ON 3-CANONICAL MAPS OF VARIETIES OF ALBANESE FIBER DIMENSION ONE

YUESEN CHEN

ABSTRACT. In the present paper, we study the (twisted) 3-canonical map of varieties of Albanese fiber dimension one. Based on a theorem about the regularity of direct image of canonical sheaves, we prove that the 3-canonical map is generically birational when the genus of a general fiber of the Albanese map is 2.

1. Introduction

Pluricanonical maps are an essential tool for understanding varieties of general type in birational geometry. As for irregular varieties, the most widely studied class is the varieties of maximal Albanese dimension (m.A.d. for short), whose pluricanonical maps have been researched extensively: Chen and Hacon [CH02] showed that the 3-canonical map of general type m.A.d. with $\chi(X,\omega_X)>0$ is birational, Jiang, Lahoz and Tirabassi [JLT13] then improved this result by eliminating the last condition. Barja et al. [BLNP12] studied the bicanonical map of m.A.d. of general type under some natural conditions. Lahoz [Lah12] and Zhang [Zha14] also proved some criteria for the birationality of bicanonical map from different points of view.

However, for a variety X of general type with Albanese fiber dimension 1, the properties of the pluricanonical map has not yet been well-established. The most updated literature is [JS15], in which Jiang and Sun proved that $|4K_X|$ is birational. It is then natural and interesting to study $|3K_X|$ for varieties of Albanese fiber dimension 1 carefully. One might conjecture that $|3K_X|$ is birational since $|3K_C|$ is birational for a curve C of genus $g \geq 2$. Even though we are unable to prove (or disprove) the conjecture, we provide some evidence towards a positive answer to this conjecture:

Theorem 1.1. Let X be a smooth complex projective variety of general type and of Albanese fiber dimension one, if the genus of a general fiber of the Albanese map is 2, then for general $P \in \text{Pic}^0(X)$, $|3K_X + P|$ is birational.

We give a summary of the proof. By a standard argument, it suffices to prove the theorem for a general $P \in \text{Pic}^0(X)_{tors}$. After constructing an intrinsic morphism, our proof is then divided into two parts: first we prove that the 3-canonical linear system separates two general fibers of the morphism; then we show that the 3-canonical linear system induces a birational map on each general fiber. Each step is heavily dependent on a decomposition theorem by Pareschi, Popa and Schnell (Theorem 2.4).

2. Preliminaries

We work over \mathbb{C} , the field of complex numbers, throughout this paper.

Let X be a smooth projective variety and $a_X : X \to A_X$ the Albanese map. We say that X is of Albanese fiber dimension l if dim $X - \dim a_X(X) = l$. In particular, X is of Albanese fiber dimension 0 if and only if X is of maximal Albanese dimension.

Given an abelian variety A, let $\hat{A} = \operatorname{Pic}^{0}(A)$ be its dual, whose elements are line bundles on A with trivial algebraic equivalence class. In some situation by an element of $\operatorname{Pic}^{0}(A)$ we also mean a divisor representing the line bundle.

For a morphism $t: X \to A$ to an abelian variety and a coherent sheaf \mathcal{F} on X, we denote by $V^i(\mathcal{F}, t)$ the *i*-th cohomological support loci:

$$\{P \in \operatorname{Pic}^0(A) \mid H^i(X, \mathcal{F} \otimes t^* P) \neq 0\}.$$

In particular, if $t = a_X$, we will simply denote $V^i(\mathcal{F}, a_X)$ by $V^i(\mathcal{F})$.

We recall the following definitions due to Pareschi, Popa [PP03] and Mukai.

Definition 2.1. Let \mathcal{F} be a coherent sheaf on a smooth projective variety X.

- (1) The sheaf \mathcal{F} is said to be a GV-sheaf if $\operatorname{codim} V^i(\mathcal{F}) > i$ for all i > 0.
- (2) If X is an abelian variety, a non-zero sheaf \mathcal{F} is said to be M-regular if $\operatorname{codim} V^i(\mathcal{F}) > i$ for every i > 0. Furthermore, a M-regular sheaf \mathcal{F} is said to be $\operatorname{IT}(0)$ if $V^i(\mathcal{F}) = \emptyset$ for all i > 0. (We set $\dim \emptyset = -\infty$.)

In the next, we introduce the notion of continuously globally generation:

Definition 2.2. Let $a: X \to A$ be a morphism from a projective variety to an abelian variety, \mathcal{F} a coherent sheaf on X and T a subset of \hat{A} . Then we have the continuous evaluation map

$$ev_{T,\mathcal{F}}: \bigoplus_{\alpha \in T} H^0(X, \mathcal{F} \otimes a^* \alpha^{-1}) \otimes a^* \alpha \to \mathcal{F}.$$

Let p be a closed point of X. The sheaf \mathcal{F} is said to be *continuously globally generated* at p (CGG at p for short), with respect to the morphism a, if the map $ev_{U,\mathcal{F}}$ is surjective at p, for all Zariski open dense subsets $U \subseteq \hat{A}$. Moreover, we will say that a sheaf is CGG, when it is CGG for all p.

M-regular sheaves have the CGG property:

Theorem 2.3 ([BLNP12]). Any M-regular sheaf on an abelian variety A is continuously globally generated. If \mathcal{F} is a M-regular sheaf on A, then $h^0(\mathcal{F} \otimes P) > 0$ for any $P \in \hat{A}$; if \mathcal{F} is IT(0) then $h^0(\mathcal{F} \otimes P)$ is independent of the choice of P.

