SCHWARZ LEMMA, AND DISTORTION FOR HARMONIC FUNCTIONS VIA LENGTH AND AREA

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This is very rough working version (Version 3, 4/27/2018).

1. Introduction and Basic definitions

We give sharp estimates for distortion of harmonic by means of area and length of the corresponding surface. In 2016 [18](a), the author has posted the current Research project Schwarz lemma, the Carathéodory and Kobayashi Metrics and Applications in Complex Analysis.[1] Various discussions regarding the subject can also be found in the Q&A section on Researchgate under the question " What are the most recent versions of The Schwarz Lemma ?",[18](b). During the fall semester 2017 at Belgrade seminar [16], we have communicated about Schwarz lemma and we have posted the arXiv paper [15], in which we have considered various version of Schwarz lemma and its relatives related to harmonic and holomorphic functions including distortion of harmonic mappings, and several variables. For the results of [15] see also [17]. For example, in Section 2 we prove several optimal versions of planar Schwarz lemma for real valued harmonic maps h from $\mathbb{U}$ into $I_0 = (-1, 1)$ (see Theorem 1 and 2, related to the case $h(0) = a$, $a \in I_0$; and Theorem 3 and 4, for the case $f(a) = b$, $a \in \mathbb{U}$). In particular if $a = 0$ a part of Theorem 1 is reduced to classical Schwarz lemma for harmonic maps.
Note that Theorem 4 yields solution of D. Khavinson extremal problem for harmonic functions in planar case, cf. [12, 13].
From Theorem 1 we also derive Theorem 6 which is a version of planar Schwarz lemma for complex valued harmonic maps h from $\mathbb{U}$ into itself, and a version of the boundary Schwarz lemma, see Theorem 5.

During my work on the subject, D. Kalaj gave an interesting communication, cf. [8](from which we have learned about his arXiv papers [6, 7]), and immediately we have realized that we can adapt our previous consideration to connect with his work. In particular, using different approach we can give new insight to these results as well as further results.

We first need some definitions.

Definition 1. a1) By $\mathbb{C}$ we denote the complex plane, by $\mathbb{U}$ the unit disk and by $\mathbb{T}$ the unit circle.
a2) In planar case $G \subset \mathbb{C}$ with euclidean norm the notation $\Lambda_f(z)$, $z \in G$, is used instead of $||(df)_z||$. For a function h, we use notation $D_1 h = h'_x$ and $D_2 h = h'_y$ for partial derivatives; $\partial h = \frac{1}{2}(h'_x - ih'_y)$ and $\overline{\partial} h = \frac{1}{2}(h'_x + ih'_y)$; we also use notations

Date: 28 Dec,2016.
1991 *Mathematics Subject Classification.* Primary 30F45; Secondary 32G15.
Key words and phrases. harmonic and holomorphic functions,hyperbolic distance.
[1]motivated by S. G. Krantz paper [11]

Dh and $\overline{D}h$ instead of ∂h and $\overline{\partial}h$ respectively when it seems convenient. We use the notation $\lambda_f(z) = \big||\partial f(z)| - |\bar\partial f(z)|\big|$ and $\Lambda_f(z) = |\partial f(z)| + |\bar\partial f(z)|$, if $\partial f(z)$ and $\bar\partial f(z)$ exist.
a3) If f is harmonic in $\mathbb{U}$ then we write $f = g+\overline{h}$, where g and h are analytic in $\mathbb{U}$. If $|g'(z)| \ge |h'(z)|$, then $\Lambda_f(z) = |g'(z)|+|h'(z)|$, $\lambda_f(z) = |g'(z)|-|h'(z)|$ and therefore $\Lambda_f(z) + \lambda_f(z) = 2|g'(z)|$. In general, $\Lambda_f(z) + \lambda_f(z) = 2\max\{|g'(z)|, |h'(z)|\}$.

Definition 2. b1) For a C^1 mapping $u : \mathbb{U} \to \mathbb{R}^m$, set $S = u(\mathbb{U})$, $D[u] = \int_{\mathbb{U}}(|D_1u|^2 + |D_2u|^2)dxdy$, $E = |D_1u|^2$, $G = |D_2u|^2$, $F = D_1u \cdot D_2u$, $J_u = \sqrt{EG - F^2}$, and $A = A(S) = A(u) = \int_{\mathbb{U}} J_u dxdy$.
b2) We say that u is $\underline{K}$-qc if $E + F \le 2\underline{K}J_u$; in planar case this definition is a small modification of the standard definition of coefficient of quasi-conformality, see Remark 23.

b3) Suppose that $u : \mathbb{U} \to \mathbb{R}^m$ is harmonic on $\mathbb{U}$. Then $u = \mathrm{Re}F$, where F is analytic. Set $\underline{D}[F] = \int_{\mathbb{U}} |F'(z)|^2 dxdy$.

b4) If f is a function on $\mathbb{T}$, we associate to f a curve $\gamma = \gamma_f$ defined by $\gamma(t) = f(e^{it})$, $t \in [0, 2\pi]$, and we denote by $L = L(f) = |\gamma_f|$ the length of γ_f.

It is easy to check that
$D_1u = \mathrm{Re}F'$ and $D_2u = -\mathrm{Im}F'$ and
(A) $|D_1u|^2 + |D_2u|^2 = |F'|^2$, and therefore
$D[u] = \underline{D}[F]$.
If in addition u is conformal at some point, $|D_1u| = |D_2u|$ and therefore
$|\mathrm{Re}F'| = |\mathrm{Im}F'|$ and (ii) $|F'|^2 = 2|D_1u|^2$.

We will use the following hypothesis in the sequel
(H_m): $u : \mathbb{U} \to \mathbb{R}^m$ is harmonic on $\mathbb{U}$, and $S = u(\mathbb{U})$,
($\underline{H}_m$): In addition to (H_m) we suppose that
(h1): u is continuous on $\overline{\mathbb{U}}$ and $\gamma = \gamma_u$ is rectifiable.
In Section 5 we prove:
(I0) If u satisfies ($\underline{H}_m$), then (i1): $4\pi A \le L^2$, where $A = A(u)$ and $L = L(u)$.
(I1) If in addition to (H_m) we suppose that u satisfies
(h2): $A(S)$ is finite and
(h3): u is $\underline{K}$-quasiconformal,
then

$$D[u] = \underline{D}[F] = \pi\Big(\sum_{k=1}^{\infty}(k|\hat{F}(k)|^2)\Big) \le 2\underline{K}A(S).$$

I2) If in addition to (H_m) and (h2) we suppose that (h4): u is conformal, then

$$A = \int_{\mathbb{U}} |D_1u|^2 dxdy = \frac{\pi}{2}\Big(\sum_{k=1}^{\infty}(k|\hat{F}(k)|^2)\Big).$$

d2) In particular, $\pi\Lambda_u^2(0) \le D[u]$ with equality iff (ii) $\gamma = \gamma_u$ is a circle given by $u_k = a_k x - b_k y$, $k = 1, 2, 3, ..., m$, where $|a| = |b|$ and $a \cdot b = 0$.
Hence using the isoperimetric inequality,
d3) $2\pi\Lambda_u(0) \le L$ with equality iff (ii).
In [6] a version of (d3) is proved.

In Section 3 we first consider the cases when $m = 2, 3$.

For $f : \mathbb{U} \to \mathbb{R}^2$ we use notation $G = f(\mathbb{U})$. For convenience of the reader we also first suppose that u is harmonic on $\overline{\mathbb{U}}$. In this case L is length of ∂G. In

Section 3 using Proposition 3.1, we consider distortion of harmonic functions on $\mathbb{U}$ related to diameter $dia(G)$ of image domain $G = f(\mathbb{U})$, see Theorem 8. Then use an inequality between $dia(G)$ and the length $L(G)$ of boundary of G we estimate distortion via $L(G)$ and prove $2\pi\lambda_f(0) \le L$, see Theorem 10. In [8] a version of the part (b) of Theorem 10 is proved for diffeomorphisms.
Under the hypothesis $(\underline{H}_m)$, $m \ge 2$, it is convenient to introduce for given $z_0 \in \mathbb{U}$ the tangent plane $Z = Z_{\mathbf{y}^0}$, $\mathbf{y}^0 = u(z_0)$. Then we use the projection p onto Z and apply planar result on $p \circ u$ to prove $2\pi(1-|z_0|^2)\lambda_u(z_0) \le L$, see Theorem 11 and Theorem 17. If in addition f is conformal at z_0, then $\Lambda_u(z_0) = \lambda_u(z_0)$ and the previous inequality holds with $\Lambda_u(z_0)$ instead of $\lambda_u(z_0)$, see Theorem 12.

