# Compactness of Kähler-Ricci solitons on Fano manifolds<sup>1</sup>

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#### Abstract

In this short paper, we improve the result of Phong-Song-Sturm on degeneration of Fano Kähler-Ricci solitons by removing the assumption on the uniform bound of the Futaki invariant. Let  $\mathcal{KR}(n)$  be the space of Kähler-Ricci solitons on *n*-dimensional Fano manifolds. We show that after passing to a subsequence, any sequence in  $\mathcal{KR}(n)$  converge in the Gromov-Hausdorff topology to a Kähler-Ricci soliton on an *n*-dimensional Q-Fano variety with log terminal singularities.

#### 1 Introduction

The Ricci solitons on compact and complete Riemannian manifolds naturally arise as models of singularities for the Ricci flow [7]. The existence and uniqueness of Ricci solitons has been extensively studied. A gradient Ricci soliton is a Riemannian metric satisfying the following soliton equation

$$Ric(g) = \lambda g + \nabla^2 u \tag{1.1}$$

for some smooth function f with  $\lambda = -1, 0, 1$ . Such a soliton is called a gradient shrinking Ricci soliton if  $\lambda > 0$ . If we let the vector field  $\mathcal{V}$  be defined by  $\mathcal{V} = \nabla u$ , the soliton equation becomes

$$Ric(g) = \lambda g + L_{\mathcal{V}}g,\tag{1.2}$$

where  $L_{\mathcal{V}}$  is the Lie derivative along  $\mathcal{V}$ .

A Kähler metric g on a Kähler manifold X is called a Kähler-Ricci soliton if it satisfies the soliton equation (1.1) or equation (1.2) for  $\mathcal{V} = \nabla u$ . Any shrinking Kähler-Ricci soliton on a compact Kähler manifold X must be a gradient Ricci soliton and such a Kähler manifold must be a Fano manifold, i.e.  $c_1(X) > 0$ . The vector field  $\mathcal{V}$  must be holomorphic and it can be expressed in terms of the Ricci potential u, with

$$R_{i\bar{j}} = g_{i\bar{j}} - u_{i\bar{j}}, \quad u_{ij} = u_{\bar{i}\bar{j}} = 0, \quad \mathcal{V}^i = -g^{i\bar{j}}u_{\bar{j}}. \tag{1.3}$$

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The well-known Futaki invariant associated to the Kähler-Ricci soliton  $(X, g, \mathcal{V})$ on a Fano manifold X is given by

$$\mathcal{F}_X(\mathcal{V}) = \int_X |\nabla u|^2 dV_g = \int_X |\mathcal{V}|^2 dV_g \ge 0.$$

Let  $\mathcal{KR}(n, F)$  be the set of compact Kähler-Ricci solitons (X, g) of complex dimension n with

$$Ric(g) = g + L_{\mathcal{V}}g, \ \mathcal{F}_X(\mathcal{V}) \leq F.$$

It is proved by Tian-Zhang [17] that  $\mathcal{KR}(n, F)$  is compact in the Gromov-Hausdorff topology with an additional uniform upper volume bound. In [10], Phong-Song-Sturm established a partial  $C^0$ -estimate on  $\mathcal{KR}(n, F)$ , generalizing the celebrated result of Donaldson-Sun [6] for the space of uniformly non-collapsed Kähler manifolds with uniform Ricci curvature bounds. An immediate consequence of the partial  $C^0$ estimate in [10] is that the limiting metric space must be a Q-Fano variety equipped with a Kähler-Ricci soliton metric.

The purpose of this paper is to remove the assumption in [10] on the bound of the Futaki invariant.

**Definition 1.1** Let  $\mathcal{KR}(n)$  be the set of compact Kähler-Ricci solitons  $(X, g, \mathcal{V})$  of complex dimension n with

$$Ric(g) = g + L_{\mathcal{V}}g.$$

The following is the main result of the paper.

**Theorem 1.1** Let  $\{(X_i, g_i, \mathcal{V}_i)\}_{i=1}^{\infty}$  be a sequence in  $\mathcal{KR}(n)$  with  $n \geq 2$ . Then after possibly passing to subsequence,  $(X_i, g_i)$  converges in the Gromov-Hausdorff topology to a compact metric length space  $(X_{\infty}, d_{\infty})$  satisfying the following.

