CONTINUOUS THEORY OF OPERATOR EXPANSIONS OF FINITE DIMENSIONAL HILBERT SPACES, CONTINUOUS STRUCTURES OF QUANTUM CIRCUITS AND DECIDABILITY

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Abstract. We consider continuous structures which are obtained from finite dimensional Hilbert spaces over $\mathbb{C}$ by adding some unitary operators. Quantum automata and circuits are naturally interpretable in such structures. We consider appropriate algorithmic problems concerning continuous theories of natural classes of these structures.

1. Introduction

Continuous logic has become the basic model theoretic tool for Hilbert spaces and $\mathbf{C}^*$-algebras: see [5], [7] and [18]. This suggests that quantum circuits, quantum automata and quantum computations in general can be defined in appropriate continuous structures and studied by means of continuous logic. The paper presents an attempt of this approach. The main object of our paper are finite dimensional Hilbert spaces in the language expanded by a finite family of unitary operators. We call them dynamical Hilbert spaces.

It is worth noting that a finite dimensional Hilbert space cannot be considered as an object interesting on its own from the point of view of continuous model theory. This case corresponds to 'finite objects' in model theory (its n-balls are compact). In our paper we study continuous theories of *classes* of these structures. This naturally leads to pseudo finite dimensional structures and to questions connected with approximations of groups by metric groups.

All necessary information on continuous logic will be described in the next section.

The main results of the paper concern decidability of continuous theories of classes of dynamical Hilbert spaces and so called 'marked dynamical Hilbert spaces'. In Section 4 we show that decidability questions for the class of finite dimensional dynamical Hilbert spaces are connected with property MF, one of the most interesting properties in the topic of approximations by metric groups [10]. Marked dynamical Hilbert spaces are defined in Section 5 as expansions of finite dimensional dynamical Hilbert spaces by unary discrete predicates. We will see that this procedure is essential for expressive power of the language. In particular there are natural subclasses of marked dynamical Hilbert spaces where undecidable first order theories of some classes of finite structures can be interpreted. These results are partially motivated by [12], where algorithmic problems for quantum automata

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were studied. In the beginning of Section 5 we give a more detailed introduction to these issues.

Section 3 contains some general observations concerning decidability. We think that this section is interesting by itself. It is naturally connected with the material of [6], [13] and [21], where decidability questions for continuous theories were initiated.

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2. Continuous structures.

2.1. **General preliminaries.** We fix a countable continuous signature

$$L = \{d, R_1, ..., R_k, ..., F_1, ..., F_l, ...\}.$$

Let us recall that a *metric L-structure* is a complete metric space (M, d) with d bounded by 1, along with a family of uniformly continuous operations on M and a family of predicates R_i, i.e. uniformly continuous maps from appropriate M^{k_i} to $[0,1]$. It is usually assumed that L assigns to each predicate symbol R_i a continuity modulus $\gamma_i : [0,1] \to [0,1]$ so that any metric structure M of the signature L satisfies the property that if $d(x_j, x'_j) < \gamma_i(\varepsilon)$ with $1 \leq j \leq k_i$ then the inequality

$$|R_i(x_1, ..., x_j, ..., x_{k_i}) - R_i(x_1, ..., x'_j, ..., x_{k_i})| < \varepsilon$$

holds for the corresponding predicate of M. It happens very often that γ_i coincides with *id*. In this case we do not mention the appropriate modulus. Similarly, the language also includes continuity moduli for functional symbols.

Note that each countable structure can be considered as a complete metric structure with the discrete $\{0,1\}$-metric.

By completeness continuous substructures of a continuous structure are always closed subsets.

Atomic formulas are the expressions of the form $R_i(t_1, ..., t_r)$, $d(t_1, t_2)$, where t_i are simply classical terms (built from functional L-symbols). We define *formulas* to be expressions built from 0,1 and atomic formulas by applications of the following functions:

$$x/2 \ , \ x \dot{-} y = \mathsf{max}(x - y, 0) \ , \ \mathsf{min}(x, y) \ , \ \mathsf{max}(x, y) \ , \ |x - y| \ ,$$

$$\neg(x) = 1 - x \ , \ x \dot{+} y = \mathsf{min}(x + y, 1) \ , \ x \cdot y \ , \ \mathsf{sup}_x \text{ and } \mathsf{inf}_x.$$

Statements concerning metric structures are usually formulated in the form

$$\phi = 0,$$

where ϕ is a formula. Sometimes statements are called an *condition* or $\bar{x}$-conditions (when ϕ depends on $\bar{x}$); we will use both names. A *theory* is a set of statements without free variables (here sup_x and inf_x play the role of quantifiers). If $\mathcal{K}$ is a class of continuous L-structures then $Th(\mathcal{K})$ denotes the set of all conditions without free variables which hold in all structures of $\mathcal{K}$.

We sometimes replace conditions of the form $\phi \dot{-} \varepsilon = 0$ where $\varepsilon \in [0,1]$ by more convenient expressions $\phi \leq \varepsilon$. When a formula ϕ is of the form $\mathsf{sup}_{x_1} \mathsf{sup}_{x_2} \dots \mathsf{sup}_{x_1} \psi$, where ψ is quantifier free, we say that ϕ is *universal*.

It is worth noting that any formula is a γ-uniformly continuous function from the appropriate power of M to $[0,1]$, where γ is the minimum of continuity moduli of L-symbols appearing in the formula.

The condition that the metric is bounded by 1 is not necessary. It is often assumed that d is bounded by some rational number d_0. In this case the (truncated) functions above are appropriately modified. Sometimes predicates of continuous structures map M^n to some $[q_1, q_2]$ where $q_1, q_2 \in \mathbb{Q}$. It is only worth noting that we always assume that when we fix an interval $[q_1, q_2]$ for values of continuous formulas, connectives are chosen so that they cannot give values outside this interval.

Following Section 4.2 of [18] we define a topology on L-formulas relative to a given continuous theory T. For n-ary formulas ϕ and ψ of the same sort set

$$\mathbf{d}^T_{\bar{x}}(\phi, \psi) = \mathsf{sup}\{|\phi(\bar{a}) - \psi(\bar{a})| : \bar{a} \in M, M \models T\}.$$

The function $\mathbf{d}^T_{\bar{x}}$ is a pseudometric.

Definition 2.1. The language L is called *separable* with respect to T if for any tuple $\bar{x}$ the density character of $\mathbf{d}^T_{\bar{x}}$ is countable.

By Proposition 4.5 of [18] when L is separable, for every $M \models T$ the set of all interpretations of L-formulas in M is separable in the uniform topology.

The paper [6] gives fourteen axioms of continuous first order logic, denoted by (A1) - (A14), and the corresponding version of *modus ponens*:

$$\frac{\phi\ ,\ \psi \dot{-} \phi}{\psi}.\text{ where } \phi, \psi \text{ are continuous formulas.}$$

Corollary 9.6 of [6] states:

> Let Γ be a set of continuous formulas of a continuous signature L with a metric. Let ϕ be a continuous L-formula. Then the following conditions are equivalent:
> (i) for any continuous structure M and any M-assignment of variables, if M satisfies all statements $\psi = 0$, $\psi \in \Gamma$, then M satisfies $\phi = 0$;
> (ii) $\Gamma \vdash \phi \dot{-} 2^{-n}$ for all $n \in \omega$.

It is called *approximated strong completeness for continuous first-order logic*. The following statement is Corollary 9.8 from [6].

> Under circumstances above the following values are the same:
> (i) $\mathsf{sup}\{\phi^M :$ for all $M \models \Gamma = 0\}$;
> (ii) $\mathsf{inf}\{p \in \mathbb{Q} : \Gamma \vdash \phi \dot{-} p\}$.

We denote this value by ϕ° and call it the *degree of truth of* ϕ *with respect to* Γ.

If the language L is computable, the set of all continuous L-formulas and the set of all L-conditions of the form

$$\phi \leq \frac{m}{n}\ \text{, where } \frac{m}{n} \in \mathbb{Q}_+,$$

are computable. Moreover if Γ is a computably enumerable set of formulas, then the set $\{\phi : \Gamma \vdash \phi\}$ is computably enumerable.

Corollary 9.11 of [6] states that when Γ is computably enumerable and $\Gamma = 0$ axiomatizes a complete theory, then the value of ϕ in models of $\Gamma = 0$ is a recursive real which is uniformly computable from ϕ. This exactly means that the corresponding complete theory is *decidable* (see Section 2). Note that in this case the value of ϕ as above coincides with ϕ°.

2.2. **Hilbert spaces.** We treat a Hilbert space over $\mathbb{R}$ exactly as in Section 15 of [5]. We identify it with a many-sorted metric structure

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r\in\mathbb{R}}, +, -, \langle\rangle),$$

where B_n is the ball of elements of norm $\leq n$, $I_{mn} : B_m \to B_n$ is the inclusion map, $\lambda_r : B_m \to B_{km}$ is scalar multiplication by r, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |r| < k$; furthermore, $+, - : B_n \times B_n \to B_{2n}$ are vector addition and subtraction and $\langle\rangle : B_n \to [-n^2, n^2]$ is the predicate of the inner product. The metric on each sort is given by $d(x, y) = \sqrt{\langle x - y, x - y\rangle}$. For every operation the continuity modulus is standard. For example in the case of λ_r this is $\frac{z}{|r|}$.

Stating existence of infinite approximations of orthonormal bases by axioms of the form

$$\mathsf{inf}_{x_1,\ldots,x_n\in B_1}\mathsf{max}_{1\leq i<j\leq n}(|\langle x_i, x_j\rangle - \delta_{i,j}|) = 0\ ,\ n \in \omega,$$
$$\delta_{i,j} \in \{0, 1\} \text{ with } \delta_{i,j} = 1 \leftrightarrow i = j,$$

we axiomatize infinite dimensional Hilbert spaces. By [5] they form the class of models of a complete theory which is κ-categorical for all infinite κ, and admits elimination of quantifiers.

When we assume that the space is finite dimensional all sorts B_n become compact. This corresponds to the case of finite structures in ordinary model theory. The statement that the dimension equals n can be described by the following statement.

$$\mathsf{inf}_{y_1,\ldots,y_n\in B_1}\mathsf{max}(\mathsf{max}_{1\leq i\leq n}(|\langle y_i, y_i\rangle - 1|),$$
$$\mathsf{sup}_{x\in B_1}(|(\langle x, x\rangle - |\langle x, y_1\rangle|^2 - \ldots\ldots - |\langle x, y_n\rangle|^2)|) = 0.$$

The corresponding continuous theory admits elimination of quantifiers. This follows by the argument of Lemma 15.1 from [5].

This approach can be naturally extended to complex Hilbert spaces,

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{C}}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}).$$

We only extend the family $\lambda_r : B_m \to B_{km}$, $r \in \mathbb{R}$, to a family $\lambda_c : B_m \to B_{km}$, $c \in \mathbb{C}$, of scalar products by $c \in \mathbb{C}$, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |c| < k$.

We also introduce Re- and Im-parts of the inner product.

If we remove from the signature of complex Hilbert spaces all scalar products by $c \in \mathbb{C} \setminus \mathbb{Q}[i]$, we obtain a countable subsignature

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}),$$

which is *dense* in the original one:
if we present $c \in \mathbb{C}$ by a sequence $\{q_i\}$ from $\mathbb{Q}[i]$ converging to c, then the choice of the continuity moduli of the restricted signature still guarantees that in any sort B_n the functions λ_{q_i} form a sequence which converges to λ_c with respect to the metric

$$\mathsf{sup}_{x\in B_n}\{|f^M(x) - g^M(x)| : M \text{ is a model of the theory of Hilbert spaces }\}.$$

This obviously implies that the original language of Hilbert spaces is *separable*.

