New methods to find patches of invisible integer lattice points

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"Mathematics is the queen of the sciences and number theory is the queen of mathematics." – Carl Friedrich Gauss (1777-1855)

Abstract

It is a surprising fact that the proportion of integer lattice points visible from the origin is exactly $\frac{6}{\pi^2}$, or approximately 60 percent. Hence, approximately 40 percent of the integer lattice is hidden from the origin. Since 1971, many have studied a variety of problems involving lattice point visibility, in particular, searching for patterns in that 40 percent of the lattice comprised of invisible points. One such pattern is a square patch, an $n \times n$ grid of n^2 invisible points, which we call a hidden forest. It is known that there exist arbitrarily large hidden forests in the integer lattice. However, the methods up to now involve the Chinese Remainder Theorem (CRT) on the rows and columns of matrices with prime number entries, and they have only been able to locate hidden forests very far from the origin. For example, using this method the closest known 4×4 hidden forest is over 3 quintillion, or 3×10^{18} , units away from the origin. We introduce the concept of quasiprime matrices and utilize a variety of computational and theoretical techniques to find some of the closest known hidden forests to this date. Using these new techniques, we find a 4×4 hidden forest that is merely 184 million units away from the origin. We conjecture that every hidden forest can be found via the CRT-algorithm on a quasiprime matrix.

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Contents

1 Introduction

Imagine the plane \mathbb{R}^2 as a forest in which each non-origin lattice point in \mathbb{Z}^2 is a tree and each tree is infinitely thin yet also opaque. In this scenario, we say that a tree is hidden if some other tree lies in your line of sight from the origin.

Consider the four lines of sight denoted by the dashed line segments emanating from the origin in Figure [1.1a.](#page-2-0) In these four lines of sight in the first quadrant, exactly four trees are visible—one per each line of sight. These visible trees are located at the black bullet points. Obscured by them are three other trees at the white bullet points, which are not visible from the origin. The tree at $(2, 6)$ is obscured by the visible tree at $(1, 3)$, while the tree at $(6, 3)$ is obscured by the tree at $(4, 2)$, which in turn is obscured by the visible tree at $(2, 1)$. The question of the visibility (or invisibility) of a lattice point from the origin can be recast in a number-theoretic setting, where it turns out that the only visible points are the points (x, y) such that $gcd(x, y) = 1$. A proof for this visibility criterion is given in Proposition [2.1.](#page-3-1)

It is well known that approximately 60% of the integer lattice is visible from the origin (see Proposition [2.4\)](#page-4-0). So a natural question to ask about the approximately 40% of the integer lattice which is hidden from view is the following:

Are there arbitrarily large square patches of invisible lattice points?

Figure 1.1

The answer to this question is yes, and in this paper we focus on invisible $n \times n$ square patches which we call *hidden forests*. An example of a 2×2 hidden forest is given in Figure [1.1b.](#page-2-0) In this figure we note the specific four visible trees that obscure this hidden forest.

Lattice point visibility is a well-studied subject that arises in a variety of areas such as number theory, integer optimization, and even theoretical physics (see Ch.10.4 of [\[2\]](#page-27-0) for a brief survey). In 1971, Herzog and Stewart studied patterns of both visible and invisible lattice points; one such invisible pattern they explored is the one we call a hidden forest [\[9\]](#page-27-1). In 1990, Schumer also examined hidden forests [\[13\]](#page-27-2). He used the Chinese Remainder Theorem (in a form similar to our Theorem [3.4\)](#page-6-1) and gave an example of a 3×3 hidden forest very far from the origin and questioned whether a closer one could be found. He then noted that finding a 4×4 forest would require solving systems of linear congruence equations modulo the product of the first 16 primes, the so-called 16th *primorial* which is approximately 32 quintillion, and declared, "Such a project is beyond the courage of this author!" In this paper we not only undertake this task of finding closer 4×4 hidden forests, but also introduce a variety of theoretical and computational techniques to aid in finding the closest known $n \times n$ hidden forests for $n \geq 4$, a task which has not yet been done to this date. The paper is broken down as follows:

- Section [2:](#page-3-0) We give a brief overview of lattice point visibility and provide a detailed proof of the well-known result that the probability of two randomly selected integers being relatively prime is $\frac{6}{\pi^2}$.
- Section [3:](#page-5-0) We give the known method of finding hidden forests in Subsection [3.1.](#page-6-0) Given $n \in \mathbb{N}$ and a prime matrix P_n , there exists an $n \times n$ hidden forest $H_{(x,y)}^n$ in the first quadrant with bottom-left corner (x, y) that is found by applying the Chinese Remainder Theorem to the rows and columns of P_n . We denote this process by the term CRT-algorithm. In Subsec-tion [3.2,](#page-7-0) we elaborate on the relationship between a prime matrix P_n and its hidden forest $H_{(x,y)}^n$ with the introduction of an object called a *gcd-matrix*. In Subsection [3.3,](#page-9-0) we apply this

method to find hidden forests $H_{(x,y)}^n$ for $n = 2, 3, 4$.

- Section [4:](#page-11-0) We introduce the concept of a *quasiprime matrix* QP_M and the QP-algorithm in Subsection [4.1](#page-12-0) and define the method of *strings of strongly composite integers* in Subsection [4.2.](#page-15-0) In Subsection [4.3,](#page-18-0) we explore the notion of an *optimal gcd-matrix* by considering the minimal number of prime factors required in a quasiprime matrix to produce an $n \times n$ hidden forest. With these three tools and some computational programming techniques, we then use the CRT-algorithm on quasiprime matrices to find $n \times n$ hidden forests. The resulting hidden forests turn out to be much closer to the origin than those found by the traditional method (given in Section [3\)](#page-5-0).
- Section [5:](#page-21-0) We combine the techniques detailed in the previous section to find the closest known (to this date) 5×5 hidden forest.
- Section [6:](#page-24-0) We give a selection of open problems. We also briefly review some recent research between second author Mbirika and collaborators Goins, Harris, and Kubik, generalizing this classic setting of straight lines of sights to curved lines of sights [\[8\]](#page-27-3).

Acknowledgments

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2 Density of visible lattice points in \mathbb{Z}^2

As mentioned in the introduction, a criterion for the visibility of an integer lattice point can be recast in the number-theoretic setting as the following proposition shows.

Proposition 2.1. *Let* $(x, y) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ *. Then* (x, y) *is visible if and only if* $gcd(x, y) = 1$ *.*

Proof. Let (x, y) be a non-origin point in \mathbb{Z}^2 . Suppose $d = \gcd(x, y)$. If $d > 1$ then $\left(\frac{x}{d}\right)$ $\frac{x}{d}, \frac{y}{d}$ *𝑑* $\sqrt{ }$ lies strictly between the points $(0, 0)$ and (x, y) , and hence (x, y) is not visible. Thus (x, y) visible implies that $gcd(x, y) = 1$.

Conversely, assume that $d = 1$ and suppose by way of contradiction that (x, y) is not visible from the origin. Then there is a point $(x_0, y_0) \in \mathbb{Z}^2$ such that $(x, y) = (cx_0, cy_0)$ for some integer $c > 1$. That is, *c* divides both *x* and *y*. But $d = 1$ is the greatest common divisor of *x* and *y*, contradicting that $c > 1$. Thus if $gcd(x, y) = 1$ then (x, y) is visible. \Box

Now that we have a simple criterion for an integer lattice point's visibility, it is natural to inquire what fraction of integer lattice points are visible from the origin. That is, we ask:

What is the density of visible lattice points in \mathbb{Z}^2 ?

Let $T(n)$ equal the total number of integer lattice points in an $n \times n$ square centered at the origin, and let $V(n)$ equal the number of these points visible from the origin. Then it suffices to compute the limit of $\frac{V(n)}{T(n)}$ as *n* approaches infinity. It turns out this limit is $\frac{6}{\pi^2}$. Proofs of this famous result are well known with the earliest proofs given in the late 19th century (see references in Remark [2.2\)](#page-4-1). Many modern solutions involve the Möbius inversion formula and Euler's totient function. In Proposition [2.4,](#page-4-0) we provide an alternative proof that is essentially an application of Euler's famous product formula and utilizes the number-theoretic criterion for the visibility of a lattice point given in Proposition [2.1.](#page-3-1)

Remark 2.2 (Historical background to the problem)**.** The historical record of the original authorship of the result in Proposition [2.4](#page-4-0) is inaccurately described on a number of occasions in the literature. Originally, the question on the probability of two random integers being coprime was raised in 1881 by Cesàro [\[3\]](#page-27-4). Two years later, he and Sylvester independently proved the result [\[4\]](#page-27-5) and [\[15\]](#page-28-0), respectively. Earlier in 1849, Dirichlet proved a slightly weaker form of the result [\[7\]](#page-27-6). The generalization to *k* coprime integers with $k > 2$ was presented again by Cesàro in 1884 [\[5\]](#page-27-7). This result was apparently proven independently in 1900 by Lehmer [\[10\]](#page-27-8).

