arXiv:1805.03461v3 [math.AG] 30 Mar 2021

PUSH-FORWARDS OF CHOW GROUPS OF SMOOTH AMPLE DIVISORS, WITH AN EMPHASIS ON JACOBIAN VARIETIES.

KALYAN BANERJEE, JAYA NN IYER AND JAMES D. LEWIS

Abstract. With a homological Lefschetz conjecture in mind, we prove the injectivity of the push-forward morphism on rational Chow groups, induced by the closed embedding of an ample divisor linearly equivalent to a higher multiple of the Theta divisor inside the Jacobian variety $J(C)$, where C is a smooth irreducible complex projective curve.

Contents

1. Introduction

Suppose X is a smooth projective variety defined over the field of complex numbers. Let $D \subset X$ be an ample smooth divisor on X. Denote the closed embedding, $j : D \hookrightarrow X$. Consider the push-forward homomorphism on Chow groups induced by j:

$$j_* : \mathrm{CH}_k(D;\mathbb{Q}) \to \mathrm{CH}_k(X;\mathbb{Q}),$$

for $k \geq 0$. In this paper, we investigate the kernel of the morphism j_*. This question is motivated by the following results and conjectures. When Chow groups are replaced by the singular homology of a smooth projective variety over $\mathbb{C}$, the (dual of the) Lefschetz hyperplane theorem gives an isomorphism of the pushforward map:

$$j_* : H_k(D,\mathbb{Z}) \to H_k(X,\mathbb{Z})$$

for $k < \dim D$, and surjectivity when $k = \dim D$. M. Nori [No, Conjecture 7.2.5] conjectured the following:

[0]Mathematics Classification Number: 14C15,14C20,15C25, 14C35,14F42
[0]Keywords: Pushforward homomorphism, Theta divisor, Jacobian varieties, Chow groups, higher Chow groups.

Conjecture 1.1. *Suppose D is a very general smooth ample divisor on X, of sufficiently large degree. Then the restriction map (the refined Gysin map,* [Fu]*):*

$$j^* : \mathrm{CH}^p(X;\mathbb{Q}) \to \mathrm{CH}^p(D;\mathbb{Q})$$

is an isomorphism, for $p < \dim D$ and is injective, for $p = \dim D$.

More generally, we have (see [Pa, Conjecture 1.5]):

Conjecture 1.2. *Let D be a smooth ample divisor on X. Then the restriction map for the inclusion of D in X:*

$$\mathrm{CH}^p(X;\mathbb{Q}) \to \mathrm{CH}^p(D;\mathbb{Q})$$

is an isomorphism, for $p \leq \frac{\dim D - 1}{2}$.

It seems reasonable to pose the following dual of above Chow Lefschetz questions:

Conjecture 1.3. *The pushforward map on the rational Chow groups, for a very general ample divisor $D \subset X$ of sufficiently large degree:*

$$j_* : \mathrm{CH}_k(D;\mathbb{Q}) \to \mathrm{CH}_k(X;\mathbb{Q})$$

is injective, whenever $k > 0$.

Similarly, we could pose the dual version of Conjecture 1.2:

Conjecture 1.4. *Let D be a smooth ample divisor on X. The pushforward map on the rational Chow groups,*

$$j_* : \mathrm{CH}_k(D;\mathbb{Q}) \to \mathrm{CH}_k(X;\mathbb{Q})$$

is injective, whenever $k > \frac{\dim D}{2}$.

In §2, we provide a motivic interpretation of Conjectures 1.2 and 1.4. If the Hodge conjecture and Bloch-Beilinson conjecture (based on the injectivity of the Abel-Jacobi map for for smooth projective varieties over $\overline{\mathbb{Q}}$) hold, then both Conjectures 1.2 and 1.4 hold. Concerning Conjecture 1.4, we prove the following generalization. (See Theorem 2.6.):

Theorem 1.5. *Assume the Hodge and Bloch-Beilinson conjectures hold. Then:*

$$k > \frac{\dim D - \nu}{2} \Rightarrow j_* : F^\nu \mathrm{CH}_k(D;\mathbb{Q}) \hookrightarrow F^\nu \mathrm{CH}_k(X;\mathbb{Q}),$$

where $\{F^\nu \mathrm{CH}^r(X;\mathbb{Q})\}_{\nu \geq 0}$ is the Bloch-Beilinson filtration on $\mathrm{CH}^r(X;\mathbb{Q})$. (The case $\nu = 0$ yields the statement of Conjecture 1.4.)

A good source for the conjectural Bloch-Beilinson filtration is [Ja], which agrees with the filtration in [JL], under the assumptions of the aforementioned Hodge and Bloch-Beilinson conjectures.

The motivation for the above dual Chow-Lefschetz conjectures, for us, arose while studying the following theorem by A. Collino.

Suppose C is a smooth projective curve of genus g over complex numbers. The symmetric power $\mathrm{Sym}^r(C)$ $(r > 0)$ is a smooth projective variety of dimension r and fix a point $p \in C$. The inclusion

$$\mathrm{Sym}^{r-1}(C) \hookrightarrow \mathrm{Sym}^r(C),\ (x_1 + x_2 + ... + x_{r-1}) + p \mapsto (x_1 + x_2 + ... + x_{r-1} + p)$$

is a smooth ample divisor [ACGH].

Theorem 1.6. [Co, Theorem 1] *The pushforward map on the Chow groups:*

$$\mathrm{CH}_k(\mathrm{Sym}^{r-1}(C); \mathbb{Q}) \to \mathrm{CH}_k(\mathrm{Sym}^r(C); \mathbb{Q})$$

is injective, $k \geq 0$.

This provides a prime example verifying Conjectures 1.3, 1.4 and the bounds in Theorem 1.5.

The next example is closely related to the above example on symmetric power.

In §3, our aim is to verify the bounds given in Theorem 1.5 when D is the Theta divisor, on the Jacobian of a smooth projective curve. It is well-known that Θ is an ample divisor on $J(C)$. We state it here, as follows.

Let C be a smooth projective curve of genus g and let Θ denote a Theta divisor inside the Jacobian $J(C)$ of C. Denote the inclusion $j : \Theta \hookrightarrow J(C)$.

Theorem 1.7. *Assume C is a non-hyperelliptic smooth projective curve of genus $g \geq 3$, over $\mathbb{C}$. The pushforward morphisms*

$$j_* : F^1\mathrm{CH}_{g-2}(\Theta; \mathbb{Q}) \to F^1\mathrm{CH}_{g-2}(J(C); \mathbb{Q})$$

and

$$j_* : F^2\mathrm{CH}_{g-3}(\Theta; \mathbb{Q}) \to F^2\mathrm{CH}_{g-3}(J(C); \mathbb{Q})$$

are injective.

See §5, Theorem 5.1.

In general, the Theta divisor is a singular variety with singular locus B, of dimension at least $g - 4$. Equality holds if C is non-hyperelliptic. Hence if $g \leq 3$ then Θ is smooth, and fulfils the above conjectural bound in Theorem 1.5. Furthermore, when $g = 4$ and C is non-hyperelliptic (this is the generic situation), then Θ is singular and B is a finite set of points. The Chow groups of Θ are taken as the usual Chow groups $\mathrm{CH}_1(\Theta - B)$. The reader should be aware of the fact that the Bloch-Beilinson filtration only applies to smooth projective varieties. However, for our purposes, there is the Abel-Jacobi map defined on the cycles homologous to zero on this group, and we define $F^2\mathrm{CH}_1(\Theta; \mathbb{Q})$ as the kernel of this map. When $g > 4$, the same convention is used for $F^2\mathrm{CH}_{g-3}(\Theta; \mathbb{Q})$, i.e.,

$$F^2\mathrm{CH}_{g-3}(\Theta; \mathbb{Q}) := F^2\mathrm{CH}_{g-3}(\Theta - B; \mathbb{Q})$$

Here the right term is the kernel of Abel-Jacobi map, see §4.6.

1.8. **Comments on Theorem 1.7.** We felt it important to incorporate some interesting comments from the referee regarding the above theorem. The assertion preceding Theorem 1.7, viz., to prove Theorem 1.5 in the case of Jacobians, can be construed as not as optimal as one would like. What is meant by this is the following:

(i) There are two parameters, k and ν in Theorem 1.5, and once one has proven the result for a given pair (k, ν), one has it for all pairs (k, ν') with $\nu' \geq \nu$. So in particular, for a fixed k, the most interesting value of ν is the minimum value, $\nu = \max\{0, \dim D - 2k + 1\}$.

(ii) At the same time, since conjecturally $F^\nu \mathrm{CH}_k(D; \mathbb{Q}) = F^\nu \mathrm{CH}^{\dim D - k}(D; \mathbb{Q}) = 0$ for $\nu > \dim D - k$ (see Theorem 2.4), the case $\nu = \dim D - k$ is the smallest term of the filtration for which the statement is conjecturally non-trivial.