In Section 4 we outline a proof of Theorem 14. Using this result one can show that some of the above described results hold under more general hypothesis then $(\underline{H}_m)$(see for example Theorem 16).

These results are communicated in November 2017,[16].

2. Schwarz lemma

For a hyperbolic plane domain D, we denote by ρ_D(or λ_D) the hyperbolic density and by abusing notation the hyperbolic metric occasionally.

Lemma 1. If G and D are simply connected domains different from $\mathbb{C}$ and $\omega \in \mathrm{Hol}(G,D)$, then $\rho_D(\omega z)|\omega'(z)| \le \rho_G(z)$, $z \in G$ and

$$\rho_D(\omega z, \omega z') \le \rho_G(z,z'), \quad z, z' \in G.$$

We denote the right half plane by Π.

Proposition 2.1. If ω is holomorphic from Π into itself, then

$$|\omega'(z)| \le \frac{\mathrm{Re}\omega(z)}{\mathrm{Re} z}.$$

If in addition ω maps $\mathbb{R}^+$ into itself, then $|\omega'(1)| \le \mathrm{Re}\omega(1) = \omega(1)$ and therefore $\omega'(1) \le \omega(1)$.

Definition 3. By $\mathbb{C}$ we denote the complex plane by $\mathbb{U}$ the unit disk and by $\mathbb{T}$ the unit circle. For $z_1 \in \mathbb{U}$, define

$$T_{z_1}(z) = \frac{z - z_1}{1 - \overline{z_1} z},$$

$\varphi_{z_1} = -T_{z_1}$.
d1) Throughout this paper by $\mathbb{S}(a,b)$ we denote the set $(a,b)\times\mathbb{R}$, $-\infty \le a < b \le \infty$, and in particular we write $\mathbb{S}_0$ for $\mathbb{S}(-1,1)$. Note that $\mathbb{S}(a,b)$ is a strip if $-\infty < a < b < \infty$ and $\mathbb{S}(a,+\infty)$ is a half-plane if a is a real number, and $\mathbb{S}(-\infty,+\infty) = \mathbb{C}$. By λ_0 and ρ_0 we denote hyperbolic metrics on $\mathbb{U}$ and $\mathbb{S}_0$ respectively.
d2) Set $I_0 = (-1,1)$, and for $a \in I_0$ define

$$s = s(a) = \tan(\frac{\pi}{4}(a+1)), \quad e = e(a) = \cot\big(\frac{\pi}{4}(a+1)\big), \quad \text{and}$$

$$X(r) = X^+(r,a) = \frac{4}{\pi}\arctan(s\frac{1+r}{1-r}) - 1, \quad X^-(r,a) = 1 - \frac{4}{\pi}\arctan\left(e\frac{1+r}{1-r}\right), z \in \mathbb{U}.$$

Further it is convenient to introduce the functions A, B, A_s and B_s by $A(r) = (1+r)(1-r)^{-1}, B(r) = (1-r)(1+r)^{-1}$, $A_s(r) = sA(r), B_s(r) = sB(r)$, and $Y(r) = X^+(|z|, |a|)$.

d3) Set $c = (a+1)/2$, $\overline{c} = 2\pi c$, $\alpha = \alpha(c) = \alpha(a) = \overline{c}/2 = (a+1)\pi/2$.

It is convenient to write $f_y(x) = f(x, y)$.

(A0) It is straightforward to check

$$X_a^-(r) = \frac{4}{\pi}\arctan(B_s(r)) - 1, \quad X_a^-(r) \le a \le X_a^+(r),$$

$X^+(r, a)$ (respectively $X^-(r, a)$) is increasing (respectively decreasing) in both variables r and a, $X_1^+ = 1$ and $X_{-1}^- = -1$.

Note that $s = s(|a|) = \tan(\frac{\pi}{4}(|a|+1))$ for $a \in \mathbb{U}$.

Since $X(r) = \frac{4}{\pi}(\arctan\circ A) - 1$ and $A_s'(r) = 2s(1-r)^{-2}$, we find

$$X'(r) = \frac{4}{\pi}\frac{2s(1-r)^{-2}}{1 + A_s^2(r)} = \frac{4}{\pi}\frac{2s}{(1-r)^2 + s^2(1+r)^2}, 0 \le r < 1. \tag{2.1}$$

In a similar way since $X_-(r) = \frac{4}{\pi}(\arctan\circ B) - 1$ and $B_s'(r) = -2s(1+r)^{-2}$, we find

$$X_-'(r) = -\frac{4}{\pi}\frac{2s}{(1+r)^2 + s^2(1-r)^2}, 0 \le r < 1. \tag{2.2}$$

Next

$$X(0) = \frac{4}{\pi}\arctan\left(\tan\frac{\alpha(c)}{2}\right) - 1 = \frac{4}{\pi}\frac{\alpha(c)}{2} - 1 = \frac{4}{\pi}(a+1)\frac{\pi}{4} - 1 = a,$$

and by (2.1),

$$X'(0) = \frac{4}{\pi}\frac{2s}{1+s^2} = \frac{4}{\pi}\frac{2\tan\frac{\alpha(c)}{2}}{1 + \left(\tan\frac{\alpha(c)}{2}\right)^2} = \frac{4}{\pi}2\tan(\alpha/2)\cos^2(\alpha/2) = \frac{4}{\pi}\sin\alpha, \tag{2.3}$$

and in a similar way using (2.2)

$$X_-'(0) = -X'(0) = -\frac{4}{\pi}\sin\alpha. \tag{2.4}$$

Suppose that f is harmonic map from U into $I_0 = (-1, 1)$ with $h(0) = a$. Using a version of Schwarz lemma [17], we will show

$$\rho_0(fz, a) = |\ln\frac{s(fz)}{s(a)}| \le \ln\frac{1+r}{1-r}, z \in \mathbb{U}. \tag{2.5}$$

This inequality is equivalent to $X^-(|z|, a) \le f(z) \le X(|z|) = X^+(|z|, a), z \in \mathbb{U}$.

Theorem 1. If $u_1, u_2 \in (-1, 1)$, then

$$\rho_0(u_1, u_2) = |\ln\frac{s(u_2)}{s(u_1)}|. \tag{2.6}$$

Let h be a real valued harmonic map from $\mathbb{U}$ into $I_0 = (-1, 1)$ with $h(0) = a$, $a \in I_0$. Then

$$X^-(|z|, a) \le h(z) \le X(|z|) = X^+(|z|, a), z \in \mathbb{U}, \tag{2.7}$$

$$\text{and} \quad |(dh)_0| \le X'(0) = \frac{4}{\pi}\sin\alpha. \tag{2.8}$$

If $a=0$, then $a_1=\tan\frac{\pi}{4}=1$ and $X(|z|,0)=\frac{4}{\pi}\arctan|z|$. Hence we get classical Schwarz lemma for harmonic maps which states $|h(z)|\le X(|z|)=X(|z|,0)=\frac{4}{\pi}\arctan|z|$.

Proof. We use

$$\sec\theta=\frac{1}{\cos\theta},\quad \tanh^{-1}z=\frac{1}{2}\ln\frac{1+z}{1-z},\quad I(x)=\int_0^x\sec t\,dt=2\tanh^{-1}(\tan\frac{x}{2})$$

and

$$\rho_0(w)=\mathrm{Hyp}_{\mathbb{S}_0}(w)=\frac{\pi}{2}\frac{1}{\cos(\frac{\pi}{2}u)},\text{for}\quad w\in\mathbb{S}_0,\quad\text{where}\quad u=\mathrm{Re}w. \tag{2.9}$$

Since $A(\tan\frac{x}{2})=\tan(\frac{\pi}{4}+\frac{x}{2})$, we find $I(x)=\ln\big(\tan(\frac{\pi}{4}+\frac{x}{2})\big)$.