Remark 2.3. Since there is no uniform distribution on the natural numbers, it is somewhat imprecise to speak about the probability that two integers chosen at random are relatively prime. However, if we consider the uniform distribution on the set $\{1, 2, \ldots, n\}$ and take the limit as *n* approaches infinity, then it is within this context that we make any probability statements in Proposition [2.4.](#page-4-0)

Proposition 2.4. The density of integer lattice points that are visible from the origin is $\frac{6}{\pi^2}$, or *approximately* 60%*.*

Proof. It suffices to show that a lattice point chosen at random has a probability of $\frac{6}{\pi^2}$ of being visible from the origin. Let *x* and *y* be randomly selected integers. Recall that (x, y) is visible if and only if $gcd(x, y) = 1$ by Proposition [2.1.](#page-3-1) Hence it suffices to compute the probability that no prime divides both *x* and *y*. The probability that *x* is divisible by the prime *p* is $\frac{1}{p}$. Similarly *y* is divisible by *p* with probability $\frac{1}{p}$. By mutual independence, the probability that both *x* and *y* are divisible by *p* is $\frac{1}{p^2}$. Hence, the probability that both integers *x* and *y* are not divisible by *p* is $1 - \frac{1}{p^2}$. For distinct primes, these divisibility events are mutually independent, thus the probability that no prime divides both x and y is the following product over the primes:

$$
\prod_{p}\left(1-\frac{1}{p^2}\right).
$$

To calculate this infinite product, it is helpful to consider the Riemann zeta function

$$
\zeta(s) = \sum_{n \ge 1} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots
$$

for $s > 1$. A result of Euler connects this infinite sum with an infinite product of infinite sums over the primes. The essence of Euler's proof is his use of the fundamental theorem of arithmetic to observe that the sum $\zeta(s)$ can be written as the following infinite product

$$
\sum_{n\geq 1} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \right).
$$
 (1)

To prove Equation [\(1\)](#page-5-1), Euler observed that since each *n* in the denominator on the left-hand side is of the form $n = p_i^{\alpha_i}$ $\frac{\alpha_{i_1}}{a_1} p_{i_2}^{\alpha_{i_2}}$ $\frac{\alpha_{i_2}}{\alpha_2} \cdots \frac{\alpha_{i_k}}{\alpha_k}$ $\frac{a_{i_k}}{b_k}$ for some *k* by the fundamental theorem of arithmetic, then by multiplying out the product on the right-hand side, each term $\frac{1}{l^s}$ on the left-hand side appears exactly once, as a product of the appropriate powers of the primes in *n*. And since each multiplicand on the right-hand side is a geometric series of the form $\frac{1}{\sqrt{1-\frac{1}{n}}}$ $1 - \frac{1}{p^s}$, Equation [\(1\)](#page-5-1) becomes

$$
\sum_{n\geq 1} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}.
$$

Setting $s = 2$ and taking reciprocals, we get

$$
\zeta(2)^{-1} = \frac{1}{\sum_{n\geq 1} \frac{1}{n^2}} = \prod_{p} \left(1 - \frac{1}{p^2} \right),
$$

where the right-hand side is the probability value we seek, and the left-hand side is the reciprocal of the well-known evaluation of the Riemann zeta function at $s = 2$, namely $\zeta(2) = \frac{\pi^2}{6}$ $\frac{t^2}{6}$ (the solution to the famous *Basel Problem*^{[1](#page-5-2)}). Hence the fraction of lattice points (x, y) visible from the origin is 6 $\frac{6}{\pi^2}$ (or approximately 60%), as desired. \Box

3 The traditional method to find hidden forests

In the previous section we showed that approximately 60% of the integer lattice is visible, and hence approximately 40% lies hidden from view. In this section, we find arbitrarily large patches of hidden square regions in \mathbb{Z}^2 using the known method. This technique is what we call the CRT-algorithm, since it is mainly an application of the Chinese Remainder Theorem (CRT). The strategy is to find two sets of *n* consecutive integers

$$
\mathbf{X} = \{x_1, x_2, \dots, x_n\} \text{ and } \mathbf{Y} = \{y_1, y_2, \dots, y_n\}
$$

such that $\mathbf{X} \cap \mathbf{Y} = \emptyset$ and $\gcd(x_i, y_j) > 1$ for all $1 \le i, j \le n$. Then it is clear that the n^2 points in the set $\{(x_i, y_j) \mid 1 \le i, j \le n\}$ yield the desired hidden square region. To this end, we first establish some necessary preliminary definitions.

¹The Basel Problem asks for the exact sum of the reciprocals of the squares of the positive integers. There are a variety of proofs of this result. Chapman in 2003 gives the details of 14 different proofs [\[6\]](#page-27-9). More recently in 2015, Moreno compiles a comprehensive list of 85 references from Euler to the present that address the Basel Problem [\[11\]](#page-27-10).

Definition 3.1 (Hidden forest). An $n \times n$ *hidden forest* in \mathbb{Z}^2 is a square patch of n^2 adjacent invisible integer lattice points. We denote this hidden forest by the symbol $H_{(x,y)}^n$ where (x, y) is the closest corner lattice point of the square to the origin. By the remark below, this closest corner point is well-defined.

Remark 3.2. Observe that the points $(x, \pm 1)$ and $(\pm 1, y)$ are visible for all $x, y \in \mathbb{Z}$ by Proposi-tion [2.1.](#page-3-1) Hence no nontrivial (that is, $n > 1$) hidden forest $H_{(x,y)}^n$ will contain any points on the *x*or *y*-axes. Hence we conclude that any $H_{(x,y)}^n$ for $n > 1$ is completely contained in the interior of one of the four quadrants.

Definition 3.3 (Prime matrix). Let $\{p_1, p_2, \ldots, p_{n^2}\}\$ be the set of the first n^2 primes. Construct an *n* × *n* matrix with these primes by filling row *i* with $p_{(i-1)n+1}$ through $p_{(i-1)n+n}$ for each $1 \le i \le n$ to yield the following:

$$
\begin{bmatrix}\np_1 & p_2 & \cdots & p_j & \cdots & p_n \\
p_{n+1} & p_{n+2} & \cdots & p_{n+j} & \cdots & p_{2n} \\
p_{2n+1} & p_{2n+2} & \cdots & p_{2n+j} & \cdots & p_{3n} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
p_{(i-1)n+1} & p_{(i-1)n+2} & \cdots & p_{(i-1)n+j} & \cdots & p_{(i-1)n+n} \\
\vdots & \vdots & & \vdots & & \vdots \\
p_{(n-1)n+1} & p_{(n-1)n+2} & \cdots & p_{(n-1)n+j} & \cdots & p_{n^2}\n\end{bmatrix}
$$

.

Note that the prime $p_{(i-1)n+j}$, boxed for visual ease, is located in row *i* and column *j* of the matrix. We call this $n \times n$ matrix a *prime matrix* and denote it P_n .

3.1 The CRT**-algorithm**

The following theorem is the primary tool used in the CRT-algorithm to find hidden forests of arbitrary size.

Theorem 3.4. For each $n \in \mathbb{N}$, there exist two sets of *n* consecutive numbers $\mathbf{X} = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$ such that $X \cap Y = \emptyset$ and $gcd(x_i, y_j) > 1$ for all $1 \le i, j \le n$.

Proof. Fix $n \in \mathbb{N}$. Consider the prime matrix P_n . Let R_i and C_j be the product of the entries in row *i* and column *j*, respectively, so we have

$$
R_i = \prod_{k=1}^n p_{(i-1)n+k} \text{ and } C_j = \prod_{k=0}^{n-1} p_{kn+j}.
$$

Since they share no primes in common, the row products R_1, R_2, \ldots, R_n are pairwise relatively prime. Similarly, the column products C_1, C_2, \ldots, C_n are pairwise relatively prime. Consider the following pair of systems of linear congruences:

$$
\begin{cases}\nx + 1 \equiv 0 \pmod{R_1} \\
x + 2 \equiv 0 \pmod{R_2} \\
\vdots \\
x + n \equiv 0 \pmod{R_n}\n\end{cases}\n\qquad\n\begin{cases}\ny + 1 \equiv 0 \pmod{C_1} \\
y + 2 \equiv 0 \pmod{C_2} \\
\vdots \\
y + n \equiv 0 \pmod{C_n}.\n\end{cases}
$$

Observe that $R_1 \cdot R_2 \cdots R_n = C_1 \cdot C_2 \cdots C_n = \prod_{i=1}^{n^2} p_i$, which we denote *M*. By the Chinese Remainder Theorem, there exist solutions x_0 and y_0 to the left and right systems, respectively, such that x_0 and y_0 are unique modulo M. Define the set $\mathbf{X} = \{x_0 + 1, x_0 + 2, \dots, x_0 + n\}$ and the set $Y = \{y_0 + 1, y_0 + 2, \ldots, y_0 + n\}$. We claim that none of the integers in **X** are pairwise relatively prime to any of the integers in **Y**. For an arbitrary $x_0 + i \in \mathbf{X}$ and $y_0 + j \in \mathbf{Y}$, these two elements by construction are multiples of R_i and C_j , respectively. Hence the prime that lies in the intersection of row *i* and column *j* in the matrix, namely $p_{(i-1)n+j}$, divides $gcd(x_0 + i, y_0 + j)$. Thus $gcd(x_0 + i, y_0 + j) > 1$ as desired.

Observe that for $n \geq 2$ the sets **X** and **Y** are necessarily disjoint. Otherwise if $X \cap Y \neq \emptyset$ then some element $a \in \mathbf{X}$ is relatively prime to some element $a \pm 1 \in \mathbf{Y}$ since $gcd(a, a \pm 1) = 1$, contradicting $gcd(x_0 + i, y_0 + j) > 1$ for all $1 \le i, j \le n$. For the trivial case when $n = 1$, the algorithm above yields $X = Y = \{2\}$. So set $Y = \{4\}$ and hence $X \cap Y = \emptyset$. \Box

The CRT-algorithm to construct a hidden forest $H_{(x,y)}^n$:

- 1. Fix a value $n \in \mathbb{N}$.
- 2. Construct the prime matrix P_n .
- 3. Apply Theorem [3.4](#page-6-1) to P_n to yield sets **X** and **Y**.
- 4. Construct the hidden forest $H_{(x,y)}^n$ from **X** and **Y**.