To study dynamical evolutions of quantum circuits we introduce the following expansion of Hilbert spaces. Let us fix a natural number t and consider the class of *dynamical Hilbert spaces* in the extended signature

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, \ldots, U_t),$$

where U_j, $1 \leq j \leq t$, are symbols of unitary operators of $\mathbb{H}$. We may assume that all U_j are defined only on B_1. For convenience we add to each U_i the symbol U'_i for the operator U_i^{-1}. Then we also add the axioms $\mathsf{sup}_{v\in B_1} d(U'_i U_i(v), v) \leq 0$ and $\mathsf{sup}_{v\in B_1} d(U_i U'_i(v), v) \leq 0$. We will not mention this below.

It is clear that this language is computable and is dense in the $\bar{U}$-extension of the standard language of the theory of Hilbert spaces. The main results of the paper concern decidability of theories in this language.

Lemma 2.2. *Assume that a structure of the form above is n-dimensional where $n \in \mathbb{N}$. Then the complete continuous theory of this structure is axiomatized by the standard axioms of Hilbert spaces, the axioms stating that each U_j is a unitary operator and the following axioms describing the matrices of U_j in some (fixed) orthogonal normal basis:*

$$\mathsf{inf}_{y_1,\ldots,y_n\in B_1}\mathsf{max}(\mathsf{max}_{1\leq i\leq n}(|\langle y_i, y_i\rangle - 1|),$$
$$\mathsf{sup}_{x\in B_1}(|(\langle x,x\rangle - |\langle x,y_1\rangle|^2 - \ldots\ldots - |\langle x,y_n\rangle|^2)|),$$
$$\mathsf{max}_{1\leq l\leq n}\mathsf{max}_{1\leq j\leq t}(\| U_j(y_l) - \sum \lambda_{c_{j,l,k}}(y_k) \| \dot{-}\varepsilon_l)) \leq 0,$$
where $\varepsilon_l \in \mathbb{Q}$ and $c_{j,l,k} \in \mathbb{Q}[i]$ are appropriate approximations of entries of matrices for $U_1, \ldots, U_t$.

Proof. Any model with these axioms is an n-dimensional space. Thus by compactness of B_1 there is an appropriate basis where the values of $U_j(y_l)$ have the coordinates described in the axioms. This model is unique up to isometry. Thus the lemma is obvious. □

2.3. **Unitary representations.** When we consider a language containing countably many operators U_i, any unitary representation of a countable group G can be considered as a dynamical Hilbert space in this language. For example we can add an operator for every element of G.

We will use several notions from the area of unitary representations. We firstly remind the reader that the left regular representation of G is obtained by the action of G on $l^2(G)$ defined by the unitary operators $U_g : f(h) \to f(g^{-1}h)$. The $*$-algebra generated by all U_g is just $\mathbb{C}G$.

The following notion is taken from Section F of [4].

Definition 2.3. Let π and ρ be unitary representations of G and H_ρ, H_π be the corresponding dynamical Hilbert spaces. Let $\varepsilon > 0$ and F be a finite subset of G. We say that ρ is (ε, F)-*contained in* π if for every $v_1, \ldots, v_n \in H_\rho$, there are $w_1, \ldots, w_n \in H_\pi$ such that

$$|\langle \rho(g)v_i, v_j\rangle - \langle \pi(g)w_i, w_j\rangle| < \varepsilon \text{ for all } g \in F.$$

We say that ρ is *weakly contained in* π and write $\rho \prec \pi$ if ρ is (ε, F)-contained in π for every ε and finite F.

When G is t-generated we apply the definition above to the corresponding homomorphisms of the form $G \to \langle U_1 \ldots, U_t\rangle$.

In the case when $t = 1$ some standard material from functional analysis can be applied. We remind the reader that a complex number λ is said to be a *regular value* of a operator U if there exists $(U - \lambda Id)^{-1}$, which is a bounded linear operator and is defined on a dense subspace of the space. The *resolvent set* of U is the set of all regular values of U. The spectrum of U, denoted by $\sigma(U)$, is the complement of the resolvent set. The set of isolated points of $\sigma(U)$ of finite multiplicity is called the *finite spectrum* and is denoted by $\sigma_{fin}(U)$. The set $\sigma_e(U) = \sigma(U) \setminus \sigma_{fin}(U)$ is called the *essential spectrum* of U.

We will use this material in combination with the following theorem of C. Ward Henson.

> Let (H, U) and (H', U') be dynamical Hilbert spaces with one operator. These spaces are elementarily equivalent in continuous logic if and only if they have the same essential spectra $\sigma_e(U)$ and $\sigma_e(U')$ and for any $r \in S^1 \setminus \sigma_e(U)$ we have
> $$\mathsf{dim}\{x \in H : Ux = rx\} = \mathsf{dim}\{c \in H' : U'x = rx\}.$$

A proof of this theorem in the case of countable spectrum can be found in [2].

3. Decidability/undecidability of continuous theories

In this section we assume that the signature L is computable and values of formulas are in $[0,1]$. The interval $[0,1]$ can be obviously replaced by any compact interval. We start with the following definition from [6].

Definition 3.1. A continuous theory T is called *decidable* if for every sentence ϕ the degree of truth
$$\phi^{\circ} = \mathsf{sup}\{\phi^M : M \models T\}$$
is a computable real which is uniformly computable from ϕ.

This exactly means that there is an algorithm which for every ϕ and a rational number δ finds a rational r such that $|r - \phi^{\circ}| \leq \delta$.

Note that decidability of T does not imply that the set of all continuous ϕ with $\phi^M = 0$ for all $M \models T$, is computable (but for a complete T this holds). On the other hand it is easy to see that decidability of T follows from this condition. This is a part of the following lemma.

Lemma 3.2. *Let T be a continuous theory in a computable language. Let a rational number q° belong to $[0,1]$.*

1. Assume that $q^{\circ} < 1$ and there is an algorithm which decides for every formula ϕ without free variables whether $\phi^{\circ} \leq q^{\circ}$. Then the theory T is decidable.

2. Assume that $q^{\circ} > 0$ and there is an algorithm which decides for every formula ϕ without free variables whether ϕ° equals q°. Then the theory T is decidable.

Proof. We start with the observation that the assumption of statement 1 with any $q^{\circ} < 1$ is equivalent to the case $q^{\circ} = 0$. This follows from the equivalence
$$\phi^{\circ} \leq q^{\circ} \Leftrightarrow (\phi \dot{-} q^{\circ})^{\circ} \leq 0.$$
In the case of statement 2 the following equivalence
$$\phi^{\circ} = q^{\circ} \Leftrightarrow (\frac{1}{q^{\circ}}\phi)^{\circ} = 1$$
shows that the assumption of statement 2 with any $q^{\circ} > 0$ is equivalent to the case $q^{\circ} = 1$.

To prove decidability of T in the case of statement 1 assume that $q^{\circ} = 0$. Given ϕ and $m > 0$ find the minimal $\frac{t}{2m}$ so that $T \models \phi \dot{-} \frac{t}{2m} \leq 0$. This defines an interval of the form $[\frac{s}{m}, \frac{s+1}{m}]$ which contains ϕ°.

In the case of statement 2 given ϕ and $m > 0$ find the minimal $\frac{t}{2m}$ so that $(\phi + \frac{t}{2m})^\circ = 1$ with respect to T. This defines an interval of the form $[\frac{s}{m}, \frac{s+1}{m}]$ which contains ϕ°. □

Remark 3.3. Lemma 3.2 will be applied in Section 3 in the situation when the segment $[0,1]$ is replaced by $[0,2]$. It obviously holds under the replacement 1 by 2 in the formulation.

3.1. **Ershov's theorem.** The following theorem is a counterpart of Ershov's decidability criterion (Theorem 6.1.1 of [17]). Here we call a sequence of complete continuous theories $\{T_i, i \in \omega\}$ *effective* if the relation

$$\{(\theta, j) : \theta \text{ is a statement so that } T_j \vdash \theta\}$$

is computably enumerable.

Theorem 3.4. *A continuous theory T is decidable if and only if T can be defined by a computably enumerable system of axioms and T can be presented $T = \bigcap_{i\in\omega} T_i$ where $\{T_i, i \in \omega\}$ is an effective sequence of complete continuous theories.*

Proof. Sufficiency. Let ϕ be a continuous sentence. For every natural n we can apply an effective procedure which looks for conditions of the form $\phi \leq \frac{k}{n}$ derived from the axioms of T and conditions of the form $\frac{l}{n} \leq \phi$ which appear in some $T_j \vdash \frac{l}{n} \leq \phi$. By Corollary 9.8 from [6] this always gives a number $k < n-1$ such that $\frac{k}{n} \leq \phi^\circ \leq \frac{k+2}{n}$.

Necessity. For every sentence ϕ we fix a computably enumerable sequence of segments $[l_{n,\phi}, r_{n,\phi}]$ converging to ϕ° so that $\phi^\circ \in [l_{n,\phi}, r_{n,\phi}]$. Then all statements $\phi \leq r_{n,\phi}$ form a computably enumerable sequence of axioms of T.

Now for every sentence ϕ we effectively build a complete theory $T_{n,\phi} \supset T$ with $T_{n,\phi} \vdash l_{n,\phi} \dot{-} \phi \leq 2^{-n}$. In fact such a construction produces an effective family T_i, $i \in \omega$, from the formulation. Indeed, then for every natural n we can find a sufficiently large m so that $T_{m,\phi} \vdash \phi^\circ \dot{-} \phi \leq 2^{-n}$ (here ϕ° is defined by T). This obviously implies that T coincides with the intersection of all $T_{m,\phi}$. Effectiveness will be verified below.

At Step 0 for every n we define $T_{n,\phi,0}$ to be the extension of T by the axiom $l_{n,\phi} \dot{-} \phi \leq 0$. At every step $m+1$ we build a finite extension $T_{n,\phi,m+1}$ of T so that each inequality $\psi \leq 0$ from $T_{n,\phi,m} \setminus T$ is transformed into an inequality $\psi \leq \varepsilon$, where $\varepsilon \leq 2^{-(2n+m+1)}$. At later steps we consider these $\psi \leq \varepsilon$ in the form $\psi \dot{-} \varepsilon \leq 0$, i.e. the next transformation of them gives inequalities $\psi \dot{-} \varepsilon \leq \varepsilon'$ (resp. $\psi \dot{-} (\varepsilon + \varepsilon') \leq 0$). In this situation we say that the original $\psi \leq 0$ is transformed into $\psi \leq \varepsilon_1$, where $\varepsilon_1 = \varepsilon + \varepsilon'$. The 'limit theory' $T_{n,\phi} = \lim_{m\to\infty} T_{n,\phi,m}$ is defined by the limits of these values $\varepsilon, \varepsilon_1, \ldots$ for all formulas ψ. Note that it can happen that $\varepsilon \leq 0$, i.e. the transformed inequality is of the form $\psi + \delta \leq 0$, with $\delta > 0$. On the other hand we will see that for every ψ the axioms of $\lim_{m\to\infty} T_{n,\phi,m}$ give an effective sequence of rational numbers which converges to the value of ψ under this theory.

Let us enumerate all triples (n, ϕ, ψ) by natural numbers > 0 so that each triple has infinitely many numbers. Assume that the number $m+1$ codes a triple (n, ϕ, ψ). For all $n' \neq n$ we put $T_{n',\phi',m+1} = T_{n',\phi',m}$. Assume that at Step m the theory $T_{n,\phi,m} \setminus T$ already contains inequalities $\frac{k_l}{l} \leq \psi_l \leq \frac{k'_l}{l}$ for some natural l and $k_l, k'_l \leq l$. We admit that the 0-th inequality $l_{n,\phi} \dot{-} \phi \leq 0$ has been already transformed into an inequality $l_{n,\phi}\dot{-}\phi \leq \varepsilon$ for some $\varepsilon \leq \sum_{i\leq m} 2^{-(2n+i)}$. It appears as one of the inequalities $\psi_l \leq \frac{k'_l}{l}$. Let θ be

$$\psi\dot{-}2^{2n+m+1}\mathsf{max}_l(\mathsf{max}(\psi_l\dot{-}\frac{k'_l}{l}, \frac{k_l}{l}\dot{-}\psi_l)).$$

Since T is decidable we compute $k_{m+1} < m$ so that $\frac{k_{m+1}}{m+1} \leq \theta^\circ \leq \frac{k_{m+1}+2}{m+1}$. Then the value of ψ under $T_{n,\phi,m}$ is equal to the value of θ under this theory and is not greater than $\frac{k_{m+1}+2}{m+1}$. This means that extending $T_{n,\phi,m}$ by $0 \leq \psi \leq \frac{k_{m+1}+2}{m+1}$ we preserve consistency of the theory. If $k_{m+1} = 0$ this finishes our construction at this step.