If $u_1,u_2\in(-1,1)$, $u_1\le u_2$, then using the change of variables $t=\frac{\pi}{2}u$, $t_k=\frac{\pi}{2}u_k$, $k=1,2$, we have

$$\rho_0(u_1,u_2)=\frac{\pi}{2}\int_{u_1}^{u_2}\frac{du}{\cos(\frac{\pi}{2}u)}=\int_{t_1}^{t_2}\frac{du}{\cos(t)}=I(t_2)-I(t_1), \tag{2.10}$$

and therefore since $I(t_k)=\ln\big(\tan(\frac{\pi}{4}+\frac{x}{2})\big)=\ln s(u_k)$, $k=1,2$, we find

$$\rho_0(u_1,u_2)=\ln\frac{\tan\frac{\pi}{4}(u_2+1)}{\tan\frac{\pi}{4}(u_1+1)}. \tag{2.11}$$

Hence (2.6) follows.

If $h(0)=a$ and recall we set $s=s(a)=\tan(\frac{\pi}{4}(a+1))$, by a version of Schwarz lemma [17], for $z\in\mathbb{U}$ we find

$$\tan(\frac{\pi}{4}(h(z)+1))\le s\frac{1+|z|}{1-|z|},\quad\text{ie.}\quad h(z)\le X(|z|)=X(|z|,a). \tag{2.12}$$

XX If $a=0$, then $s(0)=\tan\frac{\pi}{4}=1$ and $X(|z|,0)=\frac{4}{\pi}\arctan|z|$. If we set $g=-h$, then $g(0)=-a$, and by (2.12), we find $-h(z)\le X(|z|)=X(|z|,-a)$, ie. $h(z)\ge -X(|z|,-a)$. Hence one can derive (2.7).

But, we prefer the following approach. If $z\in K_r$, then

$$|\ln\frac{s(fz)}{s(a)}|\le\ln\frac{1+r}{1-r}. \tag{2.13}$$

We can rewrite this inequality as

$$\frac{s(fz)}{s(a)}\le\frac{1+r}{1-r}\quad\text{if}\quad s(a)\le s(fz),\quad\text{and}\quad\frac{s(a)}{s(fz)}\le\frac{1+r}{1-r}\quad\text{if}\quad s(a)\ge s(fz). \tag{2.14}$$

Hence if $f(z)\ge a$, we find $s(fz)\le A_s(r)$ and therefore $f(z)\le X(r)$. Further it is convenient to introduce $B(r)=s(1-r)(1+r)^{-1}$, and $X_-(r)=\frac{4}{\pi}(\arctan\circ B)-1$. Hence if $fz\le a$, we find $s(a)\le s(fz)A_s(r)$ and therefore $f(z)\ge X_-(r)$.

Hence, by (A0), we find $X_-(r)\le f(z)\le X(r)$.

Next by (2.1) and (2.2), we have

$$X'(r)=\frac{4}{\pi}\frac{2s}{(1-r)^2+s^2(1+r)^2},\quad X_-'(r)=-\frac{4}{\pi}\frac{2s}{(1+r)^2+s^2(1-r)^2},0\le r<1,$$

and in particular by (2.3) and (2.4), $X'(0)=\frac{4}{\pi}\sin\alpha,\quad X_-'(0)=-\frac{4}{\pi}\sin\alpha$, and therefore (2.8) follows. □

XX After writing the previous version we have realized that the inequality (2.7) in Theorem 1 is covered by [3], but our proof is completely different.

Definition 4. d1) For $a \in (-1,1)$, let Har^a denote the family of all real valued harmonics maps f from $\mathbb{U}$ into $(-1,1)$ with $f(0)=a$.

d2) For $a \in \mathbb{U}$ and $b \in (-1,1)$, set $L(a,b) = L(a,b) = \sup |(du)_a|$, where the supremum is taken over all real valued harmonics maps u from $\mathbb{U}$ into $(-1,1)$ with $u(a)=b$.

d3) For $a \in \mathbb{U}$ and $\ell \in T_a\mathbb{C}$ a unit vector, set $L(a) = \sup |(du)_a|$ and $L(a,\ell) = \sup |(du)_a(\ell)|$, where the supremum is taken over all real valued harmonics maps from $\mathbb{U}$ into $(-1,1)$.

Now, we can restate and strength the part of Theorem 1:

Theorem 2. If $a \in (-1,1)$ and $h \in \mathrm{Har}^a$, then
(2.15)

$$(i)\quad h(z) \le X(|z|),\quad (ii)\quad |(dh)_0| \le X'(0) = \frac{4}{\pi}\sin\alpha \quad \text{and} \quad (iii)\quad L(0,a) = \frac{4}{\pi}\sin\alpha(a).$$

Proof. We need only to prove (iii). There is a conformal mapping f of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(0)=a$ and $f'(0)>0$; then for harmonic function $u_0 = \mathrm{Re} f$ the equality holds in (iii). □

Theorem 3. Let h be a real valued harmonics map from $\mathbb{U}$ into $(-1,1)$ with $f(a)=b$, $a \in \mathbb{U}$. Then

$$h(z) \le \frac{4}{\pi}\arctan\left(\frac{1+|\varphi_a(z)|}{1-|\varphi_a(z)|}\tan\frac{\alpha(|b|)}{2}\right) - 1, \tag{2.16}$$

$$|(dh)_a| \le \frac{4}{\pi}\frac{\sin\alpha(|b|)}{1-|a|^2}. \tag{2.17}$$

Proof. Set $w=\varphi_a(z)$. Apply Theorem 2 on $h^a = h\circ\varphi_a$, we find $h^a(z) \le X(|z|)$. Hence $h(w) = h^a(z) \le X(|\varphi_a(w)|)$. Since we can identify $(d\varphi_a)_0$ with $1-|a|^2$, using $(dh^a)_0 = (dh)_a \circ (d\varphi_a)_0$ and Theorem 2 we prove (2.17). □

Further set

$$A_0(z) = \frac{1+z}{1-z}, \quad \text{and let} \quad \phi = i\frac{2}{\pi}\ln A_0;$$

that is $\phi = \phi_0 \circ A_0$, where $\phi_0 = i\frac{2}{\pi}\ln$. Let $\hat\phi$ be defined by $\hat\phi(z) = -\phi(iz)$. Note that ϕ maps $I_0 = (-1,1)$ onto y-axis and $\hat\phi$ maps I_0 onto itself.

If $\hat u = \mathrm{Re}\hat\phi$, then

$$\hat u = \frac{2}{\pi}\arg\left(\frac{1+iz}{1-iz}\right) \tag{2.18}$$

and $\hat u$ maps $I_0 = (-1,1)$ onto itself.

Let $a \in (0,1)$ and $\ell \in T_a\mathbb{C}$. There is a conformal mapping $f = f_\ell$ of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(a)=0$ and $f'(a)\ell > 0$. We will show that $u = u_\ell = \mathrm{Re} f_\ell$ is extremal. In particular, there is a conformal mapping f of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(a)=0$ and $f'(a)>0$; set $u_0 = \mathrm{Re} f$.

Theorem 4. If $a \in (-1,1)$ and $\ell \in T_a\mathbb{C}$, then

(1) $L(a) = (u_0)'_r(a) = \frac{4}{\pi}(1-|a|^2)^{-1}$ and

(2) $L(a,\ell)=L(a)=(du_\ell)_a(\ell)=\frac{4}{\pi}(1-|a|^2)^{-1}$.

This yields solution of D. Khavinson extremal problem for harmonic functions in planar case, cf. [12, 13].

Proof. (1) By hypothesis $\rho_0(f(a))|f'(a)|=2(1-|a|^2)^{-1}$, $\rho_0(f(a))=\rho_0(0)=\frac{\pi}{2}$, $(u_0)'_r(a)=f'(a)$ and $|(du_0)_a|=|\nabla u_0(0)|=\frac{4}{\pi}(1-|a|^2)^{-1}$.

