Strengthening strong immersions with Kempe chains

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Abstract

Every properly colored graph with $\chi(G) = k$ colors has edge-disjoint Kempe "backbones", Kempe chains anchored by color-critical vertices for each pair of colors. Certain color permutations arrange these backbones into a clique-like structure, a strengthening of strong immersions of complete graphs. This strengthened immersion is suggested as a template for identifying the disjoint subgraphs comprising Hadwiger's conjectured K_k minor present in k-chromatic graphs.

1 Introduction

An immersion of the complete graph K_k in a simple, undirected graph G is an injective function that maps $f : V(K_k) \to V(G)$. Each edge $(u, v) \in E(K_k)$ corresponds to a path in G with endpoints $(f(u), f(v))$. The immersion is strong if the paths are internally disjoint. Studying strong immersions of complete graphs has been motivated by their potential to make progress towards resolving Hadwiger's conjecture [1]. Here we identify additional structural elements of graphs closely aligned with Kempe chains and how they assemble into a strengthened form of K_k immersions. These *Kempe cliques* are attractive research targets because of their close alignment to the graph's chromatic number.

Let G be a simple, undirected graph with chromatic number $\chi(G) = k$, and chromatic coloring, that is, properly colored with $C = c_1, c_2, \ldots, c_k$ colors.

Definition 1. A *critical vertex*, X_i, is a vertex with color c_i that is adjacent to at least $k-1$ neighbors, colored with all remaining colors in C (Figure 1).

Definition 2. A *Kempe chain* contains vertex v colored c_i, and is the maximal connected subgraph of vertices colored either c_i or c_j. Also referred to as a (c_i, c_j)-Kempe chain.

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Definition 3. A *Kempe backbone* is a path within a $(c_i,\, c_j)$-Kempe chain leading from critical vertex X_i to critical vertex X_j. These critical vertices are the *anchors* of the backbone. Also referred to as a (c_i, c_j)-Kempe backbone (Figure 1).

Definition 4. A *Kempe swap* interchanges the colors of a (c_i, c_j)-Kempe chain. Each vertex in the chain assumes the color of its chain neighbors.

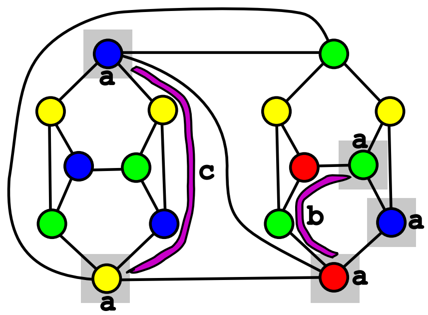

Figure 1 – Annotated structures. Critical vertices, **a**, each framed by a grey square. A (green, red)-Kempe backbone, **b**, alongside a purple ribbon. A (blue, yellow)-Kempe backbone, **c**, alongside a purple ribbon. A length-3 (yellow, green)-Kempe backbone is present but not highlighted. Three length-1 Kempe backbones, (red, yellow), (red, blue), and (blue, green) complete the list of six Kempe backbones present in a $\chi(G) = 4$ graph.

2 Kempe backbones

Observation 1. Graph G contains at least k critical vertices. Each color in C labels at least one critical vertex.

Proof. Assume that G is missing critical vertex X_a. Each c_a colored vertex may be assigned a different color because no c_a colored vertex is adjacent to all remaining colors by the definition of a critical vertex. Therefore, G is $(k-1)$-colorable, a contradiction. □

Observation 2. A Kempe swap preserves the criticality (whether a vertex is critical or non-critical) of each vertex in the Kempe chain.

Proof. Each vertex in the (c_a, c_b)-Kempe chain is adjacent to neighbors labeled with $1 \leq n < k$ colors. Interchanging the color of vertex v in the chain from c_a to c_b replaces c_b-labeled neighbors with c_a-labeled neighbors (and visa versa). Other neighbors are not affected by the swap. Therefore, n does not change for each vertex participating in the swap. □

Note that although the criticality is preserved for chain members undergoing a swap, the criticality of vertices adjacent to members of the swapped chain may change.