If $k_{m+1} > 0$ we need an additional correction. Let θ' be

$$\psi\dot{-}2^{2n+m+1}\mathsf{max}(\mathsf{max}_l(\mathsf{max}(\psi_l\dot{-}\frac{k'_l}{l}, \frac{k_l}{l}\dot{-}\psi_l)), \psi\dot{-}\frac{k_{m+1}+2}{m+1}).$$

Since T is decidable we compute $k'_{m+1} < m$ so that $\frac{k'_{m+1}}{m+1} \leq (\theta')^\circ \leq \frac{k'_{m+1}+2}{m+1}$. Then the value of ψ under the extension of $T_{n,\phi,m}$ by $\psi \leq \frac{k_{m+1}+2}{m+1}$ is not greater than $\frac{k'_{m+1}+2}{m+1}$. This means that extending $T_{n,\phi,m}$ by $\psi \leq \frac{\mathsf{min}(k_{m+1},k'_{m+1})+2}{m+1}$ we preserve consistency of the theory.

If $0 < k'_{m+1} < k_{m+1}$ we repeat this construction again. It is clear that finally we arrive at the situation when after such a repetition the number k_{m+1} does not change (or becomes 0).

Note that if the final k_{m+1} is not equal to 0, then the extension of T by $\psi \leq \frac{k_{m+1}+2}{m+1} + 2^{-(2n+m+1)}$ and all statements of the form

$$\frac{k_l}{l}-2^{-(2n+m+1)} \leq \psi_l \leq \frac{k'_l}{l}+2^{-(2n+m+1)} \text{ (for inequalities } \frac{k_l}{l} \leq \psi_l \leq \frac{k'_l}{l} \text{ from } T_{n,\phi,m})$$

is consistent and the value ψ° with respect to this extension satisfies $\frac{k_{m+1}}{m+1} \leq \psi^\circ$. Indeed, since for the final version of θ (corresponding to the final k_{m+1}) we have $\frac{k_{m+1}}{m+1} \leq \theta^\circ$ with respect to T, the following inequality must hold in any model of T where θ takes the value θ°:

$$2^{2n+m+1}\mathsf{max}(\mathsf{max}_l(\mathsf{max}(\psi_l\dot{-}\frac{k'_l}{l}, \frac{k_l}{l}\dot{-}\psi_l)), \psi\dot{-}\frac{k_{m+1}+2}{m+1}) < 1.$$

Thus the inequality $\psi \leq \frac{k_{m+1}+2}{m+1} + 2^{-(2n+m+1)}$ and the corresponding inequalities

$$\frac{k_l}{l} - 2^{-(2n+m+1)} \leq \psi_l \leq \frac{k'_l}{l} + 2^{-(2n+m+1)}$$

are satisfied in any model of T where θ takes the value θ°. Since $\theta^\circ \leq \psi^\circ$, we have the latter inequality above.

We now define $T_{n,\phi,m+1}$ as the set of so corrected statements of $T_{n,\phi,m}$ together with the statement

$$\frac{k_{m+1}}{m+1} \leq \psi \leq \frac{k_{m+1}+2}{m+1} + 2^{-(2n+m+1)}.$$

If ψ also occurs as some ψ_l above then we obviously add the strongest inequalities to $T_{n,\phi,m+1}$. By the argument of the previous paragraph the obtained extension is consistent with T.

By the choice of a repeating enumeration we see that for each sentence ψ boundaries of ψ at steps of our procedure form a Cauchy sequences with the same limit.

Thus ψ has the same value in all models of $T_{n,\phi}$. Moreover the inequality $l_{n,\phi} \dot{-} \phi \leq 0$ will be transformed into $l_{n,\phi} \dot{-} \phi \leq 2^{-n}$. We see that Step 0 guarantees that T coincides with the intersection of all $T_{n,\phi}$.

Note that after the $(m+1)$-th step we know that for every inequality $\psi' \leq \delta$ from each $T_{n',\phi',m+1} \setminus T$ the upper boundary of ψ' in the final $T_{n',\phi'}$ cannot exceed $\delta + \frac{1}{2^m}$. In particular all inequalities of this kind can be included into an enumeration of axioms of $T_{n',\phi'}$ at this step. Thus we see that by the effectiveness of our procedure the family $\{T_{n,\phi}\}$ is effective. □

3.2. **Interpretability.** In order to have a method for proving undecidability of continuous theories we now discuss interpretability of first order structures in continuous ones.

Let $L_0 = \langle P_1, ..., P_m \rangle$ be a finite relational signature. Let $\mathcal{K}_0$ be a class of finite first-order L_0-structures. Let $\mathcal{K}$ be a class of continuous L-structures, where L is as above. We say that $\mathcal{K}_0$ is *relatively interpretable* in $\mathcal{K}$ if there is a finite constant extension $L(\bar{a}) = L \cup \{a_1, ..., a_r\}$, a constant expansion $\mathcal{K}(\bar{a})$ of $\mathcal{K}$ (we admit the situation that $\bar{a}$ is empty) and there are continuous L-formulas

$$\phi^-(\bar{x}, \bar{y}) \ , \ \phi^+(\bar{x}, \bar{y}) \ , \ \theta^-(\bar{x}, \bar{y}_1, \bar{y}_2) \ , \ \theta^+(\bar{x}, \bar{y}_1, \bar{y}_2) \text{ and}$$

$$\psi_1^-(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_{l_1}) \ , \ \psi_1^+(\bar{x}, \bar{y}_1, \bar{y}_2, ..., , \bar{y}_{l_1}) \ , \ ..., \ \psi_m^-(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_{l_m}) \ , \ \psi_m^+(\bar{x}, \bar{y}_1, \bar{y}_2, ..., , \bar{y}_{l_m}),$$

$$\text{with } |\bar{y}| = |\bar{y}_1| = |\bar{y}_2| = ...|\bar{y}_{l_j}| = ... = |\bar{y}_{l_m}| \text{ , such that:}$$

(i) the L-reduct of $\mathcal{K}(\bar{a})$ coincides with $\mathcal{K}$;
(ii) the conditions $\phi^-(\bar{a}, \bar{y}) \leq 0$ and $\phi^+(\bar{a}, \bar{y}) > 0$ are equivalent in any $M \in \mathcal{K}(\bar{a})$ and the condition $\theta^-(\bar{a}, \bar{y}_1, \bar{y}_2) \leq 0$ defines an equivalence relation on the zero-set of $\phi^-(\bar{a}, \bar{y})$ (on tuples of the corresponding power M^s with $s = |\bar{y}_1|$), so that the values of any $\psi_i^\varepsilon(\bar{a}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_{l_i})$ are invariant under this equivalence relation;
(iii) the $(+)$-conditions below are equivalent to $(-)$-ones in $\mathcal{K}(\bar{a})$:

$$\theta^-(\bar{a}, \bar{y}_1, \bar{y}_2) \leq 0 \ , \ \theta^+(\bar{a}, \bar{y}_1, \bar{y}_2) > 0 \ , \ \psi_1^-(\bar{a}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_{l_1}) \leq 0 \ ,$$

$$\psi_1^+(\bar{a}, \bar{y}_1, \bar{y}_2, ..., , \bar{y}_{l_1}) > 0 \ , \ ..., \ \psi_m^-(\bar{a}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_{l_m}) \leq 0 \ , \ \psi_m^+(\bar{a}, \bar{y}_1, \bar{y}_2, ..., , \bar{y}_{l_m}) > 0;$$

(iv) for any $M \in \mathcal{K}(\bar{a})$ the conditions of (iii) define an L_0-structure from $\mathcal{K}_0$ on the θ-quotient of the zero-set of $\phi^-(\bar{a}, \bar{y})$ and any structure of $\mathcal{K}_0$ can be so realized.

Theorem 3.5. *Assume that the class of finite structures $\mathcal{K}_0$ is relatively interpretable in $\mathcal{K}$ and assume that $Th(\mathcal{K}_0)$ is undecidable. Then the continuous theory $Th(\mathcal{K}(\bar{a}))$ of the corresponding constant expansion is not a computable set.*

Proof. The proof is straightforward. To each formula ψ of the theory of $\mathcal{K}_0$ so that the quantifier-free part is in the disjunctive normal form we associate the appropriately rewritten continuous formula $\psi^-(\bar{a}, \bar{z})$ (with appropriate free variables) and the 0-statement $\psi^-(\bar{a}, \bar{z}) \leq 0$. In particular atomic formulas are written by $(-)$-conditions above, but negations of atomic formulas appear in the form of

$$\psi_i^+(\bar{a}, \bar{y}_1, \bar{y}_2, ..., , \bar{y}_{l_i}) \leq 0.$$

Condition (ii) and the condition that the θ-quotient of the zero-set of $\phi^-(\bar{a}, \bar{y})$ is always finite, allow us to use standard quantifiers in such statements $\psi^-(\bar{a}, \bar{z}) \leq 0$: the quantifier $\forall$ is written as *sup* but $\exists$ is written as *inf*.

Note that if ψ' is equivalent to $\neg\psi$ then $(\psi')^-(\bar{a}, \bar{z}) \leq 0$ is equivalent to $\psi^-(\bar{a}, \bar{z}) > 0$ for tuples from the zero-set of $\phi^-(\bar{a}, \bar{y})$ (and $\psi^-(\bar{a}, \bar{z}) > 0$ is equivalent to the corresponding $\psi^+(\bar{a}, \bar{z}) \leq 0$).

It is easy to see that this construction reduces the decision problem for $Th(\mathcal{K}_0)$ to computability of the set $Th(\mathcal{K}(\bar{a}))$. □

This theorem will be applied in Section 5 under circumstances that $\mathcal{K}(\bar{a}) = \mathcal{K}$.

It is worth noting that the theorem gives a relatively weak method of proving undecidability of continuous theories. In the classical first-order logic such a situation usually has much stronger consequences. For example Theorem 5.1.2 of [17] in a slightly modified setting (and removing the assumption that $\mathcal{K}_0$ consists of finite structures) states that hereditary undecidability of $Th(\mathcal{K}_0)$ can be lifted to $Th(\mathcal{K})$. In the following remarks we describe several difficulties arising in our approach.

Remark 3.6. As we already know the statement of Theorem 3.5 does not imply that $Th(\mathcal{K}(\bar{a}))$ is undecidable. It seems to us that it is a challenge to find a useful method of interpretability which gives undecidability of the theory.

Remark 3.7. The assumption that $\mathcal{K}_0$ consists of finite structures is essential (see the proof of Theorem 3.5). The 'positiveness' of the continuous logic does not allow stronger statements.

Remark 3.8. Assuming that $Th(\mathcal{K}_0)$ is not stable we cannot state the same for the theory of $\mathcal{K}(\bar{a})$. This follows from the requirement that in the definition of the order property for a sequence $\bar{a}_1, \ldots, \bar{a}_k, \ldots$ the inequality $\phi(\bar{a}_i, \bar{a}_j) \neq 0$ (when $i \geq j$) implies $\phi(\bar{a}_i, \bar{a}_j) = 1$ (see Section 5 of [18]).

