# A SUM OF SQUARES NOT DIVISIBLE BY A PRIME

KYOUNGMIN KIM AND BYEONG-KWEON OH

ABSTRACT. Let p be a prime. We define  $S(p)$  the smallest number k such that every positive integer is a sum of at most k squares of integers that are not divisible by p. In this article, we prove that  $S(2) = 10$ ,  $S(3) = 6$ ,  $S(5) = 5$ , and  $S(p) = 4$  for any prime p greater than 5. In particular, it is proved that every positive integer is a sum of at most four squares not divisible by 5, except the unique positive integer 79.

#### 1. Introduction

The famous four square theorem says that every non-negative integer is a sum of at most 4 squares, that is, for the quaternary quadratic form  $f(x, y, z, t)$  $x^2 + y^2 + z^2 + t^2$ , the Diophantine equation  $f(x, y, z, t) = n$  always has an integer solution for any non-negative integer  $n$ . After Lagrange [\[6\]](#page-13-0) proved this celebrated theorem, it was generalized in several directions. Ramanujan [\[13\]](#page-13-1) determined that there are exactly 55 positive definite integral diagonal quaternary quadratic forms. Later, Dickson [\[3\]](#page-13-2) confirmed Ramanujan's assertion is correct except the quaternary quadratic form  $x^2 + 2y^2 + 5z^2 + 5t^2$ , which represents all non-negative integers, except the unique integer 15. Conway and Schneeberger proved, so called, 15- Theorem which says that any positive definite integral quadratic form representing 1, 2, 3, 5, 6, 7, 10, 14, and 15 represents all non-negative integers. Recently, Bhargava [\[1\]](#page-13-3) provided a very simple and elegant proof of 15-Theorem.

Another generalization was initiated by Mordell [\[7\]](#page-13-4) and Ko [\[5\]](#page-13-5). In those papers, they proved that every positive definite integral quadratic form of rank  $n$  less than or equal to 5 is represented by a sum of  $n + 3$  squares. In fact, there is a quadratic form of rank 6 that is not represented by a sum of any number of integral squares. One of such quadratic forms is the root lattice  $E_6$ .

In this article, we generalize Lagrange's four square theorem in another direction. Let p be a prime. We say an integer n is a sum of k squares not divisible by p if there are integers  $x_1, x_2, \ldots, x_k$  such that

$$
n = x_1^2 + x_2^2 + \dots + x_k^2
$$
 and  $(p, x_1x_2 \cdots x_k) = 1$ .

<sup>2010</sup> Mathematics Subject Classification. Primary 11E25, 11E45.

Key words and phrases. A sum of squares not divisible by a prime.

This work of the first author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (NRF-2016R1A5A1008055 and NRF-2018R1C1B6007778).

This work of the second author was supported by the National Research Foundation of Korea (NRF-2017R1A2B4003758).

We define  $S(p)$  the smallest integer k such that any positive integer is a sum of at most  $k$  squares not divisible by  $p$ . In this article, we prove that

 $S(2) = 10$ ,  $S(3) = 6$ ,  $S(5) = 5$ , and  $S(p) = 4$  for any prime  $p \ge 7$ .

In particular, it is proved that every positive integer is a sum of at most four squares not divisible by 5, except the unique integer 79.

Throughout this article, we always assume that a quadratic form of rank  $n$ 

$$
f(x_1, x_2, \dots, x_n) = \sum_{i,j=1}^n a_{ij} x_i x_j \qquad (a_{ij} = a_{ji})
$$

is positive definite and integral, that is  $a_{ij} \in \mathbb{Z}$  for any  $i, j$ . The corresponding symmetric matrix  $M_f$  to the quadratic form f is defined by  $M_f = (a_{ij})$ . The discriminant  $df$  of the quadratic form  $f$  is defined by the determinant of the corresponding symmetric matrix  $M_f$ . If f is diagonal, that is,  $a_{ij} = 0$  for any  $i \neq j$ , then we write

$$
f = \langle a_{11}, a_{22}, \dots, a_{nn} \rangle.
$$

We say an integer a is represented by f if there are integers  $x_1, x_2, \ldots, x_n$  such that  $a = f(x_1, x_2, \ldots, x_n)$ . In this case, we write  $a \rightarrow f$ . In particular, we say a is a sum of k squares if a is represented by the quadratic form  $I_k = \langle 1, 1, \ldots, 1 \rangle$ . We define

$$
R(a, f) = \{(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n : a = f(x_1, x_2, \dots, x_n)\} \text{ and } r(a, f) = |R(a, f)|.
$$

Note that  $r(a, f)$  is finite, for we are assuming that f is positive definite.

For two quadratic forms  $f$  and  $g$  of rank  $n$ , we say  $f$  is *isometric* to  $g$  if there is an integral matrix  $T \in M_n(\mathbb{Z})$  such that  $T^t M_f = M_g$ . We say f is isometric to g over the *p*-adic integer ring  $\mathbb{Z}_p$  if there is a matrix  $T \in M_n(\mathbb{Z}_p)$  satisfying the above property. The isometry group  $O(f)$  of f is defined by

$$
O(f) = {T \in M_n(\mathbb{Z}) : T^t M_f T = M_f}
$$
 and  $o(f) = |O(f)|$ .

The genus gen $(f)$  of f is the set of all quadratic forms that are isometric to f over  $\mathbb{Z}_p$  for any prime p. The class number  $h(f)$  of f is the number of isometric classes in the genus of  $f$ . We say an integer  $a$  is represented by the genus of  $f$  if there is a quadratic form  $f' \in \text{gen}(f)$  that represents a. Note that a is represented by the genus of f if and only if the equation  $a = f(x_1, x_2, \ldots, x_n)$  always has a solution  $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_p^n$  for any prime p (see, for example, 102:5 of [\[10\]](#page-13-6)).

For a quadratic form  $f$  and an integer  $a$ , we define

$$
w(f) = \sum_{[g] \in \text{gen}(f)} \frac{1}{o(g)} \quad \text{and} \quad r(a, \text{gen}(f)) = \frac{1}{w(f)} \sum_{[g] \in \text{gen}(f)} \frac{r(a, g)}{o(g)},
$$

where  $[g]$  is the isometric class containing g in the genus of f. Note that if  $h(f) = 1$ , then we have  $r(a, \text{gen}(f)) = r(a, f)$ .

Any unexplained notations and terminologies can be found in [\[4\]](#page-13-7) or [\[10\]](#page-13-6).

## 2. A sum of squares not divisible by 2 or 3

Let  $p$  be a prime. We say that an integer  $n$  is a sum of  $k$  squares not divisible by p if there are integers  $x_1, x_2, \ldots, x_k$  such that

$$
n = x_1^2 + x_2^2 + \dots + x_k^2
$$
 and  $(p, x_1 x_2 \dots x_k) = 1$ .

If *n* is a sum of *k* squares not divisible by *p*, then we write  $n \xrightarrow{p} I_k$ . We further define  $S(p)$  the smallest integer k such that any positive integer is a sum of at most k squares not divisible by p. Note that  $S(p) \geq 4$  for any prime p.