3.2 The gcd-grid and gcd-matrix yielded by prime matrices

By Theorem [3.4,](#page-6-1) the prime matrix P_n yields the hidden forest $H_{(x,y)}^n$ comprised of the n^2 points (x_i, y_j) where $x_i = x_0 + i \in \mathbf{X}$ and $y_j = y_0 + j \in \mathbf{Y}$ for $1 \le i, j \le n$. The forest $H_{(x,y)}^n$ is shown in Figure [3.1a.](#page-8-0) For each $H_{(x,y)}^n$ we can write a corresponding $n \times n$ array of numbers called the *gcd-grid* where $g_{i,j} = \gcd(x_i^{(x,y)})$ for $1 \le i, j \le n$. The gcd-grid is shown in Figure [3.1b.](#page-8-0)

Figure 3.1

We may also consider the gcd-grid as a matrix if we collapse the grid structure and place the n^2 gcd-values into a matrix in the same locations that they appear in the gcd-grid as follows:

$$
\text{Gcd}_{P_n} = \begin{pmatrix} g_{1,n} & g_{2,n} & \cdots & g_{i,n} & \cdots & g_{n,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ g_{1,j} & g_{2,j} & \cdots & g_{i,j} & \cdots & g_{n,j} \\ \vdots & \vdots & & \vdots & & \vdots \\ g_{1,2} & g_{2,2} & \cdots & g_{i,2} & \cdots & g_{n,2} \\ g_{1,1} & g_{2,1} & \cdots & g_{i,1} & \cdots & g_{n,1} \end{pmatrix}
$$

.

.

We call this matrix arising from the gcd-grid the $gcd-matrix$ corresponding to P_n and denote it Gcd_{P_n}. If we denote the prime $p_{(i-1)n+j}$ in row *i* and column *j* of matrix P_n as $p_{i,j}$, then the prime matrix given in Definition [3.3](#page-6-2) can be written as follows:

$$
P_n = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,j} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,j} & \cdots & p_{2,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ p_{i,1} & p_{i,2} & \cdots & p_{i,j} & \cdots & p_{i,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,j} & \cdots & p_{n,n} \end{pmatrix}
$$

Remark 3.5. The (i, j) -entry of P_n is $p_{i,j}$. However, the (i, j) -entry of Gcd_{P_n} is not $g_{i,j}$. In fact, the entry $g_{i,j}$ is in row $n - (j - 1)$ and column *i* of Gcd_{*P_n*}.

Comparing the locations of the entries $g_{i,j}$ and $p_{i,j}$ of the matrices Gcd_{P_n} and P_n , respectively, as the values *i* and *j* vary, we observe that the subscripts of the entries in one matrix are a rotation of the subscripts of the entries in the other. In particular, the following proposition describes the relationship between the matrices Gcd_{P_n} and P_n via a third matrix which we call \widetilde{Gcd}_{P_n} .

Proposition 3.6. Let P_n be a prime matrix. A rotation by 90° counter-clockwise of the entries in P_n gives a corresponding matrix which we denote by \widetilde{Gcd}_{P_n} , and the (i, j) -entry in \widetilde{Gcd}_{P_n} divides the (i, j) -entry in the gcd-matrix Gcd_{P_n} .

Proof. The rotational relationship between P_n and \widetilde{Gcd}_{P_n} is given by a simple matrix calculation. If we let AD_n be the anti-diagonal matrix—that is, a matrix with ones in the anti-diagonal and zeroes elsewhere, then $\widetilde{Gcd}_{P_n} = (P_n \cdot AD_n)^T$, where *T* denotes the transpose of a matrix. In particular, multiplying P_n on the right by AD_n reverses the columns of P_n , and then transposing this result yields \widetilde{Gcd}_{P_n} as desired. After this rotation on P_n is performed, the entry $p_{i,j}$ of \widetilde{Gcd}_{P_n} is then located in row $n - (j - 1)$ and column *i*. In this same location in Gcd_{*P_n*} is $g_{i,j}$. Below we give an illustration of this process.

$$
P_n = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,j} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,j} & \cdots & p_{2,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ p_{i,1} & p_{i,2} & \cdots & p_{i,j} & \cdots & p_{i,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,j} & \cdots & p_{n,n} \end{pmatrix} + \frac{O}{90^{\circ} \text{ left}} \n\begin{pmatrix} \sum_{j=1}^{\infty} p_{1,n} & p_{2,n} & \cdots & p_{i,n} & \cdots & p_{n,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ p_{1,j} & p_{2,j} & \cdots & p_{i,j} & \cdots & p_{n,j} \\ \vdots & \vdots & & \vdots & & \vdots \\ p_{1,2} & p_{2,2} & \cdots & p_{i,2} & \cdots & p_{n,2} \\ p_{1,1} & p_{2,1} & \cdots & p_{i,1} & \cdots & p_{n,1} \end{pmatrix}.
$$

In the proof of Theorem [3.4,](#page-6-1) we observed that by construction the prime $p_{i,j}$ divides the value $gcd(x_0 + i, y_0 + j) = g_{i,j}$. Hence, the (i, j) -entry in \widetilde{Gcd}_{P_n} divides the (i, j) -entry in \widetilde{Gcd}_{P_n} . \Box

Remark 3.7. The rotational relationship between P_n and \widetilde{Gcd}_{P_n} proves to be very important in Section [4](#page-11-0) when we perform the reverse rotation. Starting from a gcd-matrix, a *clockwise* rotation will help us produce a *quasiprime* matrix, crucial for finding closer hidden forests.

3.3 An application: the $n = 2, 3, 4$ cases

Example 3.8. In the 2×2 case, using Theorem [3.4,](#page-6-1) we set $n = 2$ and the prime matrix is

$$
P_2 = \left(\begin{array}{cc} 2 & 3 \\ 5 & 7 \end{array}\right).
$$

The row products are $R_1 = 6$ and $R_2 = 35$, while the column products are $C_1 = 10$ and $C_2 = 21$. Hence the corresponding linear congruences we need to solve are

$$
x + 1 \equiv 0 \pmod{6}
$$
 $y + 1 \equiv 0 \pmod{10}$
\n $x + 2 \equiv 0 \pmod{35}$ $y + 2 \equiv 0 \pmod{21}$.

By the CRT-algorithm, the left and right systems have the unique solutions $x_0 = 173 \pmod{210}$ and $y_0 = 19 \pmod{210}$, respectively. Set **X** = {174*,* 175} and **Y** = {20*,* 21}. Then **X** \cap **Y** = Ø and $gcd(x_i, y_j) > 1$ for all $1 \le i, j \le 2$. Thus there is a hidden forest $H^2_{(174, 20)}$ of four trees at (174*,* 20), (174*,* 21), (175*,* 20), and (175*,* 21). In the figure below we give the hidden forest on the left and its corresponding gcd-grid on the right.

(174,21)
\n(174,20)
\n(175,21)
\n(174,20)
\n(175,20)
\ngcd(174,20) = 2
\ngcd(174,20) = 2
\n
$$
5 = \gcd(175,20)
$$

Then by Proposition [3.6,](#page-9-1) we have the following map from P_2 to \widetilde{Gcd}_{P_2} :

$$
P_2 = \left(\begin{array}{cc} 2 & 3 \\ 5 & 7 \end{array}\right) \xrightarrow[\text{90° left}]{\bigcirc} \widetilde{\text{Gcd}}_{P_2} = \left(\begin{array}{cc} 3 & 7 \\ 2 & 5 \end{array}\right).
$$

In this example, the \widetilde{Gcd}_{P_2} coincides with the gcd-matrix Gcd_{P_2} , and so the (i, j) -entry of \widetilde{Gcd}_{P_2} divides the (*𝑖, 𝑗*)-entry of Gcd*^𝑃*² as Proposition [3.6](#page-9-1) guarantees. This is an effect of the so-called *law of small numbers*, since we see in the larger *n* cases to follow that this coincidence does not occur.

Example 3.9. In the 3×3 case, using Theorem [3.4,](#page-6-1) we set $n = 3$ and the prime matrix is

$$
P_3 = \left(\begin{array}{rrr} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{array}\right).
$$

The CRT-algorithm gives the solutions $x_0 = 119,740,619$ and $y_0 = 121,379,047$, both unique modulo 223,092,870. Hence the nine coordinates (x_i, y_j) of $H^3_{(119740620, 121379048)}$ have the following values and respective prime factorizations:

It is readily verified that the corresponding 3×3 hidden forest has the following gcd-grid:

For example, the top-right node corresponds to the (x_3, y_3) -coordinate, and it is labeled by the value gcd(x_3, y_3) = 2 ⋅ 23 since x_3 = 2 ⋅ 17 ⋅ 19 ⋅ 23 ⋅ 8059 and y_3 = 2 ⋅ 5² ⋅ 13 ⋅ 23² ⋅ 353. And by Proposition [3.6,](#page-9-1) we have the following map from P_3 to \widetilde{Gcd}_{P_3} :

$$
P_3 = \left(\begin{array}{ccc} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{array}\right) + \frac{5}{90^\circ \text{ left}} \quad \widetilde{\text{Gcd}}_{P_3} = \left(\begin{array}{ccc} 5 & 13 & 23 \\ 3 & 11 & 19 \\ 2 & 7 & 17 \end{array}\right).
$$

Observe that, as expected, the (i, j) -entry of \widetilde{Gcd}_P divides the (i, j) -entry of the gcd-matrix

$$
\text{Gcd}_{P_3} = \left(\begin{array}{ccc} 2.5 & 13 & 2.23 \\ 3 & 11 & 19 \\ 2^2 & 7 & 2.17 \end{array}\right).
$$

Example 3.10. In the 4×4 case, the CRT-algorithm on the prime matrix gives the following solution:

$$
x_0 = 2,847,617,195,518,191,809
$$

$$
y_0 = 1,160,906,121,308,397,397.
$$

The absurdly large solution values in Examples [3.9](#page-10-0) and [3.10](#page-11-1) reveal that the CRT-algorithm applied to prime matrices is hardly useful for finding $n \times n$ hidden forests which are close to the origin for cases even as small as $n = 3$ and $n = 4$. For instance, we prove later that the closest $H^3_{(x,y)}$ is at $x = 1274$ and $y = 1308$. Furthermore, we reveal that there is an $H^4_{(x,y)}$ at $x = 134,043$ and $y = 184,785,885$. The *x*-value of the $H^4_{(x,y)}$ which the CRT-algorithm on a prime matrix yields is 2.12441 \times 10¹³ times larger than this *x*-value, 134,043, of the $H^4_{(x,y)}$ which we found.