4. Decidability of theories of pseudo finite dimensional Hilbert spaces

We start this section with the observation that the theory of all finite dimensional dynamical Hilbert spaces is decidable if it is computably axiomatizable (see Theorem 4.3). Connections of decidability and pseudocompactness with property MF are discussed in Section 4.2. In Section 4.3 it is proved that the universal theory of (finite dimensional) dynamical Hilbert spaces is decidable. In Section 4.4 we consider the problem when a dynamical Hilbert space is pseudo finite dimentional.

4.1. Finite dimension.

Let us now restrict the dimension of Hilbert spaces, say by N. It is natural to expect that then the theory of (dynamical) Hilbert spaces becomes decidable. In classical model theory this corresponds to the situation of a theory of structures of a fixed finite size.

Let us fix a signature

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, ..., U_t),$$

where as before we assume that U_j, $1 \leq j \leq t$, are symbols of unitary operators of $\mathbb{H}$ which are defined only on B_1. Using Theorem 3.4 we will prove that the theory of N-dimensional spaces in this language is decidable.

Remark 4.1. On the other hand since the structures are of infinite language it is not very difficult to find such a structure with undecidable continuous theory. For example one can take a dynamical 3-dimensional Hilbert space with an additional operator U such that $\mathsf{sup}_{v\in B_1} d(v, U(v)) = r$, where r is a non-computable real number which belongs to $[0, 2]$.

Let us enumerate all N-dimensional unitary matrices of algebraic complex numbers. This can be arranged by some canonical indexing of all algebraic numbers (for example see [27]) and using decidability of the theory of algebraically closed fields. This induces an enumeration Axm_j, $j \in \omega$, of systems of axioms of complete continuous theories T_j of dynamical N-dimensional spaces. Each Axm_j consists of the standard axioms of N-dimensional spaces, the axioms stating that each U_s is a unitary operator and the axioms describing the matrices of all U_s in some basis:

$$\mathsf{inf}_{y_1,\dots,y_N}\mathsf{max}(\mathsf{max}_{1\le i\le N}(|\langle y_i,y_i\rangle - 1|),$$
$$\mathsf{sup}_x(|(\langle x,x\rangle - |\langle x,y_1\rangle|^2 - \dots\dots - |\langle x,y_N\rangle|^2)|),$$
$$\mathsf{max}_{1\le l\le N}\mathsf{max}_{1\le j\le t}(\| U_j(y_l) - \sum \lambda_{c_{j,l,k}}(y_k) \| \dot{-} \varepsilon_l)) \le 0,$$
where $\varepsilon_l \in \mathbb{Q}$ and $c_{j,l,k} \in \mathbb{Q}[i]$ are appropriate approximations of entries of matrices for $U_1,\dots,U_t$.

Using Lemma 2.2 it is easy to see that each Axm_j axiomatizes a decidable theory and the enumeration Axm_j, $j \in \omega$, gives an effective indexation of complete continuous theories T_i of dynamical N-dimensional spaces in the sense of Section 2. The statement that the relation $\{(\theta,j) : \theta$ is a statement so that $T_j \vdash \theta\}$ is computably enumerable follows from the fact that this relation coincides with $\{(\theta,j) : \theta$ is a statement so that $Axm_j \vdash \theta\}$.

Theorem 4.2. *The theory of all dynamical N-dimensional Hilbert spaces with operators $U_1,\dots,U_t$ coincides with the intersection $\bigcap T_j$.*
The theory of all dynamical N-dimensional Hilbert spaces is decidable.

Proof. As we already know the theory of all dynamical N-dimensional spaces is finitely axiomatizable. Thus by Theorem 3.4 the second statement of the theorem follows from the first one. To prove it we only have to show that for any rational δ, any dynamical N-dimensional space

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1,\dots,U_t),$$

and any continuous sentence $\theta(U_1,\dots,U_t)$ over this structure there are unitary operators $\tilde{U}_1,\dots,\tilde{U}_t$ defined by matrices over $\mathbb{Q}[i]$, so that

$$|\theta(U_1,\dots,U_t) - \theta(\tilde{U}_1,\dots,\tilde{U}_t)| \le \delta.$$

Indeed this shows that when some $\theta(U_1,\dots,U_t) \le \varepsilon$ does not belong to T, then it does not belong to some T_j (defined by matrices of $\tilde{U}_1,\dots,\tilde{U}_t$).

Since any continuous formula defines a uniformly continuous function and the ball B_1 is compact it suffices to take $\tilde{U}_1,\dots,\tilde{U}_t$ so that they sufficiently approximate $U_1,\dots,U_t$. This is a folklore fact. On the other hand it is a curious place where the following fact from quantum computations can be applied (the information given in the beginning of Section 5 suffices for the terminology below).

Let $\mathcal{B}$ be a 2-dimensional space over $\mathbb{C}$. Let $(\mathcal{B})^{\otimes 2}$ be the 4-dimensional space with the (Dirac) basis

$$|00\rangle\ ,\ |01\rangle\ ,\ |10\rangle\ ,\ |11\rangle.$$

Let $CNOT$ be a 2-qubit linear operator on $(\mathcal{B})^{\otimes 2}$ defined by

$$CNOT : |00\rangle \to |00\rangle\ ,\ |01\rangle \to |01\rangle\ ,\ |10\rangle \to |11\rangle\ ,\ |11\rangle \to |10\rangle.$$

The *Toffoli gate* is a 3-qubit linear operator defined on basic vectors of $(\mathcal{B})^{\otimes 3}$ by

$$\Lambda(CNOT) : |\varepsilon_1\varepsilon_2\varepsilon_3\rangle \to |\varepsilon_1\varepsilon_2(\varepsilon_3 \oplus \varepsilon_1 \cdot \varepsilon_2)\rangle\ ,\ \text{where } \varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{0,1\}.$$

It is well-known (see [25], Section 8) that

(a) For any natural number $k \geq 2$ all unitary transformations of $(\mathcal{B})^{\otimes k}$ can be presented as products of 1-qubit unitary transformations and 2-qubit copies of CNOT at appropriate registers.

(b) The operators of the basis

$$\mathcal{Q} = \{K = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, CNOT, \Lambda(CNOT), \text{Hadamar's } H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\}$$

generate a dense subgroup of $\mathbb{U}(\mathcal{B}^{\otimes 3})/\mathbb{U}(1)$ under the operator norm.

These facts reduce the problem of construction of $\tilde{U}_1, \ldots, \tilde{U}_t$ to the case of dimension 2. The latter case follows from standard presentations of unitary 2×2-matrices. □

The method of this theorem can be easily adapted to the following statement.

Theorem 4.3. *Assume that the theory $T_{f.d}$ of all finite dimensional dynamical Hilbert spaces of the signature*

$$(\{B_l\}_{l\in\omega}, 0, \{I_{kl}\}_{k<l}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, U_2, \ldots, U_t)$$

is computably axiomatizable. Then it is decidable.

In particular assume that any dynamical Hilbert space of this signature is elementarily equivalent to an ultraproduct of finite dimensional dynamical Hilbert spaces. Then the theory of all dynamical Hilbert spaces of this signature is decidable.

Proof. We modify the proof of Theorem 4.2 starting with enumeration of all *finite* dimensional unitary matrices of algebraic complex numbers. This induces an enumeration of systems of axioms of complete continuous theories T^{fin}_j of dynamical N-dimensional spaces where N is not fixed. The axioms describing the matrices of all U_s in some basis are the same as before, where N depends on the number of T^{fin}_j. This gives an effective indexation of complete continuous theories T^{fin}_j of dynamical finite dimensional spaces in the sense of Section 2. The proof that the theory $T_{f.d}$ coincides with the intersection $\bigcap T^{fin}_j$ is the same as in Theorem 4.2. Now the first statement of the theorem follows from Theorem 3.4.

To see the second statement just note that the assumption of it says that the theory $T_{f.d}$ is axiomatizable by standard axioms of dynamical Hilbert spaces. □

The crucial point of the theorem above is the assumption that the theory $T_{f.d}$ is recursively axiomatizable. We do not know if this holds. We will see in the following section that this question is connected with an open problem in the theory of approximations by metric subgroups.

Remark 4.4. It is a folklore fact that any Hilbert space (without operators) is elementarily equivalent to an ultraproduct of finite dimensional Hilbert spaces. Thus in the case when all U_i are equal to the identity map, the argument above shows that the theory of all Hilbert spaces is decidable (which is also folklore).

4.2. **Unbounded dimension and property MF.** In this section we find a connection between the assumptions of the second statement of Theorem 4.3 and the topic of approximations by metric groups. The latter is deserved a particular attention in group theory. This is mainly motivated by investigations of *sofic and* *hyperlinear groups*. We remind the reader that a group G is called *sofic* if G embeds into a metric ultraproduct of finite symmetric groups with the normalized Hamming distance d_H, [30]:

$$d_H(g,h) = 1 - \frac{|\mathsf{Fix}(g^{-1}h)|}{n} \text{ for } g,h \in S_n.$$

A group G is called *hyperlinear* if G embeds into a metric ultraproduct of finite-dimensional unitary groups $U(n)$ with the normalized Hilbert-Schmidt metric d_{HS} (i.e. the standard l^2 distance between matrices), [16], [30]. It is an open question whether these classes are the same and whether every countable group is sofic/hyperlinear.

The use of metric ultraproducts can be replaced by the following notion of approximation, see [33] and [19] (Definition 3). In this definition and below we always assume that metric groups are considered with respect to invariant metrics.

Definition 4.5. Let $\mathcal{K}$ be a class of metric groups. We say that a group G is $\mathcal{K}$-*approximable* if there is a function $\alpha : G \to [0,\infty]$ with

$$\alpha(g) = 0 \Leftrightarrow g = 1,$$

so that for any finite $F \subset G$ and $\varepsilon > 0$ there is $(H,d) \in \mathcal{K}$ and a function $\gamma : F \to H$ so that

$$\text{if } 1 \in F \text{ then } d(1,\gamma(1)) < \varepsilon \ ,$$
$$\text{for any } g,h,gh \in F \ , \ d(\gamma(gh),(\gamma(g)\gamma(h))) < \varepsilon \text{ and}$$
$$\text{for any } g \in F \ , \ d(1,\gamma(g)) \geq \alpha(g).$$

It is known that when the metrics of $\mathcal{K}$ are bounded by some fixed number r, a group G is $\mathcal{K}$-approximable if and only if it embeds into a metric ultraproduct of groups from $\mathcal{K}$ ([33] and [19]). Moreover in the case of sofic and hyperlinear groups the function α can be taken constant on $G \setminus \{1\}$ with the value equal to any real number strictly between 0 and 1 (between 0 and $r = \sqrt{2}$ in the hyperlinear case). We develop this property of sofic and hyperlinear groups as follows.

Definition 4.6. Let G be an abstract group, $\mathcal{K}$ be a class of metric groups and α_0 be a function $G \to [0,\infty]$ with $\alpha_0(1) = 0$. Assume that G is $\mathcal{K}$-approximable. We say that α_0 is the *amplification bound* of G with respect to $\mathcal{K}$ if for any $g \neq 1$, $\alpha_0(g)$ is the supremum of all possible values $\alpha(g)$ with respect to all possible function $\alpha : G \to [0,\infty)$ satisfying the properties of Definition 4.5.

Note that in the case of sofic groups the amplification bound with respect to the class of symmetric groups with normalized Hamming metrics is the function which is 1 for all nontrivial elements.

Below instead of examples mentioned above we will consider the following one.

> Unitary groups $U(n)$ together with the metric induced by the operator norm (on $G_L(n,\mathbb{C})$) $\parallel T \parallel_{op} = \mathsf{sup}_{\parallel v \parallel = 1} \parallel Tv \parallel$.
> We put $d(T,Q) = \parallel T - Q \parallel_{op}$.

This metric is *submultiplicative*, i.e. it is defined by a norm on $M_n(\mathbb{C})$ which satisfy the property $\parallel AB \parallel \leq \parallel A \parallel \cdot \parallel B \parallel$.