<span id="page-2-0"></span>**Lemma 2.1.** Let  $f$  be a ternary quadratic form and let  $p$  be a prime not dividing 2df. Let n be a positive integer and let ord<sub>p</sub> $(n) = \lambda_p$ . If n is represented by the genus of f, then we have

$$
\frac{r(p^2 n, gen(f))}{r(n, gen(f))} = p \cdot \prod_{q|2df} \frac{\alpha_q(p^2 n, f)}{\alpha_q(n, f)} \prod_{q \nmid 2df} \frac{\alpha_q(p^2 n, f)}{\alpha_q(n, f)}
$$
\n
$$
= \left( \frac{p^{\left[\frac{\lambda_p}{2}\right]+2} - 1 - \left(\frac{-np^{-2\left[\frac{\lambda_p}{2}\right]} \cdot df}{p}\right) \left(p^{\left[\frac{\lambda_p}{2}\right]+1} - 1\right)}{p^{\left[\frac{\lambda_p}{2}\right]+1} - 1 - \left(\frac{-np^{-2\left[\frac{\lambda_p}{2}\right]} \cdot df}{p}\right) \left(p^{\left[\frac{\lambda_p}{2}\right]} - 1\right)} \right).
$$

Here  $[x]$  is the greatest integer not exceeding x and  $\left(\frac{1}{p}\right)$  is the Legendre symbol.

Proof. By the Minkowski-Siegel formula, we have

$$
r(n, \text{gen}(f)) = \pi^{\frac{3}{2}} \cdot \Gamma\left(\frac{3}{2}\right)^{-1} \cdot \sqrt{\frac{n}{df}} \cdot \prod_{q < \infty} \alpha_q(n, f),
$$

where  $\alpha_q$  is the local density over  $\mathbb{Z}_q$ . Note that by Theorem 3.1 in [\[15\]](#page-13-8), we have

$$
\alpha_p(n,f) = \begin{cases} 1 + \frac{1}{p} - \frac{1}{p^{\frac{\lambda_p+1}{2}}} - \frac{1}{p^{\frac{\lambda_p+3}{2}}} & \text{if } \lambda_p \text{ is odd,} \\ 1 + \frac{1}{p} - \frac{1}{p^{\frac{\lambda_p+2}{2}}} + \left(\frac{-p^{-\lambda_p}n \cdot df}{p}\right) \frac{1}{p^{\frac{\lambda_p+2}{2}}} & \text{otherwise.} \end{cases}
$$

Hence the lemma follows directly from this.  $\Box$ 

<span id="page-2-1"></span>**Lemma 2.2.** Let  $f$  be a ternary quadratic form and let  $n$  be a positive integer. If the class number of  $f$  is  $1$ , then for any prime  $p$  not dividing  $2df$ , we have

$$
r(p^2n, f) - r(n, f) > 0,
$$

provided that  $p^2n$  is represented by f.

*Proof.* Since we are assuming that  $h(f) = 1$ , we have by Lemma [2.1,](#page-2-0)

$$
\frac{r(p^2n, f)}{r(n, f)} = \frac{r(p^2n, gen(f))}{r(n, gen(f))} > 1.
$$

This completes the proof.

Proposition 2.3. Every positive integer is a sum of at most 10 squares of odd integers, and in fact,  $S(2) = 10$ .

*Proof.* If  $n \equiv 3 \pmod{8}$ , then by Legendre's three-square theorem, there are integers  $a_1, a_2$ , and  $a_3$  such that

$$
n = a_1^2 + a_2^2 + a_3^2
$$
 and  $(2, a_1 a_2 a_3) = 1$ .

Hence  $n \xrightarrow{2} I_3$ . Assume the  $n \equiv t \pmod{8}$  for  $3 \le t \le 8$ . Since  $n - (t - 3) \equiv 3$ (mod 8), we have  $n \xrightarrow{2} I_t$ . Next assume that  $n \equiv 1 \pmod{8}$ . If n is a square of an integer, then  $n \xrightarrow{2} I_1$ . If n is not a square, then we have  $n \xrightarrow{2} I_9$ , for  $n - 6 \equiv 3$ (mod 8). Finally, assume that  $n \equiv 2 \pmod{8}$ . If n is a sum of two squares, then  $n \xrightarrow{2} I_2$ . If n is not a sum of two squares, then  $n \xrightarrow{2} I_{10}$ . Note that any integer  $n \equiv 2 \pmod{8}$  that is not a sum of two squares is not a sum of less than 10 squares of odd integers. Therefore, we have  $S(2) = 10$ .

Proposition 2.4. Every positive integer is a sum of at most 6 squares not divisible by 3, and in fact,  $S(3) = 6$ .

*Proof.* Let n be a positive integer. First, assume that  $n \equiv 1 \pmod{3}$ . By Lagrange's four-square theorem,  $n$  is a sum of four squares, that is, there are integers  $a_1, a_2, a_3$ , and  $a_4$  such that  $n = a_1^2 + a_2^2 + a_3^2 + a_4^2$ . If  $a_1 a_2 a_3 a_4$  is not divisible by 3, then  $n \longrightarrow I_4$ . If  $a_1 a_2 a_3 a_4$  is divisible by 3, then exactly three of  $a_1, a_2, a_3$ , and  $a_4$  are divisible by 3. Without loss of generality, we assume that  $a_1, a_2$ , and  $a_3$  are divisible by 3. Since  $n \xrightarrow{3} I_1$  in the case when  $a_1 = a_2 = a_3 = 0$ , we assume that  $a_1^2 + a_2^2 + a_3^2 \neq 0$ . By applying Lemma [2.2](#page-2-1) in the case when  $f = \langle 1, 1, 1 \rangle$  and  $p = 3$ , there are integers  $b_1, b_2$ , and  $b_3$  such that

$$
a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2
$$
 and  $(3, b_1b_2b_3) = 1$ .

In fact, Lemma [2.2](#page-2-1) says that at least one of  $b_1, b_2$ , and  $b_3$  is not divisible by 3. However, in our case, this implies that none of  $b_i$ 's are divisible by 3. Hence if  $n \equiv 1 \pmod{3}$ , then  $n \stackrel{3}{\longrightarrow} I_1$  or  $n \stackrel{3}{\longrightarrow} I_4$ .

Now, assume that  $n \equiv 2 \pmod{3}$ . If n is a sum of two squares, then  $n \stackrel{3}{\longrightarrow} I_2$ . Otherwise, we have  $n \stackrel{3}{\longrightarrow} I_5$ , for  $n-1 \equiv 1 \pmod{3}$ . Finally assume that  $n \equiv 0$ (mod 3). In this case, we have  $n \xrightarrow{3} I_3$  or  $n \xrightarrow{3} I_6$ . Note that if n is not a sum of three squares, then  $n$  is not a sum of less than or equal to 5 squares not divisible by 3. Therefore, we have  $S(3) = 6$ .

### 3. WHEN  $n$  is divisible by  $p$

In this and next section, we find  $S(p)$  for a prime p greater than 3. In this section, we find the smallest number  $k$  to represent a positive integer  $n$  divisible by  $p$  as a sum of less than or equal to  $k$  squares not divisible by  $p$ .