Pareschi, Popa and Schnell have proven a theorem that provides certain M-regular properties for the pushforward of the canonical sheaf, which plays a critical role in this paper. This decomposition was originally discovered by Chen and Jiang [CJ18] in the case of maximal Albanese dimension and then extended to irregular variety and higher direct images in general. We state the simplest form of their main theorem as below:

Theorem 2.4 ([PPS17], Theorem A and [LPS20], Theorem C). Let $f: X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. Then there exist finitely many quotients of abelian varieties $p_{B_i}: A \to B_i$ with connected kernel, finitely many M-regular sheaves $\mathcal{F}_{B_i}^j$ on each B_i and torsion line bundles $Q_{B_i}^j \in \operatorname{Pic}^0(A)$ such that

$$(2.5) f_* \omega_X = \bigoplus_{\substack{p_{B_i}: A \to B_i, \\ i \in I}} \bigoplus_j p_{B_i}^* \mathcal{F}_{B_i}^j \otimes Q_{B_i}^j.$$

The following theorem by Jiang and Sun makes our conditions accessible:

Theorem 2.6 ([JS15], Theorem 3.3 and Lemma 4.1). If X is a smooth projective variety of general type and of Albanese fiber dimension one, with Albanese morphism $a_X: X \to A$. Then the translates through 0 of all irreducible components of $V^0(\omega_X, a_X)$ generate $Pic^0(A)$.

Finally, we will use the following results regarding M-regularity.

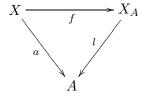
Theorem 2.7. Let $f: X \to A$ be a morphism from a smooth projective variety to an abelian variety, $Q \in Pic^0(X)$ is a torsion element, k an integer, then

- (1) If k > 0, f_{*}(ω_X^k ⊗ Q) is GV.
 (2) If X is of general type, k > 1, and f_{*}(ω_X^k ⊗ Q) is non-zero, then f_{*}(ω_X^k ⊗ Q) is IT(0).

Proof. For the proof of 1., see [PS14] and [HP12, Theorem 2.2]. For the proof of 2., see [HPS18, Theorem 10.1 and Theorem 11.2], [LPS20, Proposition 2.3 and Corollary 11.2] and [Jia11, Lemma 2.2].

3. Main Results

Let X be a smooth projective variety of general type and of Albanese fiber dimension one, with Albanese map $a: X \to A$. Take its Stein factorization



We say X is of Albanese fiber genus m if the genus of a general fiber of f is $\frac{m}{\deg l}$.

By the work of Jiang and Sun [JS15, Remark 5.4], in this paper we only consider the case that the cohomological locus is not the whole Picard zero group i.e. $V^0(K_X) \neq$ $\operatorname{Pic}^{0}(A)$, which is equivalent to the condition $\chi(a_{*}\omega_{X})=0$. In fact, this situation does happen when dim $X \geq 3$ (see Example 3.12).

Lemma 3.1. Notations as above, let B_i be the quotient abelian varieties $p_{B_i}: A \to B_i$ in the application of Theorem 2.4. Then those $p_{B_i}^* \operatorname{Pic}^0(B_i)$'s generate $\operatorname{Pic}^0(A)$ together.

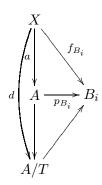
Proof. By [LPS20, Lemma 3.3] we have $V^0(K_X) = \bigcup (p_{B_i}^* \operatorname{Pic}^0(B_i) - Q_{B_i}^j)$, hence the translates through 0 of all irreducible components of $V^0(K_X)$ is just those $p_{B_i}^* \operatorname{Pic}^0(B_i)$'s, which generate $\operatorname{Pic}^0(A)$ by Theorem 2.6.

Let T_i be the subtori of A corresponding to the kernel of the quotient maps p_{B_i} : $A \to B_i$ in Theorem 2.4, then their intersection

$$T := \bigcap_{i} T_{i}$$

is a finite subgroup of A, by the lemma above.

Denote by $d: X \to A/T$ the quotient map $A \to A/T$ composed with $a: X \to A$, and $f_{B_i} = p_{B_i} \circ a: X \to B_i$ the quotient map $A \to B_i$ composed with a. Because $A \to A/T$ is an isogeny, it's enough and much natural to deal with our problem over d rather than a.



Note that by Theorem 2.6 $V^0(K_X) \neq \emptyset$, hence $a_* \omega_X$ is non-zero; by the projection formula each term \mathcal{F}_i , $i \in I$ in Theorem 2.4 is torsion-free and of rank ≥ 1 on $f_{B_i}(X)$.

We will use the notations above without any mention in this section.

We have the following M-regularity:

Lemma 3.2. Notations as above, for any integer k > 1 and any $P \in Pic^0(X)_{tors}$ (the subgroup the torsion elements), $f_{B_i*}(\omega_X^k \otimes P)$ is IT(0) (hence M-regular) on B_i .

Proof. Since $V^0(2K_X) \neq \emptyset$ (Theorem 2.6), $a_* \omega_X^2$ is non-zero therefore IT(0) (Theorem 2.7, (2.)). Take any $P \in \text{Pic}^0(X)_{tors}$, because $H^0(B_i, f_{B_i*}(\omega_X^2 \otimes P)) = H^0(A, a_*(\omega_X^2 \otimes P)) \neq 0$, $f_{B_i*}(\omega_X^k \otimes P)$ is a non-zero sheaf. Now apply Theorem 2.7.