(2) Recall there is a conformal mapping $f=f_\ell$ of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(a)=0$ and $f'(a)\ell>0$. If $u=u_\ell=\mathrm{Re}f_\ell$, then $(du)_a(\ell)=\mathrm{Re}\big(f'(a)\ell\big)$. We leave the interested reader to fill details. □

Theorem 5. Let h be a real valued harmonics map from $\mathbb{U}$ into $(-1,1)$ with $f(a)=b$, $a\in\mathbb{U}$. Then

$$(i)\quad |(dh)_a|\le\frac{4}{\pi}\frac{\sin\alpha(|b|)}{1-|a|^2},\qquad (ii)\quad L(a,b)=\frac{4}{\pi}\frac{\sin\alpha(|b|)}{1-|a|^2}. \tag{2.19}$$

Proof. There is a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(a)=b$. We leave the interested reader to show that $u_0=\mathrm{Re}f$ is extremal for (i) and therefore (ii) holds. □

2.1. Schwarz lemma at the boundary.

Theorem 6. Let h be a complex valued harmonic map from $\mathbb{U}$ into itself with $h(0)=a$, $a\in\mathbb{U}$. Then

$$|h(z)|\le X(|z|)=X^+(|z|,|a|), z\in\mathbb{U}.$$

Proof. Using rotation around 0 and Theorem 1 one can prove this result. □

Theorem 5. *Let $f:\mathbb{U}\to\mathbb{U}$ be harmonic and $s=s(f(0)$. Further assume that there is a point $b\in\mathbb{T}$ so that f extends continuously to b, $|f(b)|=1$ (say that $f(b)=b'$), and f is $\mathbb{R}$- differentiable at b. Then*

$$|\Lambda_f(b)|\ge\frac{2}{s\pi}.$$

Proof. By (2.1), we find

$$\lim_{r\to 1_-}X'(r)=\frac{2}{s\pi}.$$

The rest of proof is based on Theorem 6 and the following proposition. □

We leave the interested reader to prove the following propositions:

Proposition 2.2. (a) Let $f:\mathbb{U}\to\mathbb{U}$. Assume that there is a point $b\in\mathbb{T}$ so that f extends continuously to b, $|f(b)|=1$ (say that $f(b)=c$), and and f is $\mathbb{R}$-differentiable at b.

(b) Further assume that there is a function A such that $A:[0,1]\to[0,1]$, $A'(1)$ exists and $M_f(r)\le A(r)$.

Then $|\Lambda_f(b)|\ge|f'_r(b)|\ge A'(1)$.

Proof. Without loss of generality we can suppose that $c=b=1$. By (b),

$$|\frac{f(1)-f(r)}{1-r}|\ge\frac{1-M_f(r)}{1-r}\ge B(r):=\frac{1-A(r)}{1-r}.$$

Hence if $r\to 1_-$, we have $|f'_r(b)|\ge A'(1)$. Since by definition of $\Lambda_f(b)$, $\Lambda_f(b)\ge|f'_r(b)|$ it completes proof. □

Proposition 2.3. Under the above hypothesis, if there exists $f'(b)$, then
(i) $|f'(b)| \geq A'(1)$.

3. Distortion of harmonic functions related to diametar and length

We advise the reader to recall Definition 1. In [10] and [15] in particular, it is proved (see also [ABR], Theorem 6.16, Proposition 6.19, cf. [12, 3, 13]):

Proposition 3.1. If u is a harmonic map from $\mathbb{U}$ into $I_0 = (-1, 1)$, then

$$|\nabla u(0)| \leq \frac{4}{\pi}. \tag{3.1}$$

For convenience of the reader we outline a proof. Throughout this paper by $\mathbb{S}(a, b)$ we denote the set $(a, b) \times \mathbb{R}$, $-\infty \leq a < b \leq \infty$, and in particular by $\mathbb{S}_0 = \mathbb{S}(-1, 1)$. The mapping f_0 defined by $f_0(w) = \tan(\frac{\pi}{4} w)$ maps $\mathbb{S}_0$ onto $\mathbb{U}$.

If we denote by ρ_0 hyperbolic density on $\mathbb{S}_0$, then using f_0 we can check that for $w = u + iv \in \mathbb{S}_0$,

$$\rho_0(w) = \mathrm{Hyp}_{\mathbb{S}_0}(w) = \frac{\pi}{2} \frac{1}{\cos(\frac{\pi}{2} u)}. \tag{3.2}$$

It is known from the standard course of Complex Analysis that there is an analytic function ω on $\mathbb{U}$ such that $u = \mathrm{Re}\omega$ on $\mathbb{U}$. Since ω is holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$, then by a very special case of Schwarz-Ahlfors-Pick lemma(see also the property (I)),

$$\rho_0(\omega(z))|\omega'(z)| \leq 2(1 - |z|^2)^{-1}, \quad z \in \mathbb{U}, \tag{3.3}$$

where ρ_0 is given by (3.2).

Since $\frac{\pi}{2} \leq \rho_0(w)$ and $|\omega'| = |\overline{\nabla u}| = |\nabla u|$, we have (3.1).

In particular, if ω is a holomorphic function from the unit disk $\mathbb{U}$ into $\mathbb{S}_0$ with $\omega(0) = 0$, we have $|\omega'(0)| \leq \frac{4}{\pi}$ with the equality iff ω is a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$.

Remark 6. Note that one can derive Theorem 2 from (3.3). Namely, by the above notation $\rho_0(u(z))|\nabla u(z)| \leq 2$.

Definition 7. c1) If g is a holomorphic function on $\mathbb{U}$ by $\hat{g}_k$ we denote its Taylor coefficient and write $g(z) = \sum_{k=0}^{\infty} \hat{g}_k z^k$. Note that $k!\hat{g}_k = g^{(k)}(0)$. c2) For a set $M \subset \mathbb{R}^m$ by $\mathrm{d} = \mathrm{dia}(M)$ we denote the diameter of M.

Theorem 8. *Let $f = g + \overline{h}$ be complex valued harmonic in $\mathbb{U}$ which satisfies* (H1). *Then*
(i) $\pi \Lambda_f(0) \leq 2d$.
(ii) $2d \leq L$.
For function $u_d(z) = \frac{d}{\pi} \arg \frac{1+z}{1-z}$ the equality holds in (i).

It is interesting that $p(z) = d \cdot x/2$ is not extremal for the inequality (i).

Proof. We can suppose that $f(0) = 0$ and using rotations that $\Lambda_f(0) = |\hat{e}_1|$ and $\hat{e}_1 = df_0(e_1) = ke_1$, where $k = \Lambda_f(0)$. Set $p(w) = u$ and $F = p \circ f$. Then $p(G)$ is an interval of length equal or less then d, and by Proposition 3.1, $\pi|\nabla F(0)| \leq 2d$. Since $|\nabla F(0)| = \Lambda_f(0)$, we get the first inequality of (i). We leave to the reader to show that $2d \leq L$.

□

Definition 9. d1) For $a \in \mathbb{R}$, define $S_a = \{w : \mathrm{Re} w < a\}$ and let $\mathcal{P}_a$ denote the family of all functions f holomorphic in $\mathbb{U}$ for which $f(\mathbb{U}) \subset S_a$.

d2) If H is a holomorphic function on $\mathbb{U}$ which has zero at 0 at least of order 2 and $a \in \mathbb{C}$, it is straightforward to check that there are unique holomorphic functions $g = g_H$ and $h = h_H$ on $\mathbb{U}$ such that

(h0): XX $g(0) = h(0) = 0$ and $g' = -H + a$, $h' = z^{-2}H$.

Note that in this setting $a = g'(0)$ and set $f_{H,a} = g_H + i\overline{h_H}$. If it is not confusing we write f_H instead of $f_{H,a}$ and in this way we associate a unique harmonic function f_H to H. We say that H satisfies

(h_1'): if it satisfies (h0) with $2H \in \mathcal{P}_1$,
and that $f = g + \overline{h}$ satisfies

($\underline{\mathrm{h}}0$) with respect to H: if H satisfies (h_1').

d3) It is convenient to say that $f = g + \overline{h}$ satisfies

($\underline{\mathrm{h}}1$) with respect to ν if: $g(0) = h(0) = 0$ and

$$g' = \frac{1}{1 + z^2\nu} \quad \text{and} \quad h' = \frac{\nu}{1 + z^2\nu}, \tag{3.4}$$

where

(i3): $\nu = \omega z^{-2}$ and $\omega \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$ has zero at 0 at least of order 2.