<span id="page-3-0"></span>**Lemma 3.1.** Let p be an odd prime and let n be a positive integer. Assume that p is represented by  $\langle 1, k \rangle$ , where k is a positive integer not divisible by p. If an integer n divisible by p is represented by  $\langle 1, k \rangle$ , then there are integers u and v such that

$$
n = u^2 + kv^2 \quad and \quad (p, uv) = 1.
$$

*Proof.* See [\[9\]](#page-13-9).  $\Box$ 

<span id="page-4-0"></span>**Lemma 3.2.** Let  $p \equiv 1 \pmod{4}$  be a prime and let n be a positive integer. If n is a sum of three squares, then  $n$  is a sum of k squares not divisible by  $p$  for some integer  $k \leq 4$ .

*Proof.* From the assumption, there are integers  $a_1, a_2$ , and  $a_3$  such that  $n = a_1^2 + a_2^2$  $a_2^2 + a_3^2$ . First, assume that exactly two of  $a_1, a_2$ , and  $a_3$  are divisible by p. Without loss of generality, assume that both  $a_1$  and  $a_2$  are divisible by p. If  $a_1 = a_2 = 0$ , then  $n \stackrel{\circ}{\longrightarrow} I_1$ . If  $a_1^2 + a_2^2 \neq 0$ , then by Lemma [3.1,](#page-3-0) there are integers  $b_1$  and  $b_2$  such that

$$
a_1^2 + a_2^2 = b_1^2 + b_2^2
$$
 and  $(p, b_1b_2) = 1$ .

Therefore, we have  $n \xrightarrow{p} I_3$ . Next, assume that exactly one of  $a_1, a_2$ , and  $a_3$ , say  $a_1$ , is divisible by p. If  $a_1 = 0$ , then  $n \xrightarrow{p} I_2$ . If  $a_1 \neq 0$ , then by Lemma [3.1,](#page-3-0) there are integers  $c_1$  and  $c_2$  such that

$$
a_1^2 = c_1^2 + c_2^2
$$
 and  $(p, c_1c_2) = 1$ .

Hence we have  $n \xrightarrow{p} I_4$ . Finally, assume that  $a_i$  is divisible by p for any  $i = 1, 2, 3$ . In this case, from the above assertion, we may easily show that  $n \xrightarrow{p} I_k$  for some integer  $k \leq 4$ . This completes the proof.

<span id="page-4-1"></span>**Proposition 3.3.** Let  $p \equiv 1 \pmod{4}$  be a prime and let n be a positive integer. If *n* is divisible by p, then  $n \xrightarrow{p} I_k$  for some integer  $k \leq 4$ .

*Proof.* Without loss of generality, we may assume that  $\text{ord}_q(n) \leq 1$  for any prime  $q \neq p$ . By Lemma [3.2,](#page-4-0) we may assume that  $n \equiv 7 \pmod{8}$ . Since the class number of  $\langle 1, 1, 5 \rangle$  is one and every positive integer congruent to 7 modulo 8 is represented by  $\langle 1, 1, 5 \rangle$  over  $\mathbb{Z}_p$  for any prime p, there are integers x, y, and z such that  $n = x^2 + y^2 + 5z^2$  by 102.5 of [\[10\]](#page-13-6). If  $xyz$  is not divisible by p, then  $n = x^2 + y^2 + z^2 + (2z)^2$  and  $n \stackrel{p}{\longrightarrow} I_4$ . Assume that at least two of x, y, and z are divisible by p. Then, both x and y are divisible by p. If  $x^2 + y^2 \neq 0$ , then there are integers a and b not divisible by p such that  $x^2 + y^2 = a^2 + b^2$ . If z is not divisible by p, then  $p = 5$ . Since  $5z^2 = (2z)^2 + z^2$ , we have  $n \xrightarrow{p} I_2$  or  $n \xrightarrow{p} I_4$ . Assume that  $z$  is a non-zero integer divisible by  $p$ . Then there are integers  $c$  and  $d$ not divisible by p such that  $z^2 = c^2 + d^2$ . Hence we have

$$
5z2 = (2c + d)2 + (c - 2d)2 = (2c - d)2 + (c + 2d)2.
$$

Now, one may easily check that either  $(2c + d)(c - 2d)$  or  $(2c - d)(c + 2d)$  is not divisible by p. Therefore, we have  $n \xrightarrow{p} I_2$  or  $n \xrightarrow{p} I_4$ .

Now, assume that x is divisible by p and yz is not divisible by p. Since  $y^2 + 5z^2 \equiv$ 0 (mod p), we have  $\left(\frac{-5}{p}\right) = \left(\frac{5}{p}\right) = 1$ . Let  $x = p^t x'$  with  $(p, x') = 1$ . Since the class number of  $\langle 1, 5 \rangle$  is one, both p and  $p^{2t}$  is represented by  $\langle 1, 5 \rangle$ . Hence by Lemma [3.1,](#page-3-0) there are integers u and v such that  $p^{2t} = u^2 + 5v^2$  and  $(p, uv) = 1$ . Then we have

$$
n = x2 + y2 + 5z2 = p2tx2 + y2 + 5z2
$$
  
=  $(u2 + 5v2)x2 + y2 + 5z2$   
=  $(vx' + 2z)2 + (2vx' - z)2 + u2x2 + y2$   
=  $(vx' - 2z)2 + (2vx' + z)2 + u2x2 + y2$ ,

where  $uvx'yz$  is not divisible by p. Note that either  $(vx'+2z)(2vx'-z)$  or  $(vx'-z)$  $2z(2vx'+z)$  is not divisible by p. Hence  $n \stackrel{p}{\longrightarrow} I_k$  for some integer  $3 \le k \le 4$ . The proof of the case when  $y$  is divisible by  $p$  and  $xz$  is not divisible by  $p$  is quite similar to this. If z is divisible by p and  $xy$  is not divisible by p, then one may easily show that  $n \stackrel{p}{\longrightarrow} I_2$  or  $n \stackrel{p}{\longrightarrow} I_4$  by the similar reasoning given above.

Now, we consider the case when  $p \equiv 3 \pmod{4}$  and n is divisible by p. To deal with this case, we need some results from the theory of modular forms. For general theory of modular forms and some relation between representations of quadratic forms and modular forms, see [\[11\]](#page-13-10) and [\[14\]](#page-13-11).

For a positive integer N and a positive rational number k such that  $2k \in \mathbb{Z}$ , let  $S_k(N, \chi)$  be the space of cusp forms of weight k with character  $\chi$  for the congruence group  $\Gamma_0(N)$ .

<span id="page-5-1"></span>**Lemma 3.4.** Let  $f = \langle 1, 1, 10 \rangle$  be Ramanujan's ternary quadratic form and let n be a positive integer. For any prime  $p \neq 2, 3, 5,$  and 17, we have

$$
r(p^2n, f) - r(n, f) > 0,
$$

provided that  $p^2n$  is represented by f.