We are now devoted to prove our main result. Firstly we describe the 3-canonical linear systems over different fibers of d:

Lemma 3.3. For general $P \in \text{Pic}^0(X)_{tors}$, and any B_i appearing in Theorem 2.4, the linear system $|3K_X + P|$ separates two general different points of X lying in different fibers of f_{B_i} over $f_{B_i}(X)$. Consequently, $|3K_X + P|$ separates two general different points lying in different general fibers of d over $d(X) \subseteq A/T$.

Proof. Since $P \in \text{Pic}^0(X)_{tors}$, $f_{B_{i^*}} \mathcal{O}_X(2K_X + P)$ is M-regular by Lemma 3.2.

Let $U_0 \subseteq X$ be an open dense subset such that $f_{B_i}|_{U_0}$ and $a|_{U_0}$ are smooth, and that for any b in the open dense subset $f_{B_i}(U_0) \subseteq f_{B_i}(X)$, we have the base change isomorphism $f_{B_i*}(\mathcal{O}_X(2K_X+P)) \otimes \mathbb{C}_b \cong H^0(F_b, 2K_{F_b}+P)$, where F_b is the fiber of f_{B_i} over b. Choose $D \in |2K_X+P|$ an effective divisor on X and denote $U_1 := U_0 \cap D^c$ (here D^c is the complement of D in X).

Let $x \in U_1$ be a closed point and $b_1 = f_{B_i}(x)$. Theorem 2.4 implies that $p_{B_i}^{-1}(b_1)$ is contained in $a(X) \subseteq A$, therefore every irreducible component of the fiber F_{b_1} over b_1 is a smooth variety of Albanese fiber dimension ≤ 1 , and thus $H^0(F_{b_1}, 2K_{F_{b_1}} + P)$ is non-zero by M-regularity. Since x is not a base point of $|2K_{F_{b_1}} + P|$, we have an exact sequence (cf. [JS15, Proposition 5.2]) (here \mathcal{I}_x is the ideal sheaf of x)

$$(3.4) 0 \to f_{B,*}(\mathcal{O}_X(2K_X + P) \otimes \mathcal{I}_x) \to f_{B,*}\mathcal{O}_X(2K_X + P) \to \mathbb{C}_{b_1} \to 0.$$

By employing a trick of Tirabassi [Tir12] (see [JS15, Proposition 5.2] for our case ¹), we know that for any x in an open dense subset $U' \subseteq X$, $a_*(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X))$ is M-regular on A, and

$${R \in \operatorname{Pic}^{0}(X) | x \in Bs(|2K_{X} + R|)}$$

has codimension ≥ 2 in $Pic^0(X)$.

Therefore, fix a general $P \in \text{Pic}^{0}(X)_{tors}$, the closed subset

$$\mathcal{B}(P) := \{ R \in \hat{B}_i | x \in Bs(|2K_X + P + R|) \}$$

has codimension ≥ 2 in \hat{B}_i , for any choice of x in a dense open subset $U_2 \subseteq U_1$ within X: To see this, note that $V^1(f_{B_i*}(\mathcal{O}_X(2K_X+P)))=\emptyset$ by $\mathrm{IT}(0)$ property, and set

$$\mathcal{D} = \{(x, Q) \in U' \times \hat{B}_i | x \in Bs|2K_X + P + Q|\},\$$

which is a closed subset in $U' \times \hat{B_i}$ of codimension ≥ 2 as above, by our generic choice of $P \in \operatorname{Pic}^0(X)_{tors}$. Let $p_1 : X \times \hat{B_i} \to X$ be the first projection, then the required U_2 is nothing but the open subset of $U' \cap U_1$ over which the fiber of $p_1|_{\mathcal{D}}$ has codimension ≥ 2 in $\hat{B_i}$.

Take $x \in U_2$, since

$$V^{1}(f_{B_{i}*}(\mathcal{O}_{X}(2K_{X}+P)\otimes\mathcal{I}_{x}))=\mathcal{B}(P)\cup V^{1}(f_{B_{i}*}\mathcal{O}_{X}(2K_{X}+P))$$

and

$$V^{i}(f_{B_{i}*}(\mathcal{O}_{X}(2K_{X}+P)\otimes\mathcal{I}_{x}))=V^{i}(f_{B_{i}*}\mathcal{O}_{X}(2K_{X}+P)), i>1,$$

¹In Jiang and Sun's original paper [JS15], they only considered the subsheaf $a_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(||K_X||))$ (here $\mathcal{J}(||K_X||)$ denotes the asymptotic multiplier ideal of K_X). However, it's not hard to show that $a_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(||K_X||)) = a_*(\mathcal{O}_X(2K_X))$ (cf. [Jia, 4.2]) and hence $a_*(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X) \otimes \mathcal{J}(||K_X||)) = a_*(\mathcal{I}_x \otimes \mathcal{O}_X(2K_X))$, so their result applies.

(by (3.4)) we conclude that $f_{B_{i*}}(\mathcal{O}_X(2K_X+P)\otimes\mathcal{I}_x)$ is M-regular by Definition 2.1.