Remark 3.11. It turns out that the number 134,043 is a very interesting integer; it is the smallest positive integer *n* such that the numbers in the set $\{n, n+1, n+2, n+3\}$ have exactly four prime factors each.^{[2](#page-11-2)} We later use this value to calculate the closest known $H^4_{(x,y)}$, bearing the smallest *𝑥*-value, in Example [4.8.](#page-16-0)

4 New methods to find closer hidden forests

The previous section detailed the well-known method of using the CRT-algorithm on prime matrices to find arbitrarily large hidden forests. The main problem with that method is that for $n \geq 3$, the locations of these $H_{(x,y)}^n$ are substantially further away from the origin and thus progressively harder to compute. The aim of this section is to introduce two concepts, namely quasiprime matrices and strings of strongly composite integers, to help find substantially closer $H_{\alpha,\omega}^n$. In this section we give the closest $H_{(x,y)}^n$ for $n = 2, 3$ and the closest known (to this date) $H_{(x,y)}^n$ for $n = 4$.

Recall in Remark [3.2,](#page-6-3) we observed that $H_{(x,y)}^n$ can never contain points on the *x*- or *y*-axes when $n > 1$, and hence each $H_{(x,y)}^n$ lies completely within the interior of one of the four quadrants. Thus the closest corner point (x, y) of $H_{(x, y)}^n$ is well-defined up to quadrant selection.

Definition 4.1 (The closest $n \times n$). A hidden forest $H_{(x,y)}^n$ is said to have distance *d* from the origin where *d* is given by $d(x, y) = \sqrt{x^2 + y^2}$. We say that $H_{(x,y)}^n$ is the *closest* $n \times n$ *hidden forest* if it has the minimum distance d of all hidden $n \times n$ forests.

Convention 4.2. In searching for the closest hidden forest it suffices to search only half of Quadrant I. Observe that any $H_{(x,y)}^n$, whose lower-left corner lies in Quadrant I and above the line $y = x$, will have seven other copies up to reflectional symmetries about the lines $y = x$, $y = -x$, the *x*-axis, and the *y*-axis (see Figure [4.1\)](#page-12-1). Moreover, these seven copies are the same distance from the origin as $H_{(x,y)}^n$ is. So we focus only on $H_{(a,b)}^n$ in Quadrant I such that (a, b) lies above the diagonal $y = x$ (that is, $a < b$). Note that $\ddot{H}^n_{(a,a)}$ can never exist if $n > 1$ since $gcd(a, a + 1) = 1$ and hence $(a, a + 1)$ is a visible point.

²This is known as Problem 47 on the website <https://projecteuler.net/about> started in 2001 by Colin Hughes [\[14\]](#page-27-11). Project Euler gives a series of challenging computational problems that require more than just mathematical insights to solve. Nayuki Minase's very simple solution via Mathematica to Problem 47 is given in Listing [1.](#page-15-1)

Figure 4.1: Eight copies of the closest 2×2 hidden forest

4.1 Quasiprime matrices and the QP**-algorithm**

In Proposition [3.6,](#page-9-1) we begin with a prime matrix P_n and observe that a 90 \degree counter-clockwise rotation of P_n yields $\widetilde{\text{Gcd}}_{P_n}$ which relates very closely to the gcd-matrix Gcd_{P_n} of the corresponding $H_{(x,y)}^n$ (in particular, recall that the (i, j) -entry of \widetilde{Gcd}_{P_n} divides the (i, j) -entry of \widetilde{Gcd}_{P_n}). Now suppose instead that we start with an $H_{(x,y)}^n$ and its associated gcd-matrix, which we will denote Gcd_M. If we rotate this matrix 90° clockwise, then we get some matrix M that is not necessarily a prime matrix. Furthermore, applying the CRT-algorithm on *M* may not even be possible (see Example [4.5\)](#page-13-0). But from M , can we find a matrix M such that the (i, j) -entry of M divides the (i, j) -entry of *M* and applying the CRT-algorithm on \widetilde{M} gives the original $H_{(x,y)}^n$ from which we started? Based on much computational evidence, the answer appears to be yes. The matrix \vec{M} is what we call a quasiprime matrix QP_M to be defined in Definition [4.6,](#page-14-0) but a formal proof still awaits. For now, we proceed to give very substantial support that this conjecture holds for all $H_{(x,y)}^n$ (see Question [1\)](#page-24-2).

To find *M*, we use the matrix equality in Proposition [3.6](#page-9-1) and solve for *M* as follows:

$$
\begin{aligned} \text{Gcd}_M &= (M \cdot AD_n)^T \Rightarrow (\text{Gcd}_M)^T = M \cdot AD_n \\ &\Rightarrow M = (\text{Gcd}_M)^T \cdot AD_n, \end{aligned} \tag{2}
$$

where Equation [\(2\)](#page-12-2) follows since an anti-diagonal matrix with all ones in its nonzero entries is its

own inverse. Below we give an illustration of this process.

$$
\text{Gcd}_M = \begin{pmatrix} g_{1,n} & g_{2,n} & \cdots & g_{i,n} & \cdots & g_{n,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ g_{1,j} & g_{2,j} & \cdots & g_{i,j} & \cdots & g_{n,j} \\ \vdots & \vdots & & \vdots & & \vdots \\ g_{1,2} & g_{2,2} & \cdots & g_{i,2} & \cdots & g_{n,2} \\ g_{1,1} & g_{2,1} & \cdots & g_{i,1} & \cdots & g_{n,1} \end{pmatrix} \begin{pmatrix} \sum_{j=0 \text{ right}}^{N} g_{1,1} & g_{1,2} & \cdots & g_{1,j} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,j} & \cdots & g_{2,n} \\ \vdots & \vdots & & \vdots & & \vdots \\ g_{n,1} & g_{n,2} & \cdots & g_{n,j} & \cdots & g_{n,n} \end{pmatrix}.
$$

In the case of $n = 2$, we see that M is a prime matrix (see Example [4.3\)](#page-13-1). However in the case of $n = 3$, the matrix M can have repeated prime number entries and hence is not a prime matrix (see Example [4.5\)](#page-13-0). And in the case of $n \geq 4$, the matrix can have both repeated primes and composite number entries, and hence is not a prime matrix (see Example [4.8\)](#page-16-0). In these $n \geq 3$ cases, we construct a quasiprime version of M which we denote QP_M . And an application of the CRT-algorithm on QP_M yields the $H_{(x,y)}^n$ that has the original gcd-matrix corresponding to $H_{(x,y)}^n$. Before we present an algorithm on how to produce QP_M from M , we give two motivating examples in the $n = 2$ and $n = 3$ cases.

Example 4.3 (The closest 2×2). By examining a small grid of points in Quadrant I of \mathbb{Z}^2 , it is easy to notice that the closest $H^2_{(x,y)}$ occurs at $x = 14$ and $y = 20$. In the figure below, we give $H^2_{(14,20)}$ and to its right the gcd-grid corresponding to the four nodes.

By Equation [\(2\)](#page-12-2), we can retrieve a matrix M from the gcd-grid above as follows:

$$
\text{Gcd}_M = \begin{pmatrix} 7 & 3 \\ 2 & 5 \end{pmatrix} \xrightarrow[\text{90° right}]{\bullet} M = \begin{pmatrix} 2 & 7 \\ 5 & 3 \end{pmatrix}.
$$

Applying the CRT-algorithm to this matrix *M*, we get $x_0 = 13$ and $y_0 = 19$ as desired. Hence at the distance of $d \approx 24.4131$ we have the closest hidden forest $H^2_{(14,20)}$.

Remark 4.4. Alternate permutations of the same primes in the gcd-grid, and consequently the matrix *M*, may produce different solutions under the CRT-algorithm. This means that for each unique set of n^2 primes in the matrix P_n , the CRT-algorithm can yield up to n^2 factorial (not necessarily distinct) $H_{(x,y)}^n$. Compare the previous example with Example [3.8.](#page-9-2) Both examples use the same set of primes $\{2, 3, 5, 7\}$ but yield drastically different $H^2_{(x,y)}$.

Example 4.5 (The closest 3 \times 3). At the distance of *d* \approx 1825.91 we find the closest hidden forest $H^3_{(1274,1308)}$. Though others have cited $H^3_{(1274,1308)}$ as a hidden forest [\[9,](#page-27-1) [16\]](#page-28-1), none of these sources have asserted that it is the closest. For the $n = 3$ case, the problem of finding the closest hidden forest is computationally tractable via exhaustive means. In fact, we have written Java code^{[3](#page-13-2)} which

³See Appendix [A.1](#page-29-1) for the Java code.

exhaustively checked the square region with lower left endpoint (0*,* 0) and upper right endpoint (1308, 1308), finally confirming that this is the closest 3×3 hidden forest. Below we give the hidden forest $H^3_{(1274,1308)}$ and its corresponding gcd-grid:

By Equation (2) , we can retrieve a matrix M from the gcd-grid above as follows:

$$
\text{Gcd}_M = \begin{pmatrix} 2 & 5 & 2 \\ 7 & 17 & 11 \\ 2 & 3 & 2^2 \end{pmatrix} \xrightarrow[90^\circ \text{right}]{\bigcirc} M = \begin{pmatrix} 2 & 7 & 2 \\ 3 & 17 & 5 \\ 2^2 & 11 & 2 \end{pmatrix}.
$$

Incidentally, the *M* given here is very similar to the one given in 1971 by Herzog and Stewart [\[9\]](#page-27-1), but neither neither of these matrices can possibly produce the correct $H^3_{(x,y)}$ because the Chinese Remainder Theorem simply cannot work on such matrices. For example, since the products of row 1 and row 3 of M each have a factor of 4, then any solution x_0 to the three row equations would also have to satisfy $x + 1 \equiv 0 \pmod{4}$ and $x + 3 \equiv 0 \pmod{4}$, but the existence of such an x_0 is absurd. However, this problem is resolved by introducing the concept of a quasiprime matrix.