(iii) More succintly, in the case of a Jacobian and its Theta divisor, when one takes $k = g - 2$, the case of most interest in then $\nu = \max\{0, (g-1) - 2(g-2) + 1\} = \max\{0, 4 - g\}$. Note that if the curve is not hyperelliptic, then $g \geq 3$, so that the case of most interest is $\nu = 1$ for $g = 3$, and $\nu = 0$ for $g \geq 4$. When $k = g - 3$, then one wants to look at $\nu = \max\{0, 6 - g\}$. Therefore, for non-hyperelliptic curves of genus 3, the case $\nu = 3$ is the case of most interest, for genus 4 the case $\nu = 2$ is of most interest, for $g = 5$ the case $\nu = 1$ is of most interest, and for higher genus, $\nu = 0$.

(iv) Consequently, for the first assertion of Theorem 1.7, regarding $F^1 \mathrm{CH}_{g-2}$, for $g = 3$ the statement is sharp with respect to Theorem 1.5, but for $g > 3$ one should point out that from Theorem 1.5, one would really like to have the statement for $F^0 \mathrm{CH}_{g-2}$. And for the second assertion, regarding $F^2 \mathrm{CH}_{g-3}$, in the case $g = 3$ this choice $\nu = 2$ is stronger than what Theorem 1.5 predicts, sharp for $g = 4$, that for $g = 5$, one would want $\nu = 1$, and for $g \geq 6$ one would want $\nu = 0$. At the same time, one can say that both assertions of Theorem 1.7 are made for the smallest term of the filtration on Chow for which the statement is conjecturally non-trivial.

Acknowledgements: The first named author is grateful to Department of Atomic Energy, India for funding this project. The third named author is partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The authors owe a debt of gratitude to the referee for going through our paper meticulously, and for the many helpful comments.

Notation: Here k is an uncountable, algebraically closed field and all the varieties are defined over k. Denote

$$\mathrm{CH}_d(X; \mathbb{Q}) := \mathrm{CH}_d(X) \otimes \mathbb{Q}.$$

Here X is a variety of pure dimension n, defined over k and $\mathrm{CH}_d(X)$ denotes the Chow group of d-dimensional cycles modulo rational equivalence.

We write

$$\mathrm{CH}_d(X, s; \mathbb{Q}) := \mathrm{CH}^{\dim X + s - d}(X, s) \otimes \mathbb{Q},$$

the Bloch's higher Chow groups ([Bl]) with $\mathbb{Q}$-coefficients.

2. Motivic interpretations

We wish to provide a motivic interpretation of Conjecture 1.4. But first some terminology, and background material, which is specific to this section only. Let $\mathbb{Q}(r)$ be the Tate twist and consider the category of mixed Hodge structures over $\mathbb{Q}$ (MHS). For a $\mathbb{Q}$-MHS V, we put

$$\Gamma(V) = \hom_{\mathrm{MHS}}(\mathbb{Q}(0), V),$$
$$J(V) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), V).$$

For instance, if $X = X/\mathbb{C}$ is smooth and projective, then $\Gamma\big(H^{2r}(X, \mathbb{Q}(r))\big)$ can be identified with $\mathbb{Q}$-betti cohomology classes of Hodge type (r, r), and $J\big(H^{2r-1}(X, \mathbb{Q}(r))\big)$ can be identified (via J. Carlson) with the Griffiths jacobian (tensored with $\mathbb{Q}$). There is the cycle class map $\mathrm{CH}^r(X; \mathbb{Q}) \to \Gamma\big(H^{2r}(X, \mathbb{Q}(r))\big)$, conjecturally surjective under the classical Hodge conjecture (HC), with kernel $\mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q})$. Accordingly there is the Griffiths Abel-Jacobi map $AJ \otimes \mathbb{Q} : \mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q}) \to J\big(H^{2r-1}(X, \mathbb{Q}(r))\big)$. Beilinson and Bloch have independently conjectured the following:

Conjecture 2.1 (BBC)**.** *Let $W/\overline{\mathbb{Q}}$ be smooth and projective, and assume given an integer $r \geq 0$. Then the Abel-Jacobi map*

$$AJ \otimes \mathbb{Q} : \mathrm{CH}^r_{\mathrm{hom}}(W/\overline{\mathbb{Q}}; \mathbb{Q}) \to J\big(H^{2r-1}(W(\mathbb{C}), \mathbb{Q}(r))\big),$$

is injective.

Remark 2.2. *If one assumes the HC + BBC, then $W/\overline{\mathbb{Q}}$ can be replaced by a smooth quasi-projective variety.*

Next, we need to inform the reader of the conjectured Bloch-Beilinson (BB) filtration. First conceived by Bloch and later fortified by Beilinson in terms of motivic extension datum, the idea is to measure the complexity of $\mathrm{CH}^r(X; \mathbb{Q})$ in terms of a conjectural descending filtration. Rather than defining it here, we provide an explicit candidate which will define a Bloch-Beilinson filtration in the event that the HC and BBC hold.

2.3. **A candidate BB filtration.** We begin with the following result, by recalling:

Theorem 2.4 ([JL])**.** *Let $X/\mathbb{C}$ be smooth and projective, of dimension d. Then for all $r \geq 0$, there is a descending filtration,*

$$\mathrm{CH}^r(X; \mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^\nu \supset F^{\nu+1} \supset \cdots \supset F^r \supset F^{r+1} = F^{r+2} = \cdots,$$

which satisfies the following:

(i) $F^1 = \mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q})$.

(ii) $F^2 \subseteq \ker AJ \otimes \mathbb{Q} : \mathrm{CH}^r_{\mathrm{hom}}(X; \mathbb{Q}) \to J\big(H^{2r-1}(X(\mathbb{C}), \mathbb{Q}(r))\big)$.

(iii) $F^{\nu_1}\mathrm{CH}^{r_1}(X; \mathbb{Q}) \bullet F^{\nu_2}\mathrm{CH}^{r_2}(X; \mathbb{Q}) \subset F^{\nu_1+\nu_2}\mathrm{CH}^{r_1+r_2}(X; \mathbb{Q})$, *where $\bullet$ is the intersection product.*

(iv) *F^ν is preserved under the action of correspondences between smooth projective varieties over $\mathbb{C}$.*

(v) *Let $Gr^\nu_F := F^\nu/F^{\nu+1}$ and assume that the Künneth components of the diagonal class $[\Delta_X] = \oplus_{p+q=2d}[\Delta_X(p,q)] \in H^{2d}(X\times X, \mathbb{Q}(d)))$ are algebraic. Then*

$$\Delta_X(2d-2r+\ell, 2r-\ell)_*\big|_{Gr^\nu_F \mathrm{CH}^r(X;\mathbb{Q})} = \delta_{\ell,\nu}\cdot \text{Identity}.$$

[If we assume the conjecture that homological and numerical equivalence coincide, then (v) *says that Gr^ν_F factors through the Grothendieck motive.]*

(vi) *Let $D^r(X) := \bigcap_\nu F^\nu$. If the HC, and the Bloch-Beilinson conjecture (BBC) on the injectivity of the Abel-Jacobi map ($\otimes\mathbb{Q}$) holds for smooth projective varieties defined over $\overline{\mathbb{Q}}$, then $D^r(X) = 0$.*

It is essential to briefly explain how this filtration comes about. Consider a $\overline{\mathbb{Q}}$-spread $\rho : \mathcal{X} \to \mathcal{S}$, where ρ is smooth and proper. Let η be the generic point of $\mathcal{S}$, and put $K := \overline{\mathbb{Q}}(\eta)$. Write $X_K := \mathcal{X}_\eta$. We introduced a decreasing filtration $\mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$, with the property that $Gr^\nu_{\mathcal{F}}\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) \hookrightarrow E_\infty^{\nu,2r-\nu}(\rho)$, (no conjectures used here!), where $E_\infty^{\nu,2r-\nu}(\rho)$ is the ν-th graded piece of the Leray filtration on the lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ of Beilinson's absolute Hodge cohomology $H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ associated to ρ. That lowest weight part $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) \subset H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ is given by the image $H^{2r}_{\mathcal{H}}(\overline{\mathcal{X}},\mathbb{Q}(r)) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, where $\overline{\mathcal{X}}$ is a smooth compactification of $\mathcal{X}$. There is a cycle class map $\mathrm{CH}^r(\mathcal{X};\mathbb{Q}) := \mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}};\mathbb{Q}) \to \underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$, which is conjecturally injective under the BBC + HC conjectures, using the fact that there is a short exact sequence:

$$0 \to J\big(H^{2r-1}(\mathcal{X},\mathbb{Q}(r))\big) \to H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r)) \to \Gamma\big(H^{2r}(\mathcal{X},\mathbb{Q}(r))\big) \to 0.$$