*Proof.* Note that  $h(f) = 2$  and

$$
\mathrm{gen}(f)/\sim=\left\{f,f'=\left\langle 2\right\rangle \perp\begin{pmatrix}2&1\\1&3\end{pmatrix}\right\}.
$$

We let

$$
\phi(z) = \sum_{n=1}^{\infty} a(n)q^n = \frac{1}{4} \sum_{n=1}^{\infty} (r(n, f) - r(n, f'))q^n = q - q^3 - q^7 - q^9 + 2q^{13} + \cdots,
$$

where  $q = e^{2\pi i z}$ . Then it is known that  $\phi(z) \in S_{\frac{3}{2}}(40, \left(\frac{10}{z}\right))$  is the weight  $\frac{3}{2}$  cusp form. It is also known (see, for example, [\[12\]](#page-13-12)) that the Shimura lift of  $\phi(z)$  is a cusp form of weight 2

$$
\Phi(z) = \eta^2 (2z)\eta^2 (10z) = \sum_{n=1}^{\infty} A(n)q^n
$$
  
=  $q - 2q^3 - q^5 + 2q^7 + q^9 + 2q^{13} + 2q^{15} - 6q^{17} - 4q^{19} - 4q^{21} + 6q^{23} + \cdots$ 

Here  $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$  is the Dedekind's eta-function. Since the dimension of the space  $S_{\frac{3}{2}}(40, \left(\frac{10}{\cdot}\right))$  is one,  $\phi(z)$  is an eigenform of all Hecke operators  $T(p^2)$ . Hence for any prime p, there is a complex number  $\alpha(p)$  such that

<span id="page-5-0"></span>(3.1) 
$$
\alpha(p)a(n) = a(p^2n) + \left(\frac{-10n}{p}\right)a(n) + \left(\frac{10}{p}\right)^2 \cdot p \cdot a(n/p^2),
$$

for any positive integer *n*. Here  $a(n/p^2) = 0$  if *n* is not divisible by  $p^2$ . Since  $\Phi(z) \in$  $S<sub>2</sub>(20)$  is the newform and the Shimura lifts commute with the Hecke operators of integral and half-integral weight, we have  $\alpha(p) = A(p)$  for any prime p. By Deligne's bound on Hecke eigenvalues, we have  $|A(n)| \leq \tau(n)n^{\frac{1}{2}}$ , where  $\tau(n)$  is the number of positive divisors of n. Hence we have  $|\alpha(p)| = |A(p)| \leq 2\sqrt{p}$  for any prime p.

Let p be a prime relatively prime to 10 and let  $\text{ord}_p(n) = \lambda_p$ . From the assumption given above, we may assume that  $n$  is represented by  $f$ . Then, by Equation  $(3.1)$ , we have

$$
r(p^2n, f) - r(p^2n, f') = \left(\alpha(p) - \left(\frac{-10n}{p}\right)\right) \left(r(n, f) - r(n, f')\right) - p \cdot \left(r\left(\frac{n}{p^2}, f\right) - r\left(\frac{n}{p^2}, f'\right)\right).
$$

By Lemma [2.1,](#page-2-0) we also have

$$
r(p^{2}n, f) + 2r(p^{2}n, f') = \left(\frac{p^{\lfloor \frac{\lambda_{p}}{2}\rfloor + 2} - 1 - \left(\frac{-np^{-2\lfloor \frac{\lambda_{p}}{2}\rfloor} \cdot df}{p}\right)(p^{\lfloor \frac{\lambda_{p}}{2}\rfloor + 1} - 1)}{p^{\lfloor \frac{\lambda_{p}}{2}\rfloor + 1} - 1 - \left(\frac{-np^{-2\lfloor \frac{\lambda_{p}}{2}\rfloor} \cdot df}{p}\right)(p^{\lfloor \frac{\lambda_{p}}{2}\rfloor} - 1)}\right)
$$
  
\n
$$
= \left(\frac{p^{\lfloor \frac{\lambda_{p}}{2}\rfloor + 1}(p - 2) + 2p - 1 - \left(\frac{-np^{-2\lfloor \frac{\lambda_{p}}{2}\rfloor} \cdot df}{p}\right)(p^{\lfloor \frac{\lambda_{p}}{2}\rfloor}(p - 2) + 2p - 1)}{p^{\lfloor \frac{\lambda_{p}}{2}\rfloor + 1} - 1 - \left(\frac{-np^{-2\lfloor \frac{\lambda_{p}}{2}\rfloor} \cdot df}{p}\right)(p^{\lfloor \frac{\lambda_{p}}{2}\rfloor} - 1)}\right)
$$
  
\n
$$
\times (r(n, f) + 2r(n, f')) + 2p \cdot \left(r\left(\frac{n}{p^{2}}, f\right) + 2r\left(\frac{n}{p^{2}}, f'\right)\right).
$$
  
\nby combining two equalities given above, we have

By combining two equalities given above, we have

$$
3(r(p^2n, f) - r(n, f)) \geq (p - 5 + 2\alpha(p) - 2\left(\frac{-10n}{p}\right))r(n, f)
$$

$$
+ \left(2p - 4 - 2\alpha(p) + 2\left(\frac{-10n}{p}\right)\right)r(n, f') + 6p \cdot r\left(\frac{n}{p^2}, f'\right).
$$

If  $p \neq 3, 17$ , then  $p - 5 + 2\alpha(p) - 2\left(\frac{-10n}{p}\right) > 0$  and  $2p - 4 - 2\alpha(p) + 2\left(\frac{-10n}{n}\right)$ p  $\Big\} > 0,$ 

for any prime p. Therefore if  $p \neq 2, 3, 5$ , and 17, we have  $r(p^2n, f) - r(n, f) > 0$ . This completes the proof.

The following proposition is mentioned in Remark 3.2 of [\[8\]](#page-13-13) without proof. Here, we provide a simple proof for those who are unfamiliar with the method developed in [\[8\]](#page-13-13).

<span id="page-6-0"></span>**Proposition 3.5.** For any positive integer n such that  $n \equiv 5 \pmod{6}$ , the diophantine equation  $n = x^2 + y^2 + 10z^2$  has always an integer solution.

*Proof.* One may easily show that every integer n such that  $n \equiv 5 \pmod{6}$  is represented by the genus of Ramanujan's ternary quadratic form  $f = \langle 1, 1, 10 \rangle$ . Hence we may assume that  $n$  is represented by the other ternary quadratic form in the genus of Ramanujan's ternary quadratic form, that is, there are integers  $a, b$ , and  $c$ such that  $n = 2a^2 + 2b^2 + 2bc + 3c^2$ . Then, by a direct computation, we have

 $(a, b, c) \equiv (0, \pm 1, 0), (\pm 1, 0, 0), (1, 0, \pm 1),$ 

$$
(1, \pm 1, \mp 1), (-1, 0, \pm 1)
$$
 or  $(-1, \pm 1, \mp 1)$  (mod 3).

By changing signs of  $a, b$ , and  $c$ , if necessary, we may assume that

$$
(a, b, c) \equiv (0, 1, 0), (1, 0, 0), (1, 0, 1) \text{ or } (1, 1, -1) \pmod{3}.
$$

First, assume that  $(a, b, c) \equiv (1, 1, -1) \pmod{3}$ . Then there are integers b' and c' such that  $b - a = 3b'$  and  $c - 2a = 3c'$ . Therefore we have

$$
n = 2a2 + 2(3b' + a)2 + 2(3b' + a)(3c' + 2a) + 3(3c' + 2a)2
$$
  
= 20a<sup>2</sup> + 18b'<sup>2</sup> + 27c'<sup>2</sup> + 24ab' + 42ac' + 18b'c'  
= (a + 3b' - c')<sup>2</sup> + (3a + 3b' + 4c')<sup>2</sup> + 10(a + c')<sup>2</sup>,

which implies that n is represented by  $\langle 1, 1, 10 \rangle$ . Similarly, one may easily check that n is represented by  $\langle 1, 1, 10 \rangle$  in the cases when  $(a, b, c) \equiv (0, 1, 0) \pmod{3}$  or  $(a, b, c) \equiv (1, 0, 1) \pmod{3}$ .