Groups approximable by these metric groups are called MF (matricial field), see [10]. It is an open question if there are non-MF groups. A. Tikuisis, S. White and W. Winter proved in [34] that amenable groups are MF. A. Korchagin shows in the recent preprint [26] that in many respects property MF is similar to soficity and hyperlinearity.

Remark 4.7. It is worth mentioning that another submultiplicative metric on $U(n)$ can be defined with respect to the Frobenius norm = the unnormalized Hilbert-Schmidt norm $\parallel T \parallel_{Frob}= \sqrt{\sum_{i,j} |T_{ij}|^2}$ (i.e. just the l^2-distance). In this case the corresponding groups are called *Frobenius approximated* [11]. It is already proved in [11] that there are finitely presented groups which are not Frobenius approximated. However there is no any description of the class of Frobenius approximated groups.

The following theorem is the most important observation of this section.

Theorem 4.8. *Let $G = \langle g_1, ..., g_n \rangle$ be a finitely generated group. The group G is MF if and only if there is a dynamical Hilbert space in the signature*

$$(\{B_l\}_{l\in\omega}, 0, \{I_{kl}\}_{k<l}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, U_2, \ldots, U_n)$$

which is an ultraproduct of finite dimensional dynamical Hilbert spaces of the same signature and the group $\langle U_1, \ldots, U_n\rangle$ is isomorphic to G under the map taking U_i to g_i, $1 \leq i \leq n$.

Proof. Below use d both for metrics in Hilbert spaces and for metrics of approximating metric groups.

Sufficiency of the theorem is easy. Indeed, having a dynamical Hilbert space (say $\mathbf{H}$) as in the statement consider the family of finite dimensional dynamical Hilbert spaces occurring in the corresponding ultraproduct. To define the function α from Definition 4.5 for any $g \in G \setminus \{1\}$ just take $\alpha(g)$ to be a positive real number which is less than $\mathsf{sup}_{v\in B_1} d(v, g(v))$ computed in $\mathbf{H}$.

Since the inequalities of Definition 4.5 in the case of the operator norm can be written by formulas of continuous logic, the approximations which we need in this definition can be taken as groups generated by $U_1, U_2, \ldots, U_n$ in spaces of the family from the ultraproduct. Then the function γ appearing in such an approximation maps a word of F to the corresponding word written in $U_1, U_2, \ldots, U_n$. For an illustration we give a formula for the condition

$$\text{if } g \in F \text{ then } d(1, \gamma(g)) \geq \alpha(g).$$

Assume that g is presented by a word $w(g_1, \ldots, g_n)$. Then we formalize the condition above as follows.

$$\alpha(g) \dot{-} \mathsf{sup}_{v\in B_1} d(v, w(U_1, \ldots, U_n)v) \leq 0.$$

Let us prove the necessity of the theorem. Let $m > 0$ and let $F \subseteq G$ be the ball of elements of G presented by words of length $\leq m$. Let ε be a small real number. Since G is MF there is an embedding γ of F into some $U(l)$ which satisfies the conditions of Definition 4.5 for the corresponding metric. We may assume that the corresponding function α is greater than $|F|\varepsilon$ for non-trivial elements of F.

Let $w(x_1, \ldots, x_n)$ be a word of length $\leq m$. If we present this word in the form $(\ldots(x_{i_1}^{\delta_1} x_{i_2}^{\delta_2})\ldots)x_{i_m}^{\delta_m}$ with $\delta_i \in \{-1, 0, 1\}$, then we have

$$d(\gamma(g_{i_1}^{\delta_1})\gamma(g_{i_2}^{\delta_2}), \gamma(g_{i_1}^{\delta_1} g_{i_2}^{\delta_2})) \leq \varepsilon \ , \ d(\gamma(g_{i_1}^{\delta_1} g_{i_2}^{\delta_2})\gamma(g_{i_3}^{\delta_3}), \gamma(g_{i_1}^{\delta_1} g_{i_2}^{\delta_2} g_{i_3}^{\delta_2})) \leq \varepsilon \ , \ \ldots$$

By invariantness of d this implies that

$$d((\gamma(g_{i_1}^{\delta_1})\gamma(g_{i_2}^{\delta_2}))\gamma(g_{i_3}^{\delta_3}), \gamma(g_{i_1}^{\delta_1} g_{i_2}^{\delta_2} g_{i_3}^{\delta_2})) \leq 2\varepsilon \ ,$$

$$d(\gamma(g_{i_1}^{\delta_1})\gamma(g_{i_2}^{\delta_2})\gamma(g_{i_3}^{\delta_3})\gamma(g_{i_4}^{\delta_4}), \gamma(g_{i_1}^{\delta_1}g_{i_2}^{\delta_2}g_{i_3}^{\delta_3}g_{i_4}^{\delta_4})) \leq 3\varepsilon\ ,\ \ldots .$$

As a result we see that $d(\gamma(w(\bar{g})), w(\overline{\gamma(g)})) \leq (m-1)\varepsilon$. In particular if $G \models w(\bar{g}) \neq 1$, then $d(1, \gamma(w(\bar{g}))) \geq min(\alpha(F))$ and $d(1, w(\overline{\gamma(g)})) \geq min(\alpha(F)) - (m-1)\varepsilon$. On the other hand if $G \models w(\bar{g}) = 1$, then $d(1, \gamma(w(\bar{g}))) \leq \varepsilon$ and $d(1, w(\overline{\gamma(g)})) \leq m\varepsilon$.

Let $M_{\varepsilon,F}$ be the corresponding finite dimensional dynamical Hilbert space:

$$(\{B_n^H\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, \gamma(g_1), \gamma(g_2), \ldots, \gamma(g_n)).$$

The computations above show that for any $v \in B_1$ of norm 1 the distance $d(\gamma(w(\bar{g}))v, w(\overline{\gamma(g)})v)$ is not greater than $\leq (m-1)\varepsilon$.

Let us fix an enumeration of pairs (ε_i, F_i), $i \in \omega$, as above with $\varepsilon_i \to 0$ and $G = \bigcup F_i$. Let D be a non-principal ultrafilter on ω. We assume that $\varepsilon_i > |F_{i+1}|\varepsilon_{i+1}$ and $F_i \subset F_{i+1}$. Let us prove that in the corresponding D-ultraproduct of the structures M_{ε_i,F_i} the tuple $U_1, \ldots U_n$ corresponding to $g_1, \ldots, g_n$, generates a group naturally isomorphic to G.

Let m be a natural number and $w(x_1, \ldots, x_n)$ be a word of length $\leq m$. Assume that $G \models w(g_1, \ldots, g_n) = 1$. As we have shown above for any $\varepsilon > 0$ there is a member of the sequence (ε_i, F_i), $i \in \omega$, such that for all numbers after this pair the statement

$$\mathsf{sup}_{v\in B_1} d(v, w(\overline{\gamma_i(g)})v) \leq \varepsilon$$

holds in the corresponding structures M_{ε_i,F_i} (for appropriate γ_i).

If $w(g_1, \ldots, g_n)$ is not equal to 1, then there is a rational number q (sufficiently close to $\alpha(w(\overline{\gamma_i(g)}))$) such that almost all structures M_{ε_i,F_i} satisfy the statement

$$q \dot{-} \mathsf{sup}_{v\in B_1} d(v, w(\overline{\gamma_i(g)})v) \leq 0.$$

The rest is clear. □

Theorem 4.8 implies that the statement that any finitely generated group is MF (which is a well-known conjecture) follows from the statement that the regular representation of any finitely generated group is pseudo finite dimensional. We will discuss the latter statement in Section 4.4.

Although the following theorem is not absolute, the assumptions of it are satisfied if every countable group is MF (which is a well-known conjecture). Indeed a finitely presented group with undecidable word problem was constructed by Novikov in the 50-s, see [29].

Theorem 4.9. *Assume that there is an MF finitely presented group $G = \langle g_1, ..., g_n | \mathcal{R}\rangle$ with undecidable word problem. Let T_G be the theory of the signature*

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, U_2, \ldots, U_n)$$

axiomatized by all statements satisfied in all finite dimensional dynamical Hilbert spaces and the statements

$$\mathsf{sup}_{v\in B_1} d(v, w(\overline{U})v) \leq 0\ ,\ where\ w(\bar{g}) \in \mathcal{R}.$$

Then the set of statements of T_G is not decidable.

Before the proof we give two remarks.

Remark 4.10. In fact in the formulation of the theorem we use the conventions of Section 2. In particular we extend the signature by symbols U_i' for U_i^{-1}, $i \in \omega$, and also add axioms $\mathsf{sup}_{v\in B_1} d(v, U_i'(U_i(v))) \leq 0$ and $\mathsf{sup}_{v\in B_1} d(v, U_i(U_i'(v))) \leq 0$. The sup-formulas in the formulation are written with U_i' for U_i^{-1}.

Remark 4.11. If we do not assume in the formulation that "statements are satisfied in *finite dimensional*" members of $\mathcal{K}$, then the theorem becomes much easier. The proof is basically the same as the proof below but does not use Theorem 4.8. We just use the (infinitely dimensional) Hilbert space $l^2(G)$.

Proof. (*Theorem 4.9*) The idea of this proof is well-known. W. Baur was the first who applied it, see [3]. Let

$$(\{B_n^{\mathbf{H}}\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, U_2, \ldots, U_n)$$

be the dynamical space constructed for G in Theorem 4.8. Then for any word $w(\bar{g})$ the statement

$$\mathsf{sup}_{v\in B_1} d(v, w(\overline{U})v) \leq 0$$

is satisfied in this space if and only if $G \models w(\bar{g}) = 1$. Notice that when $G \models w(\bar{g}) = 1$ the statement above follows from T_G. This gives the reduction of the word problem to the set of 0-statements of T_G. □

Remark 4.12. It is worth noting that formulas used in the proof of Theorem 4.9 are universal. This suggests considering the decidability problem for the universal theory of all dynamical (finite dimensional) Hilbert spaces. We will study this in Section 4.3. Note that Theorem 4.9 concerns a proper extension of it.

Theorem 4.13. *Assume that there is an algorithm which decides for every formula ϕ of the signature*

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, U_2, \ldots, U_n)$$

which has atomic subformulas only for B_1-variables and does not have free variables, whether $\phi^o = 2$ with respect to the theory of finite dimensional dynamical Hilbert spaces.

Then there is not an MF finitely presented group $G = \langle g_1, ..., g_n | \mathcal{R}\rangle$ with undecidable word problem so that the amplification bound of its MF-approximations α_0 is the function having only values 0 and 2 so that $\alpha_0(g) = 0 \Leftrightarrow g = 1$.

Remark 4.14. It is worth mentioning that the maximal value of a formula ϕ of the theory of dynamical Hilbert spaces which has atomic subformulas only for B_1-variables and which does not have free variables, is 2 (see Section 1).

Proof. (*Theorem 4.13*) Assume the contrary. Let $G = \langle g_1, ..., g_n | \mathcal{R}\rangle$ be an MF finitely presented group as in the formulation and let α_0 be the corresponding amplification bound. For any word $w(\bar{g})$ consider the formula

$$\phi_w = \mathsf{sup}_{v\in B_1} d(v, w(\overline{U})v) \dot{-} \mathsf{max}\{\mathsf{sup}_{v\in B_1} d(v, w'(\overline{U})v) | \text{ where } w'(\bar{g}) \in \mathcal{R}\}.$$

If $w(\bar{g})$ is not equal to 1 in G then $\alpha_0(w(\bar{g})) = 2$. By the argument of the final paragraph of the proof of Theorem 4.8 the set of all statements $2 \dot{-} \phi_w \leq \varepsilon$ is finitely satisfiable with respect to the theory of finite dimensional dynamical Hilbert spaces. By compactness (see Section 2 of [7]) we see that $(\phi_w)^o = 2$ with respect to this theory.