- If ν satisfies (i3) there a unique f, which we denote by $f^{\underline{\nu}} = f_0^{\nu}$, such that g' and h' are given by (3.4).

Note if g' is given by (3.4), then $g'(0) = 1$, and if ν satisfies (i3), then by an application of classical Schwarz lemma, $|\nu(z)| \leq 1$, $z \in \mathbb{U}$, and the function $(1+\omega)^{-1}$ (defined by $z \mapsto (1 + z^2\nu)^{-1}$) is holomorphic on $\mathbb{U}$.

We leave to the interested reader to show that $f = g + \overline{h}$ satisfies ($\underline{\mathrm{h}}1$) with respect to ν iff it satisfies ($\underline{\mathrm{h}}0$) with respect to H with $H = \frac{\nu z^2}{1+\nu z^2}$.

Theorem 10. *Let $f = g + \overline{h}$ be complex valued continuous on $\overline{\mathbb{U}}$ and harmonic on $\mathbb{U}$ which satisfies ($\underline{H}_2$).* [2] *Then*

a) $2\pi k|\hat{g}_k| \leq L$, $k \geq 0$. *In particular*

a1) $2\pi|g'(0)| \leq L$, *with equality in the case $a = g'(0) > 0$ iff*

(h_a): $g' = -H + a$, $h' = z^{-2}H$, *and $2H \in \mathcal{P}_a$, where H has zero at 0 at least of order 2.*

a2) $2\pi \max\{|g'(0)|, |h'(0)|\} \leq L$

a3) $2\pi|g'(0)| \leq L$ *with equality iff*

(i4): $f = cf^{\nu} + c_1$, *where where ν satisfy* (i3).

b) $2\pi(1-|z|^2)|g'(z)| \leq L$, $z \in \overline{\mathbb{U}}$ *with equality iff* (i5)*: $f = cf^{\underline{\nu}} \circ \varphi_z + c_1$, where $c, c_1 \in \mathbb{C}$ and ν satisfy* (i3).

b1) $2\pi(1 - |z|^2)|\lambda_f(z)| \leq L$, $z \in \overline{\mathbb{U}}$.

As a corollary we get, $\pi(|g'(0)| + |h'(0)|) \leq L$ and since $\Lambda_f(z) + \lambda_f(z) = 2\max\{|g'(0)|, |h'(0)|\}$, $\pi(|\Lambda_f(0) + \lambda_f(0)|) \leq L$ and $2\pi\lambda_f(0) \leq L$.

Set $iX(t) = f_t' e^{-it}$. If $g'(0) > 0$ and the equality holds in a1) $X(t) \geq 0$ on $[0, 2\pi]$ and therefore, since $f_t' = iX(t)e^{it}$, $\theta = \arg(f_t') = t + \pi/2$. Hence if f is homeomorphism, γ_f is convex.

Proof. We suppose first that f is harmonic on $\overline{\mathbb{U}}$ (in general case we can apply the obtained results on f_r, $0 < r < 1$, and then pass by limit when r tends 1). Set

[2] $\gamma(t) = f(e^{it})$, $t \in [0, 2\pi]$ is a rectifiable curve and $L = |\gamma|$ is length of γ.

$z = re^{it}$. By calculation $f_t'(z) = ig're^{it} + \overline{ih're^{it}} = ig'(z)z + \overline{ih'(z)z}$. Hence $f_t'(z) = i\sum_{k=1}^{\infty} kg_k z^k - i\overline{\sum_{k=1}^{\infty} kh_k z^k}$ and $f_t'(z) = i\sum_{k=1}^{\infty} kg_k r^k e^{ikt} - i\overline{\sum_{k=1}^{\infty} kh_k r^k e^{ikt}}$ and $2\pi ikg_k = \int_0^{2\pi} f_t' e^{-ikt}dt$.
a) Since $f_t' = ig'e^{it} + \overline{ih'e^{it}}$, $2\pi ig'(0) = \int_0^{2\pi} f_t' e^{-it}dt$. Hence
$2\pi|g'(0)| \leq \int_0^{2\pi} |f_t' e^{-it}|dt = L$.
Set $iX(t) = f_t' e^{-it}$, $X(t) = g' - \overline{h'e^{2it}}$. Then (i) $2\pi g'(0) = \int_0^{2\pi} X(t)dt$.
If the equality holds in (i) and $a = g'(0) > 0$, then $X = X^+$ is a nonnegative function. Set $u = P[X]$ and $H = h'z^2$. Then $u = g' - \overline{H}$ and u is a nonnegative function. Hence $\mathrm{Im}(g') = \mathrm{Im}(\overline{H})$ and therefore $g' = -H + a$, that is (i6) $g' = -h'z^2 + a$. Since $X = -H + a - \overline{H} = a - \mathrm{Re}H$, we conclude that $2H \in \mathcal{P}_a$.
Set $M_0(w) = \frac{w}{1+w}$ and $\omega = z^2\nu$. Then $2M_0(w) \in \mathcal{P}_1$ iff $w \in \mathbb{U}$.
It is convenient to suppose for a moment that $a = 1$. Substitute $h' = \nu g'$ in (i6), we find $g' = 1 - z^2\nu g'$ and therefore

$$g' = \frac{1}{1 + z^2\nu} \quad \text{and} \quad h' = \frac{\nu}{1 + z^2\nu}.$$

Therefore $H(z) = M_0(z^2\nu)$ and $\omega \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$.
Using it one can check first that the equality holds in (a3) in the case $a = 1$ and $f(0) = 0$ iff $f = f^{\underline{\nu}}$ and in general iff f is given by (i4).

b) For $z \in \overline{\mathbb{U}}$ apply a) on $f \circ \varphi_z$. □

For the convenience of the reader we first consider harmonic maps of $\mathbb{U}$ into $\mathbb{R}^3$. Recall we will use the following hypothesis in the sequel $(\underline{H}_3')$: Suppose that $f = (f^1, f^2, f^3) : \mathbb{U} \to \mathbb{R}^3$ harmonic, $S = f(\mathbb{U})$ and the generalized length of ∂S with respect to f, $L = L^+(f) = L^+(f, \partial S)$ is finite.
In this setting, let F_k are holomorphic function in $\mathbb{U}$ such that $f_k = 2\mathrm{Re}F_k$, $k = 1, 2, 3$.
Then (A2:) $f_1 + if_2 = g + \overline{h}$, where $g = F_1 + iF_2$ and $h = F_1 - iF_2$.
(II) By $\mathbf{y} = (y_1, y_2, y_3)$ we denote coordinates in $\mathbb{R}^3$ and for $\mathbf{y}^0 \in \mathbb{R}^3$ we denote the translation $T_{\mathbf{y}^0}$ defined by $T_{\mathbf{y}^0}(\mathbf{y}) = \mathbf{y} - \mathbf{y}^0$. We use the following procedure:
(I-1) Set $p_3(\mathbf{y}) = p_3(y_1, y_2, y_3) = (y_1, y_2)$, where by $\mathbf{y} = (y_1, y_2, y_3)$ we denote coordinates in $\mathbb{R}^3$. Under the hypothesis $(\underline{H}_3)$, it is convenient to introduce for given $z_0 \in \mathbb{U}$ the tangent plane $Z = Z_{\mathbf{y}^0}$, $\mathbf{y}^0 = f(z_0)$. After rotation we can suppose that
(h_2): Z is y_1y_2-plane which we can identify with $\mathbb{C}$-plane. More precisely there is a rotation $R_{\mathbf{y}^0}$ around $\mathbf{y}^0$ such that $R = R_Z = T_{\mathbf{y}^0} \circ R_{\mathbf{y}^0}$ maps Z onto $\Pi = \{(y_1, y_2, 0) : y_1, y_2 \in \mathbb{R}\}$, with $R_Z(z_0) = 0$. Set $f^* = R \circ f$, and $\gamma^* = R \circ \gamma$. Then $f = f_Z := p_3 \circ R \circ f$ is a harmonic function from $\mathbb{U}$ into $\mathbb{C}$.
Using similar approach as in the proof Theorem 8 (ii), one can prove:

Proposition 3.2. Under the hypothesis $(\underline{H}_3)$, $2\mathrm{d} = \mathrm{dia}(G) \leq L$.

