ASYMPTOTIC LOWER BOUND OF CLASS NUMBERS ALONG A GALOIS REPRESENTATION

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ABSTRACT. Let T be a free \mathbb{Z}_p -module of finite rank equipped with a continuous \mathbb{Z}_p -linear action of the absolute Galois group of a number field K satisfying certain conditions. In this article, by using a Selmer group corresponding to T, we give a lower bound of the additive p-adic valuation of the class number of K_n , which is the Galois extension field of K fixed by the stabilizer of $T/p^n T$. By applying this result, we prove an asymptotic inequality which describes an explicit lower bound of the class numbers along a tower $K(A[p^{\infty}])/K$ for a given abelian variety A with certain conditions in terms of the Mordell–Weil group. We also prove another asymptotic inequality for the cases when A is a Hilbert–Blumenthal or CM abelian variety.

1. INTRODUCTION

Commencing with Iwasawa's class number formula ([Iw] §4.2), it is a classical and important problem to study the asymptotic behavior of class numbers along a tower of number fields. Greenberg ([Gr]) and Fukuda–Komatsu–Yamagata ([FKY]) studied Iwasawa's λ -invariant of a certain (non-cyclotomic) \mathbb{Z}_p -extension of a CM field for a prime number p: by using Mordell–Weil group of a CM abelian variety, they gave a lower bound of the λ -invariant. Sairaiji–Yamauchi ([SY1], [SY2]) and Hiranouchi ([Hi]) studied asymptotic behavior of class numbers along a p-adic Lie extension $\mathbb{Q}(E[p^{\infty}])/\mathbb{Q}$ generated by coordinates of all p-power torsion points of an elliptic curve E defined over \mathbb{Q} satisfying certain conditions, and obtained results analogous to those in $|\mathbf{Gr}|$ and [FKY]. In this article, by using the terminology of Selmer groups, we generalize their results to the p-adic Lie extension of a number field K along a p-adic representation of the absolute Galois group $G_K := \operatorname{Gal}(\overline{K}/K)$ (Theorem 1.1). As an application of this theory, we prove an asymptotic inequality which gives a lower bound of the class numbers along a tower $K(A[p^{\infty}])/K$ for a given abelian variety A with certain conditions (Corollary 1.3). We also prove another asymptotic inequality for the cases when A is a Hilbert–Blumenthal or CM abelian variety (Corollary 1.5).

Let us introduce our notation. Fix a prime number p, and $\operatorname{ord}_p: \mathbb{Q}^{\times} \longrightarrow \mathbb{Z}$ the additive p-adic valuation normalized by $\operatorname{ord}_p(p) = 1$. Let K/\mathbb{Q} be a finite extension, and Σ a finite set of places of K containing all places above p and all infinite places. We denote by K_{Σ} the maximal Galois extension field of K unramified outside Σ , and put $G_{K,\Sigma} := \operatorname{Gal}(K_{\Sigma}/K)$. Let $d \in \mathbb{Z}_{>0}$, and suppose that a free \mathbb{Z}_p -module T of rank dequipped with a continuous \mathbb{Z}_p -linear $G_{K,\Sigma}$ -action $\rho: G_{K,\Sigma} \longrightarrow \operatorname{Aut}_{\mathbb{Z}_p}(T) \simeq \operatorname{GL}_d(\mathbb{Z}_p)$. We put $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $W := V/T \simeq T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Let $n \in \mathbb{Z}_{\geq 0}$. We denote the $\mathbb{Z}_p[G_{K,\Sigma}]$ -submodule of W consisting of all p^n -torsion elements by $W[p^n]$.

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define $K_n := K(W[p^n])$ to be the maximal subfield of K_{Σ} fixed by the kernel of the continuous group homomorphism $\rho_n \colon G_{K,\Sigma} \longrightarrow \operatorname{Aut}_{\mathbb{Z}/p^n\mathbb{Z}}(W[p^n])$ induced by ρ . We denote by h_n the class number of K_n . In this article, we study the asymptotic behavior of the sequence $\{\operatorname{ord}_p(h_n)\}_{n>0}$ by using a Selmer group of W.