(Injectivity would imply $D^r(X) = 0$.) Regardless of whether or not injectivity holds, the filtration $\mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$ is given by the pullback of the Leray filtration on $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}(r))$ to $\mathrm{CH}^r(\mathcal{X};\mathbb{Q})$. The term $E_\infty^{\nu,2r-\nu}(\rho)$ fits in a short exact sequence:

$$0 \to \underline{E}_\infty^{\nu,2r-\nu}(\rho) \to E_\infty^{\nu,2r-\nu}(\rho) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) \to 0,$$

where

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) = \Gamma\big(H^\nu(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\big),$$

$$\underline{E}_\infty^{\nu,2r-\nu}(\rho) = \frac{J\big(W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\big)}{\Gamma\big(Gr^0_W H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\big)} \subset J\big(H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\big).$$

[Here the latter inclusion is a result of the short exact sequence:

$$W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \hookrightarrow W_0H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \twoheadrightarrow Gr^0_W H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)).]$$

One then has (by definition)

$$F^\nu\mathrm{CH}^r(X_K;\mathbb{Q}) = \varinjlim_{U\subset S/\overline{\mathbb{Q}}} \mathcal{F}^\nu\mathrm{CH}^r(\mathcal{X}_U;\mathbb{Q}), \quad \mathcal{X}_U := \rho^{-1}(U)$$

$$F^\nu\mathrm{CH}^r(X_{\mathbb{C}};\mathbb{Q}) = \varinjlim_{K\subset\mathbb{C}} F^\nu\mathrm{CH}^r(X_K;\mathbb{Q})$$

Further, since direct limits preserve exactness,

$$Gr_F^\nu \mathrm{CH}^r(X_K;\mathbb{Q}) = \varinjlim_{U\subset S/\overline{\mathbb{Q}}} Gr_{\mathcal{F}}^\nu \mathrm{CH}^r(\mathcal{X}_U;\mathbb{Q}),$$

$$Gr_F^\nu \mathrm{CH}^r(X_{\mathbb{C}};\mathbb{Q}) = \varinjlim_{K\subset\mathbb{C}} Gr_F^\nu \mathrm{CH}^r(X_K;\mathbb{Q})$$

2.5. Now let $j : D \hookrightarrow X$ be an inclusion of smooth irreducible projective varieties, with D ample and of codimension 1. The weak Lefschetz theorem implies that $j^* : H^i(X,\mathbb{Z}) \to H^i(D,\mathbb{Z})$ is an isomorphism if $i < \dim D$ and injective for $i = \dim D$. If we set $i = 2r-\nu$, then the statement $2r < \dim D$ implies that $2r-\nu \le \dim D - 1$ for $0 \le \nu \le r$. Then by Theorem 2.4, and under the assumption of the HC and BBC:

$$r \le \left[\frac{\dim D - 1}{2}\right] \Rightarrow j^* : Gr_F^\nu \mathrm{CH}^r(X;\mathbb{Q}) \xrightarrow{\sim} Gr_F^\nu \mathrm{CH}^r(D;\mathbb{Q}),\ \forall \nu = 0,...,r$$

$$\Rightarrow j^* : \mathrm{CH}^r(X;\mathbb{Q}) \xrightarrow{\sim} \mathrm{CH}^r(D;\mathbb{Q}),$$

by downward induction. This incidentally, provides the motivic interpretation of Conjecture 1.2.[1]

Let $(j^*)^{-1} : \mathrm{CH}^r(D;\mathbb{Q}) \xrightarrow{\sim} \mathrm{CH}^r(X;\mathbb{Q})$ be the inverse map. It is clearly cycle induced by the HC applied to the isomorphism of Hodge structures:

$$[j^*]^{-1} : \bigoplus_{\nu=0}^{r} H^{2r-\nu}(D,\mathbb{Q}) \xrightarrow{\sim} \bigoplus_{\nu=0}^{r} H^{2r-\nu}(X,\mathbb{Q}).$$

[Explicit: Apply the Hodge conjecture to

$$\Gamma\left(\bigoplus_{\nu=0}^{r} H^{2\dim D-2r+\nu}(D,\mathbb{Q}) \otimes H^{2r-\nu}(X,\mathbb{Q})\big(\dim D\big)\right).]$$

One clearly has a commutative diagram;

$$\begin{array}{ccc} \mathrm{CH}^r(D;\mathbb{Q}) & \underset{\xleftarrow{\quad j^* \quad \sim}}{\xrightarrow{(j^*)^{-1} \quad \sim}} & \mathrm{CH}^r(X;\mathbb{Q}) \\ & j_* \searrow \qquad \swarrow j_* \circ j^* & \\ & \mathrm{CH}^{r+1}(X) & \end{array} \tag{1}$$

[1]We also remark in passing that under the same conjectural assumptions and argument, we have

$$r \le \left[\frac{\dim D - 1 + \nu}{2}\right] \Rightarrow j^* : Gr_F^\ell \mathrm{CH}^r(X;\mathbb{Q}) \xrightarrow{\sim} Gr_F^\ell \mathrm{CH}^r(D;\mathbb{Q}),\ \forall \ell = \nu,...,r$$

$$\Rightarrow j^* : F^\nu \mathrm{CH}^r(X;\mathbb{Q}) \xrightarrow{\sim} F^\nu \mathrm{CH}^r(D;\mathbb{Q}).$$

moreover $j_* \circ j^* = \cup\{D\}$. Since $j : D \hookrightarrow X$ is ample, it follows that for $2r < \dim X$, $j_* \circ j^* : H^{2r-\nu}(X, \mathbb{Q}) \to H^{2(r+1)-\nu}(X, \mathbb{Q})$ is injective. Now working with the diagram:

$$
\begin{array}{ccccccccc}
0 & \to & F^{\nu+1}\mathrm{CH}^r(X;\mathbb{Q}) & \to & F^{\nu}\mathrm{CH}^r(X;\mathbb{Q}) & \to & Gr_F^{\nu}\mathrm{CH}^r(X;\mathbb{Q}) & \to & 0 \\
 & & j_* \circ j^* \Big\downarrow & & j_* \circ j^* \Big\downarrow & & j_* \circ j^* \Big\downarrow & & \\
0 & \to & F^{\nu+1}\mathrm{CH}^{r+1}(X;\mathbb{Q}) & \to & F^{\nu}\mathrm{CH}^{r+1}(X;\mathbb{Q}) & \to & Gr_F^{\nu}\mathrm{CH}^{r+1}X;\mathbb{Q}) & \to & 0
\end{array}
\tag{2}
$$

it follows that if the left and right vertical arrows in diagram (2) are injective, then so is the middle. By downward induction on ν, we deduce from the BB filtration that $j_* \circ j^*$ in diagram (1) is injective, *a fortiori* j_* is injective in (1). Now let $k = d-1-r = \dim D - r$. Then we have $j_* : \mathrm{CH}_k(D;\mathbb{Q}) \to \mathrm{CH}_k(X;\mathbb{Q})$ injective, provided $k > \dim D/2$. Quite generally, one can show the following:

Theorem 2.6. *Assume the Hodge (HC) and Bloch-Beilinson (BBC) conjectures. Then:*

$$k > \frac{\dim D - \nu}{2} \Rightarrow j_* : F^{\nu}\mathrm{CH}_k(D;\mathbb{Q}) \hookrightarrow F^{\nu}\mathrm{CH}_k(X;\mathbb{Q}).$$

Recall that under the assumptions, the BB-filtration is the same as Lewis' filtration.

Now if we allow the injective statement $j^* : H^{d-1}(X, \mathbb{Q}) \hookrightarrow H^{d-1}(D, \mathbb{Q})$, then in diagram (1), j^* is injective with left inverse $(j^*)^{-1}$. Then $2k = 2\dim D - 2r \geq \dim D > \dim D - 1$, i.e. $k > \frac{\dim D-1}{2}$, but a caveat is in order here as $(j^*)^{-1}$ is not injective. We can get around this by restricting to null-homologous cycles, via the above theorem for $\nu = 1$.

The next 3 examples illustrate what can happen if

$$\frac{\dim D - 1}{2} < k \leq \frac{\dim D}{2},$$

thus indicating that the inequality in Conjecture 1.4, is effective.