- 1. The singular set  $\Sigma_{\infty}$  of the metric space  $(X_{\infty}, d_{\infty})$  is a closed set of Hausdorff dimension no greater than 2n 4.
- 2.  $(X_i, g_i, \mathcal{V}_i)$  converges smoothly to a Kähler-Ricci soliton  $(X_{\infty} \setminus \Sigma_{\infty}, g_{\infty}, \mathcal{V}_{\infty})$ satisfying

$$Ric(g_{\infty}) = g_{\infty} + L_{\mathcal{V}_{\infty}}g_{\infty}, \qquad (1.4)$$

where  $\mathcal{V}_{\infty}$  is a holomorphic vector field on  $X_{\infty} \setminus \Sigma_{\infty}$ .

(X<sub>∞</sub>, d<sub>∞</sub>) coincides with the metric completion of (X<sub>∞</sub> \ Σ<sub>∞</sub>, g<sub>∞</sub>) and it is a projective Q-Fano variety with log terminal singularities. The soliton Kähler metric g<sub>∞</sub> extends to a Kähler current on X<sub>∞</sub> with bounded local potential and V<sub>∞</sub> extends to a global holomorphic vector field on X<sub>∞</sub>.

The assumption on the bound of the Futaki invariant in [10] is used to obtain a uniform lower bound of Perelman's  $\mu$ -functional. We use the recent deep result of Birkar [1] in birational geometry and show that there exists  $\epsilon(n) > 0$  such that for any *n*-dimensional Fano manifold X, there exists a Kähler metric g with  $Ric(g) \ge \epsilon g$ . In particular, the  $\mu$ -functional for (X, g) is bounded below by a uniform constant that only depends on n. Then for any Kähler-Ricci soliton  $(X, g) \in \mathcal{KR}(n)$ , the  $\mu$ -functional for (X, g) is uniformly bounded below because the soliton metric is the limit of the Kähler-Ricci flow. The proof of Theorem 1.1 also implies a uniform bound for the scalar curvature and the Futaki invariant for all  $(X, g) \in \mathcal{KR}(n)$ .

**Corollary 1.1** There exist F = F(n), D = D(n) and K = K(n) > 0 such that for any  $(X, g, u) \in \mathcal{KR}(n)$ , the Futaki invariant, the diameter and scalar curvature R of (X, g) satisfy

$$\mathcal{F}_X \leq F, \ diam(X,g) \leq D, \ 0 < R \leq K.$$

We also derive some general compactness for compact or complete gradient shrinking solitons assuming a uniform lower bound of Perelman's  $\mu$ -functional (see Section 3). For any closed or complete gradient shrinking soliton (M, g, u), one can always normalize u such that  $\int_M e^{-u} dV_g = 1$ . We define  $\mathcal{RS}(n, A)$  to be the space of closed or complete shrinking gradient soliton (M, g, u) of real dimension  $n \geq 4$ satisfying

$$\mu(g) \ge -A. \tag{1.5}$$

Then for any  $A \ge 0$  and any sequence  $(M_j, g_j, u_j, p_j) \in \mathcal{RS}(n, A)$  with  $p_j$  being the minimal point of  $u_j$ , after passing to a subsequence, it converges in the pointed Gromov-Hausdorff topology to a compact or complete metric space  $(M_{\infty}, d_{\infty})$  of dimension n with smooth convergence to a shrinking gradient Ricci soliton outside the closed singular set of dimension no greater than n - 4.

## 2 Proof of Theorem 1.1

Let us first recall the  $\alpha$ -invariant introduced by Tian on a Fano manifold [14].

**Definition 2.1** On a Fano manifold  $(X, \omega)$  with  $\omega \in c_1(X)$ , the  $\alpha$ -invariant is defined as

$$\alpha(X) = \sup\{\alpha > 0 \mid \exists C_{\alpha} < \infty \text{ such that } \int_{X} e^{-\alpha(\varphi - \sup_{X} \varphi)} \omega^{n} \le C_{\alpha}, \, \forall \varphi \in PSH(X, \omega)\}$$

It is obvious that the  $\alpha(X)$  does not depend on the choice  $\omega \in c_1(X)$ .