Finally, assume that  $(a, b, c) \equiv (1, 0, 0) \pmod{3}$ . Let  $\mathfrak{G}_{-20}$  be the set of all proper classes of primitive binary quadratic forms with discriminant  $-20$ . Then it is well known that  $\mathfrak{G}_{-20}$  forms an abelian group with the composition law (for details, see [\[2\]](#page-13-14)). In fact,  $\mathfrak{G}_{-20} = \{ [x^2 + 5y^2], [2x^2 + 2xy + 3y^2] \}$  and  $[x^2 + 5y^2]$  is the identity class. Since  $3^2$  is primitively represented by the identity class  $[x^2 + 5y^2]$ , every integer that is represented by  $2x^2 + 2xy + 3y^2$  is 3-primitively represented by it by Theorem 4.1 of [\[9\]](#page-13-9). This implies that there are always integers  $d, e$  such that

$$
2b2 + 2bc + 3c2 = 2d2 + 2de + 3e2 \text{ and } (3, d, e) = 1,
$$

unless  $b = c = 0$ . This implies that  $(a, d, e) \equiv (1, \pm 1, \mp 1)$  or  $(1, 0, \pm 1)$  (mod 3). Therefore *n* is represented by  $\langle 1, 1, 10 \rangle$  from the above argument. Note that any integer of the form  $2a^2$  is represented by Ramanujan's ternary quadratic form  $\langle 1, 1, 10 \rangle$ . This completes the proof.

<span id="page-7-0"></span>**Lemma 3.6.** Let  $p \equiv 3 \pmod{4}$  be a prime and let n and a be positive integers such that

- (i) the diagonal ternary quadratic form  $\langle 1, 1, a \rangle$  has class number 1;
- (ii) the integer  $a$  is either  $a$  square or  $a$  sum of  $2$  squares not divisible by  $p$ ;
- (iii)  $\left(\frac{a}{p}\right) = 1$  and n is divisible by p;
- (iv) *n* is represented by  $\langle 1, 1, a \rangle$ .

Then,  $n \longrightarrow I_3$  if a is a square, and  $n \longrightarrow I_4$  if a is a sum of 2 squares not divisible by p.

*Proof.* By condition (iv), there are integers  $x_1, x_2$ , and  $x_3$  such that  $n = x_1^2 + x_2^2 +$  $ax_3^2$ . Since the proofs are quite similar to each other, we assume that a is a sum of 2

squares not divisible by p. Then, there are integers  $a_1$  and  $a_2$  such that  $a = a_1^2 + a_2^2$ and  $(p, a_1 a_2) = 1$ . Hence we have

$$
n = x_1^2 + x_2^2 + (a_1x_3)^2 + (a_2x_3)^2.
$$

If one of  $x_i$ 's is divisible by p, then by condition (iii), all of the  $x_i$ 's are divisible by p. Therefore, if  $\text{ord}_p(n) = 1$ , then  $n \xrightarrow{p} I_4$ . If  $\text{ord}_p(n) \geq 2$ , then we may take  $x_i$ 's such that at least one of them is not divisible by  $p$  by Lemma [2.2.](#page-2-1) This implies that all of the  $x_i$ 's are not divisible by p. This completes the proof.  $\Box$ 

<span id="page-8-0"></span>Remark 3.7. Note that the class number of Ramanujan's ternary quadratic form  $\langle 1, 1, 10 \rangle$  is two. However, we may apply the above lemma to the case when  $a = 10$ and  $p > 3$  by using Lemma [3.4](#page-5-1) instead of Lemma [2.2.](#page-2-1) In this case, it is not easy to check whether condition (iv) of Lemma [3.6](#page-7-0) holds or not without Generalized Riemann Hypothesis(GRH) (see [\[12\]](#page-13-12)).

<span id="page-8-2"></span>**Proposition 3.8.** Let  $p \equiv 3 \pmod{4}$  be a prime greater than 3 and let n be a positive integer divisible by p. Then  $n \stackrel{p}{\longrightarrow} I_k$  for some integer  $k = 3$  or 4.

*Proof.* We may assume, without loss of generality, that  $\text{ord}_q(n) \leq 1$  for any prime  $q \neq p$ . If n is a sum of three squares, then  $n \xrightarrow{p} I_3$  by Lemma [3.6.](#page-7-0) Assume that  $n \equiv 7 \pmod{8}$ . Then n is represented by both  $\langle 1, 1, 2 \rangle$  and  $\langle 1, 1, 5 \rangle$ . If either  $\left(\frac{2}{p}\right) = 1$  or  $\left(\frac{5}{p}\right) = 1$ , then  $n \stackrel{p}{\longrightarrow} I_4$  by Lemma [3.6.](#page-7-0)

Now, assume that  $\left(\frac{2}{p}\right) = \left(\frac{5}{p}\right) = -1$ . Then  $\left(\frac{10}{p}\right) = 1$ . Assume further that  $\text{ord}_p(n) \geq 2$ . Then *n* is represented by Ramanujan's ternary quadratic form  $\langle 1, 1, 10 \rangle$  (see Theorem 1 of [\[12\]](#page-13-12)). Hence we may still apply Lemma [3.6](#page-7-0) to show that  $n \stackrel{p}{\longrightarrow} I_4$ , as stated in Remark [3.7.](#page-8-0)

Finally, assume that  $\text{ord}_p(n) = 1$ . If  $n \equiv 2 \pmod{3}$ , that is,  $n \equiv 5 \pmod{6}$ , then *n* is represented by  $\langle 1, 1, 10 \rangle$  by Proposition [3.5,](#page-6-0) which implies that  $n \longrightarrow I_4$ by Lemma [3.6.](#page-7-0) Next, assume that  $n \equiv 1 \pmod{3}$ . Then exactly one of  $n-1, n-4$ , and  $n - 25$  is divisible by 9. Let  $s_0 \in \{1, 2, 5\}$  be the integer such that  $n - (s_0)^2 \equiv 0$ (mod 9). Since  $n - (s_0)^2 \equiv 3$  or 6 (mod 8), the integer  $\frac{n - (s_0)^2}{9}$  $\frac{(80)}{9}$  is a sum of three squares, whereas it is not a sum of two squares. Hence there are non-zero integers  $a, b$ , and c such that

$$
n - (s_0)^2 = (\pm a)^2 + (\pm b)^2 + (\pm c)^2
$$
 and  $a \equiv b \equiv c \equiv 0 \pmod{3}$ .

Since  $n - (s_0)^2$  is not divisible by p, at least one of a, b, and c is not divisible by p. If abc is not divisible by p, then  $n \stackrel{p}{\longrightarrow} I_4$ . Hence we may assume that abc is divisible by p. Since  $-1$  is not a square modulo p, we may assume that exactly one of  $a, b$ , and  $c$  is divisible by  $p$ . Without loss of generality, assume that  $c$  is divisible by  $p$ . By choosing signs suitably, we further assume that

(3.2) 
$$
a \not\equiv 2b \pmod{p}
$$
,  $2a \not\equiv b \pmod{p}$ , and  $a \not\equiv -b \pmod{p}$ .