Let us introduce some notation related to Selmer groups in our setting briefly. (For details, see §2.1.) Let $\mathcal{F} = \{H^1_{\mathcal{F}}(L_w, V) \subseteq H^1(L_w, V)\}_{L,w}$ be any local condition on (V, Σ) in the sense of Definition 2.1. For instance, we can set \mathcal{F} to be Bloch– Kato's finite local condition f. Let $v \in \Sigma$ be any element, and $H^1_{\mathcal{F}}(K_v, W)$ be the \mathbb{Z}_p -submodule of $H^1(K_v, W)$ attached to \mathcal{F} . Since the Galois cohomology $H^1(K_v, W)$ is a cofinitely generated \mathbb{Z}_p -module, so is the subquotient

$$\mathcal{H}_v := H^1_{\mathcal{F}}(K_v, W) / (H^1_{\mathcal{F}}(K_v, W) \cap H^1_{\mathrm{ur}}(K_v, W))$$

where $H^1_{ur}(K_v, W)$ denotes the unramified part of $H^1(K_v, W)$. We denote by the corank of the \mathbb{Z}_p -module \mathcal{H}_v by $r_v = r_v(T, \mathcal{F})$. We define the Selmer group of W over K with respect to the local condition \mathcal{F} by

$$\operatorname{Sel}_{\mathcal{F}}(K,W) := \operatorname{Ker}\left(H^{1}(K_{\Sigma}/K,W) \longrightarrow \prod_{v \in \Sigma} \frac{H^{1}(K_{v},W)}{H^{1}_{\mathcal{F}}(K_{v},W)}\right).$$

Since $H^1(K_{\Sigma}/K, W)$ is a cofinitely generated \mathbb{Z}_p -module, so is $\operatorname{Sel}_{\mathcal{F}}(K, W)$. We define $r_{\operatorname{Sel}} := r_{\operatorname{Sel}}(T, \mathcal{F})$ to be the corank of the \mathbb{Z}_p -module $\operatorname{Sel}_{\mathcal{F}}(K, W)$.

For two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ of real numbers, we write $a_n \succ b_n$ if we have $\liminf_{n\to\infty}(a_n-b_n) > -\infty$, namely if the sequence $\{a_n-b_n\}_{n\geq 0}$ is bounded below. The following theorem is the main result of our article.

Theorem 1.1. Assume that T satisfies the following two conditions.

(Abs) The representation W[p] of $G_{K,\Sigma}$ over \mathbb{F}_p is absolutely irreducible.

(NT) If d = 1, then $G_{K,\Sigma}$ acts on W[p] non-trivially.

Then, we have

$$\operatorname{ord}_p(h_n) \succ d\left(r_{\operatorname{Sel}} - \sum_{v \in \Sigma} r_v\right) n.$$

Remark 1.2. In this article, we also show a stronger assertion than Theorem 1.1 which describes not only asymptotic behavior but also a lower bound of each h_n in the strict sense. (See Theorem 3.4.)

Let A be an abelian variety defined over a number field K, and put $g := \dim A$. For each $N \in \mathbb{Z}_{\geq 0}$, we denote by A[N] the N-torsion part of $A(\overline{K})$. Put $r_{\mathbb{Z}}(A) := \dim_{\mathbb{Q}}(A(K) \otimes_{\mathbb{Z}} \mathbb{Q})$. For each $n \in \mathbb{Z}_{\geq 0}$, we denote by $h_n(A; p)$ the class number of $K(A[p^n])$, which is the extension field of K generated by the coordinates of elements $A[p^n]$. By applying Theorem 1.1, and we obtain the following corollary. (See §4.)