**Definition 2.2** Let X be a normal projective variety and  $\Delta$  an effective  $\mathbb{Q}$ -Cartier divisor, the pair  $(X, \Delta)$  is said to be log canonical if the coefficients of components

of  $\Delta$  are no greater than 1 and there exists a log resolution  $\pi : Y \to X$  such that  $\pi^{-1}(\operatorname{supp}\Delta) \cup \operatorname{exc}(\pi)$  is a divisor with normal crossings satisfying

$$K_Y = \pi^*(K_X + \Delta) + \sum_j a_j F_j, \quad \mathbb{Q} \ni a_j \ge -1, \forall j.$$

**Definition 2.3** Let X be a projective manifold and D be a  $\mathbb{Q}$ -Cartier divisor. The log canonical threshold of D is defined by

$$lct(X, D) = \sup\{t \in \mathbb{R} \mid (X, tD) \text{ is log canonical}\}.$$

It is proved by Demailly that the  $\alpha$ -invariant is related to the log canonical thresholds of anti-canonical divisors through the following formula (see Theorem A.3. in the Appendix A of [5]).

**Theorem 2.1** For any Fano manifold X,

$$\alpha(X) = \inf_{m \in \mathbb{Z}_{>0}} \inf_{D \in |-mK_X|} \operatorname{lct}(X, m^{-1}D)$$

Recently Birkar (Theorem 1.4 of [1]) obtains a uniform positive lower bound of the log canonical threshold and the following is an immediate corollary of Birkar's result.

**Theorem 2.2** There exists  $\varepsilon_0 = \varepsilon_0(n) > 0$  such that for any n-dimensional Fano manifold X

$$\alpha(X) \ge \varepsilon_0(n).$$

From the Harnack inequality in [14], for any fixed Kähler metric  $\omega \in c_1(X)$ , the curvature equation for  $\omega_t$  along the continuity method

$$Ric(\omega_t) = t(\omega_t) + (1-t)\omega \tag{2.1}$$

can be solved for all  $t \in [0, (n+1)\alpha(X)/n)$ . As a consequence, we have the following corollary.

**Corollary 2.1** There exists  $\varepsilon_1 = \varepsilon_1(n) > 0$  such that for any n-dimensional Fano manifold X, there exists a Kähler metric  $\hat{\omega} \in c_1(X)$  satisfying

$$Ric(\hat{\omega}) \ge \varepsilon_1 \hat{\omega}.$$
 (2.2)

We can also assume that  $\hat{\omega}$  is invariant under the group action of the maximal compact subgroup G of Aut(X) by choosing a G-invariant Kähler metric  $\omega$  in the equation (2.1).

The greatest Ricci lower bound R(X) for a Fano manifold X is introduced in [16, 12] and is defined by

$$\mathcal{R}(X) = \sup\{t \in \mathbb{R} \mid \exists \ \omega \in c_1(X) \text{ such that } Ric(\omega) \ge t\omega\}.$$

Immediately one has the following corollary.

**Corollary 2.2** There exists an  $r_0 = r_0(n) > 0$  such that for any n-dimensional Fano manifold X,

$$\mathcal{R}(X) \ge r_0. \tag{2.3}$$

We are informed by Xiaowei Wang that Corollary 2.2 is already a consequence of results in [8]. In fact, Theorem 5.2 and Proposition 5.1 in [8] will imply that there exist m = m(n) > 0 and  $\beta = \beta(n) \in (0, 1]$  such that for any *n*-dimensional Fano manifold X, there exists a smooth divisor  $D \in |-mK_X|$  and a conical Kähler-Einstein metric  $\omega \in c_1(X)$  satisfying

$$Ric(\omega) = \beta\omega + (1-\beta)m^{-1}[D].$$

Then Corollary 2.2 immediately follows by the relation between  $\mathcal{R}(X)$  and the existence of conical Kähler-Einstein metric established in [11].