Theorem 11. *Suppose that* $f = (f^1, f^2, f^3)$ *satisfies* $(\underline{H}_3)$. *Then*
a) (i) $\pi\Lambda_f(0) \leq 2d$, *where* $\mathrm{d} = \mathrm{dia}(S)$.
b) $\pi(1 - |z|^2)\Lambda_f(z) \leq 2d$, $z \in \overline{\mathbb{U}}$.

Proof. Apply Theorem 8 on f_Z. □

For a fixed z, set $f_H^z = f_H \circ \varphi_z - f_H(z)$.

Theorem 12. *Under the hypothesis* ($\underline{H}'_3$),
b) *If f is conformal at z, then* (ii) $2\pi(1-|z|^2)|f'_x(z)| \le L$, $z \in \overline{\mathbb{U}}$.
b1) *The equality holds in* b) *for some $z \in \mathbb{U}$ iff* (iii)*: $f(\mathbb{U})$ is in the tangent plane $Z = Z_{f(z)}$ and $f_Z = c f^z_H$ or $f_Z = c\overline{f^z_H}$, $c \in \mathbb{C}$, and H satisfy* (h0) *with $2H \in \mathcal{P}_1$ and $H(z) = 0$, where $|c| = |f'_x(z)|(1-|z|^2)$.*

For a fixed z set $Z = Z_{f(z)}$. If f is conformal at z, it is easy to check that $L = L(f) = L(f_Z)$ and (b2): $2\pi|(f_Z)'_x(0)| \le L$. The equality holds in (b2) iff f satisfies (iii).

To get filling about Theorem 12, we give some comments in the following remark.

Remark 13. (i) Using a similar procedure one can show that the corresponding version of Theorem 11 and Theorem 12 hold under hypothesis ($\underline{H}_3$), that is $m \ge 2$.
(ii) The equality case in b).
Note that the equality holds in b) for some $z \in \mathbb{U}$, if, for example, $f(\mathbb{U})$ is in a plane say Z and $f = f(z) + R$, where $R : \mathbb{U} \to Z$ is a composition of a rotation in Z around $f(z)$ and homotety wrt $f(z)$. It is interspersed that the family of extremal maps is much larger then the family described in the previous sentence. Suppose that the equality holds in b) for some $z \in \mathbb{U}$. After rotation we can suppose that $Z = Z_{f(z)}$ is y_1y_2-plane which we can identify with $\mathbb{C}$-plane. Then $F'_1(z) = iF'_1(z)$ or $F'_1(z) = -iF'_1(z)$. In the case $F'_1(z) = iF'_1(z)$, the equality holds in b) iff $f(\mathbb{U})$ is in the tangent plane $Z = Z_{f(z)}$ and (i5): $f = cf^{\underline{\nu}} \circ \varphi_z + c_1$, where $c, c_1 \in \mathbb{C}$, and ν satisfy (i3) with $\nu(z) = 0$. In the case $F'_1(z) = -iF'_1(z)$ we leave the reader to state the corresponding statement.

Proof. In particular if ($\underline{H}_3$)(for dimension $m = 3$) holds [3], then the theorem holds. We will prove the theorem under this assumption. By application this case to f_r, $0 < r < 1$, and letting r tends to 1, one can get general result.

a) Let $S = f(\overline{\mathbb{U}})$, and $M_0 = f(0)$. Since f is conformal at 0, then
(c1): $f'_x(0) = 0$ or (c2): $f'_x(0) \times f'_y(0) \ne 0$.
In the case (c1), (i) is clear. In the case (c2) there is the tangent plane Z of S at M_0.[4] Set $\tilde{f} = p_3 \circ f^*$, $\tilde{f} = (f^1, f^2)$ and $\tilde{\gamma}(t) = \tilde{f}(e^{it})$. Then $\tilde{f} = \tilde{g} + \overline{\tilde{h}}$.

Recall by notation in (II), Π is tangent plane of $S^* = f^*(\overline{\mathbb{U}})$ at 0, so that

$$(f^*)^3(x) = o(\tilde{f}(x)) = \epsilon(x)\tilde{f}'_x(0)x$$

and therefore $((f^*)^3)'_x(0) = 0$. Hence $\tilde{f}'_x(0) = f'_x(0)$. Since f is conformal at 0, then $\tilde{f}$ is conformal at 0, we can suppose wlg that $\tilde{h}'_x(0) = 0$. Thus, since R_Z is an euclidean isometry in this case we have
(ii1) $|(\tilde{g})'(0)| = |\tilde{f}'_x(0)| = |f'_x(0)|$.
If $\tilde{L} = |\tilde{\gamma}|$ and $L^* = |\gamma^*|$ are lengths of $\tilde{\gamma}$ and γ^* respectively, then by Theorem 10a)
(ii2) $2\pi|(\tilde{g})'(0)| \le \tilde{L}$.
Since $\tilde{\gamma}$ is the projection of γ^*, we first conclude that (ii3) $\tilde{L} \le L^* = L$, and now by (ii1),(ii2) and (ii3), we get
(ii4) $2\pi|f'_x(0)| = 2\pi|(\tilde{g})'(0)| \le \tilde{L} \le L^* = L$,
which yields the part a).
b) Apply a) on $f \circ \varphi_z$. Note that $\tilde{L} \le L$ with equality iff $f(\mathbb{U})$ is in a plane.

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[3]($\underline{H}_3$): f is continuous on $\overline{\mathbb{U}}$, harmonic on $\mathbb{U}$ and γ_f is a rectifiable curve.
[4]Note that after rotation we can suppose that Z is y_1y_2-plane which we can identify with $\mathbb{C}$-plane and $f(0) = 0$.

If for some $z \in \mathbb{U}$ the equality holds in b), then $\tilde{L} = L$ and the equality holds in b) for the function $f \circ \varphi_z$ at 0, that is the equality holds in (b2). In particular $\tilde{L} = L^*$. Therefore $tr(\gamma^*)$ is in a plane Π^* parallel to Π. By an application of the mean value theorem to $f^* \circ \varphi_{z_0}$, we conclude that $0 = f^* \circ \varphi_{z_0}(0)$ belongs Π^* and therefore $\Pi^* = \Pi$. Then f^* is a planar mapping and we can apply Theorem 10. □

4. Harmonic and analytic disks

4.1. **Harmonic disks.** If $f : \mathbb{U} \to \mathbb{R}^m$ is a vector harmonic on $\mathbb{U}$, we call $S = f(\mathbb{U})$ a harmonic disk with center at $f(0)$ (defined by f).

Theorem 14. *If $f : \mathbb{U} \to \mathbb{R}^m$ is a vector harmonic on $\mathbb{U}$, then*
(a) $|f|$ *and* $|f'_t|$ *are subharmonic.*
(b) $L(r)$ *and* $d(r)$ *are increasing in* $r \in [0, 1)$.

Proof. For $z_0 \in \mathbb{U}$ and $r > 0$ small enough, by the mean value theorem, $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt$, and therefore

$$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + re^{it})dt\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|dt.$$

□

The following example shows how the boundary behavior of harmonic mappings may differ from that of conformal mappings. [5]

Example 1 ([4]). Let $l(z) = \frac{z}{1-z}$, $s(z) = \frac{1}{2}\ln\frac{1+z}{1-z}$ and $f(z) = \mathrm{Re}\, l(z) + i\mathrm{Im}\, s(z)$. Observe that $f(e^{it}) = w_0$ on $0 < t < \pi$ and $f(e^{it}) = \overline{w_0}$ on $\pi < t < 2\pi$, where $w_0 = -\frac{1}{2} + i\frac{\pi}{4}$. In particular, f collapses the upper and lower semicircles to single points. In fact, it can be proved that l, s, and f map the disk onto $S_1 = \{\mathrm{Re} w > -\frac{1}{2}\}$, $S_2 = \{|\mathrm{Im} w| < \frac{\pi}{4}\}$ and $S_3 = \{S_1 \cap S_2\}$ respectively.