Corollary 1.3. Suppose that A[p] is an absolutely irreducible representation of G_K over \mathbb{F}_p . Then, it holds that

$$\operatorname{ord}_p(h_n(A; p)) \succ 2g\left(r_{\mathbb{Z}}(A) - g[K:\mathbb{Q}]\right) n.$$

Remark 1.4. Suppose that $K = \mathbb{Q}$, and A is an elliptic curve over \mathbb{Q} . Then, it follows from Corollary 1.3 that $\operatorname{ord}_p(h_n(A; p)) \succ 2(r_{\mathbb{Z}}(A) - 1)n$. This asymptotic inequality coincides with that obtained by Sairaiji–Yamauchi ([SY1], [SY2]) and Hi-ranouchi ([Hi]). Moreover, if p is odd, then Theorem 3.4, which is a stronger result

than Theorem 1.1 mentioned in Remark 1.2, implies strict estimates obtained in [SY1], [SY2] and [Hi]. For details, see Example 4.4.

We have another application of Theorem 1.1 for the cases when A is a Hilbert– Blumenthal abelian variety or a CM abelian variety. Suppose that K is a totally real or CM field which is Galois over K, and denote by \mathcal{O}_K the ring of integers in K. We denote by K^+ the maximal totally real subfield of K, and put $g := [K^+ : \mathbb{Q}]$. Let pbe a prime number which splits completely in K, and has prime ideal decomposition $p\mathcal{O}_K = \prod_{\sigma \in \text{Gal}(K/\mathbb{Q})} \sigma(\pi)\mathcal{O}_K$ for some element $\pi \in \mathcal{O}_K$. Let A be a g dimensional Hilbert–Blumenthal (resp. CM) abelian variety over K which has good reduction at every places above p, and satisfies $\text{End}_K(A) = \mathcal{O}_K$ if K is totally real (resp. CM). For each $n \in \mathbb{Z}_{\geq 0}$, we denote by $h_n(A; \pi)$ the class number of $K(A[\pi^n])$. We put $r_{\mathcal{O}_K}(A) := \dim_K(A(K) \otimes_{\mathcal{O}_K} K)$. Then, the following holds.

Corollary 1.5. Suppose that $A[\pi]$ is an absolutely irreducible non-trivial representation of G_K over \mathbb{F}_p . Then, we have

$$\operatorname{ord}_p(h_n(A;\pi)) \succ \frac{2}{[K:K^+]} (r_{\mathcal{O}_K}(A) - g) n.$$

Remark 1.6. When K is a CM field, the asymptotic inequality in Corollary 1.5 coincides with that obtained by Greenberg [Gr] and Fukuda–Komatsu–Yamagata [FKY].

The strategy for the proof of our main result, namely Theorem 1.1, is quite similar to those established in earlier works [Gr], [FKY], [SY1], [SY2] and [Hi]. For each $n \in \mathbb{Z}_{\geq 0}$, by using elements of the Selmer group, we construct a finite abelian extension L_n/K_n which is unramified outside Σ , and whose degree is a power of p. Then, we compute the degree $[L_n : K_n]$ and the order of inertia subgroups at ramified places.

In §2, we introduce notation related to Galois cohomology, and prove some preliminary results. In §3, we prove our main theorem, namely Theorem 1.1. In §4, we apply Theorem 1.1 to the Galois representations arising from abelian varieties, and prove Corollary 1.3 and Corollary 1.5. We also compare our results with earlier works by Fukuda–Komatsu–Yamagata [FKY], Sairaiji–Yamauchi [SY2] and Hiranouchi [Hi].

Notation. Let L/F be a Galois extension, and M a topological abelian group equipped with a \mathbb{Z} -linear action of G. Then, for each $i \in \mathbb{Z}_{\geq 0}$, we denote by $H^i(L/F, M) :=$ $H^i_{\text{cont}}(\text{Gal}(L/F), M)$ the *i*-th continuous Galois cohomology. When L is a separable closure of F, then we write $H^i(F, M) := H^i(L/F, M)$.