For any $y \in U_2 \subseteq X$ with $b_2 := f_{B_i}(y) \neq b_1 = f_{B_i}(x)$, the map

$$f_{B_i*}(\mathcal{O}_X(2K_X+P)\otimes\mathcal{I}_x)\twoheadrightarrow\mathbb{C}_{b_2}$$

which is the pushforward of the natural surjection

$$m_y: \mathcal{O}_X(2K_X+P) \otimes \mathcal{I}_x \to \mathbb{C}_y \cong \mathcal{O}_X(2K_X+P) \otimes \mathcal{I}_x \otimes \mathcal{O}_y$$

via f_{B_i*} , is again surjective, since y is not a base point of $H^0(F_{b_2}, 2K_{F_{b_2}} + P) \cong f_{B_i*}(\mathcal{O}_X(2K_X + P) \otimes \mathcal{I}_x) \otimes \mathbb{C}_{b_2}$. Then by the CGG property (Theorem 2.3) of $f_{B_i*}(\mathcal{O}_X(2K_X + P) \otimes \mathcal{I}_x)$, there is an open dense subset $V \subseteq \hat{B}_i$, such that for any $R \in V$, the map

$$H^0(X, \mathcal{O}_X(2K_X + P + R) \otimes \mathcal{I}_x) \xrightarrow{H^0(m_y)} \mathbb{C}_y$$

is surjective.

On the other hand, denote by Q the torsion line bundle as in Theorem 2.4 respect to a direct summand on B_i . Note that $f_{B_{i*}} \mathcal{O}_X(K_X - Q)$ is GV and $V^0(f_{B_{i*}} \mathcal{O}_X(K_X - Q)) = \hat{B}_i$, hence for a general $y \in X$ with

$$y \notin Z := \bigcap_{L \in V^1(f_{B_{i*}} \mathcal{O}_X(K_X - Q))^c} Bs(|K_X - Q + L|),$$

the set

$$\mathcal{B}_{f_{B_i}}(y) = \{ L \in \hat{B}_i | \ y \notin Bs(|K_X - Q + L|) \}$$

contains an open dense subset of \hat{B}_i (cf. [BLNP12, 4.12]), say W. Taking the following groups into an evaluation map:

$$\bigoplus_{R\in W} H^0(X, \mathcal{O}_X(2K_X+P)\otimes R^{-1})\otimes H^0(X, \mathcal{O}_X(K_X-Q+R))\to H^0(3K_X+P-Q),$$

since $V + W = \hat{B}_i$, by (3) it's easy to see that $H^0(3K_X + P - Q)$ has a section that passes through x but does not pass through y. Therefore if we fix a general $P \in \text{Pic}^0(X)_{tors}$, $|3K_X + P - Q|$ separates x and y, for any $x, y \in U_2 \cap Z^c$ with $f_{B_i}(x) \neq f_{B_i}(y)$.

The last assertion now follows since for any $x, y \in X$, the equality $f_{B_i}(x) = f_{B_i}(y)$ for every B_i implies that d(x) = d(y), by Lemma 3.1.

Next we investigate the behavior under the fiber restriction map.

Lemma 3.6. Notations as in Theorem 2.4, assume B_i is a quotient abelian variety, $\mathcal{F} := \mathcal{F}_{B_i}^j$ is a M-regular sheaf of rank 1 on its support, with the associated torsion line bundle Q. Take any $P \in \text{Pic}^0(X)_{tors}$. Let F be a general fiber of the composition $f_{B_i} := p_{B_i} \circ a : X \to B := f_{B_i}(X)$, then the image of the natural restriction map:

$$H^0(X, 3K_X + P) \to H^0(F, 3K_F + P)$$

contains

$$\mathcal{F} \otimes \mathbb{C}_{f_{B_i}(F)} \subseteq f_{B_i*}(\mathcal{O}_X(K_X - Q)) \otimes \mathbb{C}_{f_{B_i}(F)} \cong H^0(F, \mathcal{O}_F(K_F - Q))$$

(via the natural base change isomorphism) considered as a subvector space of $H^0(F, \mathcal{O}_F(3K_F+P))$ by the inclusion induced by any nonzero section in the subvector space $H^0(F, \mathcal{O}_F(2K_F+Q+P))$. (The inclusion $\mathcal{F} \hookrightarrow f_{B_{i*}}(\mathcal{O}_X(K_X-Q))$ is due to a version of the projection formula; we always identify P, Q with its restriction to F.)

Proof. We just prove the statement for $P = \mathcal{O}_X$, the same argument works for any $P \in \text{Pic}^0(X)_{tors}$.

By construction $f_{B_{i*}} \mathcal{O}_X(2K_X + Q)$ is a M-regular sheaf on B_i , therefore continuously globally generated (Theorem 2.3).

This implies

(3.7)
$$\bigoplus_{R\in U} H^0(B_i, f_{B_{i*}} \mathcal{O}_X(2K_X + Q) \otimes R) \otimes R^{-1} \to f_{B_{i*}} \mathcal{O}_X(2K_X + Q) \otimes \mathbb{C}_b$$

is surjective for any $b \in B$ and any dense open $U \subseteq Pic^0(B_i)$.

The same argument for the M-regular sheaf \mathcal{F} gives that \mathcal{F} is continuously globally generated at any $b \in B_i$, and we have a surjection:

$$\bigoplus_{R \in \operatorname{Pic}^0(B_i)} H^0(\mathcal{F} \otimes R) \otimes R^{-1} \to \mathcal{F} \otimes \mathbb{C}_b.$$

As \mathcal{F} is of rank one on its support, take a general $b \in f_{B_i}(X)$ we may assume $\mathcal{F} \otimes \mathbb{C}_b \cong \mathbb{C}$, and then there is a nonempty open $U_b \subseteq \operatorname{Pic}^0(B_i)$ such that

$$(3.8) H^0(\mathcal{F} \otimes R) \otimes R^{-1} \to \mathcal{F} \otimes \mathbb{C}_b$$

is surjective for any $R \in U_h$.