If the map has no continuous extension to $\overline{\mathbb{U}}$ (in particular the boundary map collapses) at first sight the following definitions seems convenient.

Definition 15. Let $S \subset \mathbb{R}^m$ be a harmonic disk defined by f.

Let h^1_L denote the family of vector harmonic function $f : \mathbb{U} \to \mathbb{R}^m$ for which $L^+(f) = \sup\{L(f, r) : r \in [0, 1)\} < \infty$. If $L^+(f) < \infty$, then there is a boundary function f^*, but in general $L^+(f) > L(f^*)$.

It seems that the above described result hold under each of the following hypothesis:

($\underline{H}'_m$): Suppose that $u = (u^1, u^2, ..., u^m) : \mathbb{U} \to \mathbb{R}^m$ is harmonic, $S = u(\mathbb{U})$ and the generalized length of ∂S with respect to u, $L = L^+(u) = L^+(u, \partial S)$ is finite. If ($\underline{H}_m$) holds then the generalized length L is reduced to the length of ∂S.

We plan in a forthcoming paper to consider the above discussed results in connections with hypothesis ($\underline{H}'_m$).

Here we only show that that the corresponding version of Theorem 8 holds under hypothesis (H0).

[5]According to the Caratheodory extension theorem, a conformal mapping between two Jordan domains always extends to a homeomorphism of the closures.

Theorem 16. *Let $f = g + \overline{h}$ be complex valued harmonic in $\mathbb{U}$ which satisfies* $(\underline{H}_m)$.
Then
(i) $\pi\Lambda_f(0) \leq 2d$.
(ii) $2d \leq L^+(f)$.
For function

$$u_d(z) = \frac{d}{\pi}\arg\frac{1+z}{1-z}$$

the equality holds in (i).

It is interesting that $p(z) = d \cdot x/2$ is not extremal for the inequality (i).

Proof. Set $\hat{e}_1 = df_0(e_1)$. We can suppose that $f(0) = 0$ and using rotations that $\Lambda_f(0) = |\hat{e}_1|$ and $\hat{e}_1 = df_0(e_1) = ke_1$, where $k = \Lambda_f(0)$. Set $p(w) = u$ and $F = p \circ f$. Then $p(G)$ is an interval of length equal or less then d, and by Proposition 3.1, $\pi|\nabla F(0)| \leq 2d$. Since $|\nabla F(0)| = \Lambda_f(0)$, we get the first inequality of (i). We leave to the reader to show that $2d \leq L$.
For $0 < r < 1$ set $G_r = f_r(\mathbb{U})$, $|\gamma_{f_r}| = L_f(r)$ and denote with d_r the diameter of G_r.
By Theorem 8, (iii) $\pi r\Lambda_f(0) \leq 2d(r) \leq L(r)$, $0 < r < 1$. Since $|f_t'|$ and $|f|$ are subharmonic L_r and d_r are increasing functions in $r \in [0, 1)$. Hence by letting r to 1 in (iii), one can prove the result. □

4.2. **Harmonic and analytic disks.** For $\mathbf{z} = (z_1, z_2, ..., z_m) \in \mathbb{C}^m$, set $\operatorname{Re}\mathbf{z} = (\operatorname{Re}z_1, \operatorname{Re}z_2, ..., \operatorname{Re}z_m)$ and $\operatorname{Im}\mathbf{z} = (\operatorname{Im}z_1, \operatorname{Im}z_2, ..., \operatorname{Im}z_m)$. Recall we will use the following hypothesis in the sequel (H_m): Suppose that $u = (u^1, u^2, ..., u^m) : \mathbb{U} \to \mathbb{R}^m$ is harmonic, $S = u(\mathbb{U})$ and the generalized length of ∂S wrt u, $L = L(u) = L_-(u, \partial S)$ is finite.
(B0)In this setting, there are are holomorphic functions F_k in $\mathbb{U}$ such that $u_k = \operatorname{Re}F_k$, $k = 1, 2, 3, ..., m$. Set $F = (F^1, F^2, ..., F^m)$. We say shortly that holomorphic function F is associated to u. Then $u_x' = \frac{1}{2}(F_x' + \overline{F_x'}) = \operatorname{Re}F'(z)$ and therefore
(B) $u_x' - iu_y' = F'$.
If $f : G \to \mathbb{R}^2$, recall then
(B1:) $f_1 + if_2 = g + \overline{h}$, where $g = (F_1 + iF_2)/2$ and $h = (F_1 - iF_2)/2$.
(B2:) If $p = f_z$ and $q = f_{\bar{z}}$, then $p = f_z = g_z = g'$, $q = f_{\bar{z}} = \bar{h}'$, $J_f = \operatorname{Re}(i\overline{F_1'}F_2')$ and
$4|p|^2 = |F'|^2 + 2J_f$, $4|q|^2 = |F'|^2 - 2J_f$, $2(|g'|^2 + |h'|^2) = |F'|^2$ and
if $|g'| \geq |h'|$, then $2|g'| \geq |F'| \geq 2|h'|$.
(B3:) If in addition u is conformal at 0, then $h'(0) = 0$, and $|F'(0)| = \sqrt{2}|g'|$.

Theorem 17. *Suppose the hypothesis* $(\underline{H}_m)$. [6]
d1) *Then there is a holomorphic function $F : \mathbb{U} \to \mathbb{C}^m$ such that $u = \operatorname{Re}F$; and in this setting $\pi|F'(0)| \leq L$, where $L = |\gamma|$ is length of γ.*
d2) *Then* (i): $2\pi|D_z u(0)| \leq L$.
d3) *If in addition u is conformal at 0, then $2\pi\Lambda_u(0) \leq L$.*

Proof. Since $2u_t' = F'(z)ie^{it} + \overline{izF'(z)}$, $2\pi\hat{F}(0) = \int_0^{2\pi} F'(z)dt$, $F'(0) = \hat{F}(1)$ $\pi iF'(0) = \int_0^{2\pi} u_t'e^{-it}dt$, we find $\pi|F'(0)| \leq \int_0^{2\pi} |u_t'e^{-it}|dt = L$. Then $\pi|F'(0)| \leq L$, and since $2D_z u(0) = F'(0)$, we get (i).

[6] $(\underline{H}_m)$: $u : \mathbb{U} \to \mathbb{R}^m$ is continuous on $\overline{\mathbb{U}}$ and harmonic on $\mathbb{U}$, and $\gamma = \gamma_u$ is rectifiable.

Using similar approach as in the proof of Theorem 11(the procedure described in (I-1)) and apply Theorem 10(planar case), one can prove d3). □

Remark 18. By (B), since $u'_x, u'_y \in \mathbb{R}^m$, $|F'|^2 = |u'_x - iu'_y|^2 = |u'_x|^2 + |u'_y|^2$. Since $2u'_t = izF'(z) + \overline{izF'(z)}$, $L \le |F'|_1$.

Question 1. What is relation between L and $|F'|_1$?

Note that $2D_z u(0) = F'(0)$ in $\mathbb{C}^m$. If $m = 2$ then $D_z u = g'$ in $\mathbb{C}$. Here we need to be careful because we identify $(u_1, u_2) \in \mathbb{C}^2$ with $u_1 + iu_2 \in \mathbb{C}$(but the corresponding norms in $\mathbb{C}^2$ and $\mathbb{C}$ are not equal in general). Therefore $2|g'| \ne |F'|$ in general. (B3) shows that the estimat (i) in d2) is not optimal in general.

Question 2. Can we modify our procedure to get an optimal estimate?

5. AREA ESTIMATE

We advise the reader to recall Definition 2.