Example 2.7. *Let $j : D \hookrightarrow X$ be a finite set of points defining an ample divisor on a smooth curve X. We assume that D supports a zero cycle that is rationally equivalent to zero on X. Obviously $j_* : \mathrm{CH}_0(D;\mathbb{Q}) \to \mathrm{CH}_0(X;\mathbb{Q})$ is not injective, and yet $k = 0 = (\dim D)/2$.*

Example 2.8. *Let $j : D \hookrightarrow X := \mathbb{P}^3$ be a smooth surface with Picard rank $\rho > 1$, such as a Fermat surface of degree ≥ 2. Note that $\mathrm{CH}_1(D;\mathbb{Q}) \simeq \mathbb{Q}^{\rho}$ and $\mathrm{CH}_1(X;\mathbb{Q}) \simeq \mathbb{Q}$. Thus $j_* : \mathrm{CH}_1(D;\mathbb{Q}) = F^0\mathrm{CH}_1(D;\mathbb{Q}) \to F^0\mathrm{CH}_1(X;\mathbb{Q}) = \mathrm{CH}_1(X;\mathbb{Q})$ is not injective. Here $k = 1 = (\dim D)/2$ and $\nu = 0$. If $k = 1 = \nu$, then $j_* : F^1\mathrm{CH}_1(D;\mathbb{Q}) = \mathrm{CH}_{1,\mathrm{hom}}(D;\mathbb{Q}) \to \mathrm{CH}_{1,\mathrm{hom}}(X;\mathbb{Q}) = F^1\mathrm{CH}_1(X;\mathbb{Q})$ is trivially injective since $\mathrm{CH}_{1,\mathrm{hom}}(D;\mathbb{Q}) = 0$. Here $k = 1 > (\dim D - 1)/2$.*

Example 2.9. *Let $D =$ Fermat quintic in $\mathbb{P}^5 =: X$. Let $\xi = L_1 - L_2 \in \mathrm{CH}_2(D;\mathbb{Q})$, a difference of two nonhomologous planes in D. Then $j_*(\xi) = 0$. Here $k = 2 = (\dim D)/2$.*

Regarding Conjecture 1.1, if $n = \dim X$ then we require $p < n-1$ for an isomorphism and $p = n-1$ for an injection. [Consider the fact that $\mathrm{CH}^n(D) = 0$, and yet $\mathrm{CH}^n(X)$ can be highly nontrivial.]

2.10. **Higher Chow analogues.** From the works of M. Saito and M. Asakura (see [AS]), Theorem 2.4 naturally extends to the higher Chow groups. In particular, if one assumes the HC, together with a generalized version of the BBC, viz.,

Conjecture 2.11. *Let $W/\overline{\mathbb{Q}}$ be a smooth projective variety. Then the Abel-Jacobi map*

$$\mathrm{CH}^r_{\mathrm{hom}}(W/\overline{\mathbb{Q}}, m; \mathbb{Q}) \to J\big(H^{2r-m-1}(W, \mathbb{Q}(r))\big),$$

is injective;

then for $X/\mathbb{C}$ smooth projective of dimension d, there is a (unique) BB filtration

$$\{F^\nu \mathrm{CH}^r(X, m; \mathbb{Q})\}^r_{\nu=0},$$

for which the ν-th graded piece

$$Gr^\nu_F \mathrm{CH}^r(X, m; \mathbb{Q}) \simeq \Delta_X(2d - 2r + m + \nu, 2r - m - \nu)_* \mathrm{CH}^r(X, m; \mathbb{Q}).$$

Theorem 2.12. *Let us assume Conjecture 2.11 and the HC. Then*

$$j^* : \mathrm{CH}^r(X, m; \mathbb{Q}) \xrightarrow{\sim} \mathrm{CH}^r(D, m; \mathbb{Q}),$$

for

$$r \leq \frac{\dim D + m - 1}{2}; \quad \text{moreover,}$$
$$k > \frac{\dim D - \nu + m}{2} \Rightarrow j_* : F^\nu \mathrm{CH}_k(D, m; \mathbb{Q}) \hookrightarrow F^\nu \mathrm{CH}_k(X, m; \mathbb{Q}).$$

Proof. (Sketch.) Using the theory of mixed Hodge modules [AS], the idea of proof is virtually the same as when $m = 0$, with a modification of indices. For instance, one is now dealing with a short exact sequence

$$0 \to \underline{E}_\infty^{\nu, 2r-\nu-m}(\rho) \to E_\infty^{\nu, 2r-\nu-m}(\rho) \to \underline{\underline{E}}_\infty^{\nu, 2r-\nu-m}(\rho) \to 0,$$

where

$$\underline{\underline{E}}_\infty^{\nu, 2r-\nu-m}(\rho) = \Gamma\big(H^\nu(\mathcal{S}, R^{2r-\nu-m}\rho_* \mathbb{Q}(r))\big),$$
$$\underline{E}_\infty^{\nu, 2r-\nu-m}(\rho) = \frac{J\big(W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big)}{\Gamma\big(Gr^0_W H^{\nu-1}(\mathcal{S}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big)} \subset J\big(H^{\nu-1}(\mathcal{S}, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))\big).$$

The statement $j^* : H^{2r-\nu-m}(X, \mathbb{Q}) \xrightarrow{\sim} H^{2r-\nu-m}(D, \mathbb{Q})$ holds for all $\nu = 0, ..., r$ provided that $2r - m \leq \dim D - 1$, i.e. $r \leq \frac{\dim D + m - 1}{2}$. Quite generally

$$r \leq \frac{m + \nu + \dim D - 1}{2} \Rightarrow j^* : F^\nu \mathrm{CH}^r(X, m; \mathbb{Q}) \xrightarrow{\sim} F^\nu \mathrm{CH}^r(D, m; \mathbb{Q}).$$

For the latter part of the theorem, observe that $\mathrm{CH}^r(D, m) = \mathrm{CH}_k(D, m)$, where $k = \dim D + m - r$. Then

$$r \leq \frac{m + \nu + \dim D - 1}{2} \Leftrightarrow k \geq \frac{\dim D + m - \nu + 1}{2} \Leftrightarrow k > \frac{\dim D + m - \nu}{2}.$$

One then argues, as in the case $m = 0$, that

$$k > \frac{\dim D + m - \nu}{2} \Rightarrow j_* : F^\nu \mathrm{CH}_k(D, m; \mathbb{Q}) \hookrightarrow F^\nu \mathrm{CH}_k(X, m; \mathbb{Q}).$$

□

Example 2.13. *Let $X = \mathbb{P}^2$ and $j : D \hookrightarrow X$ an elliptic curve. We consider the map $j_* : \mathrm{CH}_1(D, 2; \mathbb{Q}) \to \mathrm{CH}_1(\mathbb{P}^2, 2; \mathbb{Q})$. In this case $k = 1$ is almost, but not quite in the range of the above theorem, even in the event that $\nu = 1$, where it is well-known that for $m \geq 1$ that $F^0\mathrm{CH}^r(X, m; \mathbb{Q}) = F^1\mathrm{CH}^r(X, m; \mathbb{Q})$, as $\Gamma H^{2r-m}(W, \mathbb{Q}(r)) = 0$, for any projective algebraic manifold W. Note that $\mathrm{CH}_1(D, 2; \mathbb{Q}) = \mathrm{CH}^2(D, 2; \mathbb{Q})$ and $\mathrm{CH}_1(\mathbb{P}^2, 2; \mathbb{Q}) = \mathrm{CH}^3(\mathbb{P}^2, 2; \mathbb{Q})$. We need the following terminology. Given a variety $Y/\mathbb{C}$, we denote by $\pi_Y : Y \to \mathbf{Spec}(\mathbb{C})$ the structure map, and where appropriate, L_Y is the operation of taking the intersection product with a hyperplane section of Y. Note that by a slight generalization of the Bloch-Quillen formula, $\mathrm{CH}^1(Y, 2) = 0$ for smooth Y, and for dimension reasons, $\mathrm{CH}^3(\mathbf{Spec}(\mathbb{C}), 2) = 0$. Thus by the projective bundle formula, $\mathrm{CH}^3(\mathbb{P}^2, 2) = L_{\mathbb{P}^2} \cup \pi^*_{\mathbb{P}^2}\mathrm{CH}^2(\mathbf{Spec}(\mathbb{C}), 2) \simeq \mathrm{CH}^2(\mathbf{Spec}(\mathbb{C}), 2)$. Note that $\pi^*_D : \mathrm{CH}^2(\mathbf{Spec}(\mathbb{C}), 2; \mathbb{Q}) \to \mathrm{CH}^2(D, 2; \mathbb{Q})$ is injective. This is because, up to multiplication by some $N \in \mathbb{N}$, the left inverse is given by $\pi_{D,*} \circ L_D$. There is a commutative diagram:*

$$\begin{array}{ccc}
0 & & \\
\uparrow & & \\
\mathbf{cok} & & \\
\uparrow & & \\
\mathrm{CH}^2(D, 2; \mathbb{Q}) & \xrightarrow{j_*} & \mathrm{CH}^3(\mathbb{P}^2, 2; \mathbb{Q}) \\
\pi^*_D \uparrow & & \wr \downarrow \\
\mathrm{CH}^2(\mathbf{Spec}(\mathbb{C}), 2; \mathbb{Q}) & = & \mathrm{CH}^2(\mathbf{Spec}(\mathbb{C}), 2; \mathbb{Q}) \\
\uparrow & & \\
0 & &
\end{array}$$

It is obvious that $\mathbf{cok} \neq 0$ is the obstruction to j_* being injective, and yet that is the case if D is an elliptic curve. Note that if we accommodate the situation where $k = 2$, then we are looking at $j_* : 0 = \mathrm{CH}^1(D, 2) = \mathrm{CH}_2(D, 2) \to \mathrm{CH}_2(\mathbb{P}^2, 2) = \mathrm{CH}^2(\mathbb{P}^2, 2) \simeq \mathrm{CH}^2(\mathbf{Spec}(\mathbb{C}), 2) = K_2(\mathbb{C})$, which is clearly injective, albeit not surjective.