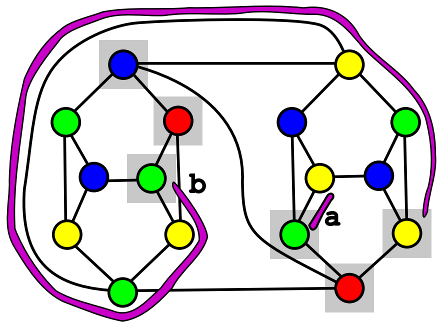

Figure 2 – A chromatically colored $k = 4$ graph. Critical vertices are framed by grey squares. Swapping the (green, yellow)-Kempe chain, **a**, will eliminate the green critical vertex. The second (green, yellow)-Kempe chain, **b**, includes a (green, yellow)-Kempe backbone following the purple ribbon. Swapping this chain merely interchanges the critical vertices of the backbone.

The following theorem describes the Kempe backbone characteristic of G. There are $(k^2 - k)/2$ unique pairs of colors. For each unique color pair (c_i, c_j), there exist two critical vertices, (X_i, X_j), anchoring a (c_i, c_j)-Kempe backbone.

Theorem 1. For each pair of colors (c_i, c_j) with $c_i, c_j \in C$, $i \neq j$ there is a critical vertex pair (X_i, X_j) that anchors a (c_i, c_j)-Kempe backbone.

Proof. Let (a, b) be such a color pair and assume instead that no (X_a, X_b) pair of critical vertices anchors an (a, b)-Kempe backbone. By Observation 1, critical vertices X_a and X_b exist. So, X_a and X_b are members of different, disconnected (a, b)-Kempe chains. Swap the colors of each (a, b)-Kempe chain containing a critical vertex X_a. By Observation 2, this replaces each X_a with a critical vertex of color b. This process may be repeated to eliminate all critical vertices of color a, which is a contradiction by Observation 1. □

Graph G may have more than one critical vertex of a given color; not all critical vertices are required to participate in a Kempe backbone to satisfy Theorem 1. Also, more than one (c_a, c_b)-Kempe backbone may exist in G.

Theorem 1 describes the architecture that prevents G from using fewer than k colors. Any attempt to use (a, b)-Kempe swaps to remove all X_a critical vertices will encounter at least one X_a that has a Kempe backbone to X_b (Figure 2). A (c_a, c_b)-Kempe swap on the backbone anchored by X_a and X_b will merely interchange the critical vertices, failing to eliminate X_a.

3 Kempe cliques

Definition 5. A *correctly colored* graph is properly colored with $q \geq \chi(G)$ colors such that it includes q critical vertices, $\{X_1, X_2, \ldots, X_q\}$, each labeled with a different color, anchoring $(q^2 - q)/2$ Kempe backbones.

Definition 6. A *Kempe clique*, Q_q, is the collection of $(q^2 - q)/2$ Kempe backbones anchored by q critical vertices in a correct coloring.

The presence of a Kempe clique does not directly follow from Theorem 1. Figures 1 and 2 are examples of proper (and chromatic) colorings that are not correct colorings. A different color permutation of a properly colored graph may be required to reveal a Kempe clique. Figure 3 shows labeled examples of correctly colored graphs and their Kempe cliques.