Now let us recall Perelman's entropy functional for a Fano manifold (X, g) with the associated Kähler form  $\omega_g \in c_1(X)$ . The  $\mathcal{W}$ -functional is defined by

$$\mathcal{W}(g,f) = \frac{1}{V} \int_X (R + |\nabla f|^2 + f - 2n)e^{-f}dV_g,$$

where  $V = c_1^n(X)$ , and the  $\mu$ -functional is defined by

$$\mu(g) = \inf_{f} \left\{ \mathcal{W}(g, f) \mid \frac{1}{V} \int_{X} e^{-f} dV_g = 1 \right\}.$$

**Lemma 2.1** There exists A = A(n) > 0 such that for the Riemannian metric  $\hat{g}$  associated to the form  $\hat{\omega}$  in (2.2)

$$\mu(\hat{g}) \ge -A.$$

**Proof** Since  $Ric(\hat{g})$  is bounded from below by a uniform positive constant  $\varepsilon_1(n)$ , by Myers' theorem and volume comparison,

$$\operatorname{Vol}(X, \hat{g}) \le C(n), \quad \operatorname{diam}(X, \hat{g}) \le C(n).$$

On the other hand, since  $\hat{\omega} \in c_1(X)$  is in an integral cohomology class, in particular  $\operatorname{Vol}(X, \hat{g}) \geq c(n) > 0$ . By Croke's theorem, the Sobolev constant  $C_S$  of  $(X, \hat{g})$  is uniformly bounded. It is well-known that a Sobolev inequality implies the lower bound of  $\mu$ -functional. For completeness, we provide a proof below.

For any  $f \in C^{\infty}$  with  $\int_X e^{-f} dV_{\hat{g}} = V$ , we write  $e^{-f/2} = \phi$ . By Jensen's inequality

$$\frac{1}{V} \int_X \phi^2 \log \phi^{\frac{2}{n-1}} \le \log \left( \frac{1}{V} \int_X \phi^{\frac{2n}{n-1}} \right)$$
$$\le \log \left( C_S \int_X (|\nabla \phi|^2 + \phi^2) \right)$$
$$\le \frac{4}{n-1} \int_X |\nabla \phi|^2 + C(n).$$

So

$$\mathcal{W}(\hat{g}, f) = \frac{1}{V} \int_X (R\phi^2 + 4|\nabla\phi|^2 - \phi^2 \log \phi^2) dV_{\hat{g}} - 2n \ge -C(n).$$

Let  $(X, g) \in \mathcal{KR}(n)$  be a gradient shrinking Kähler-Ricci soliton which satisfies the equation

$$Ric(\omega_g) + \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} u = \omega_g, \quad \nabla \nabla u = 0.$$
(2.4)

Let  $G \subset Aut(X)$  be the compact one-parameter subgroup generated by the holomorphic vector field  $\operatorname{Im}(\nabla u)$ . As we mentioned before, the metric  $\hat{\omega}$  in (2.2) can be taken to be *G*-invariant.

**Corollary 2.3** For any  $(X,g) \in \mathcal{KR}(n)$ , we have

 $\mu(g) \ge -A,$ 

where A = A(n) is the constant in Lemma 2.1.

**Proof** We consider the normalized Kähler-Ricci flow with initial metric  $\hat{\omega}$  in (2.2) *G*-invariant.

$$\frac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) + \omega(t), \quad \omega(0) = \hat{\omega}.$$

By the convergence theorem for Kähler-Ricci flow ([17, 19]),  $\omega(t)$  converges smoothly to  $\omega_g$ , modulo some diffeomorphisms. So  $\lim_{t\to\infty} \mu(g(t)) = \mu(g)$ .

On the other hand,  $\mu(g(t))$  is monotonically non-decreasing along the Kähler-Ricci flow ([9]). The lower bound of  $\mu(g)$  follows from this monotonicity and the lower bound of  $\mu(\hat{g})$  established in Lemma 2.1.

Now we can apply the same argument as in [10] because the assumption of the uniform bound for the Futaki invariant in [10] is to obtain a uniform lower bound for the  $\mu$ -functional. This will complete the proof of Theorem 1.1. The argument in [10] also implies the uniform bound for the scalar curvature and diameter of  $(X, g, u) \in \mathcal{KR}(n)$  as well as  $|\nabla u|^2$  and hence the Futaki invariant of (X, g). This implies Corollary 1.1.