Example 2.14. *Let $X = \mathbb{P}^3$, and $j : D \hookrightarrow \mathbb{P}^2$ a general $K3$ surface. The map $j_* : \mathrm{CH}^2(D, 1; \mathbb{Q}) = \mathrm{CH}_1(D, 1; \mathbb{Q}) \to \mathrm{CH}^3(\mathbb{P}^3, 1; \mathbb{Q})$ is not injective, due to the presence of "indecomposables" in $\mathrm{CH}^2(D, 1; \mathbb{Q})$* [C-L]. *Notice that $k = 1 \leq \frac{\dim D - \nu + m}{2} = \frac{3-\nu}{2}$, for $\nu = 0, 1$. If we conside a $k = 2$ example, then we are looking at $j_* : \mathrm{CH}^1(D, 1) = \mathbb{C}^\times \xrightarrow{=} \mathbb{C}^\times \simeq \mathrm{CH}^2(\mathbb{P}^3, 1)$, which is an isomorphism in this case, a fortiori j_* is injective.*

3. Inclusion of Theta divisor into the Jacobian

In this section we investigate the kernel of the push-forward homomorphism, induced by the closed embedding j of the Theta divisor inside the Jacobian of a smooth projective

curve C of genus g. Recall that Θ is an ample divisor on $J(C)$. Consider the induced pushforward map on the rational Chow groups:

$$j_* : \mathrm{CH}_k(\Theta; \mathbb{Q}) \to \mathrm{CH}_k(J(C); \mathbb{Q})$$

for $k \geq 0$.

To investigate the map j_*, we use a similar comparison theorem ([Co]) on symmetric products of the curve C.

Fix a point P in C. Consider the following map j_C from $\mathrm{Sym}^{g-1}C$ to $\mathrm{Sym}^g C$ defined by

$$P_1 + \cdots + P_{g-1} \mapsto P_1 + \cdots + P_{g-1} + P \ .$$

Here the sum denotes the unordered set of points of lengths $(g-1)$ and (g).

With this definition of j_C the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Sym}^{g-1}C & \xrightarrow{j_C} & \mathrm{Sym}^g(C) \\ \downarrow{\scriptstyle q_\Theta} & & \downarrow{\scriptstyle q} \\ \Theta & \xrightarrow{j} & \mathrm{Pic}^g(C) \end{array}$$

We recall the structure of the birational morphisms q_Θ and q.

Lemma 3.1. *Suppose C is a smooth projective curve over the complex numbers.*

1) The morphism q is a blow-up along the subvariety

$$W_g^1 := \{l \in \mathrm{Pic}^g(C) : h^0(l) \geq 2\}.$$

Furthermore, the singular locus of W_g^1 is

$$W_g^2 = \{l \in \mathrm{Pic}^g(C) : h^0(l) \geq 3\}.$$

This is a Cohen-Macaulay and a normal variety. Hence codimension of W_g^2 in W_g^1 is at least two.

Denote $B = Sing(\Theta)$, the singular locus of Θ.

2) Then $\dim B = g-4$, *when C is non-hyperelliptic and is equal to $g-3$ if C is hyperelliptic.*

3) We have the equality:

$$B = W_{g-1}^1 = \{l \in \mathrm{Pic}^{g-1}(C) : h^0(l) \geq 2)\}$$

Furthermore, B is a Cohen-Maculay, and a normal variety.

In particular $\mathrm{codim}(Sing(B)) \geq 2$, *i.e., the singular locus of B has codimension at least* 2.

4) The morphism q is an isomorphisms on $\mathrm{Sym}^g(C) - q^{-1}(W_g^1)$ onto $J(C) - W_g^1$. The fibres over W_g^1 are projective spaces.

5) The morphism q_Θ is an isomorphism on $\mathrm{Sym}^{g-1}(C) - q_\Theta^{-1}(W^1_{g-1})$. The fibres over W^1_{g-1} are projective spaces.

Proof. See [ACGH, p.190, Proposition 4.4, Corollary 4.5] and [Mu, Theorem 2.3 and Lemma 1.2]. □

We identify $\mathrm{Pic}^r(C)$ with $\mathrm{Pic}^0(C)$ via the map $l \mapsto l \otimes \mathcal{O}_C(-r.p)$. Apply this to $r = g-1, g$, and we obtain the commutative diagram on the rational Chow groups:

$$\begin{array}{ccc} \mathrm{CH}_k(\mathrm{Sym}^{g-1}C;\mathbb{Q}) & \xrightarrow{j_{C*}} & \mathrm{CH}_k(\mathrm{Sym}^{g};\mathbb{Q}) \\ \downarrow{\scriptstyle q_{\Theta *}} & & \downarrow{\scriptstyle q_*} \\ \mathrm{CH}_k(\Theta;\mathbb{Q}) & \xrightarrow{j_*} & \mathrm{CH}_k(J(C);\mathbb{Q}) \end{array} \tag{3}$$

3.2. $k = 0$**.** We start by looking at the case when $k = 0$.

Proposition 3.3. *Let C be a smooth projective curve of genus g. Let Θ be a Theta divisor embedded inside $J(C)$ and let j denote the embedding. Then the push-forward homomorphism*

$$j_* : \mathrm{CH}_0(\Theta;\mathbb{Q}) \to \mathrm{CH}_0(J(C);\mathbb{Q})$$

is injective.

Proof. Refer to the commutative diagram (3). Consider the pushforward map:

$$j_* : \mathrm{CH}_0(\Theta;\mathbb{Q}) \to \mathrm{CH}_0(J(C);\mathbb{Q}).$$

By Collino's theorem [Co, Theorem 1], the map $(j_C)_*$ is injective. Since the morphisms q and q_C are birational morphisms (see [Mu]), and CH_0 is a birational invariant for smooth varieties, we have the equality:

$$\mathrm{CH}_0(J(C);\mathbb{Q}) \;=\; \mathrm{CH}_0(\mathrm{Sym}^g(C);\mathbb{Q}).$$

We refer to Lemma 3.1, and consider $B := Sing(\Theta)$, the singular locus of Θ and $U := \Theta - B$. Now q_Θ is an isomorphism outside B. Consider the localization maps:

$$\mathrm{CH}_0(q_\Theta^{-1}B) \to \mathrm{CH}_0(\mathrm{Sym}^{g-1}(C);\mathbb{Q}) \to \mathrm{CH}_0(U;\mathbb{Q}) \to 0$$

and

$$\mathrm{CH}_0(B) \to \mathrm{CH}_0(\Theta;\mathbb{Q}) \to \mathrm{CH}_0(U;\mathbb{Q}) \to 0.$$

There is a stratification of B, on which the restriction of q_Θ is given by projective bundles. By the projective bundle formula, we conlcude that

$$\mathrm{CH}_0(q_\Theta^{-1}(B);\mathbb{Q}) = \mathrm{CH}_0(B;\mathbb{Q}).$$

Hence we conclude the injectivity of j_*. □

3.4. **Case $g = 3$, $k = 1$.** Suppose genus(C) $= 3$ and C is non-hyperelliptic. Here $\mathrm{Sym}^3(C)$ is the blow-up of $J(C)$ along the curve C, i.e.

$$C = \{L \in \mathrm{Pic}^3(C) : h^0(L) \geq 2\}.$$

Furthermore, $\Theta = \mathrm{Sym}^2(C)$. See Lemma 3.1. Here $C \hookrightarrow J(C) \simeq \mathrm{Pic}^3(C)$, via $\mathcal{O}(x) \mapsto K_C \otimes \mathcal{O}_C(-x)$, where K_C is the canonical line bundle of C.

Hence we can write

$$q : \mathrm{Sym}^3(C) = Bl_C(J(C)) \to J(C).$$

Let E_C denote the exceptional surface inside $\mathrm{Sym}^3(C)$.

By the blow-up formula:

$$\mathrm{CH}_1(\mathrm{Sym}^3(C); \mathbb{Q}) \;=\; \mathrm{CH}_1(J(C); \mathbb{Q}) \oplus \mathrm{CH}_1(E_C; \mathbb{Q}).$$

Proposition 3.5. *Assume $g = 3$ and C is non-hyperelliptic. The pushforward map*

$$j_* : F^1\mathrm{CH}_1(\Theta; \mathbb{Q}) \to F^1\mathrm{CH}_1(J(C); \mathbb{Q})$$

is injective.