Theorem 19. *Suppose that u satisfies the hypothesis $(\underline{H}_m)$. Then*
(i1)*: $4\pi A(S) \le L^2$, where $A = A(u)$ and $L = L(u)$.*
(I1) *If in addition to (H_m) we suppose that*
(h2): *$A(S)$ is finite and*
(h3): *u is $\underline{K}$-quasiconformal and $F = \underline{F}$ is a corresponding holomorphic function associated to u, then*

$$D[u] = \underline{D}[F] = \pi\Big(\sum_{k=1}^{\infty}(k|\hat{F}(k)|^2)\Big) \le 2\underline{K}A(S).$$

(I2) *In particular under (H_m) and* (h2),
(i2): *$|D_1u(0)|^2 + |D_2u(0)|^2 \le 2\underline{K}A(S)$ with equality iff*
(i3): *$u(z) = ax + by$, where $a = D_1u(0)$, $b = D_2u(0)$ with $\underline{K} = \frac{|a|^2+|b|^2}{J}$, where $J = \sqrt{|a|^2|b|^2 - (a\cdot b)^2}$. In the case* (i2)*, $u(\mathbb{U})$ is a planar domain bounded by an ellipse.*

Proof. Since the Gaussian curvature of S is negative, by a version of isoperimetric inequality (see for example Theorem 3.4 [9]), we get (i1). Set $J_u = \sqrt{EG - F^2}$. Then, by (h3): $|F'|^2 \le \underline{K}J_u$ on $\mathbb{U}$. By (A) we have $D[u] = \underline{D}[F]$ and hence by Parseval's formula we get (I1). If equality holds in i2) then $\hat{F}(k) = 0$ for $k > 1$ and therefore $F(z) = cz$, where $c = a + ib$, $a, b \in \mathbb{R}^m$. Hence we get (i3).

□

Theorem 20. d1) *If in addition to (H_m) we suppose that* (h2): *$A(S)$ is finite and* (h4): *u is conformal, then*

$$A = \int_{\mathbb{U}} |D_1u|^2 dxdy = \frac{\underline{D}[F]}{2} = \frac{\pi}{2}\Big(\sum_{k=1}^{\infty}(k|\hat{F}(k)|^2)\Big).$$

d2) *In particular, $\pi\Lambda_u^2(0) \le D[u]$ with equality iff* (ii): *$\gamma = \gamma_u$ is a circle given by $u_k = a_kx - b_ky$, $k = 1, 2, 3, ..., m$, where $|a| = |b|$ and $a\cdot b = 0$.*
d3) *$2\pi\Lambda_u(0) \le L$ with equality iff (ii).*

Examples $u(z) = z + \overline{z}$ and $u_n(z) = nz + \overline{z}/n$ show that i3) is not true in general without hypothesis that the mapping is qc.

Proof. By (A) we have $2A = D[u] = \underline{D}[F]$ and hence by Parseval's formula we get d1). If equality holds in d2) then $\hat{F}(k) = 0$ for $k > 1$ and therefore $F(z) = cz$, where $c = a + ib$, $a, b \in \mathbb{R}^m$, and since u is conformal at 0 (ii) holds.
By the isoperimetric inequality $2\pi D[u] = 4\pi A \leq L^2$ and therefore d2) implies d3). □

6. Appendix

Let $f : \Omega \to f(\Omega)$ be a C^1-diffeomorphism. We write $df = pdz + qd\overline{z}$, where $p = f_z$ and $q = f_{\overline{z}}$.
$J_f = |f_z|^2 - |f_{\overline{z}}|^2$.
Let f be a diffeomorphism in a neighborhood U of a point z_0. Then f is orientation preserving mapping in U if and only if $J_f(z_0) > 0$.
If f is orientation preserving mapping in U at z_0, then df maps the tangent space T_{z_0} into T_{w_0}, where $w_0 = f(z_0)$, and circles K_r with center at z_0 of radius r onto ellipses E_r with center at w_0 and with major axis of length $\Lambda_f r$ and minor axis of length $\lambda_f r$. The dilatation (or distortion) at z_0 is defined to be

$$D_f := \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \geq 1. \tag{6.1}$$

The complex dilatation at z_0 is

$$\mu_f = \frac{f_{\overline{z}}}{f_z}. \tag{6.2}$$

It is often more convenient to consider

$$d_f = |\frac{f_{\overline{z}}}{f_z}|.$$

The dilatation and distortion are related by

$$D_f = \frac{1 + |\mu_f|}{1 - |\mu_f|}.$$

Let $f \in C^1$ be orientation preserving mapping. Then f is conformal iff $q = f_{\overline{z}} \equiv 0$ (Cauchy-Riemann equations). If f is conformal, $D_f = 1$ and $q = 0$, so $df = pdz$ maps circles to circles.

Definition 21 (Grotzsch analytic definition for regular mappings)**.** Let $f : \Omega \to \mathbb{C}$ be a diffeomorphism. We say that f is a *quasiconformal map* if $D_f(z)$ is bounded in Ω. We say f is a K-quasiconformal map if $D_f(z) \leq K$ for all $z \in \Omega$.

$K(f) = ess\sup_{z\in\Omega} D_f(z)$ is called the coefficient of quasi-conformality (or linear dilatation) of f in the domain Ω.

Definition 22. b4) For a planar domain D and C^1 mapping $u : D \to \mathbb{R}^m$, set $S = u(D)$,

$$K_*(f, z) = \frac{E + G}{2J_u}$$

and $K_*(f) = ess\sup_{z\in D} K_*(f, z)$ which is called the coefficient of quasi-conformality (or linear dilatation) of f in the domain D.

Remark 23. If S is in a plane and K the standard coefficient of quasi-conformality, then $K_* = \frac{K^2+1}{2K}$,that is $K = K_* + \sqrt{K_*^2 - 1}$, where $K_* = K_*(f)$ and $K = K(f)$. Motivated by this we give an alternative definition: u is K-qc if $E+F \le \underline{K} J_u$, where $\underline{K} = K + \frac{1}{K}$; in planar case this definition is reduced to the standard definition of coefficient of quasi-conformality.

If D, G are domains in $\mathbb{R}^n$, by $\mathrm{Har}(D, G)$ denote the family of all vector valued harmonics maps f from D into G.

Definition 24 ($\mathrm{Har}(p)$,$\mathrm{Har}_c(p)$)**.** For $p \in \mathbb{B}$, let $\mathrm{Har}(p) = \mathrm{Har}(\mathbb{B}, \mathbb{B}; p)$ (respectively $\mathrm{Har}_c(p)$) denote the family of all vector valued harmonics maps f from $\mathbb{B}$ into itself with $f(0) = p$ (respectively which are conformal at 0 respectively).

Set $L_h(p) = \sup\{|f'(0)| : f \in \mathrm{Har}(p)\}$ and $K_h(p) = \frac{L(p)}{\sqrt{1-|p|^2}}$, $L_c(p) = \sup\{|f'(0)| : f \in \mathrm{Har}_c(p)\}$ and $K_c(p) = \frac{L_c(p)}{1-|p|^2}$.

For planar domains D and G and given $z \in D$ and $q \in G$ denote by $L_h(z, p; D, G) = \sup\{|f'(z)|\}$, where the supremum is taken over all $f \in \mathrm{Har}(D, G)$ with $f(z) = p$. If $D = \mathbb{U}$ we write $\mathrm{Har}(G)$ instead of $\mathrm{Har}(\mathbb{U}, G)$ and if in addition $z = 0$, we write simply $L_h(p, G)$ (or $L_{\mathrm{har}}(p, G)$) and if in addition $G = \mathbb{U}$, $L_h(p)$.

Problem 1 (Extremal)**.** For given $p \in \mathbb{B}$ find $K_h(p)$ and $K_c(p)$.

For given $p \in \mathbb{B}$, find $\sup\{|f'(p)| : f \in \mathrm{Har}(\mathbb{B}, \mathbb{B})\}$. For given $p, q \in \mathbb{B}$, find $\sup\{|f'(p)| : f \in \mathrm{Har}(\mathbb{B}, \mathbb{B}), f(p) = q\}$.

The editors of JMAA paid my attention to [12] and the book [13].

Acknowledgement. We are indebted to Shi Qingtian, who has been reading very carefully several versions, for useful discussions and useful comments which improved the exposition.

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