Let F be a non-archimedean local field. We denote by F^{ur} the maximal unramified extension of F. For any topological abelian group M equipped with continuous \mathbb{Z} linear action of G_F , we define $H^1_{\text{ur}}(F, M) := \text{Ker}(H^1(F, M) \longrightarrow H^1(F^{\text{ur}}, M)).$

Let R be a commutative ring, and M an R-module M. We denote by $\ell_R(M)$ the length of M. For each $a \in R$, we denote M[a] by the R-submodule of M consisting of elements annihilated by a.

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2. Preliminaries on Galois Cohomology

Here, we introduce some notation related to Galois cohomology, and prove preliminary results. Let K, Σ and T be as in §1, and assume that T satisfies the conditions (Abs) and (NT) in Theorem 1.1. We denote by $\operatorname{Fin}(K; \Sigma)$ be the set of all intermediate fields L of K_{Σ}/K which are finite over K. For each $L \in \operatorname{Fin}(K; \Sigma)$, we denote by P_L the set of all places of L, and by Σ_L the subset of P_L consisting of places above an element of Σ .

2.1. Local conditions. In this subsection, let us define the notion of local conditions and Selmer groups in our article.

Definition 2.1. Recall that we put $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and W := V/T.

- (i) A collection $\mathcal{F} := \{ H^1_{\mathcal{F}}(L_w, V) \subseteq H^1(L_w, V) \mid L \in \operatorname{Fin}(K; \Sigma), w \in \Sigma_L \}$ of \mathbb{Q}_{p^-} subspaces is called *a local condition* on (V, Σ) if the following (*) is satisfied.
 - (*) Let $\iota: L_1 \hookrightarrow L_2$ be an embedding of fields belonging to $\operatorname{Fin}_K(L; \Sigma)$ over K. Then, for any $w_1 \in P_{L_1}$ and $w_2 \in P_{L_2}$ satisfying $\iota^{-1}w_2 = w_1$, the image of $H^1_{\mathcal{F}}(L_{1,w_1}, V)$ via the map $H^1(L_{1,w_1}, V) \longrightarrow H^1(L_{2,w_2}, V)$ induced by ι is contained in $H^1_{\mathcal{F}}(L_{w_2}, V)$.
- (ii) Let $L \in Fin(K; \Sigma)$ and $w \in P_L$. Then, we define $H^1_{\mathcal{F}}(L_w, W)$ to be the image of $H^1_{\mathcal{F}}(L_w, V)$ via the natural map $H^1(L_w, V) \longrightarrow H^1(L_w, W)$. For any $n \in \mathbb{Z}_{\geq 0}$, we define $H^1_{\mathcal{F}}(L_w, W[p^n])$ to be the inverse image of $H^1_{\mathcal{F}}(L_w, W)$ via the natural map $H^1(L_w, W[p^n]) \longrightarrow H^1(L_w, W)$.
- (iii) Let $L \in Fin(K; \Sigma)$, and $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then, we define

$$\operatorname{Sel}_{\mathcal{F}}(L, W[p^n]) := \operatorname{Ker}\left(H^1(K_{\Sigma}/L, W[p^n]) \longrightarrow \prod_{w \in \Sigma_L} \frac{H^1(L_w, W[p^n])}{H^1_{\mathcal{F}}(L_w, W[p^n])}\right)$$

Remark 2.2. Let \mathcal{F} be a local condition on (V, Σ) . Then, by definition, for any $L \in \operatorname{Fin}(K; \Sigma)$ and $w \in P_L$, the \mathbb{Z}_p -module $H^1_{\mathcal{F}}(L_w, W)$ is divisible.

