

Some Order-Theoretic Properties of the Zeros of the Zeta Function

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Abstract

The (partially) ordered set of the non-trivial zeros of the zeta function with positive imaginary parts is considered. The order is the coordinatewise order inherited from \mathbb{C} . Some interesting properties regarding the minimal elements of this poset are proven.

1 Introduction

To the best knowledge of the author the set of zeros of the zeta function has not been considered from an order-theoretic perspective. It seems such a viewpoint could prove useful, however, and this paper aims to draw the attention towards the goal of understanding the structure of said set using order theory.

2 Definitions

We will consider the poset (\mathbb{C}, \leq) , where the order relation \leq is the coordinatewise order, defined by: $a + ib \leq c + id$ if and only if $a \leq c$ and $b \leq d$. Let Z denote the non-trivial roots of $\zeta(s)$ with non-negative imaginary parts, i. e.

$$Z = \{s \in \mathbb{C} : 0 \leq \Im(s), \zeta(s) = 0\}. \quad (1)$$

Then we can consider (Z, \leq) with the inherited order relation from (\mathbb{C}, \leq) . Since the zeros of a meromorphic function are isolated and there are upper

bounds on the number of zeros in a region of the critical strip (see e. g. [1, 2]) we can, for convenience, index the distinct imaginary parts of the elements of Z and they will form a strictly increasing sequence $\{t_i\}_{i \in \mathbb{N}}$. If t_1 is the first member of this sequence (and thus the smallest one), the root $\sigma_1 + it_1 \in Z$ is known to lie on the critical line, i. e. $\sigma_1 = \frac{1}{2}$, and is trivially a minimal element of Z .

Definiton 2.1. *Let*

$$Z_n := \{s \in Z : \Im(s) = t_n\}. \quad (2)$$

Define the diameter of Z_n by

$$d(Z_n) = \max_{s_1, s_2 \in Z_n} (\Re(s_1) - \Re(s_2)). \quad (3)$$

Every (Z_n, \leq) is a totally ordered set. It has a least and a greatest element which we will denote by $\hat{\sigma}_n + it_n$ and $\tilde{\sigma}_n + it_n$, respectively.

3 Results

First we will state the following lemma:

Lemma 3.1. *The Riemann hypothesis is true if and only if (Z, \leq) is a totally ordered set.*

Proof. 1.) Assume RH. (Z, \leq) is a partially ordered set by definition. Take $\sigma_n + it_n, \sigma_m + it_m \in Z$. Then $\sigma_n = \sigma_m = \frac{1}{2}$. Now we have $t_n \leq t_m$ or $t_m \leq t_n$ since $t_n, t_m \in \mathbb{R}$. Thus $\sigma_n + it_n \leq \sigma_m + it_m$ or $\sigma_m + it_m \leq \sigma_n + it_n$, i. e. (Z, \leq) is a totally ordered set.

2.) Assume that (Z, \leq) is a totally ordered set.

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Let $\sigma_1 + it_1, \sigma_k + it_k \in Z$ be such that $\sigma_1 + it_1$ is the same as in section 2 and $\sigma_k \neq \frac{1}{2}$.

If $\sigma_k < \frac{1}{2}$, then $\sigma_k < \sigma_1$ and $t_k > t_1$. Thus $\sigma_k + it_k \parallel \sigma_1 + it_1$, but (Z, \leq) is a totally ordered set which is a contradiction.

If $\sigma_k > \frac{1}{2}$, then from the symmetry of the zeros about the $\Re(s) = \frac{1}{2}$ line there must exist $\sigma_l + it_l \in Z$ such that $\sigma_l = 1 - \sigma_k$ and $t_l = t_k$. We now have $\sigma_l < \sigma_1$ and $t_l > t_1$. Thus $\sigma_l + it_l \parallel \sigma_1 + it_1$ and again we reach a contradiction. This means that $\sigma_k = \frac{1}{2}$ and the proof is complete. \square

Next is a result concerning the number of minimal elements of (Z, \leq) . Denoting the set of said elements by $\text{Min}(Z, \leq)$, we have:

Theorem 3.1. *There is a bijection between $\text{Mn} := \{Z_n : n > 1, d(Z_n) > d(Z_i) \forall i \ 0 < i < n\}$ and $\text{Min}(Z, \leq) \setminus \{\sigma_1 + it_1\}$.*

Proof. Let $f : \text{Mn} \rightarrow Z$ be such that $f(Z_n) = \hat{\sigma}_n + i\hat{t}_n$.