Take $U = -U_b$ in (3.7) and compose it with (3.8), we get a surjection

$$\bigoplus_{R\in U_b} H^0(B_i, f_{B_i*} \mathcal{O}_X(2K_X + Q) \otimes R^{-1}) \otimes H^0(B_i, \mathcal{F} \otimes R) \to$$

$$f_{B_{i^*}}\mathcal{O}_X(2K_X+Q)\otimes\mathbb{C}_b\cdot\mathcal{F}\otimes\mathbb{C}_b\subseteq f_{B_{i^*}}\mathcal{O}_X(3K_X)\otimes\mathbb{C}_b.$$

Meanwhile, the inclusion and evaluation

$$\bigoplus_{R\in U_b} H^0(B_i, f_{B_i*} \mathcal{O}_X(2K_X + Q) \otimes R^{-1}) \otimes H^0(B_i, \mathcal{F} \otimes R) \hookrightarrow$$

$$\bigoplus_{R \in U_b} H^0(X, \mathcal{O}_X(2K_X + Q) \otimes R^{-1}) \otimes H^0(X, \mathcal{O}_X(K_X - Q) \otimes R) \to H^0(X, 3K_X)$$

give the relation

$$\operatorname{Im}\{H^0(X,3K_X)\to H^0(X_F,3K_{X_F})\}\supseteq f_{B_{i*}}\mathcal{O}_X(2K_X+Q)\otimes\mathbb{C}_b\cdot\mathcal{F}\otimes\mathbb{C}_b$$

for any general $b \in f_{B_i}(X)$ and $F := f_{B_i}^{-1}(b)$. After base change this is nothing but the statement of the lemma.

We are now able to prove:

Theorem 3.9. Notations as above, if X is of Albanese fiber genus 2, then $|3K_X + P|$ is birational, for general $P \in \text{Pic}^0(X)_{tors}$.

Proof. The proof is divided into five steps. In step 1, we show that $|3K_X + P|$ is of degree at most 2 on a general Albanese fiber. In step 2, we show that the Albanese of the morphism induced by $|3K_X + P|$ is an isogeny. These two obstructions of birationality are reduced to two Galois morphisms in step 3. In step 4 and 5, we apply an argument of Jiang and Sun to show that both two Galois morphisms are in fact birational, which finishes the proof.

Step 1. Take a general $P \in \operatorname{Pic}^0(X)_{tors}$ such that Lemma 3.3 holds. Because X is of general type, a general fiber of a is a hyperelliptic curve of genus 2. Moreover, Theorem 2.4 tells us that for any $x \in X$, we have $a(x) + T \subseteq a(X)$.

By the work of Jiang and Sun [JS15], we only need to consider the case that the cohomological loci is not the whole Picard zero group i.e. $V^0(K_X) \neq \operatorname{Pic}^0(A)$. In this case the sheaves $f_{B_{i*}}\omega_X$ are torsion-free on the image of X, and by Theorem 2.4 the decomposition of $a_*\omega_X$ consists of exactly two pull-back term \mathcal{F}_i from the quotients B_i , with corresponding torsion elements $Q_i \in \operatorname{Pic}^0(X)_{tors}$, such that $\operatorname{rank}(\mathcal{F}_i|_{f_{B_i}(X)}) = 1$ (i = 1, 2). By Lemma 3.1 we have $\dim B_i \geq \dim f_{B_i}(X) > 0$, since $f_{B_i}(X)$ generates B_i . (We remark that clearly $B_1 \neq B_2$.)

Because \hat{B}_i 's generate $\operatorname{Pic}^0(X)$ (Lemma 3.1), there exist two elements $P_i \in \hat{B}_i \subseteq \operatorname{Pic}^0(X)$, i = 1, 2 such that

$$P_1 + Q_1 = P_2 + Q_2.$$

Let T_{B_i} be a general fiber of $q_i: A \to B_i$ over $f_{B_i}(X)$, note that the sheaf $q_i^* \mathcal{F}_i$ is trivial restricted on T_{B_i} , and hence we have the natural surjections

$$(3.10) (\mathcal{F}_i \otimes \mathbb{C}_{q_i(T_{B_i})} \cong H^0(T_{B_i}, q_i^* \mathcal{F}_i \mid_{T_{B_i}})) \otimes \mathcal{O}_{T_{B_i}} \longrightarrow q_i^* \mathcal{F}_i \mid_{T_{B_i}}.$$

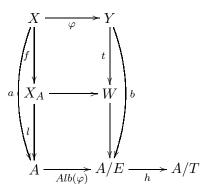
Take a general point $a_0 \in T_{B_i} \subseteq a(X)$, let F_a be the fiber of $a: X \to A$ over $a_0 \in A$, and set $F_{B_i} := a^{-1}(T_{B_i})$. By (3.10) above, the image of the natural restriction map (via base change)

$$H^0(F_{B_i}, \omega_{F_{B_i}} \otimes Q_i^{-1}) \supseteq \mathcal{F}_i \otimes \mathbb{C}_{q_i(T_{B_i})} \to H^0(F_a, \omega_{F_a} \otimes Q_i^{-1}) \cong a_* \omega_X \otimes \mathbb{C}_{a_0}$$

contains $q_i^* \mathcal{F}_i \otimes \mathbb{C}_{a_0}$, regarding the isomorphism $\bigoplus_{i=1,2} q_i^* \mathcal{F}_i \otimes \mathbb{C}_{a_0} \cong a_* \omega_X \otimes \mathbb{C}_{a_0}$ induced by Theorem 2.4.