Proof. Let Θ be denoted by $H := [\Theta] \in \mathrm{CH}^1(J(C))$. Consider the maps given by intersection with the Θ, in $J(C)$:

$$\mathrm{CH}^1(J(C); \mathbb{Q}) \stackrel{\cap H}{\to} \mathrm{CH}^1(\Theta; \mathbb{Q}) \to \mathrm{CH}^2(J(C); \mathbb{Q}).$$

This map restricts on the F^1-piece, and is compatible with the Abel-Jacobi maps. Hence we get a commutative diagram:

$$\begin{array}{ccccc}
F^1\mathrm{CH}^1(J(C); \mathbb{Q}) & \xrightarrow{\cap H} & F^1\mathrm{CH}^1(\Theta; \mathbb{Q}) & \xrightarrow{j_*} & F^1\mathrm{CH}^2(J(C); \mathbb{Q}) \\
\downarrow{\scriptstyle AJ_J^1} & & \downarrow{\scriptstyle AJ_\Theta^1} & & \downarrow{\scriptstyle AJ_J^2} \\
IJ(H^1(J(C); \mathbb{Q}) & \xrightarrow{\cap H} & IJ(H^1(\Theta; \mathbb{Q})) & \xrightarrow{h} & IJ(H^3(J(C); \mathbb{Q})
\end{array}$$

Now AJ_J^1 and AJ_Θ^1 are isomorphisms. Since H is an ample divisor, $h \circ (\cap H)$ is injective by the hard Lefschetz theorem, and $\cap H$ on $IJ(H^1(J(C); \mathbb{Q})$ is an isomorphism, by Lefschetz hyperplane theorem. This implies that h is injective. Hence j_* is injective. □

4. Abel-Jacobi maps on $F^1\mathrm{CH}_r(\Theta; \mathbb{Q})$

When $g \geq 4$, the theta divisor is singular and the singular locus B has dimension at least $g - 4$. When C is non-hyperelliptic the dimension is equal to $g - 4$. See Lemma 3.1. We will consider a non-hyperelliptic curve C, which is the generic situation.

Consider the localization sequence:

$$\mathrm{CH}_k(B; \mathbb{Q}) \to \mathrm{CH}_k(\Theta; \mathbb{Q}) \to \mathrm{CH}_k(\Theta - B) \to 0.$$

We would like to know how F^1 behaves with localization and associate Abel-Jacobi maps to the F^1-terms.

4.1. **General Abel-Jacobi maps.** Suppose X is a smooth quasi-projective variety defined over the complex numbers. Let $X \subset \overline{X}$ be a smooth compactification of X. Let MHS denote the category of $\mathbb{Q}$-mixed Hodge structures. There is an Abel-Jacobi map:

$$\begin{aligned} \mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q}) &\to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathrm{H}^{2\mathrm{m}-1}(\mathrm{X}, \mathbb{Q}(\mathrm{m}))) \\ &= \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathrm{W}_0\mathrm{H}^{2\mathrm{m}-1}(\mathrm{X}, \mathbb{Q}(\mathrm{m}))) \end{aligned}$$

We are interested in the Abel-Jacobi map restricted to the image:

$$\mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q})^\circ := \mathrm{Im}\big(\mathrm{CH}^m_{\mathrm{hom}}(\overline{X};\mathbb{Q}) \to \mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q})\big).$$

The conjectured equality

$$\mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q})^\circ = \mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q}),$$

is a consequence of the Hodge conjecture. In fact, we have:

Proposition 4.2. *Let $Y \subset \overline{X}$ be a subvariety, where $\overline{X}$ is smooth projective of dimension n, and let $X = \overline{X}\backslash Y$. For $m \leq 2$ and $m \geq n-1$ (and more generally for all m if one assumes the Hodge conjecture), there is an exact sequence:*

$$\mathrm{CH}^m_Y(\overline{X};\mathbb{Q})^\circ \to \mathrm{CH}^m_{\mathrm{hom}}(\overline{X};\mathbb{Q}) \to \mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q}) \to 0,$$

where

$$\mathrm{CH}^m_Y(\overline{X};\mathbb{Q}) = \mathrm{CH}_{n-m}(Y;\mathbb{Q}),$$

and

$$\mathrm{CH}^m_Y(\overline{X};\mathbb{Q})^\circ := \{\xi \in \mathrm{CH}^m_Y(\overline{X};\mathbb{Q}) \mid j(\xi) \in \mathrm{CH}^m_{\mathrm{hom}}(\overline{X};\mathbb{Q})\}.$$

Here j is the map

$$\mathrm{CH}^m_Y(\overline{X};\mathbb{Q}) \xrightarrow{j} \mathrm{CH}^m(\overline{X};\mathbb{Q}) \to \mathrm{CH}^m(X;\mathbb{Q}) \to 0.$$

Proof. Let $\xi \in \mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q})$, and choose $\overline{\xi} \in \mathrm{CH}^m(\overline{X};\mathbb{Q})$ which maps to ξ. By construction, the fundamental class $[\overline{\xi}] \in H^{2m}(\overline{X}, \mathbb{Q}(m))$ lies in the image $H^{2m}_Y(\overline{X}, \mathbb{Q}(m)) \to H^{2m}(\overline{X}, \mathbb{Q}(m))$. Let $q = \mathrm{codim}_{\overline{X}} Y$, and let $\sigma : \tilde{Y} \to Y$ be a desingularization. By a weight argument and mixed Hodge theory, $[\overline{\xi}]$ lies in the image $H^{m-q,m-q}(\tilde{Y}, \mathbb{Q}(m-q)) \to H^{2m}(\overline{X}, \mathbb{Q}(m))$, which will come from the fundamental class of an algebraic cycle $\gamma \in \mathrm{CH}^{m-q}(\tilde{Y};\mathbb{Q})$, provided that the Hodge conjecture holds for $\tilde{Y}$. Assuming this, then $\overline{\xi} - j(\sigma_*(\gamma)) \in \mathrm{CH}^m_{\mathrm{hom}}(\overline{X};\mathbb{Q})$ maps to $\xi \in \mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q})$. The rest is clear. □

Thus we get a map $\mathrm{CH}^m(X;\mathbb{Q})^\circ_{\mathrm{hom}} \to$

$$\mathrm{Im}\big(\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathrm{W}_{-1}\mathrm{H}^{2\mathrm{m}-1}(\mathrm{X}, \mathbb{Q}(\mathrm{m})) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathrm{W}_0\mathrm{H}^{2\mathrm{m}-1}(\mathrm{X}, \mathbb{Q}(\mathrm{m}))\big), \tag{4}$$

where we use the fact that

$$W_{-1}H^{2m-1}(X, \mathbb{Q}(m)) = \mathrm{Image}\big(H^{2m-1}(\overline{X}, \mathbb{Q}(m)) \to H^{2m-1}(X, \mathbb{Q}(m))\big),$$

together with the Abel-Jacobi image of a class in $\mathrm{CH}^m_{\mathrm{hom}}(X;\mathbb{Q})$ being in the Abel-Jacobi image of a class in $\mathrm{CH}^m_{\mathrm{hom}}(\overline{X};\mathbb{Q})$. Note that the term in (4) above can be identified with

$$\frac{\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathrm{W}_{-1}\mathrm{H}^{2\mathrm{m}-1}(\mathrm{X}, \mathbb{Q}(\mathrm{m}))}{\delta\, \mathrm{hom}_{\mathrm{MHS}}(\mathbb{Q}(0), Gr_0^W H^{2m-1}(X, \mathbb{Q}(m)))},$$

where δ is the connecting homomorphism in the long exact sequence associated to

$$0 \to W_{-1}H^{2m-1}(X, \mathbb{Q}(m)) \to W_0 H^{2m-1}(X, \mathbb{Q}(m)) \to Gr_0^W H^{2m-1}(X, \mathbb{Q}(m)) \to 0.$$

In case, $Gr_0^W H^{2m-1}(X, \mathbb{Q}(m)) = 0$ then the target of the Abel-Jacobi map is the group $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathrm{W}_{-1}\mathrm{H}^{2\mathrm{m}-1}(\mathrm{X}, \mathbb{Q}(\mathrm{m}))$.

4.3. **F^1-term of $\mathrm{CH}_k(\mathrm{Sym}^{g-1};\mathbb{Q})$ and $\mathrm{CH}_k(\Theta;\mathbb{Q})$.** In this subsection, we consider the situation $B \subset \Theta$. Denote the complement $U := \Theta - B$.

Recall from Lemma 3.1, that the morphism $q_\Theta : \mathrm{Sym}^{g-1}C \to \Theta$ is a birational morphism and is a smooth resolution of Θ ([Mu]).

Denote $U_{\mathrm{sym}} := {q_\Theta}^{-1}(U)$ and $Y := \mathrm{Sym}^{g-1}(C) - U_{\mathrm{sym}}$. Note $U_{\mathrm{sym}} \simeq U$.

Lemma 4.4. *The restriction map*

$$F^1\mathrm{CH}_k(\mathrm{Sym}^{g-1}(C);\mathbb{Q}) \to F^1\mathrm{CH}_k(U_{\mathrm{sym}};\mathbb{Q})$$

is surjective, when $k = g-2,\ g-3$.