Definition 4.6. Given a matrix *M* arising from a Gcd_M via Equation [\(2\)](#page-12-2), we produce a *quasiprime* matrix QP_M defined by the QP-algorithm given below.

The QP-algorithm to construct a quasiprime matrix QP_M **:**

- 1. Construct matrix M arising from a Gcd_M via Equation [\(2\)](#page-12-2).
- 2. Let ${p_i}_{i=1}^s$ be the union of the sets of all primes appearing in the prime factorizations of each entry of M.
- 3. For a fixed p_i with $1 \le i \le s$, locate the entry in *M* which contains p_i^k $\frac{k}{i}$ for $k \ge 1$ such that *k* is largest. If there is more than one entry which contains p_i^k $\frac{k}{i}$, then choose exactly one.
- 4. Place the selected p_i^k $\frac{k}{i}$ in QP_M in the same location where it appears in *M*. Place the value 1 in QP_M in every location where a p_i^j ^{*l*}_{*i*} appears in *M* for each $j \leq k$.
- 5. Repeat the previous two steps for each p_i with $1 \leq i \leq s$.

Example 4.7 (The closest 3×3 via a quasiprime matrix). From the matrix *M* in Example [4.5,](#page-13-0) we can produce the quasiprime matrix as follows using the QP-algorithm:

$$
M = \begin{pmatrix} 2 & 7 & 2 \\ 3 & 17 & 5 \\ 2^2 & 11 & 2 \end{pmatrix} \xrightarrow[\text{algorithm}]{\text{QP}} \quad QP_M = \begin{pmatrix} 1 & 7 & 1 \\ 3 & 17 & 5 \\ 2^2 & 11 & 1 \end{pmatrix}.
$$

By use of the CRT-algorithm on QP_M , we solve the following system of linear congruences

$$
x + 1 \equiv 0 \pmod{7}
$$
 $y + 1 \equiv 0 \pmod{2^2 \cdot 3}$
\n $x + 2 \equiv 0 \pmod{3 \cdot 5 \cdot 17}$ $y + 2 \equiv 0 \pmod{7 \cdot 11 \cdot 17}$
\n $x + 3 \equiv 0 \pmod{2^2 \cdot 11}$ $y + 3 \equiv 0 \pmod{5}$

which has solutions $x_0 = 1273$ and $y_0 = 1307$. Hence the QP_M yields the closest 3×3 hidden forest $H^3_{(1274,1308)}$.

4.2 Computer-heavy approach: Strings of strongly composite integers

Another technique we implement that proves very powerful in finding hidden forests involves using strings of consecutive integers each with several prime factors. In Section [5,](#page-21-0) we find that combining the technique below with a clever computational use of quasiprime matrices yields the closest known $n \times n$ hidden forests for $n \geq 4$.

In 1990, Schumer proved that there exist strings of *n* consecutive integers each divisible by at least *k* distinct primes, which he calls strings of strongly composite integers [\[13\]](#page-27-2). The proof uses the Chinese Remainder Theorem, and hence like Theorem [3.4](#page-6-1) it produces very large numbers for $n \geq 3$. However, there is an efficient way to find the smallest set of *n* consecutive integers each with at least *k* prime factors each (ignoring multiplicity). We can then use these values as our *x*-values in our hunt for closer hidden forests for $n \geq 4$.

The following Mathematica code easily produces the very first number *n* in a sequence of four consecutive integers each with four prime factors (ignoring multiplicity):

Listing 1: Nayuki Minase's solution to Project Euler Problem 47

Has4PrimeFactors[n] := Length[FactorInteger[n]] == 4
$i = 2$:
While ^[] (Has4PrimeFactors ^[1] && Has4PrimeFactors ^{[1} + 1] &&
Has4PrimeFactors[i + 2] && Has4PrimeFactors[i + 3]), $i++1$]

The value that it yields is 134,043. This number and the next three consecutive integers have the following prime factorizations:

$$
134,043 = 3 \cdot 7 \cdot 13 \cdot 491
$$

$$
134,044 = 2^2 \cdot 23 \cdot 31 \cdot 47
$$

$$
134,045 = 5 \cdot 17 \cdot 19 \cdot 83
$$

$$
134,046 = 2 \cdot 3^2 \cdot 11 \cdot 677.
$$

Example 4.8 (The closest 4×4 to date with the smallest *x*-value). Using the four values 134,043 to 134,046 as the *𝑥*-values of 4 × 4 hidden forest we seek, we exhaustively searched for the very first set of four consecutive integers such that all four values share at least one prime factor with each of the four values 13[4](#page-16-1),043 to 134,046. After running for only two minutes⁴, the Java program which we wrote outputs the value 184,785,885. This number and the next three consecutive integers have the following prime factorizations:

$$
184,785,885 = 32 \cdot 5 \cdot 312 \cdot 4273
$$

$$
184,785,886 = 2 \cdot 17 \cdot 491 \cdot 11,069
$$

$$
184,785,887 = 11 \cdot 13 \cdot 19 \cdot 23 \cdot 2957
$$

$$
184,785,888 = 25 \cdot 3 \cdot 7 \cdot 83 \cdot 3313.
$$

Using these four numbers as the *y*-values of our $H^4_{(x,y)}$ and the four values 134,043 to 134,046 as the *x*-values, we get a hidden forest $H^4_{(134043,184785885)}$ with the following gcd-grid:

By Equation (2) , we retrieve a matrix M from the gcd-grid above as follows:

$$
\text{Gcd}_M = \begin{pmatrix} 3 \cdot 7 & 2^2 & 83 & 2 \cdot 3 \\ 13 & 23 & 19 & 11 \\ 491 & 2 & 17 & 2 \\ 3 & 31 & 5 & 3^2 \end{pmatrix} + \frac{0}{90^\circ \text{ right}} \quad M = \begin{pmatrix} 3 & 491 & 13 & 3 \cdot 7 \\ 31 & 2 & 23 & 2^2 \\ 5 & 17 & 19 & 83 \\ 3^2 & 2 & 11 & 2 \cdot 3 \end{pmatrix}.
$$

Applying the QP-algorithm to M , we get the following quasiprime matrix

$$
QP_M = \left(\begin{array}{rrrr} 1 & 491 & 13 & 7 \\ 31 & 1 & 23 & 2^2 \\ 5 & 17 & 19 & 83 \\ 3^2 & 1 & 11 & 1 \end{array}\right).
$$

Then applying the CRT-algorithm to QP_M does indeed yield the forest $H^4_{(134043,184785885)}$ as desired.

Remark 4.9. The hidden forest $H^4_{(134043,184785885)}$ is distance $d \approx 1.84786 \times 10^8$ from the origin. In comparison recall Example [3.10,](#page-11-1) where we used the only known method to date in the literature

⁴The run times for the Java code are based on the code running on the Blugold Supercomputing Cluster of UWEC, while the run times for the Mathematica code are based on the code running on a standard office computer.

(that is, the prime matrix P_4 and Theorem [3.4\)](#page-6-1). Using that traditional method we found the hidden forest $H^4_{(x_1, y_1)}$ with

$$
x_1 = 2,847,617,195,518,191,810
$$

$$
y_1 = 1,160,906,121,308,397,398
$$

at a distance $d \approx 3.07516 \times 10^{18}$ which is 1.66418×10^{10} times farther than the hidden forest which we found in Example $4.8!$ The matrix P_4 and its associated gcd-grid of our closer hidden forest $H^4_{(134043,184785885)}$ is as follows:

Hence the gcd-matrix of this 4×4 hidden forest is

$$
\text{Gcd}_M = \left(\begin{array}{rrrr} 7 & 19 & 37 & 53 \\ 2.5 & 17 & 2^2.31 & 47 \\ 3 & 13 & 29 & 3.43 \\ 2 & 11 & 2.23 & 41 \end{array} \right).
$$

Remark 4.10. Other researchers have found 4×4 hidden forests of distances relatively close to the one shown in Remark [4.9.](#page-16-2) In 2002 Pighizzini and Shallit addressed the issue of finding the closest $n \times n$ hidden forests [\[12\]](#page-27-12). For a positive integer *n*, they define a function $S(n)$, which is the least positive integer *r* such that there exists $m \in \{0, 1, \ldots, r\}$ with $gcd(r - i, m - j) > 1$ for $0 \le i, j \le n$. This is equivalent to finding the closest $n \times n$ hidden forest. They were only successful in finding this value for $n = 1, 2, 3$, but for $n = 4$ they were able to give bounds $450000 < S(4) \le 172379781$ by finding a hidden forest $H^4_{(x,y)}$ with $x = 172,379,778$ and $y = 153,132,342$. An even closer 4×4 hidden forest was later revealed in 2013 in a book by Baake and Grimm [\[1,](#page-27-13) pg.422]. The forest they find has bottom left corner $x = 13,458,288$ and $y = 13,449,225$ however no proof or justification of how this was found is given. Moreover, they give no assertion regarding whether this is the closest known 4×4 hidden forest. The following table gives the distances of the three closest known 4×4 hidden forests in the literature.