On the other hand note that in the case $G \models w(\bar{g}) = 1$ the equality $(\phi_w)^o = 2$ does not hold. Indeed if a dynamical Hilbert space as in the formulation realizes this equality then all the statements

$$\mathsf{sup}_{v\in B_1} d(v, w'(\overline{U})v) \leq 0 \text{ , where } w'(\bar{g}) \in \mathcal{R},$$

are also realized in it (by $\mathsf{sup}_{v\in B_1} d(v, w(\overline{U})v) \leq 2$). Then by the definition of G the map $w(\overline{U})$ defines the identity operator in this structure, a contradiction.

We now see that the problem of the equality $(\phi_w)^o = 2$ with respect to the theory of finite dimensional dynamical Hilbert spaces is not decidable. This is a contradiction with our assumption. □

4.3. **Universal theory.** In this section we study the universal theory of finite dimensional dynamical Hilbert spaces with unitary operators $U_1, \ldots U_t$. The following proposition is the crucial observation of this section. It is obviously related to Theorem 4.8.

Proposition 4.15. *Any dynamical Hilbert space is embeddable into a metric ultraproduct of finite dimensional Hilbert spaces.*

Proof. Let M be a dynamical Hilbert space. It suffices to show that for every rational ε, every quantifier-free formula $\psi(\bar{x})$ and every tuple $\bar{c} \in M$ there is a finite dimensional N and $\bar{c}' \in N$ such that $|\psi(\bar{c})^M - \psi(\bar{c}')^N| \leq \varepsilon$. Indeed having this we can approximate all $\psi(\bar{c})^M$ by values of some $\psi(\bar{c}')$ in finite dimensional spaces and then just apply the version of Łoś's theorem for metric ultraproducts.

Applying the Löwenheim-Skolem theorem if necessary we can arrange that $\bar{c}$ is taken from a separable M. We may assume that all terms appearing in $\psi(\bar{c})$ belong to a finite dimensional subspace $L < M$. Applying arguments of Section 7 of [32] we find a countable algebraically closed subfield $\mathcal{Q} < \mathbb{C}$ which is closed under complex conjugation, and a countable dense $\mathcal{Q}$-subspace $M' < M$ containing L such that the inner product and the norm on M' takes values in $\mathcal{Q}$. Since the formula $\psi(\bar{z})$ is a uniformly continuous function on M we may approximate the operators $U_1, \ldots, U_t$ on L by unitary operators say $U'_1, \ldots, U'_t$ on M' so that $\psi(\bar{c})^M$ is sufficiently close to $\psi(\bar{c})$ in M'. We now apply Lemma 7.4 of [32] and make $U'_1, \ldots, U'_t$ finitary. As a result we obtain a finite dimensional dynamical $\mathcal{Q}$-subspace $M'' < M'$ so that the value of $\psi(\bar{c})$ in M'' belongs to $[\psi(\bar{c})^M - \varepsilon, \psi(\bar{c})^M + \varepsilon]$. It is densely contained in a finite dimensional dynamical Hilbert space over $\mathbb{C}$, say N. This finishes the proof. □

Preserving the notation of Section 3.1 let $T_{f.d}$ be the theory of all finite dimensional dynamical Hilbert spaces with unitary operators $U_1, \ldots U_t$. Let us consider the universal (i.e. sup)-sentences of this theory. The following corollary of Proposition 4.15 states that their values coincide with ones corresponding to the theory of all dynamical Hilbert spaces.

Proposition 4.16. *Let ϕ be a universal sentence of the language of dynamical Hilbert spaces. Then ϕ° with respect to the theory of all dynamical Hilbert spaces coincides with ϕ° with respect to $T_{f.d}$.*

Proof. We may assume that all possible values of ϕ in dynamical Hilbert spaces belong to $[0, 2]$. Let r be the value of ϕ° with respect to the theory of all dynamical Hilbert spaces and r' be the value of ϕ° with respect to the theory $T_{f.d}$. It is clear that $r' \leq r$. To see $r \leq r'$ assume the contrary and find a rational number $q \in [r', r]$. Then there is a separable dynamical Hilbert space M such that $q < \phi^M$. This is equivalent to the condition that in M the existential formula $2 \dot{-} \phi$ has the value which is less than $2 - q$. Since M is embeddable into a metric ultraproduct of finite dimensional dynamical Hilbert spaces (by Proposition 4.15), it is clear that

$2\dot{-}\phi$ has value $< 2-q$ in some finite dimensional space, i.e. ϕ takes value $> r'$ in this space, a contradiction. □

Using decidability of the theory of algebraically closed fields we fix an effective indexation of all t-tuples of unitary matrices of algebraic numbers so that any tuple consists of matrices of the same dimension. If $\bar{C}$ is such a tuple let $H_{\bar{C}}$ be the corresponding dynamical Hilbert space of the same dimension as the dimension of matrices in $\bar{C}$, say n. The following statements describe $\bar{C}$ (corresponding to $\bar{U}$):

$$\mathsf{inf}_{x_1,\ldots,x_n\in B_1}\mathsf{max}(\mathsf{max}_{1\le i<j\le n}(|\langle x_i,x_j\rangle-\delta_{i,j}|)$$

$$(\text{ where } \delta_{i,j}\in\{0,1\} \text{ with } \delta_{i,j}=1\leftrightarrow i=j)\ ,$$

$$\mathsf{max}_{1\le l\le n}\mathsf{max}_{1\le j\le t}(\|\ U_j(x_l)-\sum\lambda_{c_{j,l,k}}(x_k)\ \|\ \dot{-}\varepsilon_l))\le 0,$$

$$\text{where } \varepsilon_l\in\mathbb{Q} \text{ and } c_{j,l,k}\in\mathbb{Q}[i] \text{ are appropriate approximations}$$

$$\text{of entries of matrices for } C_1,\ldots,C_t.$$

For every $\bar{C}$ we fix a computable sequence of such axioms, say $\Sigma_{\bar{C}}$. Let $\hat{T}$ be the extension of the theory of all dynamical Hilbert spaces obtained by the additional family axioms consisting of the union of all $\Sigma_{\bar{C}}$. We see that $\hat{T}$ is computably axiomatizable. Let

$$\hat{H}=\bigoplus\{H_{\bar{C}}:\bar{C} \text{ is a tuple of unitary matrices of algebraic numbers }\}.$$

Then $\hat{H}\models\hat{T}$. Since any finite dimensional dynamical Hilbert space is embeddable into a metric ultraproduct of spaces of the form $H_{\bar{C}}$, it is also embeddable into an ultrapower of $\hat{H}$. On the other hand $\hat{H}$ is embeddable into a ultraproduct of all $H_{\bar{C}}$. Thus applying Proposition 4.16 we have the first statement of the following lemma.

Lemma 4.17. *(a) For any universal sentence ϕ the value ϕ° with respect to $\hat{T}$ coincides with the value ϕ° with respect to the theory of all finite dimensional dynamical Hilbert spaces. The latter value coincides with the value ϕ° with respect to the theory $Th(\hat{H})$.*

(b) For every existential sentence ψ all values of ψ in all models of $\hat{T}$ are the same.

Proof. (b) Since any dynamical Hilbert space is embeddable into a ultrapower of $\hat{H}$ the value of ψ in $\hat{H}$ is minimal among all possible values. On the other hand since any model $M\models\hat{T}$ contains all $H_{\bar{C}}$ the space $\hat{H}$ is embeddable into a ultrapower of M. This shows that the values of ψ in M and $\hat{H}$ are the same. □

We can now prove the main result of Section 3.3.

Theorem 4.18. *The universal theory of all dynamical Hilbert spaces is decidable.*

Proof. According Lemma 4.17 if ϕ is a universal sentence, then for the theory $\hat{T}$ the value of the existential sentence $2\dot{-}\phi$ coincides with $2\dot{-}\phi^\circ$. Moreover the value ϕ° is the same for $\hat{T}$ and for the theory of all dynamical Hilbert spaces. Thus the following algorithm always gives the result. Given universal ϕ and a rational ε, run all proofs from $\hat{T}$ and wait until you see that $\hat{T}\vdash\phi\dot{-}r$ and $\hat{T}\vdash(2\dot{-}\phi)\dot{-}s$, where r and s are rational numbers so that $r-(2-s)\le\varepsilon$. As a result one of the numbers r or s belongs to the interval of length ε which contains ϕ° with respect to the theory of all dynamical Hilbert spaces. □

4.4. **Pseudocompactness.** The author does not know if any dynamical Hilbert space is elementarily equivalent to an ultraproduct of finite dimensional dynamical Hilbert spaces (i.e. if its unit ball is pseudocompact). In this subsection we discuss some natural cases where we prove/expect the positive answer. Note that by the results of Section 4.1 the global positive solution of this problem implies decidability of the theory of all dynamical Hilbert spaces (and the theory of pseudo finite dimensional ones too). Moreover it also implies that any finitely generated group is MF (see the comment after Theorem 4.8).

According to [9] for any countable group G the theory of all unitary G-representations has a model completion. It can be described as follows.

Let ∞H_ρ be a dynamical Hilbert space corresponding to a representation of G which is maximal with respect to almost containedness $\prec$ (see Definition 2.3). Theorem 2.11 of [9] states that the class of existentially closed unitary representations of G is axiomatized by all 0-inf-statements which hold in ∞H_ρ. Moreover by the amalgamation property the theory of these representations is complete.

Note all dynamical Hilbert spaces with t operators form the class of all representations of the free t-generated group F_t. Thus the class of all existentially closed representations of F_t coincides with the model completion of the theory of all dynamical Hilbert spaces. The following observation was pointed out to the author by E. Hrushovski (who applied a different argument).

Proposition 4.19. *Existentially closed dynamical Hilbert spaces are pseudo finite dimensional.*

Proof. By Proposition 4.15 any existentially closed dynamical Hilbert space is contained in a pseudo finite dimensional one. All 0-inf-statements which hold in ∞H_ρ also hold in the corresponding pseudo finite dimensional dynamical Hilbert space. By Theorem 2.11 of [9] it is also existentially closed and by completeness of the theory is elementarily equivalent to ∞H_ρ. □

This proposition justifies the following question.

> *Given finitely generated group G are existentially closed dynamical Hilbert spaces corresponding to representations of G pseudo finite dimensional?*

By Theorem 2.15 of [9] a countable group G is amenable if and only if the direct sum of countably many copies of $l^2(G)$, the regular unitary representation of G, is existentially closed. Thus it is easy to see that in the amenable case the positive answer to this question follows from the statement that for any finitely generated group G the dynamical space $l^2(G)$ is pseudo finite dimensional. As we already know the latter statement implies that any finitely generated group is MF.

We have the following partial result. This can be considered as a stronger version of the statement that any finitely generated LEF group is MF, see [10].

Proposition 4.20. *Let G be a finitely generated LEF group. Then the dynamical G-space $l^2(G)$ is pseudo finite dimensional.*

Before the proof of the proposition we remind the reader that a group H is called LEF [35] if for every finite subset $F \subseteq H$ there is a finite group S containing F so that for any $x, y, z \in F$ the equality $x \cdot y = z$ holds in H if and only if it holds in S. Residually finite groups are LEF. This proposition together with Theorem

2.15 of [9] imply that when G is an amenable LEF group generated by t elements, all dynamical Hilbert spaces which are existentially closed G-representations are pseudo finite dimensional.

Proof. Let G be t-generated. We fix a tuple $g_1, \ldots, g_t$ of generators and consider Γ_G, the corresponding coloured Cayley graph of G. By the condition LEF for every natural k there is a finite t-generated group G_k such that the k-ball of 1 in the Cayley graph of G_k coincides with the k-ball of 1 in Γ_G. It is also worth mentioning that for any two elements of Γ_{G_k} their k-balls are naturally isomorphic.

Let $\hat{H}_G$ be a non-principal metric ultraproduct of all dynamical spaces $l^2(G_k)$. Then identifying any word $w(\bar{g})$ from G with the sequence of the corrresponding elements from all G_k we consider $l^2(G)$ as a substructure of $\hat{H}_G$. Moreover each g_i naturally defines a unitary transformation of $\hat{H}_G$. So the regular representation of G naturally extends to a representation on $\hat{H}_G$. The $\mathbf{C}^*$-algebra generated by G in the algebra of all bounded operators of $l^2(G)$ is the operator norm closure of the $*$-algebra generated by the regular action of G, i.e. by $\mathbb{C}G$. We denote it by $C^*(G)$.