Accordingly, the results above together with Lemma 3.6 imply that the image of the restriction map of vector spaces $H^0(3K_X + P) \to H^0(3K_{F_a})$ contains the subvector space $s \cdot H^0(K_{F_a})$, where s is a nonzero section in $H^0(2K_{F_a})$ coming from a section in $H^0(2K_X + P_1 + Q_1 + P) = H^0(2K_X + P_2 + Q_2 + P)$. Hence we have showed that $|3K_X + P|$ is birational or of degree 2 on F_a because so is $|K_{F_a}|$.

Step 2. After some birational modifications, $|3K_X + P|$ induces a map $\varphi : X \to Y$ onto a smooth variety Y and hits into a commutative diagram:



Where a and b are Albanese maps, f and t are their Stein factorizations. Let $G \subseteq X \times Y \times A/T$ be the graph of $id_X \times \varphi \times (h \circ b \circ \varphi) : X \to X \times Y \times A/T$, and $p_{23}(G)$ be its projection in $Y \times A/T$. Then Lemma 3.3 implies $p_{23}(G) \subseteq Y \times A/T$ is birational to Y via the first projection, hence $Alb(\varphi)$ is an isogeny over A/T (with kernel E) by the universal property of the Albanese map.

Step 3. We decompose φ into two Galois morphisms.

We first assume that the linear system $|3K_X + P|$ is of degree 2 on F_a , thus it induces the quotient of the hyperelliptic involution. The birational case could be treated similarly.

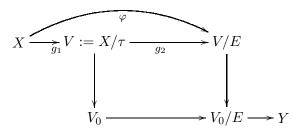
Let H be the graph of X in $X \times Y \times X_A$, then the projection $g_1 : H \to p_{23}(H) \subseteq Y \times X_A$ is of degree 2 with Galois group μ_2 , and the further projection $g_2 : p_{23}(H) \to Y$ is of degree $\sharp E$ (cardinality of E) with Galois group E, such that the composition $g_2 \circ g_1$ is identified with φ up to an isomorphism. Note that E does not lift to a subgroup of Bir(X) in general.

Let us do some birational modifications.

Let V_0 be a regularization of $p_{23}(H)$, i.e V_0 is a projective variety with an action of E and a E-equivariant birational map $\xi: V_0 \dashrightarrow p_{23}(H)$ (see [Sum74, Theorem 3]). Taking an E-equivariant resolution of singularities (see [AW97]), one can assume that V_0 is smooth. By the universal property of Albanese map we see $V_0 \to A$ is E-equivariant, therefore E acts freely on V_0 . Replace $p_{23}(H)$ by V_0 and Y by V_0/E , we may assume g_2 is an E-étale covering.

Doing the same process for X, we may assume X is smooth and μ_2 acts on X, with an E-equivariant morphism $X/\tau \to V_0$. Where τ is the generator of μ_2 , which is a biregular involution on X. After a further modification (cf. [JS15, proof of Theorem 5.3, Step3]), we assume that each connected component of the fixed loci of τ is a smooth divisor of X, thus X/τ is smooth. Replace V_0 by $V := X/\tau$ and V_0/E by V/E, we may assume g_1 is an ramified cover of degree 2, which is even flat

([Eis95, Corollary 18.17]):



Step 4.

At this step, we show that $|3K_X + P|$ induces a degree 1 map on F_a (i.e. g_1 is birational), using an argument of [JS15, Proof of Theorem 5.3].

Assume the contrary. Since $g_1: X \to V$ is a finite Galois flat morphism, $g_{1*} \mathcal{O}_X$ is decomposed into a direct sum of rank one flat (invertible) eigensheaves. Then there is a divisor D on V with ([Kol13, 2.44])

$$g_{1*} \mathcal{O}_X = \mathcal{O}_V \oplus \mathcal{O}_V(-D).$$

The multiplication $\mathcal{O}_V(-D) \otimes \mathcal{O}_V(-D) \to \mathcal{O}_V$ of the \mathcal{O}_V algebra $g_{1*} \mathcal{O}_X$ gives an effective divisor $B \in |2D|$. By Stein factorization $X \cong \mathbf{Spec}_V(g_{1*} \mathcal{O}_X)$, hence g_1 is a double covering branched along B, and we have:

$$\omega_X = g_1^*(\omega_V \otimes \mathcal{O}_V(D)).$$

Moreover, a general fiber T' of f' (the Stein factorization of a', see the diagram below) is isomorphic to \mathbb{P}^1 . Applying the Hurwitz formula for $F_a \to T'$, we have $\deg'_T(D) = g_{F_a} + 1 = 3$.

By the projection formula we write the decomposition of $g_{1*}(\omega_X^3 \otimes P)$:

$$g_{1*}(\mathcal{O}_X(3K_X+P)) = \mathcal{O}_V(3K_V+3D+P) \oplus \mathcal{O}_V(3K_V+2D+P).$$

Because $\deg_T(\mathcal{O}_T(3K_V+2D+P)) \geq 0$, both $a'_*(\mathcal{O}_V(3K_V+3D))$ and $a'_*(\mathcal{O}_V(3K_V+2D+P))$ are non-zero sheaves on X_V .

On the other hand, since $a_*(\mathcal{O}_X(3K_X+P))$ is a M-regular sheaf (Lemma 3.2), so are the both sheaves $a'_*(\mathcal{O}_V(3K_V+3D))$ and $a'_*(\mathcal{O}_V(3K_V+2D))$, which means

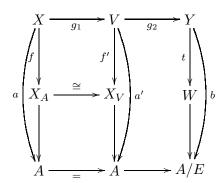
$$H^0(a'_*(\mathcal{O}_V(3K_V+3D))\otimes Q)\neq 0$$

and

$$H^0(a'_*(\mathcal{O}_V(3K_V+2D))\otimes Q)\neq 0$$

for all $Q \in \hat{A}$, by Lemma 2.3.