Now, we have

<span id="page-8-1"></span>
$$
n - (s_0)^2 = (2m - a)^2 + (2m - b)^2 + (2m - c)^2 \quad \text{and} \quad a + b + c = 3m,
$$

where m is an integer. By [\(3.2\)](#page-8-1), the integer  $(2m-a)(2m-b)(2m-c)$  is not divisible by p, which implies that  $n \stackrel{p}{\longrightarrow} I_4$ .

Now, assume that  $n \equiv 0 \pmod{3}$ . Since  $\frac{n}{3} \equiv 5 \pmod{8}$ , we have  $\frac{n}{3}$  $\stackrel{p}{\longrightarrow} I_3$  by Lemma [3.6.](#page-7-0) Hence there are integers  $a, b$ , and  $c$  such that

$$
n = 3(a2 + b2 + c2)
$$
 and  $(p, abc) = 1$ .

By Euler's four-square identity, we have

$$
n = (12 + 12 + 12 + 02)(a2 + b2 + c2 + 02)
$$
  
= (a - b - c)<sup>2</sup> + (a + b)<sup>2</sup> + (a + c)<sup>2</sup> + (b - c)<sup>2</sup>  
= (a + b + c)<sup>2</sup> + (a - b)<sup>2</sup> + (a - c)<sup>2</sup> + (b - c)<sup>2</sup>  
= (a + b - c)<sup>2</sup> + (a - b)<sup>2</sup> + (a + c)<sup>2</sup> + (b + c)<sup>2</sup>  
= (a - b + c)<sup>2</sup> + (a + b)<sup>2</sup> + (a - c)<sup>2</sup> + (b + c)<sup>2</sup>.

Assume that  $a - b - c \equiv 0 \pmod{p}$ . Then clearly,  $(a + b + c)(a - b)(a - c)$  is not divisible by p. If  $b - c$  is divisible by p, then  $a \equiv 2b \pmod{p}$ . This implies that  $n = a^2 + b^2 + c^2 \equiv 6b^2 \not\equiv 0 \pmod{p}$ , which is a contradiction to the fact that  $n \equiv 0 \pmod{p}$ . Hence  $b - c$  is not divisible by p and  $n \stackrel{p}{\longrightarrow} I_4$ . Similarly, if one of  $(a + b + c), (a + b - c)$ , and  $(a - b + c)$  is divisible by p, then  $n \xrightarrow{p} I_4$ . Therefore we may assume that

$$
(a-b-c)(a+b+c)(a+b-c)(a-b+c) \not\equiv 0 \pmod{p}.
$$

Since  $a$  is not divisible by  $p$ , we have

(3.3) 
$$
a+b \not\equiv 0 \pmod{p}
$$
 or  $a-b \not\equiv 0 \pmod{p}$ .

If both  $(a + c)(b - c)$  and  $(a - c)(b + c)$  are divisible by p, then

<span id="page-9-0"></span>
$$
a \equiv b \equiv -c \pmod{p}
$$
 or  $a \equiv b \equiv c \pmod{p}$ ,

which implies that  $n = 3(a^2 + b^2 + c^2) \equiv 9c^2 \not\equiv 0 \pmod{p}$ . This is a contradiction. Similarly, one may easily show that either  $(a - c)(b - c)$  or  $(a + c)(b + c)$  is not divisible by *p*. From these and Equation (3.3), we have  $n \xrightarrow{p} I_4$ . divisible by p. From these and Equation [\(3.3\)](#page-9-0), we have  $n \stackrel{p}{\longrightarrow} I_4$ .

4. WHEN  $n$  is not divisible by  $p$ 

In this section, we consider the case when a positive integer  $n$  is not divisible by  $p$ , where  $p$  is a prime greater than 3, as in the previous section. In this case, we may assume that  $n$  is square-free.

<span id="page-9-1"></span>**Lemma 4.1.** Let n be a positive integer and let p be a prime greater than 3.

- (i) If  $5 \leq p \leq 13$  and  $n < (16p)^2$ , then  $n \xrightarrow{p} I_k$  for some integer  $k \leq 4$ , except the case when  $p = 5$  and n is 79. In fact, 79 is a sum of 5 squares not divisible by 5.
- (ii) If  $p \geq 17$  and  $n < (10p)^2$ , then  $n \stackrel{p}{\longrightarrow} I_k$  for some integer  $k \leq 4$ .

*Proof.* To prove that n is a sum of k squares not divisible by p for some positive integer k, we may assume that  $0 \leq \text{ord}_q(n) \leq 1$  for any prime  $q \neq p$ . By Lagrange's four square theorem, we may also assume that  $n \geq p^2$ .

First, assume that  $p^2 \le n < (10p)^2$ . Then there are integers  $u$   $(1 \le u \le 9)$  and  $a (0 \le a \le p-1)$  such that  $(up+a)^2 \le n < (up+a+1)^2$ . We assume that  $p \ge 73$ . If  $a = 0$ , then

$$
n - (up - 1)^2 \le n - (up - 2)^2 < (up + 1)^2 - (up - 2)^2 \le 6up - 3 < p^2.
$$

If  $a = 1$ , then

$$
n - (up + 1)^2 \le n - (up - 2)^2 < (up + 2)^2 - (up - 2)^2 \le 8up < p^2,
$$

and if  $a \ge 2$ , then

$$
n - (up + a)^2 \le n - (up + a - 1)^2 < (up + a + 1)^2 - (up + a - 1)^2 \le 4up + 4a < p^2.
$$

Therefore, there is a positive integer  $k$  less than or equal to 3 such that

$$
\begin{cases}\nn - (up - 1)^2 \text{ or } n - (up - 2)^2 \xrightarrow{p} I_k & \text{if } a = 0, \\
n - (up - 2)^2 \text{ or } n - (up + 1)^2 \xrightarrow{p} I_k & \text{if } a = 1, \\
n - (up + a - 1)^2 \text{ or } n - (up + a)^2 \xrightarrow{p} I_k & \text{if } a \ge 2.\n\end{cases}
$$

Therefore if  $p \ge 73$  and  $n < (10p)^2$ , then  $n \xrightarrow{p} I_k$  for some integer  $k \le 4$ .

For the case when  $5 \leq p \leq 71$ , one may check by a direct computation that  $n \xrightarrow{p} I_k$  for some integer  $k \leq 4$ , except the case when  $p = 5$  and  $n = 79$ . Note that essentially different representations of 79 by  $I_4$  are

$$
79 = 12 + 22 + 52 + 72 = 22 + 52 + 52 + 52 + 52 + 32 + 52 + 62.
$$

Hence 79 is not represented by a sum of 4 squares not divisible by 5. Since  $79 = 1^2 + 1^2 + 2^2 + 3^2 + 8^2$ . 79 is a sum of 5 squares not divisible by 5.  $1^2 + 1^2 + 2^2 + 3^2 + 8^2$ , 79 is a sum of 5 squares not divisible by 5.

<span id="page-10-1"></span>**Theorem 4.2.** If  $n \equiv 0$  or 2 (mod 3), then  $n \stackrel{p}{\longrightarrow} I_k$  for some integer  $k \leq 4$ .