	Year	Distance of H^4 (x,y)	Proof/method given?
Pighizzini and Shallit	2002	2.30574×10^8	No
Baake and Grimm	2013	1.90265×10^7	Nο
Goodrich, Mbirika, and Nielsen	2014^5	1.84786×10^8	Yes

⁵We discovered this forest in 2014; however, it is in this 2020 paper in which we give its existence and proof.

4.3 Computer-free approach: Minimum prime factors in an optimal gcdmatrix

The concept of an "optimal" gcd-matrix for a hidden $n \times n$ forest $H_{(x,y)}^n$ depends on n and is based on the minimal number of prime factors required in the gcd-grid of $H^{n}(\alpha,y)$. We find that minimizing the number of primes used in the gcd-matrix while simultaneously maximizing the number of locations in the gcd-grid where a prime can be used again leads to a closer $H_{(x,y)}^n$ than the traditional method given in Section [3.](#page-5-0)

Observe that the gcd-matrix of the $H_{x,y}^{(4)}$ in Remark [4.9](#page-16-2) is hardly optimal in the sense that if the corner entries were all multiples of 3, then we immediately get the four corners "hidden for free", as in the forest in Example [4.8—](#page-16-0)that is, the values x_1, x_4, y_1 and y_4 would all be divisible by 3 and hence none of the four points (x_1, y_1) , (x_4, y_1) , (x_1, y_4) , or (x_4, y_4) would be visible. An optimal situation is to have one corner, for example, the bottom-left coordinate (x_1, y_1) to be divisible by both 2 and 3. Then we would have a forest where the gcd of the following 16 coordinates are divisible by 2, 3, and nine other primes p_1, \ldots, p_9 as in Figure [4.2.](#page-18-1)

This leads one to consider a different type of gcd-matrix that does not give the exact gcd $g_{i,i}$ (recall Figure [3.1b\)](#page-8-0) for each coordinate (x_i, x_j) (recall Figure [3.1a\)](#page-8-0) of $H_{(x,y)}^n$. But on the other hand, this new matrix would simply give the smallest prime divisor of the gcd for each coordinate. We make this more precise in Definition [4.12.](#page-18-2) But first we need to recall the following number-theoretic function.

Definition 4.11. The *prime counting function* $\pi : \mathbb{R} \to \mathbb{N}$ counts the number of primes less than or equal to a given real number.

Definition 4.12. Construct an *optimal gcd-matrix* as follows. Let one of the four corner entries of the *n* × *n* matrix contain the product of the first $k_n := \pi(n)$ primes (where π is the prime counting function). Without loss of generality, choose the bottom-left corner for this value. Denote these first k_n primes as $q_1, q_2, \ldots, q_{k_n}$. For each q_i with $1 \le i \le k_n$, any entry in the matrix that is a multiple of q_i rows to the right of the bottom-left corner and/or a multiple of q_i columns above the bottom-left corner must be filled with the value q_i . If more than one prime fits this criteria for a specific matrix entry, then simply multiply the primes in that entry together. In the remaining unfilled entries, place one prime in each entry from the set of the next smallest primes larger than the prime q_{k_n} . Denote this set of primes by $\{p_1, p_2, \ldots\}$. We denote this optimal gcd-matrix by the symbol opt- Gcd_M .

Example 4.13. An optimal 4×4 gcd-matrix is

opt-Gcd_M =
$$
\begin{pmatrix} 3 & p_9 & p_8 & 3 \ 2 & p_5 & 2 & p_7 \ p_3 & p_2 & p_4 & p_6 \ 2 \times 3 & p_1 & 2 & 3 \end{pmatrix}
$$

where the entries p_1, \ldots, p_9 are the nine smallest prime numbers other than 2 or 3. Observe that the locations of the primes 2 and 3 correspond exactly to their location in the gcd-grid in Figure [4.2.](#page-18-1) The manner in which the p_1 through p_9 are distributed in this particular matrix is the $n = 4$ case that arises from the grid in Figure [4.3.](#page-19-0)

The grid in Figure [4.3](#page-19-0) shows us the minimum number of primes and their relative locations in a candidate for an optimal gcd-matrix for an $n \times n$ hidden forest. In this grid, we choose the bottomleft corner (denoted with the symbol •) to contain the product of powers of the first k_n primes where k_n is the value given in Definition [4.12.](#page-18-2) In the far-left shaded column, each entry refers to the size *n* of the corresponding $n \times n$ grid. In the bottom shaded row, in each box we give the number of additional primes that are needed to go from an $n \times n$ grid to an $(n + 1) \times (n + 1)$ grid. For example, for $n = 5$, we need a minimum $k_5 + 3 + 2 + 4 + 4 = 15$ distinct primes in the optimal gcd-matrix for an $H_{x,y}^{(5)}$. Indeed in Section [5,](#page-21-0) we see that this minimum is achieved.

Figure 4.3: An optimal gcd-grid with bottom left corner having the product of k_n primes

Example 4.14. If $n = 4$ then $k_4 = 2$, and hence we place 2×3 in a corner location. This is because the 4×4 portion of the grid in Figure [4.3](#page-19-0) says that we need a minimum of 9 primes, not counting the primes 2 and 3 which are placed in the locations where they appear in the grid. Hence an *optimal gcd-matrix* might be as follows:

$$
opt-Gcd_M = \begin{pmatrix} 3 & \frac{29}{19} & \frac{31}{11} & \frac{3}{12} \\ \frac{2}{11} & \frac{11}{11} & \frac{13}{11} & \frac{17}{17} \\ 2 \times 3 & 5 & 2 & 3 \end{pmatrix}
$$

.

In the boxed entries in the matrix above, we place the 9 smallest primes larger than 3 where the grid in Figure [4.3](#page-19-0) places p_1, \ldots, p_9 .

Observe that the forest $H^4_{(134043,184785885)}$ found in Example [4.8](#page-16-0) is the closest known 4 \times 4 forest and is attained by cleverly using the method of *strings of strongly composite integers*. That is, using computer computation to find the smallest four consecutive values x_1, \ldots, x_4 which each have at least 4 primes factors each, and then using computer computation again to compute the next set of four values y_1, \ldots, y_4 each of which is not relatively prime to all four *x*-values. However, the QP_M associated to this closest forest is not optimal in the sense that it uses 10 primes (not including 2 and 3), whereas an optimal QP_M uses at most 9 primes (not including 2 and 3).

Note that much computer assistance was required to generate $H^4_{(134043,184785885)}$, however no computer assistance whatsoever is required to create the optimal gcd-matrix, $opt\text{-}Gcd_M$. From the matrix opt-Gcd_M in this example, we can produce the quasiprime matrix as follows using the QPalgorithm:

$$
opt-Gcd_M = \begin{pmatrix} 3 & 29 & 31 & 3 \\ 2 & 19 & 2 & 23 \\ 7 & 11 & 13 & 17 \\ 2 \times 3 & 5 & 2 & 3 \end{pmatrix} \xrightarrow[\text{algorithm}]{\text{QP}} \quad QP_M = \begin{pmatrix} 1 & 29 & 31 & 1 \\ 1 & 19 & 1 & 23 \\ 7 & 11 & 13 & 17 \\ 6 & 5 & 1 & 1 \end{pmatrix}.
$$

Applying the CRT-algorithm on QP_M , we then get the forest $H^4_{(x,y)}$ with $x = 153,630,616,137$ and $y = 116,380,988,514$ and the following prime factorizations of the 16 coordinates (x_i, y_j) for all $1 \leq i, j \leq 4$:

Below we summarize the distances of the 4×4 hidden forests found by the traditional method versus the two new methods given in this paper.

So we can easily see that the two new methods produce substantially closer hidden forests than the traditional methods. However, we find that merging the method of strings of composite integers with the method of the optimal matrix is an even better idea. And that is precisely what we do in the 5×5 case in the following section.

5 An application: the closest known 5 × 5 **hidden forest**

We employ a combination of the techniques from both strings of strongly composite integers and an optimal quasiprime matrix to find the closest known 5×5 hidden forest to date. We first calculate a length 5 analogue of the Project Euler Problem 47 (recall the footnote given in Remark [3.11\)](#page-11-3). By slightly altering Minase's solution (see Listing [1\)](#page-15-1), we find the smallest set of five consecutive integers each with at least five prime factors. Mathematica completed this computation in 36 minutes. These five integers and their prime factorizations are

$$
x_1 = 129,963,314 = 2 \cdot 13 \cdot 37 \cdot 53 \cdot 2549
$$

\n
$$
x_2 = 129,963,315 = 3 \cdot 5 \cdot 31 \cdot 269 \cdot 1039
$$

\n
$$
x_3 = 129,963,316 = 2^2 \cdot 7 \cdot 97 \cdot 109 \cdot 439
$$

\n
$$
x_4 = 129,963,317 = 11^2 \cdot 17 \cdot 23 \cdot 41 \cdot 67
$$

\n
$$
x_5 = 129,963,318 = 2 \cdot 3 \cdot 89 \cdot 199 \cdot 1223.
$$

In Example [4.8,](#page-16-0) it took the Java code only 2 minutes to find the smallest four consecutive values which are each not relatively prime to the four values $134,043$ through $134,046$. However in this $n = 5$ case, it is not as simple. After the Java code ran continuously for four days $\frac{6}{3}$ $\frac{6}{3}$ $\frac{6}{3}$, it had checked up to the *y*-value of 500 billion and still did not find an $H^5_{(x,y)}$ with the *x*-values x_1, \ldots, x_5 given earlier. So we approached this problem from a more theoretical perspective instead.