Proof. Apply the localization sequence and Proposition 4.2, to the triple

$$(Y \subset \mathrm{Sym}^{g-1}(C) \supset U_{\mathrm{sym}}).$$

This gives a surjective map, when $k = g-2,\ g-3$:

$$F^1\mathrm{CH}_k(\mathrm{Sym}^{g-1}(C);\mathbb{Q}) \to F^1\mathrm{CH}_k(U_{\mathrm{sym}};\mathbb{Q}) \to 0. \tag{5}$$

□

Consider the pushforward $q_{\Theta *} : \mathrm{CH}_k(\mathrm{Sym}^{g-1}(C);\mathbb{Q}) \to \mathrm{CH}_k(\Theta;\mathbb{Q})$.

Lemma 4.5. *For $k = g-2, g-3$, the restriction map in (5) induces a map*

$$\frac{F^1\mathrm{CH}_k(\mathrm{Sym}^{g-1}(C);\mathbb{Q})}{F^1 \cap \ker(q_{\Theta *})} \to F^1\mathrm{CH}_k(U_{\mathrm{sym}};\mathbb{Q})$$

which is an isomorphism.

Proof. Since C is a non-hyperelliptic curve, $dim(B) = g-4$. Hence if $k > g-4$, the restriction

$$h_* : \mathrm{CH}_k(\Theta;\mathbb{Q}) \to \mathrm{CH}_k(U;\mathbb{Q})$$

is an isomorphism. Consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}_k(\mathrm{Sym}^{g-1}C;\mathbb{Q}) & \xrightarrow{j_{u*}} & \mathrm{CH}_k(U_{\mathrm{sym}};\mathbb{Q}) \\ \downarrow{\scriptstyle q_{\Theta *}} & & \downarrow{\scriptstyle q_{u*}} \\ \mathrm{CH}_k(\Theta;\mathbb{Q}) & \xrightarrow{h_*} & \mathrm{CH}_k(U;\mathbb{Q}) \end{array}$$

This induces a corresponding diagram on the F^1-terms. Note that q_{u*} is an isomorphism, since q_Θ is an isomorphism outside B. Hence, we obtain an isomorphism:

$$\frac{F^1\mathrm{CH}_k(\mathrm{Sym}^{g-1}(C);\mathbb{Q})}{F^1 \cap \ker(q_{\Theta *})} \to F^1\mathrm{CH}_k(U_{\mathrm{sym}};\mathbb{Q}) = F^1\mathrm{CH}_k(\Theta;\mathbb{Q}) \tag{6}$$

□

4.6. **Abel-Jacobi maps on F^1-terms.** When $k = g-2, g-3$, denote $l = 1, 2$ the corresponding codimension. Using §4.1 and purity of Hodge structures (here $\dim B = g-4$), there is an Abel-Jacobi map:

$$AJ_\Theta : F^1\mathrm{CH}_k(\Theta;\mathbb{Q}) = F^1\mathrm{CH}_k(U;\mathbb{Q}) \to IJ(H^l(U);\mathbb{Q}), \tag{7}$$

where we recall that:

$$F^2\mathrm{CH}_k(X;\mathbb{Q}) = F^2\mathrm{CH}_k(U;\mathbb{Q}) = \mathrm{kernel}(\mathrm{AJ}_\Theta).$$

We recall the following, which will be used in the next section.

Lemma 4.7. *The Abel-Jacobi map*

$$AJ : F^1CH^1(U,\mathbb{Q}) \to IJ(H^1(U,\mathbb{Q}))$$

is an isomorphism.

Proof. See [JL, Proposition 2.5]. □

5. Main Theorem

Now we can state our main theorem.

Theorem 5.1. *Assume C is a non-hyperelliptic smooth projective curve of genus $g \geq 3$, over $\mathbb{C}$. The pushforward morphisms*

$$j_* : F^1\mathrm{CH}_{g-2}(\Theta;\mathbb{Q}) \to F^1\mathrm{CH}_{g-2}(J(C);\mathbb{Q})$$

and

$$j_* : F^2\mathrm{CH}_{g-3}(\Theta;\mathbb{Q}) \to F^2\mathrm{CH}_{g-3}(J(C);\mathbb{Q})$$

are injective.

Note that the first injectivity statement generalizes Proposition 3.5.

Proof. Consider the birational morphisms

$$q_\Theta : \mathrm{Sym}^{g-1}(C) \to \Theta$$

and

$$q : \mathrm{Sym}^{g}(C) \to J(C).$$

These maps induce the commutative diagram on the F^1-terms of the rational Chow groups:

$$\begin{array}{ccc} F^1\mathrm{CH}_k(\mathrm{Sym}^{g-1}(C);\mathbb{Q}) & \xrightarrow{j_{C*}} & F^1\mathrm{CH}_k(\mathrm{Sym}^{g}(C);\mathbb{Q}) \\ \downarrow{\scriptstyle q_{\Theta *}} & & \downarrow{\scriptstyle q_*} \\ F^1\mathrm{CH}_k(\Theta;\mathbb{Q}) & \xrightarrow{j_*} & F^1\mathrm{CH}_k(J(C);\mathbb{Q}) \end{array}$$

Denote $l = g - 1 - k$. The above diagram is compatible via Abel-Jacobi maps to the corresponding commutative digram of intermediate Jacobians:

$$\begin{array}{ccc} IJ(H^l(\mathrm{Sym}^{g-1}(C);\mathbb{Q})) & \xrightarrow{j_{C*}} & IJ(H^{l+2}((\mathrm{Sym}^{g}(C);\mathbb{Q})) \\ \downarrow{\scriptstyle q_{\Theta *}} & & \downarrow{\scriptstyle q_*} \\ IJ(H^l(\Theta;\mathbb{Q})) & \xrightarrow{h} & IJ(H^{l+2}((J(C);\mathbb{Q})) \end{array}$$

In the above diagram, $l = 1, 2$. Since $\mathrm{codim}(B) \geq 2$ in Θ, without any confusion, we write $IJ(H^l(\Theta;\mathbb{Q})) = IJ(H^l(U;\mathbb{Q}))$.

Case 1) $l = 1$.

Denote H the ample divisor Θ on $J(C)$, Consider the diagram obtained by intersecting with H in $\mathrm{CH}^*(J(C))$ (resp. in $H^*(J(C),\mathbb{Q})$).

$$\begin{array}{ccccc} F^1\mathrm{CH}^1(J(C);\mathbb{Q}) & \xrightarrow{\cap H} & F^1\mathrm{CH}^1(\Theta;\mathbb{Q}) & \xrightarrow{j_*} & F^1\mathrm{CH}^2(J(C);\mathbb{Q}) \\ \downarrow{\scriptstyle AJ_J^1} & & \downarrow{\scriptstyle AJ_\Theta} & & \downarrow{\scriptstyle AJ_J^2} \\ IJ(H^1(J(C);\mathbb{Q}) & \xrightarrow{\cap H} & IJ(H^1(\Theta;\mathbb{Q})) & \xrightarrow{h} & IJ(H^3(J(C);\mathbb{Q}) \end{array}$$

Now AJ_J^1 and AJ_Θ are isomorphisms. In particular AJ_Θ is defined in terms of $U := \Theta - B$ (see Lemma 4.7). Since H is an ample class, $h \circ (\cap H)$ is injective by the hard Lefschetz theorem, and $\cap H$ on $IJ(H^1(J(C);\mathbb{Q})$ is an isomorphism, by the Lefschetz hyperplane theorem. This implies that h is injective. Hence j_* is injective.

Case 2): $l = 2$.

Now $q : \mathrm{Sym}^g(C) \to J(C)$ is a blow-up morphism along a codimension two subvariety

$$W = W_g^1 = \{L \in \mathrm{Pic}^g(C) : h^0(L) \geq 2\}.$$

(See Lemma 3.1). Denote $E_W \subset \mathrm{Sym}^g(C)$ the exceptional locus of the blow-up morphism q. In particular, we can write a decomposition:

$$\mathrm{CH}_k(\mathrm{Sym}^g(C); \mathbb{Q}) = \mathrm{CH}_k(J(C); \mathbb{Q}) \oplus \mathrm{CH}_k(E_W; \mathbb{Q}) \tag{8}$$

Denote H the ample divisor $\mathrm{Sym}^{g-1}(C)$ on $\mathrm{Sym}^g(C)$, in $\mathrm{CH}^*(\mathrm{Sym}^g(C); \mathbb{Q})$ (resp. in $H^*(\mathrm{Sym}^g(C), \mathbb{Q})$).