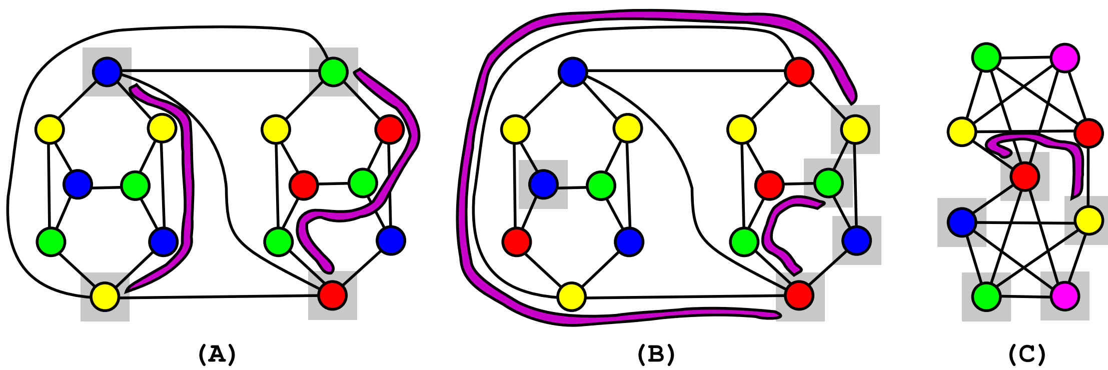

Figure 3 – Examples of correct coloring. Critical vertices are framed by grey squares. Kempe backbones greater than length-1 follow purple ribbons. (A) A graph with $\chi(G) = 4$. The Kempe clique includes four length-1 Kempe backbones, one of length-3 and one of length-5. (B) A graph with $\chi(G) = 4$. In addition to the Kempe backbones comprising a Kempe clique, a correctly colored graph may have additional critical vertices and additional Kempe backbones. (C) A correctly colored graph with $\chi(G) = 5$. The Kempe clique includes a length-3 Kempe backbone.

Kempe cliques strengthen strong immersions of complete graphs [7, 3]. Every Kempe clique, Q_q, is a strong immersion of the complete graph, K_q, but the converse does not follow. For example, Koester's planar graph [6] with $\chi(G) = 4$ (Figure 4) has a strongly immersed K_5, as do all 4-regular graphs [7]. However, there is no correct coloring of Koester's graph when properly colored with $q = 5$ colors. It is straightforward to show that Q_5 is non-planar. In fact, a proof that all $\chi(G) = 5$ graphs admit a correct coloring would serve as an alternate proof of the four-color theorem.

Conjecture 1. Every simple, undirected graph can be correctly colored.

Note that a correct coloring uses $q \geq \chi(G)$ colors; G need only be properly-colored, not chromatically-colored. An example of a correct coloring requiring $q > \chi(G)$ can be found in Catlin, 1979 [2]. Catlin constructs a counterexample to the Hajós Conjecture that every