But the non-zero group $H^0(\mathcal{O}_V(3K_V+2D+P))$ is the involution-anti-invariant part of $H^0(g_{1*}(\mathcal{O}_X(3K_X+P)))$, under the action of μ_2 , which means $H^0(X,\mathcal{O}_X(3K_X+P))$ has a section that separates $x \in F_a$ and its involution. Thus $|3K_X+P|$ must induce a birational map on F_a .



Step 5. We now use the same argument to gurantee the birationality of $|3K_X + P|$. Since $\deg(g_1) = 1$, we may assume φ is identified with g_2 , by the construction above.

Since φ is an étale covering, we write

$$\varphi_* \mathcal{O}_X = \bigoplus_i \mathcal{O}_Y \otimes R_i.$$

Here $R_i \in \hat{E} \subseteq \hat{A/E}$, $1 \leq i \leq \sharp E$ (by the dual exact sequence of abelian varieties) are torsion elements respect to the decomposition of eigensheaves under Galois morphism: $Alb(\varphi)_* \mathcal{O}_A = \bigoplus \mathcal{O}_{A/T_0}(R_i)$.

Fix a $P_0 \in \hat{A/E}$ with $\varphi^* P_0 = P$. By the projection formula we write $\varphi_*(\omega_X^3 \otimes P)$:

$$\varphi_*(\mathcal{O}_X(3K_X+P)) = \bigoplus_i \mathcal{O}_Y(3K_Y+P_0) \otimes R_i.$$

Because $\deg_{F_t}(\mathcal{O}_{F_t}(3K_Y+P_0+R_i))>0$, all $b_*(\mathcal{O}_Y(3K_Y+P_0)\otimes R_i)$ are non-zero sheaves on W.

Since $(Alb(\varphi) \circ a)_*(\mathcal{O}_X(3K_X+P))$ is a M-regular sheaf, so are the sheaves $b_*(\mathcal{O}_Y(3K_Y+P_0) \otimes R_i)$, which means

$$H^0(b_*(\mathcal{O}_Y(3K_Y+P_0)\otimes R_i)\otimes Q)\neq 0$$

for all $Q \in A/E$.

Because the non-zero groups $H^0(\mathcal{O}_Y(3K_Y+3D+P_0)\otimes\mathcal{O}_Y(R_i))$ are the eigenspaces respect to the action of E in $H^0(X,3K_X+P)=H^0(Y,\varphi_*(\omega_X^3\otimes P)), H^0(X,\mathcal{O}_X(3K_X+P))$ has sections that separate a general orbit of E i.e. $\deg(g_2)=1$ and $E=\{0\}$. Thus $|3K_X+P|$ is birational.

Proof of Theorem 1.1: Let p_1, p_2 be the first and second projection of $X \times \text{Pic}^0(X)$, denote by $\mathcal{F} := (p_1^* \omega_X^3) \otimes \mathcal{P}$, where \mathcal{P} is the Poincaré bundle on $X \times \text{Pic}^0(X)$. Then

the surjection of the counit map $\mu: p_2^* p_{2*} \mathcal{F} \to \mathcal{F}$ onto its image induces a rational map (cf.[Har77, ChII, Prop7.12]):

$$\Phi: X \times \operatorname{Pic}^{0}(X) \dashrightarrow \mathbf{Proj}_{\operatorname{Pic}^{0}(X)} \bigoplus_{d>0} \operatorname{S}^{d}(p_{2*}\mathcal{F}).$$

By generic flatness, there is a nonempty open $U_0 \subseteq \operatorname{Pic}^0(X)$ over which \mathcal{F} is flat, by Grauert's theorem there is a nonempty open $U \subseteq U_0$ over which $p_{2*}\mathcal{F}$ is locally free and has isomorphic base change map. We assume further that the map μ is surjective at the generic point of any fiber of p_2 over U. Then for any $P \in U$, the map $\Phi|_{X \times \{P\}}$ coincides with $|3K_X + P|$ on a nonempty open set of X. Moreover, by Theorem 2.6 and the M-regularity of $a_* \omega_X^2 \otimes Q$, we have $\dim |3K_X + P| \geq 1$ for any $P \in \operatorname{Pic}^0(X)$.

Let $W_0 \subseteq X \times \operatorname{Pic}^0(X)$ be the open set of definition of Φ . Then $W := \Phi(W_0)$ is a constructible subset of $\operatorname{\mathbf{Proj}}_{\operatorname{Pic}^0(X)} \bigoplus_{d>0} \operatorname{S}^d(p_{2*}\mathcal{F})$, which carries an integral scheme structure. Denote $\pi: W \to \operatorname{Pic}^0(X)$ the natural projection. Take a general fiber $W_P := \pi^{-1}P$ over $P \in U$. As $\operatorname{Pic}^0(X)_{tors}$ is dense in $\operatorname{Pic}^0(X)$, by the semicontinuity of fiber dimension([Har77, ChII, Exercise 3.22]) and Theorem 3.9 we have

$$\dim W_P = \dim W_{P_0} = \dim X,$$

where $P_0 \in \operatorname{Pic}^0(X)_{tors}$ is general. This implies $\dim W = 2\dim X - 1$ and Φ is generically finite.