1.) Consider some $Z_k \in \text{Mn}$. Then $f(Z_k) = \hat{\sigma}_k + i\hat{t}_k$. From the symmetry of the zeros about the $\Re(s) = \frac{1}{2}$ line follows that

$$\hat{\sigma}_k = \frac{1}{2} - \frac{d(Z_k)}{2}. \quad (4)$$

Take $\sigma_l + it_l \in Z$, such that $\sigma_l + it_l \leq \hat{\sigma}_k + i\hat{t}_k$. Then

$$t_l \leq \hat{t}_k, \quad \sigma_l \leq \hat{\sigma}_k. \quad (5)$$

Also there exists some set Z_l of the form (2), such that $\sigma_l + it_l \in Z_l$.

If $t_l = \hat{t}_k$, then $Z_l = Z_k$. Since $\hat{\sigma}_l + i\hat{t}_l$ is the least element in Z_l , $\hat{\sigma}_k = \hat{\sigma}_l \leq \sigma_l$. Thus using (5) we get $\sigma_l = \hat{\sigma}_k$ and $\sigma_l + it_l = \hat{\sigma}_k + i\hat{t}_k$.

If $t_l < \hat{t}_k$, then $d(Z_l) < d(Z_k)$, since $Z_k \in \text{Mn}$. Similarly to (4), $\hat{\sigma}_l = \frac{1}{2} - \frac{d(Z_l)}{2}$. Then $\hat{\sigma}_k < \hat{\sigma}_l \leq \sigma_l$, which is a contradiction with (5).

Thus $\hat{\sigma}_k + i\hat{t}_k$ is a minimal element of Z , i. e. $f(Z_k) \in \text{Min}(Z, \leq) \setminus \{\sigma_1 + it_1\}$.

2.) Take $Z_k, Z_l \in \text{Mn}$, such that $Z_k \neq Z_l$. Then $\hat{t}_k \neq \hat{t}_l$ and $f(Z_k) \neq f(Z_l)$. Thus f is injective.

3.) Consider some $\sigma_k + it_k \in \text{Min}(Z, \leq) \setminus \{\sigma_1 + it_1\}$. There exists some Z_k , such that $\sigma_k + it_k \in Z_k$. Then $\hat{\sigma}_k + i\hat{t}_k \leq \sigma_k + it_k$, but since $\sigma_k + it_k$ is a minimal element, we have

$$\hat{\sigma}_k + i\hat{t}_k = \sigma_k + it_k. \quad (6)$$

Now suppose that for some $l < k$ there exists Z_l , such that $d(Z_l) \geq d(Z_k)$. Then $\hat{t}_l < \hat{t}_k$ and from (4) $\hat{\sigma}_l \leq \hat{\sigma}_k$. Thus

$$\hat{\sigma}_l + i\hat{t}_l < \hat{\sigma}_k + i\hat{t}_k = \sigma_k + it_k, \quad (7)$$

but $\sigma_k + it_k$ is minimal and we reach a contradiction. It follows that $d(Z_l) < d(Z_k)$, i. e. $Z_k \in \text{Mn}$. From (6) $f(Z_k) = \sigma_k + it_k$, so f is surjective. This completes the proof. \square

A dual result can be stated for the number of maximal elements of (Z, \leq) :

Theorem 3.2. *There is a bijection between $\text{Mx} := \{Z_n : d(Z_n) > d(Z_i) \forall i \ i > n\}$ and $\text{Max}(Z, \leq)$.*

The bijection here is given by $g : \text{Mx} \rightarrow Z$, such that $g(Z_n) = \tilde{\sigma}_n + i\tilde{t}_n$. After that the proof is analogous to the previous one, considering that

$$\tilde{\sigma}_n = \frac{1}{2} + \frac{d(Z_n)}{2}. \quad (8)$$

Corollary 3.1. *RH is not true if and only if (Z, \leq) has at least 2 minimal elements.*

Proof. RH not true implies that there is a set Z_n with $d(Z_n) > 0$, i. e. $Z_n \in \text{Mn}$, and by theorem 3.1 (Z, \leq) has a minimal element distinct from $\sigma_1 + it_1$.

Conversely, if (Z, \leq) has at least 2 minimal elements, then there exists a set $Z_n \in \text{Mn}$ with $d(Z_n) > 0$, again by theorem 3.1, and RH is false. \square

Similarly:

Corollary 3.2. *RH is not true if (Z, \leq) has at least 1 maximal element.*

This follows from theorem 3.2 analogously to the previous corollary.

We should note here that corollary 3.2 doesn't include an "only if" statement. This is the case, because for example the sequence $\{d(Z_i)\}_{i \in \mathbb{N}}$ can be monotonically increasing and have a strictly increasing subsequence, in which case Mx is empty and so is $\text{Max}(Z, \leq)$. Then RH is not true, but there are 0 maximal elements.

References

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