Consider the Abel-Jacobi maps on the F^1-terms of the Chow groups of the symmetric products, which are compatible with the intersection $\cap H$:

$$\begin{array}{ccccc}
F^1\mathrm{CH}^2(\mathrm{Sym}^g(C); \mathbb{Q}) & \xrightarrow{\cap H} & F^1\mathrm{CH}^2(\mathrm{Sym}^{g-1}(C); \mathbb{Q}) & \xrightarrow{j_*} & F^1CH^3(\mathrm{Sym}^g(C); \mathbb{Q}) \\
\downarrow{\scriptstyle AJ^1_{sym}} & & \downarrow{\scriptstyle AJ_{sym}} & & \downarrow{\scriptstyle AJ^2_{sym}} \\
IJ(H^3(\mathrm{Sym}^g(C); \mathbb{Q}) & \xrightarrow{\cap H} & IJ(H^2(\mathrm{Sym}^{g-1}(C); \mathbb{Q})) & \xrightarrow{h} & IJ(H^5(\mathrm{Sym}^g(C); \mathbb{Q}).
\end{array}$$

By [Co, Theorem 1], the Chow restriction map $\cap H$ is surjective and j_* is injective. This implies that using the decomposition in (8), we can write the above commutative diagram as

$$\begin{array}{ccccc}
F^1\mathrm{CH}^2(J(C); \mathbb{Q}) \oplus F^1\mathrm{CH}^1(E_W; \mathbb{Q}) & \xrightarrow{\cap H} & H.F^1\mathrm{CH}^2(\mathrm{Sym}^g(C); \mathbb{Q}) \oplus H.F^1\mathrm{CH}^1(E_W; \mathbb{Q}) & \xrightarrow{j_*} & F^1\mathrm{CH}^3(\mathrm{Sym}^g(C); \mathbb{Q}) \\
\downarrow{\scriptstyle AJ^1_{sym}} & & \downarrow{\scriptstyle AJ_{sym}} & & \downarrow{\scriptstyle AJ^2_{sym}} \\
IJ(H^3(\mathrm{Sym}^g(C); \mathbb{Q}) & \xrightarrow{\cap H} & IJ(H^3(\mathrm{Sym}^{g-1}(C); \mathbb{Q})) & \xrightarrow{h} & IJ(H^5(\mathrm{Sym}^g(C); \mathbb{Q})
\end{array}$$

A similar decomposition exists for the intermediate Jacobians. This implies that we have the equality

$$\mathrm{Kernel}(\mathrm{AJ}_{\mathrm{sym}}) = \mathrm{Kernel}(\mathrm{AJ}^{\mathrm{J}}_{\mathrm{sym}}) \oplus \mathrm{Kernel}(\mathrm{AJ}^{\mathrm{W}}_{\mathrm{sym}}).$$

Here AJ^J_{sym} is the restriction of AJ_{sym} on the first summand and AJ^W_{sym} is the restriction of AJ_{sym} on the second summand.

However,

$$AJ^W_{\mathrm{sym}} : H.F^1\mathrm{CH}^1(E_W; \mathbb{Q}) \to H.IJ(H^1(E_W))$$

has no kernel.

Hence,

$$\mathrm{Kernel}(\mathrm{AJ}_{\mathrm{sym}}) = \mathrm{Kernel}(\mathrm{AJ}^{\mathrm{J}}_{\mathrm{sym}})$$

In other words, if we consider the composed map

$$F^1\mathrm{CH}^2(\mathrm{Sym}^{g-1}(C); \mathbb{Q}) \to F^1\mathrm{CH}^2(\mathrm{Sym}^g(C); \mathbb{Q}) \to F^1\mathrm{CH}^3(J(C); \mathbb{Q})$$

(the second map is the projection to its first summand), then it induces an injective map

$$F^2\mathrm{CH}^2(\mathrm{Sym}^{g-1}(C);\mathbb{Q}) \hookrightarrow F^2\mathrm{CH}^3(J(C);\mathbb{Q})$$

Now observe that

$$F^2\mathrm{CH}^2(\mathrm{Sym}^{g-1}(C);\mathbb{Q}) = F^2\left(\frac{\mathrm{CH}^2(\mathrm{Sym}^{g-1}(C);\mathbb{Q})}{\ker(q_{\Theta *})}\right).$$

This is because $\ker(q_{\Theta *})$ is supported on $\Theta - B$, and

$$q_{\Theta *} : \mathrm{CH}^2(\mathrm{Sym}^{g-1}(C);\mathbb{Q}) \to \mathrm{CH}^2(\Theta;\mathbb{Q})$$

is injective on the first summand $H.F^1\mathrm{CH}^2(\mathrm{Sym}^g(C);\mathbb{Q})$.

It now suffices to show that $F^2(H.\mathrm{CH}^1(E_W;\mathbb{Q})) = 0$, to conclude

$$F^2\mathrm{CH}^2(\Theta;\mathbb{Q}) \hookrightarrow F^2\mathrm{CH}^2(J(C);\mathbb{Q})$$

is injective, for $g \geq 4$ and C non-hyperelliptic.

□

Lemma 5.2.

$$F^2\mathrm{CH}_{g-2}(E_W;\mathbb{Q}) = 0$$

Proof. Now $\dim(E_W) = g-1$, which is a bundle of projective spaces over W.

Using Lemma 3.1 and $\mathrm{codim}(W) = 2$ in $J(C)$, (for a hyperplane class h on E_W) we can write:

$$\mathrm{CH}_{g-2}(E_W;\mathbb{Q}) = \mathrm{CH}_{g-2}(W;\mathbb{Q}).h \oplus \mathrm{CH}_{g-3}(W;\mathbb{Q}) \text{ (modulo the image } \mathrm{CH}_{g-2}(\mathrm{E}_{\mathrm{W}_g^2};\mathbb{Q})).$$

Since $dim(E_{W_g^2}) \leq g-2$, $F^1\mathrm{CH}_{g-2}(E_{W_g^2};\mathbb{Q}) = 0$.

Restricting to F^1-terms gives:

$$F^1\mathrm{CH}_{g-2}(E_W;\mathbb{Q}) = F^1\mathrm{CH}_{g-3}(W) = F^1\mathrm{CH}^1(W;\mathbb{Q}).$$

Furthermore, $\mathrm{codim}(Sing(W)) \geq 2$ (see Lemma 3.1,1)).

Now we are reduced to the case 1) situation when $l = 1$. Namely, there is an Abel-Jacobi map $W - Sing(W)$, which is an isomorphism onto $IJ(H^1(W - Sing(W)))$ (see Lemma 4.7). This shows the kernel of the Abel-Jacobi map is trivial, and $F^2 \subseteq \ker AJ$, one has that $F^2 = 0$. This suffices to conclude the proof. □

References

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, *Geometry of algebraic curves. Vol. I.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 267. Springer-Verlag, New York, 1985. xvi+386 pp.

[AS] M. Asakura, *Motives and algebraic de Rham cohomology.* In: Gordon, B., Lewis, J., Müller-Stach, S., Saito, S., Yui, N. (eds.) The Arithmetic and Geometry of Algebraic Cycles, Proceedings of the CRM Summer School, 7-19 June, 1998, Banff, Alberta. NATO Science Series, vol. **548**. Kluwer Academic Publishers, Berlin (2000)

[BG] K. Banerjee and V. Guletskii, *Rational equivalence for line configurations on cubic hypersurfaces in* $\mathbb{P}^5$., `arXiv:1405.6430v1`, 2014.

[Be] A. Beauville, *Sur l'anneau de Chow d'une variété abélienne*, Mathematische Annalen, 1986.
[BL] C. Birkenhake and H. Lange, *Complex abelian varieties*, Springer-Verlag Berlin, Heidelberg, New York, 2003.
[Bl] S. Bloch, *Algebraic cycles and higher K-theory*, Advances in Mathematics, **61**, 267-304, 1986.
[Co] A. Collino, *The rational equivalence ring of symmetric product of curves*, Illinois journal of mathematics,19,no.4, 567-583, 1975.
[C-L] X. Chen and J. D. Lewis, *The Hodge-$\mathcal{D}$-conjecture for K3 and Abelian surfaces,* J. Algebraic Geom. **14**(2), 213-240 (2005).
[Fu] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 2. Springer-Verlag, Berlin, 1984.
[Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.
[Ja] U. Jannsen, *Motivic sheaves and filtrations on Chow groups,* Proc. Symposia in Pure Math.,Vol. **55** (1994), 245–302.
[JL] J. D. Lewis, *A filtration on the Chow groups of a complex projective variety,* Compositio Math. **128**, (2001), 299–322.
[Mu] Munoz-Porras, *On the structure of the birational Abel morphism*, Math. Ann. 281 (1988), no. 1, 1–6.
[No] M.V. Nori, *Algebraic cycles and Hodge-theoretic connectivity*, Invent. Math. 111 (1993), no. **2**, 349–373.
[Pa] Paranjape, *Cohomological and cycle-theoretic connectivity*, Ann. of Math. (2) 139 (1994), no. **3**, 641–660.
[Vo] C. Voisin, *Complex algebraic geometry and Hodge theory II*, Cambridge University Press, Cambridge, 2003.

School of Mathematics, Tata Institute of Fundamental research, Homi Bhabha Road, Colaba, Mumbai 400001

Email address: kalyan@math.tifr.res.in

The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India

Email address: jniyer@imsc.res.in

Department of Mathematics, University of Alberta, Edmonton, AB, Canada T6G 2G1

Email address: lewisjd@ualberta.ca