# 3 Generalizations

We generalize our previous discussion to Riemannian complete gradient shrinking Ricci solitons  $(M^n, g, u)$  satisfying the equation

$$Ric(g) + \nabla^2 u = \frac{1}{2}g.$$

By [3] we can always normalize u such that  $\int_M e^{-u} dV_g = 1$ .

**Definition 3.1** We denote  $\mathcal{RS}(n, A)$  to the set of n-dimensional closed or complete shrinking gradient Ricci solitons (M, g, u) satisfying

$$\mu(g) \ge -A$$

with the normalization condition  $\int_M e^{-u} dV_g = 1$ .

The following proposition is the main result of this section and most results in the proposition are straightforward applications of the compactness results [20, 21] with Bakry-Emery Ricci curvature bounded below.

**Proposition 3.1** Let  $\{(M_i, g_i, u_i, p_i)\}_{i=1}^{\infty}$  be a sequence in  $\mathcal{RS}(n, A)$  with  $n \geq 4$ , where  $p_i$  be a minimal point of  $u_i$ . Then after possibly passing to subsequence,  $(M_i, g_i, u_i, p_i)$  converges in the Gromov-Hausdorff topology to a metric length space  $(M_{\infty}, d_{\infty}, u_{\infty})$  satisfying the following.

- 1. The singular set  $\Sigma_{\infty}$  of the metric space  $(M_{\infty}, d_{\infty})$  is a closed set of Hausdorff dimension no greater than n 4.
- 2.  $(M_i, g_i, u_i)$  converges smoothly to a gradient shrinking Ricci soliton  $(M_{\infty} \setminus \Sigma_{\infty}, g_{\infty}, u_{\infty})$ satisfying

$$Ric(g_{\infty}) = \frac{1}{2}g_{\infty} + \nabla^2 u_{\infty}.$$

3.  $(M_{\infty}, d_{\infty})$  coincides with the metric completion of  $(M_{\infty} \setminus \Sigma_{\infty}, g_{\infty})$ .

Furthermore, if there exists V > 0 such that  $Vol_{g_i}(M_i) \leq V$  for all i = 1, 2, ..., the limiting metric space  $(M_{\infty}, d_{\infty})$  is compact.

**Proof** For any  $(M, g, u) \in \mathcal{RS}(n, A)$ ,  $R = n/2 - \Delta u \ge 0$  ([23]), the potential function u satisfies

$$\Delta u - |\nabla u|^2 + u = a, \quad a = \int_M u e^{-u} dV_g.$$

We denote  $\tilde{u} = u - a$ . From  $\Delta u \leq n/2$  and immediately we have  $|\nabla \tilde{u}|^2 \leq n/2 + \tilde{u}$ . By [3], the minimum of  $\tilde{u}$  is achieved at some finite point  $p \in M$ , so  $\min \tilde{u} = \tilde{u}(p) \geq -n/2$ . Applying maximum principle to  $\tilde{u}$  which satisfies  $\Delta \tilde{u} - |\nabla \tilde{u}|^2 + \tilde{u} = 0$  at a minimum point  $p \in M$ , we obtain that  $\min_M \tilde{u} = \tilde{u}(p) \leq 0$ .

From  $|\nabla \tilde{u}|^2 \leq \tilde{u} + n/2$ , we have  $|\nabla \sqrt{\tilde{u} + n/2}| \leq \frac{1}{2}$ . Thus for any  $x \in M$ 

$$\tilde{u}(x) \le \frac{1}{2}d(p,x)^2 + \tilde{u}(p) + C(n) \le \frac{1}{2}d(p,x)^2 + C(n).$$
 (3.1)

Immediately we have

$$|\nabla \tilde{u}|^2(x) \le \frac{1}{2}d(p,x)^2 + C(n),$$
(3.2)

and

$$-n/2 \le -\Delta \tilde{u}(x) \le \frac{1}{2}d(p,x)^2 + C(n).$$
 (3.3)