By semicontinuity, generic base change ([LPS20, Proposition 4.1]) and Theorem 3.9 again, it's clear that the sheaf $\Phi_* \mathcal{O}_{W_0}$ is of rank 1 on W, the statement follows. \square

Remark 3.11. When X is of Albanese fiber genus > 2, one can still use the argument above to show that, for a general $P \in \text{Pic}^0(X)$, $|3K_X + P|$ is not generically finite of degree 2 (because degree 2 morphisms are always Galois).

At last, we give an example following the ideas of [EL97, Example 1.13] and [JS15, Example 3.4].

Example 3.12. (Ein-Lazarsfeld type threefold) Let E be a elliptic curve and $C \to E$ a double covering by a curve C of genus ≥ 2 , and ι the corresponding involution on C. Let $\mathbb{P}^1 \to \mathbb{P}^1$ be the 2nd power morphism with involution σ_0 , $p: C' \to \mathbb{P}^1$ be a σ_0 -equivariant double covering with the involution τ from a genus 2 curve C', such that σ_0 lifts to another involution σ on C'. Take $X_0 := \frac{C \times C \times C'}{\iota \times \iota \times \tau}$, then there is an involution ξ on X_0 induced by id $\times \iota \times \sigma$. Consider $X_1 := X_0/\xi$ into a diagram

$$C \times C \times C' \xrightarrow{f} X_0 \xrightarrow{g} X_1 \to A := E \times E.$$

We note that both f and g are finite morphisms of degree 2 and unramified in codimension 1, hence X_1 has only canonical singularities (cf. [Kol13, Theorem 3.21]). Let X be a desingularization of X_1 , then the natural morphism $a: X \to A$ is the Albanese map, hence X is of Albanese fiber genus 2 and of general type. Moreover,

$$V^{0}(K_{X}) = (\hat{E} \times \{\hat{0}\}) \bigcup (\{\hat{0}\} \times \hat{E}).$$

Acknowledgements

The author thanks greatly to Zhi Jiang who, besides giving this problem, pointed out several significant mistakes in the original draft of this paper and made a variety of useful suggestions. The author also thanks his advisor Meng Chen for constant support and encouragement.

4. References

References

- [AW97] D. Abramovich and J. Wang, Equivariant resolution of singularities in characteristic 0, Math. Res. Lett **4(2-3)** (1997), 427–433.
- [BLNP12] M. Barja, M. Lahoz, J. Naranjo, and G. Pareschi, On the bicanonical map of irregular varieties, Journal of Algebraic Geometry 21 (2012), 445–471.
 - [CH02] J. A. Chen and C. Hacon, Linear series of irregular varieties, Algebraic geometry in East Asia (Kyoto, 2001) (2002), 143–153.
 - [CJ18] J. A. Chen and Z. Jiang, Positivity in varieties of maximal Albanese dimension, J. reine angew. Math 736 (2018), 225–253.
 - [Eis95] D. Eisenbud, Commutative algebra: with a view toward algebraic geometry, Springer-Verlag, New York, 1995.
 - [EL97] L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, Journal of the American Mathematical Society 10 (1997), 243–258.
 - [Har77] R. Hartshorne, Algebraic geometry, Springer-Verlag, New York, 1977.
 - [HP12] C. Hacon and R. Pardini, Birational characterization of products of curves of genus 2, Mathematical Research Letters 12 (2012), no. 1, 129–140.
 - [HPS18] C. Hacon, M. Popa, and C. Schnell, Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Paun, Contemporary Mathematics 712 (2018), 143–195.
 - [Jia11] Z. Jiang, An effective version of a theorem of Kawamata on the Albanese map, Communications in Contemporary Mathematics 13 (2011), 509–532.
 - [Jia] _____, Recent advances in generic vanishing, preprint.
 - [JLT13] Z. Jiang, M. Lahoz, and S. Tirabassi, On the Iitaka fibration of varieties of maximal Albanese dimension, International Mathematics Research Notices 83(13) (2013), 2984– 3005.
 - [JS15] Z. Jiang and H. Sun, Cohomological support loci of varieties of Albanese fiber dimension one, Transactions of the American Mathematical Society 367 (2015), 103–119.
 - [Kol13] J. Kollár, Singularities of the minimal model program, Cambridge University Press, Cambridge, 2013.
 - [Lah12] M. Lahoz, Generic vanishing index and the birationality of the bicanonical map of irregular varieties, Mathematische Zeitschrift 272 (2012), 1075–1086.
 - [LPS20] L. Lombardi, M. Popa, and C. Schnell, Pushforwards of pluricanonical bundles under morphisms to abelian varieties, Journal of the European Mathematical Society 22 (2020), 2511—2536.
 - [PP03] G. Pareschi and M. Popa, Regularity on abelian varieties I, Journal of the American Mathematical Society 16 (2003), 285–302.
 - [PPS17] G. Pareschi, M. Popa, and C. Schnell, Hodge modules on complex tori and generic vanishing for compact Kähler manifolds, Geometry and Topology 21 (2017), 2419–2460.
 - [PS14] M. Popa and C. Schnell, On direct images of pluricanonical bundles, Algebra and Number Theory (2014), 2273–2295.
 - [Sum74] H. Sumihiro, Equivariant completion, J. Math. Kyoto Univ 14 (1974), 1–28.
 - [Tir12] S. Tirabassi, On the tetracanonical map of varieties of general type and maximal Albanese dimension, Collectanea Mathematica 64(3) (2012), 345–349.

[Zha14] L. Zhang, On the bicanonical map of primitive varieties with $q(X)=\dim X$: the degree and the Euler number, Mathematische Zeitschrift 277 (2014), 575–590.

Department of Mathematics, Fudan University , Shanghai, China

Email address: 14110180003@fudan.edu.cn