*Proof.* Reacle that we are assuming that  $p$  is a prime greater than 3 and  $n$  is a square-free positive integer not divisible by  $p$ . By Lemma [4.1,](#page-9-1) we may further assume that

(4.1) 
$$
\begin{cases} n \geqslant (16p)^2 & \text{if } 5 \leqslant p \leqslant 13, \\ n \geqslant (10p)^2 & \text{if } p \geqslant 17. \end{cases}
$$

First, we will prove that there exist integers k and  $s_0 \in \{0, 1, 2, 3\}$  such that

- <span id="page-10-0"></span>(1)  $n - (6k + s_0)^2 \neq 4^{\alpha}(8\beta + 7)$  for any non-negative integers  $\alpha$  and  $\beta$ ;
- (2)  $n (6k + s_0)^2 \equiv 2 \pmod{3};$ (3)  $n - (6k + s_0)^2 > 0;$ (4)  $\left(\frac{n - (6k + s_0)^2}{n}\right)$ p ˙  $\neq$  $\sqrt{5}$ p  $\Big)$ , 0; (5)  $6k + s_0 \not\equiv 0 \pmod{p}$ .

We choose an integer  $s_0$  such that

$$
s_0 = \begin{cases} 0 & \text{if } n \neq 3 \pmod{4} \text{ and } n \neq 0 \pmod{3}, \\ 1 & \text{if } n \neq 1 \pmod{4} \text{ and } n \equiv 0 \pmod{3}, \\ 2 & \text{if } n \equiv 1 \pmod{4} \text{ and } n \equiv 0 \pmod{3}, \\ 3 & \text{if } n \equiv 3 \pmod{4} \text{ and } n \neq 0 \pmod{3}. \end{cases}
$$

Then clearly, the first and the second conditions hold for any integer  $k$ . Now, we will find an integer k satisfying the above conditions  $(3) \sim (5)$  for this integer  $s_0$ . If  $p$  is 5 or 7, then one may easily find an integer  $k$  such that the above conditions  $(3)$   $\sim$  (5) are all satisfied. Hence we may assume that p is greater than 7. In fact, we will choose an integer k in the set  $T = \{1, 2, \dots, \frac{p+9}{2}\}\.$  Since n satisfies [\(4.1\)](#page-10-0), we have  $n - (6k + s_0)^2 > 0$  for any  $k \in T$  and any  $s_0 \in \{0, 1, 2, 3\}$ . It is well known that the number of solutions  $(x, y)$  of the equation  $x^2 + y^2 = n$  over  $\mathbb{F}_p$  is greater than or equal to  $p-1$ . Hence the number of  $x_0$ 's such that  $n - x_0^2$  is a zero or a square in  $\mathbb{F}_p$  is at least  $\frac{p-1}{2}$ . Since  $|T| = \frac{p+9}{2}$ , there are at least four  $k \in T$  such that  $n - (6k + s_0)^2$  is a zero or a square in  $\mathbb{F}_p$ . Similarly, there are at least four  $k \in T$ such that  $n - (6k + s_0)^2$  is a zero or a non-square in  $\mathbb{F}_p$ . Therefore there exists a positive integer  $k \in T$  such that

<span id="page-11-0"></span>
$$
\left(\frac{n - (6k + s_0)^2}{p}\right) \neq \left(\frac{5}{p}\right), 0 \text{ and } 6k + s_0 \neq 0 \pmod{p}.
$$

Now, by  $(1)$  and  $(3)$ , there are integers a, b, and c such that

(4.2) 
$$
n - (6k + s_0)^2 = (\pm a)^2 + (\pm b)^2 + (\pm c)^2.
$$

Since  $n - (6k + s_0)^2 \equiv 2 \pmod{3}$ , we may suitably choose signs in Equation [\(4.2\)](#page-11-0) so that  $a + b + c$  is divisible by 3. If  $a + b + c = 3m$  for some integer m, then we have

<span id="page-11-1"></span>(4.3) 
$$
n - (6k + s_0)^2 = a^2 + b^2 + c^2 = (2m - a)^2 + (2m - b)^2 + (2m - c)^2.
$$

If abc is not divisible by p, then  $n \stackrel{p}{\longrightarrow} I_4$ . Now, assume that at least two of a, b, and c are divisible by p. Without loss of generality, we assume that both a and b are divisible by p. Since c is not divisible by p by (4), m is not divisible by p. Therefore,  $(2m - a)(2m - b)(2m - c)$  is not divisible by p. This implies that  $n \stackrel{p}{\longrightarrow} I_4$ .

Finally, assume that exactly one of  $a, b$ , and  $c$  is divisible by  $p$ . Without loss of generality, we assume that c is divisible by p and ab is not divisible by p. We will show that there are integers a, b, and c satisfying  $(4.3)$  such that m is not divisible by p. Suppose, on the contrary, that m is divisible by p. Since  $n - (6k + s_0)^2 \equiv 2$ (mod 3), exactly one of a, b, and c is divisible by 3. If a is divisible by 3, then

$$
n - (6k + s_0)^2 = (-a)^2 + b^2 + c^2
$$
 and  $-a + b + c = 3\left(m - \frac{2}{3}a\right)$ ,

where  $(m - \frac{2}{3}a)$  is not divisible by p. The same argument can be applied to the case when b is divisible by 3. Hence we may assume that c is divisible by 3. Since  $a + b \equiv 0 \pmod{3}$ , there are integers  $b_1$  and  $c_1$  such that  $a + b = 3b_1$  and  $c = 3c_1$ . Then, we have

$$
n - (6k + s_0)^2 = a^2 + (-a + 3b_1)^2 + (3c_1)^2
$$
  
=  $(a - 2b_1 + 2c_1)^2 + (-a + b_1 + 2c_1)^2 + (-2b_1 - c_1)^2$   
=  $(a - 2b_1 - 2c_1)^2 + (-a + b_1 - 2c_1)^2 + (-2b_1 + c_1)^2$ .

Note that both  $b_1$  and  $c_1$  are divisible by p. By applying the similar argument given above to this situation, we may assume that  $-2b_1 + c_1 \equiv 0 \pmod{3}$  and  $-2b_1 - c_1 \equiv 0 \pmod{3}$ . This implies that  $b_1 \equiv c_1 \equiv 0 \pmod{3}$ . Now, suppose that  $t, b_t$ , and  $c_t$  are integers such that

 $a + b = 3<sup>t</sup>b<sub>t</sub>$ ,  $c = 3<sup>t</sup>c<sub>t</sub>$ , and either  $b<sub>t</sub>$  or  $c<sub>t</sub>$  is not divisible by 3.

Note that both  $b_t$  and  $c_t$  are divisible by p. Let  $x_t, y_t$  be integers such that

 $x_t^2 + 2y_t^2 = 3^{2t}$  and  $x_t y_t \neq 0$  (mod 3).