It is easy to see that the $*$-algebra $\mathbb{C}G$ naturally acts on $\hat{H}_G$. Let us note that $C^*(G)$ also has a natural action on $\hat{H}_G$. Indeed let $p_1, \ldots, p_i, \ldots$ be a sequence from $\mathbb{C}G$ which is norm convergent. For every i let l_i be the maximal length of words from Γ_G which appear in $p_1, \ldots, p_i$. Then for every natural number m there is a natural number n such that for all $k > n$ all the $(l_m + 2m)$-balls of 1 in Γ_{G_k} are naturally isomorphic. In particular for any element $v \in \Gamma_{G_k}$ the partial actions of $p_1, \ldots, p_m$ inside the subspace of $l^2(G_k)$ supported by the $(l_m + 2m)$-ball of v correspond to the actions defined by $p_1, \ldots, p_m$ in the subspace of $l^2(G)$ supported by the $(l_m + 2m)$-ball of 1. This obviously implies that for any i and j

$$\lim_{k\to\infty} \| p_i - p_j \|_{l^2(G_k)} = \| p_i - p_j \|_{l^2(G)} .$$

In particular the sequence $p_1, \ldots, p_i, \ldots$ is norm convergent in $\hat{H}_G$. As a result we see that $\hat{H}_G$ is a $C^*(G)$-module.

We now apply Theorem 2.20 of [1]. It states that two representations of a $\mathbf{C}^*$-algebra $\mathcal{A}$ are elementarily equivalent in continuous logic if and only if for any $a \in \mathcal{A}$ the ranks of the corresponding elements are finite and the same or are infinite. It is easy to see that the arguments above can be applied for a verification that the $C^*(G)$-representations $l^2(G)$ and $\hat{H}_G$ satisfy the conditions of this theorem. □

We finish this section by the observation that in the case of a single operator pseudocompactness follows from the spectral decomposition theorem.

Proposition 4.21. *Any dynamical Hilbert space with a single unitary operator is pseudo finite dimensional.*

This observation was suggested to the author by E. Hrushovski. Before the proof we remind the reader the *spectral decomposition theorem.*

> Let U be a unitary operator. Then there is a unique resolution of the identity $\{E_r | r \in [0, 2\pi]\}$ such that $U^k = \int_0^{2\pi} e^{ikt} dE_t$ for all $k \in \mathbb{Z}$.

In this formulation each E_r is a projection operator, $E_r = 0$ for $r \leq 0$, $E_r = Id$ for $r > 2\pi$, and each $E_s - E_r$ is positive for $r < s$. The theorem should be interpreted as follows. Given $\varepsilon > 0$ and $\delta > 0$ there is n_0 such that for any partition

$0 = r_0 < s_0 = r_1 < s_1 = r_2 < \ldots < s_{n_0} = 2\pi + \varepsilon$ with $\mathsf{max}\{s_l - r_l\} < \frac{2(2\pi+\varepsilon)}{n_0}$ we have $\| U - \sum_{l=1}^{n_0} e^{ikr_l}(E_{s_l} - E_{r_l}) \| < \delta$.

Proof. Apply the spectral decomposition theorem to a dynamical Hilbert space (H, U). All operators E_r are self-ajoint, their images are closed subspaces and $E_r(H) \subset E_s(H)$ for $r < s$. In particular $E_s - E_r$ is an orthogonal projection operator. Given $\varepsilon > 0$, $\delta > 0$ and $n_0(\varepsilon, \delta)$ as above for each natural number n one can define a subspace of $H' < H$ of dimension $\leq n(n_0 + 1)$, where

$$\mathsf{dim} E_{r_1}(H') = \mathsf{min}(n, \mathsf{dim} E_{r_1}(H)), \ldots,$$

$$\mathsf{dim}(E_{s_l} - E_{r_l})(H') = \mathsf{min}(n, \mathsf{dim}(E_{s_l} - E_{r_l})(H)), \ldots.$$

Then the formula $\sum_{l=1}^{n_0} e^{ikr_l}(E_{s_l} - E_{r_l})$ defines an operator on H'. We denote it by $U_{\varepsilon,\delta,n}$.

Take a sequence $(\varepsilon_i, \delta_i, n_i)$ such that $\varepsilon_i \to 0$, $\delta_i \to 0$ and $n_i \to \infty$. Let $(\hat{H}, \hat{U})$ be a non-principal metric ultraproduct of the corresponding structures $(H', U_{\varepsilon,\delta,n})$. In order to prove that $(\hat{H}, \hat{U})$ is elementarily equivalent to (H, U) we apply the Henson's theorem (Theorem 3.1 of [2]) cited in Section 2.3 above. We also apply some part of the proof of Theorem 3.3 from [2].

Let $\mu \in S^1 \setminus \sigma(U)$ and $\eta = d(\mu, \sigma(U))$. Then the statement

$$\mathsf{sup}_u(\eta \parallel u \parallel \dot{-} \parallel U(u) - \mu u \parallel) \leq 0$$

holds in (H, U) and so holds its 2δ-approximation in $(H', U_{\varepsilon,\delta,n})$ as above. Now it is easy to see that the exact statement holds in $(\hat{H}, \hat{U})$. Therefore $\sigma(\hat{U}) \subseteq \sigma(U)$. The same argument proves $\sigma(U) \subseteq \sigma(\hat{U})$.

For each $\lambda \in \sigma(U)$ let us consider statements of the form

$$\mathsf{inf}_{u_1} \ldots \mathsf{inf}_{u_m} \mathsf{max}_{i,j}(|\langle u_i, u_j\rangle|, |\parallel u_i \parallel -1|, |U(u_i) - \lambda u_i|) \leq 0.$$

Repeating the argument above we see that they do not distinguish (H, U) and $(\hat{H}, \hat{U})$. When λ is an isolated point in $\sigma(U)$ this implies that the dimensions of $\{x \in H : U(x) = \lambda x\}$ and $\{x \in \hat{H} : \hat{U}(x) = \lambda x\}$ are the same. Thus the conditions of the Henson's theorem are satisfied. □

5. Dynamical n-qubit spaces

In this section we demonstrate how the method of interpretability works in some expansions of dynamical spaces. In paragraphs (A) - (C) below we describe why these expansions are natural from the point of view of quantum computations.

5.1. **Preliminaries.**

(A). We remind the reader that states of quantum systems are represented by normed vectors of tensor products

$$(...(\mathcal{B}_1 \bigotimes \mathcal{B}_2) \bigotimes ....) \bigotimes \mathcal{B}_k,$$

where $\mathcal{B}_i \cong \mathbb{C} \bigoplus \mathbb{C}$ under isomorphisms of Hilbert spaces, $i \leq k$.

In Dirac's notation elements of $\mathcal{B}_i$ are denoted by $|h\rangle$ and tensors

$(...(|h_1\rangle \otimes |h_2\rangle)...) \otimes |h_k\rangle$ are denoted by $|h_1 h_2 ... h_k\rangle$.

Any normed $h \in \mathcal{B}_i$ is called a *qubit*; it is a linear combination of $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$.

The probability amplitude $a(\phi \to \psi)$ is defined as the inner product $\langle \psi | \phi \rangle$ and the probability $p(\phi \to \psi)$ is $|a(\phi \to \psi)|^2$. Dynamical evolutions of the quantum system are represented by unitary operators on $\mathcal{B}^{\otimes k}$.

(B). It is worth noting that continuous logic can be considered as a theory in some extension (**RPL**$\forall$) of Łukasiewicz logic (see [13]). The latter is traditionally linked with quantum mechanics, [8], [31]. Thus the idea that continuous logic should enter into the field is quite natural.

We will consider dynamical n-qubit spaces in continuous logic as follows. Firstly we extend structures of complex Hilbert spaces by additional discrete sort Q with $\{0,1\}$-metric and a map $qu : Q \to B_1$ so that the set $qu(Q)$ is an orthonormal basis of $\mathbb{H}$.

When Q consists of 2^n elements we may denote them by $|i_0...i_{n-1}\rangle$ with $i_j \in \{0,1\}$. In Quantum Computations this set is called the *computational basis of the system* and $\mathcal{B}^{\otimes n}$ is called the *n-qubit space*. Secondly we enrich the structure $(qu, Q, \mathcal{B}^{\otimes n})$ by unitary operators $U_1, ..., U_t$. We call it a *dynamical n-qubit space*. It turns out that the condition $|Q| = 2^n$ is not essential. For example one can consider subspaces generated by arbitrary subsets of the computational basis. Therefore we will consider *marked Hilbert spaces* and *marked dynamical Hilbert spaces*, i.e. (dynamical) Hilbert spaces expanded by a discrete sort Q and a map qu which injectively maps Q onto an orthonormal basis of the space. In this section we study the following problem.

> *Describe classes of marked dynamical Hilbert spaces having decidable continuous theory.*

Below we give examples of classes of marked dynamical spaces with undecidable sets of 0-statements. This material is based on the method of interpretability described in the second part of Section 3. Comparing these results with Section 4 it is worth noting that in fact (following the approach of Quantum Computations) we extend the language used in Section 4 by a discrete unary predicate Q. We will see that this procedure is essential for the expressive power of the language: there are natural subclasses of marked dynamical Hilbert spaces where undecidable first order theories of some classes of finite structures can be interpreted on Q.

(C). It is worth noting that a dynamical n-qubit space defines a family of quantum automata over the language $\{1, ..., t\}^*$, where each automaton is determined by the 2^n-dimensional diagonal matrix P of the projection to final states. Fixing $\lambda \in \mathbb{Q}$ we say that a word $w = i_1...i_k$ is *accepted* by the corresponding P-automaton if

$$ACC_w = \| PU_{i_k}...U_{i_1}|0^{\otimes n}\rangle \|^2 > \lambda.$$

These issues are described in [22], [28] and [12]. The corresponding algorithmic problems were in particular studied in the paper of H. Derksen, E. Jeandel, P. Koiran [12]. They have proved that the following problems are decidable for $U_1, ..., U_t$ over finite extensions of $\mathbb{Q}[i]$:
(i) Is there w such that $ACC_w > \lambda$?
(ii) Is a threshold λ isolated, i.e. is there ε that for all w, $|ACC_w - \lambda| \geq \varepsilon$?
(iii) Is there a threshold λ which is isolated?

The observation that given P each statement $ACC_w \leq \lambda$ or $|ACC_w - \lambda| \geq \varepsilon$ can be rewritten as a continuous statement of the theory of dynamical n-qubit spaces partially motivated our research in this paper.

5.2. **Interpretations.** We start this section with an undecidability result of some classes of constant expansions of marked Hilbert spaces. Then we apply the idea of the proof to a more interesting example of a class of marked dynamical spaces. In Remark 5.2 we comment how these classes are natural.

Theorem 5.1. *There is a class of marked Hilbert spaces expanded by four constants, i.e. structures of the form*

$$(Q, qu, \{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, a_1, a_2, b_1, b_2),$$

which is distinguished in the class of all marked Hilbert spaces with $\langle b_1, b_2\rangle \neq 0$ by a single continuous statement and which has undecidable set of all 0-statements.

Proof. Consider the following formula:

$$\psi(x, y_1, y_2) = |\langle qu(y_1), x\rangle - \langle qu(y_2), x\rangle|,$$

where y_1, y_2 are variables of the sort Q and x is of the sort B_1.

In any marked Hilbert space any equivalence relation on Q can be realised by $\psi(a, y_1, y_2) \leq 0$ for appropriately chosen $a \in B_1$: define a to be a linear combination of $qu(q_l)$, so that for equivalent q_j and q_k the coefficients of $qu(q_j)$ and $qu(q_k)$ in a are the same.

