with an element of "large" order in its automorphism group*. *Internat. J. Algebra Comput.* **26** (2016), no. 2, 399-433.
- [4] E. Badr and F. Bars, *On the locus of smooth plane curves with a fixed automorphism group*, *Mediterr. J. Math.* **13** (2016), 3605-3627. doi: 10.1007/s00009-016-0705-9.
- [5] E. Badr, F. Bars and E. Lorenzo García, *On twists of smooth plane curves*. To appear in *Mathematics of Computation*, doi: <https://doi.org/10.1090/mcom/3317>
- [6] F. Bars, *On the automorphisms groups of genus 3 curves*. *Surv. Math. Sc.* **2** (2012), no. 2, 83-124.
- [7] F. Bars, *On quadratic points of classical modular curves* Momose Memorial Volume. Number theory related to modular curves Momose memorial volume, 17-34, *Contemp. Math.*, 701, Amer. Math. Soc., Providence, RI, 2018.
- [8] A. Bremner, J. W. S. Cassels, *On the equation $Y^2 = X(X^2 + p)$* . *Math. Comput.* **42**, 257-264 (1984).

- [9] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the Modularity of Elliptic Curves Over Q : Wild 3-Adic Exercises*. J. Amer. Math. Soc. **14**, 843-939, (2001).
- [10] B. Conrad, F. Diamond and R. Taylor, *Modularity of Certain Potentially Barsotti-Tate Galois Representations*. J. Amer. Math. Soc. **12**, 521-567, (1999).
- [11] G. Faltings, *Endlichkeitsstze fr abelsche Varieteten ber Zahlkrpern [Finiteness theorems for abelian varieties over number fields]*. Inventiones Mathematicae (in German) **73** (3): 349-366. doi:10.1007/BF01388432, (1983).
- [12] G. Faltings, *Erratum: Endlichkeitsstze fr abelsche Varieteten ber Zahlkrpern*. Inventiones Mathematicae (in German) **75** (2): 381, (1984).
- [13] GAP Group, *Gapgroups, algorithms, and programming system for computational discrete algebra (2008), available at <http://www.gap-system.org>. version 4.4.11*.
- [14] H. Grauert, *Mordells Vermutung ber rationale Punkte auf algebraischen Kurven und Funktionenkrper*. Publications Mathematiques de l'IHS (25): 131-149, (1965).
- [15] J. Harris and J. H. Silverman, *Bielliptic curves and symmetric products*. Proc. Am. Math. Soc. **112**, 347-356 (1991).
- [16] P. Henn, *Die Automorphismengruppen dar algebraischen Funktionenkorper vom Geschlecht 3*, Inagural-dissertation, Heidelberg, 1976.
- [17] M. Homma, *Funny plane curves in characteristic $p > 0$* . Comm. Algebra, **15** (1987), 1469-1501.
- [18] F.A. Izadi, F. Khoshnam, and K. Nabardi, *Sums of two biquadrates and elliptic curves of rank ≥ 4* . Math. J. Okayama Univ. **56**, 51-63 (2014).
- [19] T. Kudo, K. Motose, *On group structure of some special elliptic curve*. Math. J. Okayama Univ. **47**, 81-84 (2005).
- [20] R. Lercier, C. Ritzenthaler, F. Rovetta, and J. Sijsling, *Parametrizing the moduli space of curves and applications to smooth plane quartics over finite fields*, LMS J. Comput. Math. **17** (2014), no. suppl. A, 128-147. MR 3240800.
- [21] E. Lorenzo García, *Arithmetic properties of non-hyperelliptic genus 3 curves*. PhD dissertation, Universitat Politècnica de Catalunya (2015), Barcelona.
- [22] M. Maenishi, *On the rank of elliptic curves $y^2 = x^3 - pqx$* . Kumamoto J. Math. **15**, 1-5 (2002).
- [23] L. Mai and M. R. Murty, *A note on quadratic twists of an elliptic curve, Elliptic curves and related topics*, Ed. H. Kisilevsky and M. R. Murty, Amer. Math. Soc. CRM Proceedings and Lecture Notes (1994), 121-124.
- [24] B. Mazur, *Modular curves and the Eisenstein ideal*. Publ. Math. I.H.E.S., 47, 1978.
- [25] M. Namba, *Families of meromorphic functions on compact Riemann surfaces*. Lect. Notes Math. 767, Springer-Verlag, Berlin, 1979.
- [26] K. Ono, *Rank zero quadratic twists of modular elliptic curves*. Compositio Mathematica **104**, 293-304 (1996).
- [27] P. Samuel *Compléments à un article de Hans Grauert sur le conjecture de Mordell*, Publ. I.H.E.S., tome 29, 55-62. (1966).
- [28] A. Schweizer, *Bielliptic Drinfeld modular curves*. Asian J. Math. **5**, (2001), 705-720.
- [29] J. P. Serre, *Cohomologie galoisienne*. fifth ed., Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin, 1994. MR 1324577.
- [30] J. H. Silverman, *The Arithmetic of elliptic curves*, GTM 106 Springer, Ed.(1992).
- [31] H. Stichtenoth, *Algebraic Function Fields and Codes*, Springer Verlar, GTM254, 2nd Edition, (2008).
- [32] J. T. Tate, *Genus change in inseparable extensions of function fields*, Proc. Amer. Math. Soc. **3**, (1952), 400-406.
- [33] R. Taylor and A. Wiles, *Ring-Theoretic Properties of Certain Hecke Algebras*. Ann. Math. **141**, 553-572, (1995).
- [34] A. Wiles, *Modular Elliptic-Curves and Fermat's Last Theorem*. Ann. Math. **141**, 443-551, (1995).
- [35] H. Yoshihara, *Function field theory of plane curves by dual curves*, J. Algebra **239** (1), (2001), 340-355.

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