Note that such integers always exist by Lemma [3.1.](#page-3-0) Then we have

$$
n - (6k + s_0)^2 = a^2 + (-a + 3^t b_t)^2 + (3^t c_t)^2
$$
  
=  $\left(a + \frac{(x_t - 3^t)}{2}b_t + y_t c_t\right)^2 + \left(-a + \frac{(x_t + 3^t)}{2}b_t + y_t c_t\right)^2 + (y_t b_t - x_t c_t)^2$   
=  $\left(a + \frac{(x_t - 3^t)}{2}b_t - y_t c_t\right)^2 + \left(-a + \frac{(x_t + 3^t)}{2}b_t - y_t c_t\right)^2 + (y_t b_t + x_t c_t)^2.$ 

Now, by applying the same argument given above, we may assume that

 $y_t b_t - x_t c_t \equiv 0 \pmod{3}$  and  $y_t b_t + x_t c_t \equiv 0 \pmod{3}$ ,

which implies that  $b_t \equiv c_t \equiv 0 \pmod{3}$ . This is a contradiction. Therefore, we may assume that the integer m given in  $(4.3)$  is not divisible by p. Since c is divisible by p and abm is not divisible by p,  $2m-c$  is not divisible by p. If  $2m-a$  is divisible by p, then  $a \equiv 2m \pmod{p}$  and  $b \equiv m \pmod{p}$ . This implies that  $n - (6k + s_0)^2 \equiv 5m^2$ (mod p), which is a contradiction to the fact that  $\left(\frac{n-(6k+s_0)^2}{p}\right)$  $\left(\frac{p}{p}\right)^2 \neq \left(\frac{5}{p}\right)$ . Therefore  $2m - a$  is not divisible by p. By similar reasoning,  $2m - b$  is not divisible by p.<br>Therefore, by (4.3), we have  $n \xrightarrow{p} I_A$ . This completes the proof. Therefore, by [\(4.3\)](#page-11-1), we have  $n \xrightarrow{p} I_4$ . This completes the proof.

<span id="page-12-0"></span>**Theorem 4.3.** If  $n \equiv 1 \pmod{3}$ , then  $n \xrightarrow{p} I_k$  for some integer  $k \leq 4$ , except the case when  $p = 5$  and  $n = 79$ .

*Proof.* Recall that we are assuming that  $p$  is a prime greater than 3 and  $n$  is a square-free positive integer not divisible by  $p$ . By Lemma [4.1,](#page-9-1) we may further assume that [\(4.1\)](#page-10-0) holds. Then, similarly to Theorem [4.2,](#page-10-1) one may easily show that there exist integers k and  $s_0 \in \{1, 2, 4, 5, 7, 8\}$  such that

- (1)  $n (18k + s_0)^2 \neq 4^{\alpha}(8\beta + 7)$  for any integers  $\alpha$  and  $\beta$ ;
- (2)  $n (18k + s_0)^2 \equiv 0 \pmod{9};$ (3)  $n - (18k + s_0)^2 > 0;$ (4)  $\left(\frac{n-(18k+s_0)^2}{n}\right)$ p ˙ ‰  $\sqrt{5}$ p  $\Big)$ , 0; (5)  $18k + s_0 \not\equiv 0 \pmod{p}$

Since  $\frac{n-(18k+s_0)^2}{9}$  $\frac{k+s_0}{9}$  is a sum of three squares, there are integers  $a, b, c$ , and m such that  $a + b + c = m$  and

$$
n - (18k + s_0)^2 = (3a)^2 + (3b)^2 + (3c)^2 = (2m - 3a)^2 + (2m - 3b)^2 + (2m - 3c)^2.
$$

Note that at least one of a, b, and c is not divisible by p. If abc is not divisible by p, then  $n \stackrel{p}{\longrightarrow} I_4$ . Assume that exactly two of a, b, and c are divisible by p. Without loss of generality, assume that both  $a$  and  $b$  are divisible by  $p$ . Then neither m nor  $(2m - 3a)(2m - 3b)(2m - 3c)$  is divisible by p, which implies that  $n \stackrel{p}{\longrightarrow} I_4$ . Assume that exactly one of a, b, and c is divisible by p. Without loss

of generality, we assume that a is divisible by p and bc is not divisible by p. By changing a sign of b, if necessary, we may assume that  $a+b+c = m$  is not divisible by p. Then clearly,  $2m - 3a$  is not divisible by p. If  $2m - 3b$  is divisible by p, then  $n - (18k + s_0)^2 \equiv 5m^2 \pmod{p}$ . This is a contradiction to (4). Hence  $2m - 3b$  is not divisible by p. Similarly, we may also show that  $2m - 3c$  is not divisible by p.<br>Therefore  $n \xrightarrow{p} I_A$ . This completes the proof. Therefore  $n \xrightarrow{p} I_4$ . This completes the proof.

By combining Propositions [3.3](#page-4-1) and [3.8,](#page-8-2) Theorems [4.2](#page-10-1) and [4.3,](#page-12-0) we have the following:

**Theorem 4.4.** Let p be a prime greater than or equal to 5. Any positive integer n is a sum of at most 4 squares not divisible by p, except the case when  $p = 5$  and  $n = 79$ . In the exceptional case, 79 is a sum of 5 squares not divisible by 5.

### **REFERENCES**

- <span id="page-13-14"></span><span id="page-13-3"></span>[1] M. Bhargava, On the Conway-Schneeberger fifteen theorem, Contem. Math. 272(2000), 27-38.
- <span id="page-13-2"></span>[2] J. K. S. Cassels, Rational quadratic forms, Academic Press, 1978.
- <span id="page-13-7"></span>[3] L. E. Dickson, Quaternary quadratic forms representing all integers, Amer. J. Math. 49(1927), 39-56.
- <span id="page-13-5"></span>[4] Y. Kitaoka, Arithmetic of quadratic forms, Cambridge University Press, 1993.
- [5] C. Ko, On the representation of a quadratic form as a sum of squares of linear forms, Quart. J. Math. Oxford 8(1937), 81-98.
- <span id="page-13-4"></span><span id="page-13-0"></span>[6] J. L. Lagrange, *Nouveau*, Mém. Acad. Roy. Sci. Berlin (1772) 123; Oevres. vol. 3 1869, pp. 189-201.
- <span id="page-13-13"></span>[7] L. J. Mordell, A new Waring's problem with squares of linear forms, Quart. J. Math. Oxford 1(1930), 276-288.
- <span id="page-13-9"></span>[8] B.-K. Oh, Regular positive ternary quadratic forms, Acta Arith. 147(2011), 233-243.
- <span id="page-13-6"></span>[9] B.-K. Oh and Hoseog Yu, Completely p-primitive binary quadratic forms, submitted.
- <span id="page-13-10"></span>[10] O. T. O'Meara, Introduction to quadratic forms, Springer Verlag, New York, 1963.
- <span id="page-13-12"></span>[11] K. Ono, The web of modularity: Arithmetic of the coefficients of modular forms and qseries, CBMS Regional Conference Series in Mathematics, 102, 2004.
- [12] K. Ono and K. Soundararajan, Ramanujan's ternary quadratic form, Invent. math. 130(1997), 415-454.
- <span id="page-13-1"></span>[13] S. Ramanujan, On the expression of a number in the form  $ax^2 + by^2 + cz^2 + du^2$ , Proc. Camb. Phil. Soc. 19(1916), 11-21.
- <span id="page-13-11"></span>[14] X. Wang and D. Pei, Modular forms with integral and half-integral weights, Science Press Beijing, Beijing; Springer, Heidelberg, 2012.
- <span id="page-13-8"></span>[15] T. Yang, An explicit formula for local densities of quadratic forms, J. Number Theory 72(1998), 309-356.

Department of Mathematics, Sungkyunkwan University, Suwon 16419, korea E-mail address: kiny30@skku.edu

Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Korea

E-mail address: bkoh@snu.ac.kr