Let us introduce the following formula:

$$\psi^c(x, z_1, z_2, y_1, y_2) = |\langle z_1, z_2\rangle| \dot{-} |\langle qu(y_1), x\rangle - \langle qu(y_2), x\rangle|,$$

where y_1, y_2 are variables of the sort Q and x, z_1, z_2 are of B_1.

If a defines an equivalence relation on Q as above then there are $b_1, b_2 \in B_1$ with sufficiently small $|\langle b_1, b_2\rangle| \neq 0$ (in fact here we only need

$$|\langle b_1, b_2\rangle| < \mathsf{min}(|\langle qu(y_1), a\rangle - \langle qu(y_2), a\rangle| : y_1 \text{ and } y_2 \text{ are not equivalent })$$

and satisfying

$$\mathsf{sup}_{y_1,y_2}\mathsf{min}(\psi(a, y_1, y_2), \psi^c(a, b_1, b_2, y_1, y_2)) \leq 0$$

$$\mathsf{sup}_{y_1,y_2}(|\langle b_1, b_2\rangle| \dot{-} (\psi(a, y_1, y_2) + \psi^c(a, b_1, b_2, y_1, y_2))) \leq 0.$$

We see that the formula $\psi^c(a, b_1, b_2, y_1, y_2)$ can be interpreted as the complement of the equivalence relation defined by $\psi(a, y_1, y_2)$ in the class of these marked spaces.

This allows us to define interpretability of the first-order theory of finite structures of two equivalence relations (which is undecidable by Proposition 5.1.7 from [17]) in the class, say $\mathcal{K}$, of marked Hilbert spaces extended by constants a_1, a_2, b_1, b_2 where b_1, b_2 satisfy the statements above for both a_1 and a_2 instead of a.

In fact we axiomatize $\mathcal{K}$ in the class of all marked Hilbert spaces with $\langle b_1, b_2\rangle \neq 0$ by the following continuous statements:

$$\mathsf{sup}_{y_1,y_2}\mathsf{min}(\psi(a_i, y_1, y_2), |\langle b_1, b_2\rangle| \dot{-} |\langle qu(y_1), a_i\rangle - \langle qu(y_2), a_i\rangle|) \leq 0 \text{ , where } i = 1, 2.$$

In terms of Theorem 3.5 the formulas $\phi^+, \phi^-, \theta^+, \theta^-$ become degenerate: ϕ^- can be taken as $d(y, y)$ for the (descrete) sort Q, then

$$\phi^+(y) = 1 \dot{-} d(y, y) \text{ , } \theta^-(y_1, y_2) = d(y_1, y_2) \text{ , } \theta^+(y_1, y_2) = 1 - d(y_1, y_2).$$

Formulas $\psi(a_1, y_1, y_2)$ and $\psi^c(a_1, b_1, b_2, y_1, y_2)$ play the role of ψ_1^- and ψ_1^+. Then $\psi(a_2, y_1, y_2)$ and $\psi^c(a_2, b_1, b_2, y_1, y_2)$ play the role of ψ_2^- and ψ_2^+.

To each formula $\rho(\bar{y})$ of the theory of two equivalence relations so that the quantifier-free part is in the disjunctive normal form we associate the appropriately rewritten continuous formula $\rho^*(a_1, a_2, b_1, b_2, \bar{y})$ (where we use min and max instead of $\vee$ and $\wedge$, and we exchange $+$ and $-$ when $\neg$ appears). Since the free variables of the latter, $\bar{y}$, are of the sort Q, when ρ is quantifier-free the values $\rho^*(a_1, a_2, b_1, b_2, \bar{c})$ belong to $\{0\} \cup [|\langle b_1, b_2\rangle|, 1]$. It is easy to see that in structures of $\mathcal{K}$ the same property holds for any formula $\rho(\bar{y})$.

This obviously implies that when ρ is a sentence, the sentence

$$\mathsf{min}(|\langle b_1, b_2\rangle|, \rho^*(a_1, a_2, b_1, b_2))$$

has the following property:

> ρ is satisfied in all finite models of two equivalence relations if and only if all structures of $\mathcal{K}$ satisfy $\mathsf{min}(|\langle b_1, b_2\rangle|, \rho^*(\bar{a}, \bar{b})) = 0$.

By Theorem 3.5 this gives a required reduction. □

Remark 5.2. The class of marked Hilbert spaces considered in Theorem 5.1 is the intersection of an axiomatizable class of continuous structures with the complement of an axiomatizable class (defined by the inequality $\langle b_1, b_2\rangle \neq 0$). Both classes of this intersection are natural and easily defined. It is also clear that the axiomatizable closure of this class still has undecidable set of 0-statements. On the other hand we do not have any reasonable description of the members of this closure. The following theorem concerns a very similar situation (in a different language). The author would very like to find an easily described axiomatizable class of marked dynamical Hilbert spaces with undecidable theory (or at least with undecidable set of 0-statements).

Theorem 5.3. *There is a class of marked dynamical Hilbert spaces in the signature*

$$(Q, qu, \{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{Q}[i]}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}, U_1, U_2, U_3, U_4, U_5),$$

which is distinguished in the class of all marked dynamical Hilbert spaces with

$$\mathsf{sup}_v d(U_3(v), v) \neq 0$$

by a single continuous statement and the set of 0*-statements of the continuous theory of which is not computable.*

Proof. We will use the construction of Theorem 5.1 with some necessary changes. For example we replace the value $|\langle b_1, b_2\rangle|$ from that theorem by $\mathsf{sup}_v d(U_3(v), v)$. The constants a_1 and a_2 will appear as the normed vectors fixed by U_1 and U_2 respectively. Although we choose U_i, $i = 1, 2$, so that the subspace of fixed vectors of U_i coincides with $\mathbb{C}a_i$, we cannot define these constants by a continuous formula. This is why some additional values will be used in the proof. The values $\mathsf{sup}_v d(U_3(v), v)$ and $\mathsf{sup}_v d(U_4(v), v)$ will appear in 'fuzzy' versions of formulas from the proof of Theorem 5.1.

Let:

$$\psi_i(y_1, y_2) = \mathsf{sup}_u \mathsf{min}(\mathsf{sup}_{v_1}(d(U_3(v_1), v_1)) \dot{-} \mathsf{max}(d(U_i(u), u), |1- \parallel u \parallel |),$$
$$(|\langle qu(y_1), u\rangle - \langle qu(y_2), u\rangle| \dot{-} \mathsf{sup}_{v_2} d(U_4(v_2), v_2))),$$

where y_1, y_2 are variables of the sort Q, $i \in \{1, 2\}$ and u, v_1, v_2 are of the sort B_1.

To see that *in any marked dynamical Hilbert space any equivalence relation on* Q *can be realized by* $\psi_1(y_1, y_2) \leq 0$ let us define a_1 to be a linear combination of $qu(q_l)$ of length 1, so that for equivalent q_j and q_k the coefficients of $qu(q_j)$ and $qu(q_k)$ in a_1 are the same (the case of ψ_2 and a_2 is similar). We also fix a rational number r (for both a_1 and a_2) so that for non-equivalent q_j and q_k the coefficients of $qu(q_j)$ and $qu(q_k)$ in a_1 are distant by $> r$. Note that any $e^{i\phi}a_1$ has the same properties as a_1 with respect to elements of $qu(Q)$.

We extend a_1 to an orthonormal basis of the space and define U_1 to be a unitary operator having the vectors of the basis as eigenvectors so that $\mathbb{C}a_1$ is the subspace of fixed points. The remaining eigenvalues are chosen in the form $e^{i\varphi}$ so that the corresponding eigenvectors are taken by U_1 at the distance $\geq 1/10$ (i.e.$|1-e^{i\varphi}| \geq 1/10$).

We will assume that $\mathsf{sup}_{v\in B_1} d(U_3(v), v) > 0$ in our structures. Choosing U_4 we demand that $\mathsf{sup}_{v\in B_1}(d(U_4(v), v))$ is much less than r. It follows that when q_i and q_j are not equivalent the value $u = a_1$ realizes the inequality $\psi_1(q_i, q_j) > 0$.

Having U_4 we construct U_3 so close to Id (with respect to the operator norm) that the statement $\psi_1(y_1, y_2) \leq 0$ indeed realizes the equivalence relation we consider. For this we only need the condition that if q_i and q_j are equivalent and a vector c satisfies

$$|\langle qu(q_i), c\rangle - \langle qu(q_j), c\rangle| > \mathsf{sup}_{v\in B_1} d(U_4(v), v)) \text{ and } |1- \parallel c \parallel | < \mathsf{sup}_{v\in B_1}(d(U_3(v), v))$$

i.e. the projection of c to $qu(q_i) - qu(q_j)$ and the length of c are sufficiently large (i.e. c is sufficiently distant from a_1), then $\mathsf{sup}_{v\in B_1}(d(U_3(v), v)) \leq d(U_1(c), c)$.

Let us now introduce U_5 with $r = \mathsf{sup}_v(d(U_5(v), v)$ and consider the following formulas for $i = 1, 2$:

$$\psi_i^c(y_1, y_2) = \mathsf{sup}_u \mathsf{min}(\mathsf{sup}_{v_1}(d(U_3(v_1), v_1)) \dot{-} \mathsf{max}(d(U_i(u), u), |1- \parallel u \parallel |),$$
$$(\mathsf{sup}_{v_2} d(U_5(v_2), v_2) \dot{-} |\langle qu(y_1), u\rangle - \langle qu(y_2), u\rangle|)),$$

where y_1, y_2 are variables of the sort Q and u, v_1, v_2 are of B_1. If necessary we may correct U_3 making $\mathsf{sup}_v d(U_3(v), v)$ smaller so that the following statement holds.

$$\mathsf{sup}_{y_1,y_2} \mathsf{min}(\psi_i(y_1, y_2), \psi_i^c(y_1, y_2)) \leq 0,$$

Verifying this one can apply the argument of the previous paragraph. Similar reasoning implies that

$$\mathsf{sup}_{y_1,y_2}(\mathsf{sup}_v d(U_3(v), v) \dot{-} (\psi_i(y_1, y_2) + \psi_i^c(y_1, y_2))) \leq 0.$$

As before the formula $\psi_i^c(y_1, y_2)$ will be interpreted as the complement of the equivalence relation defined by $\psi_i(y_1, y_2)$ in the class of these qubit spaces.

This allows us to define interpretability of the (undecidable) first-order theory of finite structures of two equivalence relations in the class, say $\mathcal{K}$, of marked dynamical spaces with respect to operators U_1, U_2, U_3, U_4, U_5. The formulas $\phi^+, \phi^-, \theta^+, \theta^-$ (see Theorem 3.5) are taken as in Theorem 5.1 (i.e. $\phi^-(y) = d(y, y)$ and $\theta^-(y_1, y_2) = d(y_1, y_2)$). Formulas $\psi_i(y_1, y_2)$ and $\psi_i^c(y_1, y_2)$ play the role of ψ_i^- and ψ_i^+ for $i = 1, 2$.

To each formula $\rho(\bar{y})$ of the theory of two equivalence relations so that the quantifier-free part is in the disjunctive normal form we associate the appropriately rewritten continuous formula $\rho^*(\bar{y})$. Since the free variables of the latter $\bar{y}$ are of the sort Q, when ρ is quantifier-free, the values $\rho^*(\bar{c})$ belong to $\{0\} \cup [\mathsf{sup}_v d(U_3(v), v), 1]$. Thus we see that in structures of $\mathcal{K}$ the same property holds for any formula $\rho(\bar{y})$.

This obviously implies that when ρ is a sentence, the sentence

$$\mathsf{min}(\mathsf{sup}_v d(U_3(v), v), \rho^*)$$

has the following property:

> ρ is satisfied in all finite models of two equivalence relations if and only if all structures of $\mathcal{K}$ satisfy $\mathsf{min}(\mathsf{sup}_v d(U_3(v), v), \rho^*) = 0$.

This finishes the proof. □

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