When (M,g) is closed and  $\operatorname{Vol}(M,g) \leq V$ . We note by Jensen's inequality  $a \leq \log V$ . The Ricci soliton (M,g,u) gives rise to a Ricci flow  $g(t) = \varphi_t^* g$  with initial metric g(0) = g, where  $\varphi_t$  is the diffeomorphism group generated by  $\nabla u$ ,  $\frac{\partial g(t)}{\partial t} = -2Ric(g(t)) + g(t)$ . Combining with the fact that  $R(g) \geq 0$  and Perelman's non-collapsing theorem, we see that (M,g) is non-collapsed in the sense that if  $R \leq r^{-2}$  on  $B_r(x)$ , then  $\operatorname{Vol}(B_r(x)) \geq \kappa(n,A)r^n$ , for all  $r \in (0, \overline{r}(n,A)]$ . With this non-collapsing and equations (3.1), (3.2) and (3.3), we can apply the same argument of Perelman as in Section 3 of [13] to show that there exists a uniform constant C(n, A, V) > 0 such that for any closed  $(M, g, u) \in \mathcal{RS}(n, A)$  with the additional assumption  $\operatorname{Vol}(M,g) \leq V$ ,

$$||u||_{L^{\infty}} + ||\nabla u||_{L^{\infty}(M,g)} + ||R||_{L^{\infty}} + \operatorname{diam}(M,g) \le C(n,A,V).$$
(3.4)

The non-collapsing of (M, g) also implies a uniform lower bound on Vol(M, g). Now we can apply the main theorem of [22].

In general, when (M, g) is complete, applying [9] to the Ricci flow associated to (M, q), there exists a  $\kappa = \kappa(A, n)$  such that (M, q) is  $\kappa$ -noncollapsed. In particular,  $Vol(B(p,1)) \ge c(A,n) > 0$ . On any geodesic ball B(p,r) with p being the minimal point of  $u, |\nabla u| \leq \frac{1}{2}r^2 + C(n, A)$ . By the Cheeger-Colding theory for Bakry-Emery Ricci tensor  $Ric(g) + \nabla^2 u$  ([20, 21]), for any sequence of  $(M_i, g_i, u_i, p_i) \in \mathcal{RS}(n, A)$ converges (up to a subsequence) in pointed Gromov-Hausdorff topology to a metric space  $(M_{\infty}, d_{\infty}, p_{\infty})$ . Here we choose  $p_i$  to be a minimum point of  $u_i$ .  $M_{\infty}$  has the regular-singular decomposition  $M_{\infty} = \mathcal{R} \cup \Sigma$ . Recall a point  $y \in \mathcal{R}$  if all tangent cone of  $(M_{\infty}, d_{\infty})$  at y is isometric to  $\mathbb{R}^n$ . From [21] we know the singular set  $\Sigma$  is closed and of Hausdorff dimension at most n-4 and  $d_{\infty}$  on  $\mathcal{R}$  is induced by a  $C^{\alpha}$ metric  $g_{\infty}$ . For any  $y \in \mathcal{R}$  and  $M_i \ni y_i \xrightarrow{GH} y$ , when *i* is large enough there exists a uniform  $r_0 = r_0(y)$  such that  $(B_{q_i}(y_i, r_0), g_i)$  has uniform  $C^{\alpha}$  bound (Theorem 1.2) of [21]). By choosing  $r_0$  even smaller if possible, we may assume the isoperimetric constant of  $(B_{g_i}(y_i, r_0), g_i)$  is very small so that we can apply Perelman's pseudolocality theorem ([9]) to the associated Ricci flow to derive uniform higher order estimates of  $g_i$  nearby  $y_i$ , which in turn gives local estimates of  $u_i$ . So locally near  $y_i$ , the convergence is smooth and we conclude that the metric  $g_{\infty}$  in a small ball around y is a Ricci soliton.

We remark that in the compact case, a compactness result is obtained earlier by Zhang [22] assuming a uniform upper bound for the diameter and a uniform lower bound for the volume. Acknowledgements: The authors would like to thank Xiaowei Wang for valuable discussions and for teaching us his proof of Corollary 2.2 in his work [8] with Li and Xu.

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