Consider the list of five consecutive integers x_1, \ldots, x_5 . Observe that x_1, x_3 , and x_5 are divisible by 2 and that x_2 and x_5 are divisible by 3. Hence a hidden 5×5 forest bearing these *x*-values would be "optimal" if the corresponding five *y*-values (which we denote y_1, \ldots, y_5) have the property that y_1 , y_3 , and y_5 are divisible by 2 and that y_2 and y_5 are divisible by 3. The benefit of this optimal situation is that 12 of the 25 coordinates will automatically have $gcd(x_i, y_j) > 1$ and hence these 12 points are hidden. In the matrices below, we represent each of these 12 points with the symbol ∙ in the gcd-matrix Gcd*^𝑀* on the left, and to its right we give the 90◦ clockwise rotation matrix *𝑀* from which we construct a quasiprime matrix.

$$
\text{Gcd}_M = \begin{array}{c} y_5 \\ y_4 \\ y_3 \\ y_1 \\ \end{array} \rightarrow \begin{pmatrix} \cdot & \cdot & \cdot & d_5 \\ a_2 & b_3 & c_2 & d_4 & e_1 \\ \cdot & b_2 & \cdot & d_3 & \cdot \\ a_1 & \cdot & c_1 & d_2 & \cdot \\ \cdot & b_1 & \cdot & d_1 & \cdot \end{pmatrix} \xrightarrow{\text{O} \atop 90^\circ \text{right}} M = \begin{array}{c} y_1 \\ y_2 \\ x_3 \\ x_4 \\ \end{array} \rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \end{pmatrix} \xrightarrow{\text{V}_4} \begin{array}{c} y_2 \\ y_3 \\ y_4 \\ \end{array} \xrightarrow{\text{V}_5} \begin{array}{c} y_3 \\ y_4 \\ y_5 \\ \end{array}
$$

⁶Recall the footnote regarding computation times in Example [4.8.](#page-16-0)

Since we know that x_5 and y_5 are both divisible by 2 and 3 in this optimal case, we place a 6 in this entry, and the QP_M matrix has the following abstract form

$$
QP_M = \begin{pmatrix} 1 & a_1 & 1 & a_2 & 1 \\ b_1 & 1 & b_2 & b_3 & 1 \\ 1 & c_1 & 1 & c_2 & 1 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ 1 & 1 & 1 & e_1 & 6 \end{pmatrix}
$$
 (3)

where

$$
x_1 = 2 \cdot 13 \cdot 37 \cdot 53 \cdot 2549 \implies a_1, a_2 \in \{13, 37, 53, 2549\},
$$

\n
$$
x_2 = 3 \cdot 5 \cdot 31 \cdot 269 \cdot 1039 \implies b_1, b_2, b_3 \in \{5, 31, 269, 1039\},
$$

\n
$$
x_3 = 2^2 \cdot 7 \cdot 97 \cdot 109 \cdot 439 \implies c_1, c_2 \in \{7, 97, 109, 439\},
$$

\n
$$
x_4 = 11^2 \cdot 17 \cdot 23 \cdot 41 \cdot 67 \implies d_1, d_2, d_3, d_4, d_5 \in \{11, 17, 23, 41, 67\},
$$
and
\n
$$
x_5 = 2 \cdot 3 \cdot 89 \cdot 199 \cdot 1223 \implies e_1 \in \{89, 199, 1223\}.
$$

Observation 5.1. Consider the QP_M matrix in [\(3\)](#page-22-0). Then the following hold.

- (1) There are 1,244,160 distinct ways to produce a quasiprime matrix QP_M .
- (2) Applying the CRT-algorithm to any of the QP_M yields the same solution values x_1, \ldots, x_5 as the *x*-values of the hidden forest $H^5_{(x_1, y_1)}$. In particular, this unique x_1 value is 129,963,314.
- (3) The *y*-value solutions have the property that $y_1, y_3, y_5 \in 2\mathbb{Z}$ and $y_2, y_5 \in 3\mathbb{Z}$.

Proof of (1): There are $P(4, 2) = \frac{4!}{(4-2)!} = 12$ possible 2-permutations of a 4-element set. So since a_1 and a_2 must be distinct elements of {13*,* 37*,* 53*,* 2549}, the ordered tuple $(a_i)_{i=1}^2$ can be chosen in 12 ways. Applying a similar argument to count the possible $(b_i)_{i=1}^3$, $(c_i)_{i=1}^2$, $(d_i)_{i=1}^5$, and the (e_1) , we see that the ordered tuple $(b_i)_{i=1}^3$ can be chosen in 12 ways, the $(c_i)_{i=1}^2$ in 12 ways, the $(d_i)_{i=1}^5$ in 120 ways, and (e_1) in 3 ways. Thus there are 1,244,160 distinct ways to produce a quasiprime matrix QP_M , which proves (1).

Proof of (2): Unfortunately, we only proved this by computational exhaustion using Mathematica. See part (a) of Question [1.](#page-24-2)

Proof of (3): Consider an arbitrary QP_M . Suppose y_0 is a solution to the five linear congruences $y + k \equiv 0 \pmod{C_k}$ where C_k equals the product of the column entries of QP_M for $1 \leq k \leq 5$. Setting $y_k = y_0 + k$, we observe that $y_5 \equiv 0 \pmod{6 \cdot d_5}$, and thus $y_5 \equiv 0 \pmod{2}$ and $y_5 \equiv 0$ (mod 3). Hence $y_5 \in 2\mathbb{Z} \cap 3\mathbb{Z}$. Since $y_5 \in 2\mathbb{Z}$, it follows that $y_3 = y_5 - 2$ implies $y_3 \in 2\mathbb{Z}$, and *y*₁ = *y*₅ − 4 implies *y*₁ ∈ 2ℤ. Moreover since *y*₅ ∈ 3ℤ, it follows that *y*₂ = *y*₅ − 3 implies *y*₂ ∈ 3ℤ. Thus (3) holds.

We wrote a program in Mathematica which applies the CRT-algorithm to each of the possible 1,244,160 matrices. Four minutes later, the program yields that the smallest *𝑦*-value solution is given by the following quasiprime matrix:

$$
QP_M = \begin{pmatrix} 1 & 37 & 1 & 13 & 1 \\ 31 & 1 & 5 & 269 & 1 \\ 1 & 109 & 1 & 7 & 1 \\ 67 & 17 & 41 & 23 & 11 \\ 1 & 1 & 1 & 89 & 6 \end{pmatrix} . \tag{4}
$$

This *y*-value and the next four consecutive integers have the following prime factorizations (with commas omitted in the factorizations for readability):

$$
y_1 = 2,546,641,254,872,348 = 2^2 \cdot 31 \cdot 67 \cdot 461 \cdot 664921471
$$

\n
$$
y_2 = 2,546,641,254,872,349 = 3^2 \cdot 17 \cdot 37 \cdot 109 \cdot 8681 \cdot 475421
$$

\n
$$
y_3 = 2,546,641,254,872,350 = 2 \cdot 5^2 \cdot 41 \cdot 11113 \cdot 111784759
$$

\n
$$
y_4 = 2,546,641,254,872,351 = 7^2 \cdot 13 \cdot 23 \cdot 73 \cdot 89 \cdot 269 \cdot 271 \cdot 367
$$

\n
$$
y_5 = 2,546,641,254,872,352 = 2^5 \cdot 3 \cdot 11 \cdot 2411592097417.
$$

Comparing the x_1, \ldots, x_5 with the y_1, \ldots, y_5 we see that $gcd(x_i, y_j) > 1$ for all $1 \le i, j \le 5$ and in fact $H^5_{(x_1, y_1)}$ has the following gcd-matrix Gcd_M and corresponding matrix M :

$$
\text{Gcd}_{M} = \begin{pmatrix} 2 & 3 & 2^{2} & 11 & 2 \cdot 3 \\ 13 & 269 & 7 & 23 & 89 \\ 2 & 5 & 2 & 41 & 2 \\ 37 & 3 & 109 & 17 & 3 \\ 2 & 31 & 2^{2} & 67 & 2 \end{pmatrix} + \sum_{\substack{90^{\circ} \text{ right} \\ 90^{\circ} \text{ right}}} M = \begin{pmatrix} 2 & 37 & 2 & 13 & 2 \\ 31 & 3 & 5 & 269 & 3 \\ 2^{2} & 109 & 2 & 7 & 2^{2} \\ 67 & 17 & 41 & 23 & 11 \\ 2 & 3 & 2 & 89 & 2 \cdot 3 \end{pmatrix}.
$$

Remark 5.2. If we apply the QP-algorithm to the M above, then we are forced to place a 2^2 in either the (3, 1)- or (3, 5)-entry of QP_M , and consequently the 6 in the (5, 5) entry becomes a 3. Hence this new QP_M differs from the quasiprime matrix in [\(4\)](#page-23-0). However, applying the CRT-algorithm to this new QP_M gives the same hidden forest as expected.

Remark 5.3. The forest $H^5_{(x,y)}$ with $x = 129,963,314$ and $y = 2,546,641,254,872,348$ is at a distance $d \approx 2.54664 \times 10^{15}$ from the origin. Had we used the only known method until now (that is, Theorem [3.4\)](#page-6-1), then we get a forest $H^5_{(x,y)}$ with the following *x* and *y* values:

$$
x = 251,080,644,933,696,940,130,615,676,720,763,950
$$

$$
y = 108,580,359,501,475,197,963,484,708,875,960,338.
$$

This forest is at a distance $d \approx 2.73553 \times 10^{35}$ from the origin, and hence is 1.07417×10^{20} times farther than the forest we reveal in this paper! We have not found a computationally tractable method to find the closest 5×5 hidden forest, nor do we believe that anyone else has. So for the time being, the $H^5_{(x,y)}$ we present in this paper is the closest 5×5 hidden forest to date.

6 Open problems and progress on recent research

There are many avenues for further research motivated from the work in this present paper. In this section, we give not only some open problems identified during our research process, but also some recent progress in generalizations of lattice point visibility.