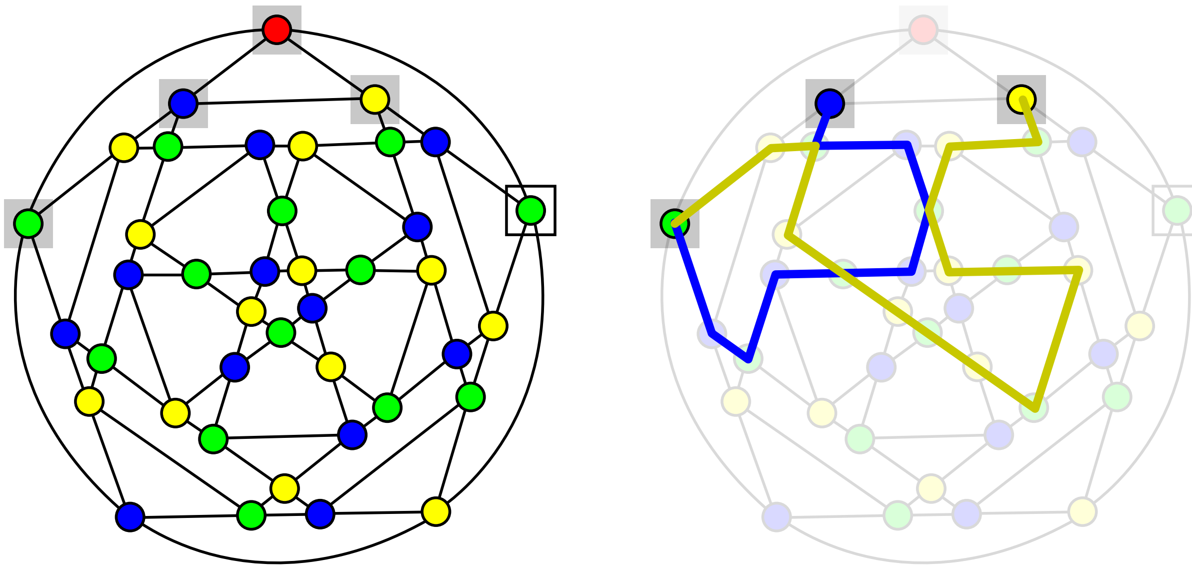

Figure 4 – (A) A correct coloring of Koester's graph. (B) Highlighted are the Kempe backbones with length $l > 1$.

simple graph has a K_k subdivision by taking the crossproduct of a cycle and complete graph (Figure 5). Given cycle length $2n + 1$ and complete graph order k, the construction's chromatic number is $\chi(G) = 2k + \lceil k/n \rceil$ [2]. A correct coloring of the same construction requires $q = 3k$ colors.

A proof of Conjecture 1 would allow the possibility that the Kempe clique itself serves as the template for the K_k minor conjectured by Hadwiger [4]. In particular, each critical vertex of the Kempe clique "seeds" a different, disjoint connected subgraph that identifies to form a complete graph. For example, the critical vertices in Figure 5B each seed a different, disjoint subgraph that forms a K_6 minor. Critical vertices 1 and 2 are sole members of their subgraphs. Critical vertices 3, 4, 5, and 6 each include a second vertex in their respective subgraphs.

Conjecture 2. The critical vertices of the Kempe clique in a correctly colored graph with q colors are the seeds of a K_q minor. That is, each is a member of a different, disjoint connected subgraph. Identifying each subgraph forms K_q. Given h, the Hadwiger number, $\chi(G) \leq q \leq h$.

There are some graph families that are known to be correctly colorable.

Theorem 2. A k-critical graph with minimum degree $\delta(G) = k-1$ admits a correct coloring.

Proof. Select vertex v with $k - 1$ neighbors. Since $G - v$ is $(k - 1)$-colorable, v is the only vertex labeled c_a. By Observation 1, v is a critical vertex. By the same Observation, the graph must contain $k-1$ additional critical vertices labeled with the remaining $C - c_a$ colors.

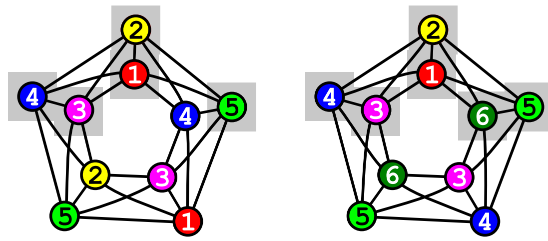

Figure 5 – Catlin's counterexample construction, the crossproduct of C_5 and K_2, $\chi(G) = 5$. (A) Graph G is chromatically-colored. Although all vertices are critical, only a subset are framed in grey to ease identification of Kempe backbones. No correct coloring is possible with 5 colors; there is no set of 10 Kempe backbones anchored by 5 critical vertices. (B) Graph G is correctly colored with 6 colors.

To be critical, each of these $k-1$ additional critical vertices must be adjacent to a c_a-labeled vertex. Therefore, the $k-1$ neighbors of v comprise the remaining critical vertices in the graph. Because the graph has only k critical vertices, these comprise all anchors for the graph's Kempe backbones, forming a Kempe clique. □

In Abu-khzam and Langston, 2003 [1], Corollary 2 similarly identifies an immersed K_k in a color critical graph with $\delta(G) = k - 1$.

Theorem 3. Uniquely-colorable graphs are correctly colored.

Proof. Because the subgraph induced by the union of two color classes is connected in a uniquely-colored graph [5], the Kempe backbones form a Kempe clique. □

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