6.1 Open problems

Question 1. Is it true that for every hidden forest $H_{(x,y)}^n$, there exists a quasiprime matrix QP_M in $Mat_n(\mathbb{Z})$ such that the CRT-algorithm applied to QP_M yields $H_{(x,y)}^n$? Related to this question are the following subquestions:

- (a) Why do all 1,244,160 distinct quasiprime matrices in Matrix [\(3\)](#page-22-0) yield exactly the same *𝑥*-value solution under the CRT-algorithm?
- (b) Do all distinct quasiprime matrices produce unique solutions?
- (c) Can one code a computationally efficient method to search for the closest $H_{(x,y)}^n$ for $n \geq 4$?

Question 2. Higher dimensional analogues of patches of invisible points can be found. Observe that our proof of Proposition [2.4](#page-4-0) can easily be extended to higher dimensions by setting the value *s* (in the proof) to the appropriate dimension. That is, the probability that (x_1, x_2, \ldots, x_s) is visible in \mathbb{Z}^s is $\frac{1}{\zeta(s)}$. In Example [6.1,](#page-24-3) we find a hidden $2 \times 2 \times 2$ forest using a 3-dimensional analogue of the CRT-algorithm, and we see that the forest found by this method is very far from the origin. Can we generalize the quasiprime matrix to these higher dimensional settings and find closer hidden *n*-dimensional forests?

Example 6.1. In Figure [6.1,](#page-24-4) we give an example of a hidden $2 \times 2 \times 2$ forest with corner point (x_1, y_1, z_1) at $x_1 = 9,126,194, y_1 = 8,286,564,$ and $z_1 = 8,822,099$.

Figure 6.1: A $2 \times 2 \times 2$ hidden forest

To find this 3-dimensional hidden forest, we considered a 3-dimensional version of the prime matrix as a cube whose corners contain the first 8 prime numbers. Then to each face of the cube, we multiplied the four numbers in each corner as the following image illustrates.

Solving the following three pairs of systems of congruences

$$
\begin{cases}\nx + 1 \equiv 0 \pmod{\text{Back}} \\
x + 2 \equiv 0 \pmod{\text{Front}}\n\end{cases}\n\quad\n\begin{cases}\ny + 1 \equiv 0 \pmod{\text{Left}} \\
y + 2 \equiv 0 \pmod{\text{Right}}\n\end{cases}\n\quad\n\begin{cases}\nz + 1 \equiv 0 \pmod{\text{Bottom}} \\
z + 2 \equiv 0 \pmod{\text{Top}}\n\end{cases}
$$

yields the three simultaneous solutions $x_0 = 9,126,193, y_0 = 8,286,563$, and $z_0 = 8,822,098$. Then the following values x_1 , y_1 , z_1 , x_2 , y_2 , and z_2 have the prime factorizations

It is readily verified from these factorizations that each of the eight tuples of coordinates (x_i, y_j, z_k) for $1 \le i, j, k \le 2$ have the property $gcd(x_i, y_j, z_k) > 1$. Hence this 3-dimensional forest is indeed hidden from the origin.

Question 3. What can be said about hidden forests in the $\mathbb{Z}[i] \times \mathbb{Z}[i]$ lattice? What is meant by the coordinate values $(x, y) \in \mathbb{Z}[i] \times \mathbb{Z}[i]$ being relatively prime? Recall that if *R* is a Euclidean domain (as is the case for the ring $\mathbb{Z}[i]$ of Gaussian integers), then greatest common divisors can be computed using the Euclidean algorithm. Can we apply methods in this paper to the visibility of points in the lattice $\mathbb{Z}[i] \times \mathbb{Z}[i]$?

6.2 Progress on recent research

The following open problem was initially started by the second author Mbirika and his colleagues Pamela Harris and Bethany Kubik during their Visiting Assistant Professor appointments at West Point Military Academy in the summer of 2015. This question below has been recently explored in 2018 by Goins, Harris, Kubik, and Mbirika in [\[8\]](#page-27-3).

Question 4 (Harris, Kubik, Mbirika)**.** The classic setting focuses on integer lattice points which lie on straight lines through the origin with rational slopes. We generalize this notion of lines of sights to include all curves through the origin given by power functions of the form $f(x) = ax^b$ where $a \in \mathbb{Q}$ and $b \in \mathbb{N}$. What can we conclude about lattice point visibility in this generalized setting? To begin to answer this question, we establish the following criterion for *b*-(in)visibility.

Definition 6.2 (Visible and invisible lattice points). Fix $b \in \mathbb{N}$. A point $(r, s) \in \mathbb{N} \times \mathbb{N}$ is said to be *b*-invisible if the following two conditions hold:

- (1) The point (r, s) lies on the graph of $f(x) = ax^b$ for some $a \in \mathbb{Q}$. That is $s = ar^b$.
- (2) There exists an integer $k > 1$ such that *k* divides *r* and k^b divides *s*.

The point is said to be *b-visible* if it satisfies Condition (1) but fails to satisfy Condition (2).

To speak about the *b*-visibility of a lattice point in this new setting, we develop a generalization of the greatest common divisor.

Definition 6.3. Fix $b \in \mathbb{N}$. The *generalized greatest common divisor* of *r* and *s* with respect to *b* is denoted gcd*^𝑏* and is defined as

 $gcd_b(r, s) := \max\{k \in \mathbb{N} \mid k \text{ divides } r \text{ and } k^b \text{ divides } s\}.$

The following result gives a necessary and sufficient condition to determine *b*-visibility.

Proposition 6.4. *A point* $(r, s) \in \mathbb{N} \times \mathbb{N}$ *is b-visible if and only if* $gcd_b(r, s) = 1$ *.*

Figure [6.2](#page-26-0) demonstrates both the classic and generalized setting. The **red** curve $f(x) = 7x$ represents the classic setting while the **blue** and **green** curves $g(x) = x^2$ and $h(x) = \frac{1}{7}x^3$, respectively, represent the generalized setting.

Figure 6.2: Three lines of sight $f(x) = 7x$, $g(x) = x^2$, and $h(x) = \frac{1}{7}x^3$.

Observe that the point (7, 49) is not 1-visible since $gcd(7, 49) = 7$ and is not 2-visible since $\gcd_2(7, 49) = 7$. However the point (7, 49) is 3-visible since $\gcd_3(7, 49) = 1$.

Theorem 6.5 (Goins, Harris, Kubik, Mbirika [\[8\]](#page-27-3)). *Fix an integer* $b \in \mathbb{N}$. *Then the proportion of points* $(r, s) \in \mathbb{N} \times \mathbb{N}$ *that are b-visible is* $\frac{1}{\epsilon}$ $\zeta(b+1)$ *.*

Theorem 6.6 (Goins, Harris, Kubik, Mbirika [\[8\]](#page-27-3)). *For every* $m, n, b \in \mathbb{N}$ *, there exists b-invisible* $n \times m$ *forests.*

Question 5. Can we apply the new techniques detailed in this paper to find the closest *b*-invisible $n \times n$ forests?

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Appendix

A.1 Java code to verify closest hidden forests

In this appendix we provide the Java code that we wrote to exhaustively search the integer lattice for the closest hidden forests.

Listing 2: Java code to search the integer lattice for hidden forests

```
package project1;
2
 import java. util .Scanner;
4
  public class Compiler {
6
       public static void main(String [] args) {
            Scanner in = new Scanner(System.in);
            // long is a number;
10 // get bottom, leftmost, and rightmost from user input;
11
12 System.out. println ("What<sub>u</sub>is<sub>u</sub>the<sub>u</sub>bottom?");
\log bottom = in.nextLong();
14
15 System.out. println ("What<sub>u</sub>is<sub>u</sub>the<sub>u</sub>leftmost?");
\log long leftmost = in.nextLong();
17
18 System.out. println ("What<sub>u</sub>is<sub>u</sub>the<sub>u</sub>rightmost?");
19 long rightmost = in.nextLong();
20
|21| long boxWidth = rightmost – leftmost;
|22| long count = 0;
|23| long found Count = 0;
24
25 System.out. println ("How<sub>L</sub>often<sub>Li</sub>do<sub>Li</sub>you<sub>Li</sub>want<sub>Li</sub>to<sub>Li</sub>check?");
26 long check = in.nextLong();
27
28 System.out. println ("Starting");
_{29} boolean exit = false;
\overline{30}31 while (bottom < Long.MAX VALUE – boxWidth && !exit) {
32 boolean equalsOne = false;
\log \text{startY} = \text{bottom}+1;34 for (long y = bottom; y < (startY + boxWidth) && lequalsOne
35 && !exit; ++y} {
36 ++count;
37 for (long x = leftmost; (x \lt = rightmost) && !equalsOne && !exit; ++x) {
38 ++foundCount;
39 if ( test (x, y) == 1) {
40 \mid bottom = y;
\begin{aligned} \mathcal{A}_1 \vert \quad \text{equalsOne} = \text{true}; \end{aligned}42 foundCount = 0;
\mathsf{a}_3 x = leftmost;
```

```
^{44} }// test end
45 if (\text{foundCount} == ((\text{boxWidth} + 1) * (\text{boxWidth} + 1)))^{46} system.out. println ("found\_one!!\_upper\_right\_corner\_=\_x:"
+ x + " \cup y: " + y);48 exit = true;
49 foundCount = 0;
50 }// foundCount end
51 }// for x end
\overrightarrow{f} if (count % check == 0) {
53 System.out. println (y + "<sub>u</sub>is<sub>u</sub>current<sub>uy</sub>");
54 }// count%check end
55 }// for y end
56 ++bottom;
\frac{57}{ } \frac{1}{ } \frac{1}{ } while 1 end
58 }// main end
59
60 private static long test (long x, long y) {
61 if (y == 0)\begin{array}{c|c}\n\text{62} \\
\text{63}\n\end{array} return x;
\begin{bmatrix} 63 \end{bmatrix} return test (y, x, \%);
64 }
65 | }// compiler end
```