Remark 2.3. Let $L \in Fin(K; \Sigma)$ any element, and $w \in \Sigma_L$ an infinite place. Then, we note that $H^1(L_w, V) = 0$. Thus for any local condition \mathcal{F} on (V, Σ) , it clearly holds that $H^1_{\mathcal{F}}(L_w, V) = 0$. We also note that $H^1(L_w, W)$ is annihilated by 2. In particular $H^1(L_w, W)$ never has a non-trivial divisible \mathbb{Z}_p -submodule, So, the corank of $H^1(L_w, W)$ is zero. When we treat a local condition, we may not care infinite places.

Example 2.4 ([BK] §3). For $L \in Fin(K; \Sigma)$ and finite place $w \in P_L$, we put

$$H^{1}_{f}(L_{w}, V) := \begin{cases} H^{1}_{\mathrm{ur}}(L_{w}, V) & \text{(if } w \nmid p), \\ \operatorname{Ker} \left(H^{1}(L_{w}, V) \longrightarrow H^{1}(L_{w}, V \otimes_{\mathbb{Q}_{p}} B_{\mathrm{crys}}) \right) & \text{(if } w \mid p), \\ 0 & \text{(if } w \mid \infty) \end{cases}$$

where B_{crys} is Fontaine's *p*-adic period ring introduced in [Fo] and [FM]. Then, we can easily verify that the collection $\{H_f^1(L'_{w'}, V) \mid L' \in \text{Fin}(K; \Sigma), w' \in \Sigma_{L'}\}$ forms a local condition on (V, Σ) . We call this collection *Bloch-Kato's finite local condition*.

2.2. Global cohomology. In this subsection, we introduce some preliminaries on global Galois cohomology. We put $K_{\infty} := \bigcup_{n\geq 0} K_n$. For each $m, n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $n \geq m$, we put $G_{n,m} := \operatorname{Gal}(K_n/K_m)$, and $G_m := \operatorname{Gal}(K_{\Sigma}/K_m)$.

First, we control the kernel of the restriction map

$$\operatorname{res}_{n,W} \colon H^1(K, W[p^n]) \longrightarrow H^1(K_n, W[p^n])$$

for every $n \in \mathbb{Z}_{\geq 0}$. In order to do it, the following fact and the irreducibility of V, which follows from (Abs) for T become a key.

Theorem 2.5 ([Ru] Theorem C.1.1 in Appendix C). Let V' be a finite dimensional \mathbb{Q}_p -vector space, and suppose that a compact subgroup H of $\operatorname{GL}(V') := \operatorname{Aut}_{\mathbb{Q}}(V')$ acts irreducibly on V' via the standard action. Then, we have $H^1(H, V') = 0$.

By using Theorem 2.5, let us prove the following lemma.

Lemma 2.6. There exist a positive integer ν_{im} such that for any $n \in \mathbb{Z}_{\geq 0}$, we have $\ell_{\mathbb{Z}_p}(\text{Ker res}_{n,W}) \leq \nu_{im}$.

Proof. Let us show that $\#H^1(K_{\infty}/K, W) < \infty$. By (Abs) and Theorem 2.5, we have $H^1(K_{\infty}/K, V) = 0$. So, we obtain an injection $H^1(K_{\infty}/K, W) \hookrightarrow H^2(K_{\infty}/K, T)$. Note that $G_{\infty,0} = \operatorname{Gal}(K_{\infty}/K)$ is a compact *p*-adic analytic group since $G_{\infty,0}$ can be regarded as a compact subgroup of $\operatorname{GL}(V)$. So, by Lazard's theorem (for instance, see [DDMS] 8.1 Theorem), it holds that $G_{\infty,0}$ is topologically finitely generated. This implies that the order of $H^2(K_{\infty}/K, W[p])$ is finite. Thus $H^2(K_{\infty}/K, T)$ is finitely generated over \mathbb{Z}_p . (See [Ta] Corollary of (2.1) Proposition.) Since $H^1(K_{\infty}/K, W)$ is a torsion \mathbb{Z}_p -module, we deduce that the order of $H^1(K_{\infty}/K, W)$ is finite.

Let ν be the length of the \mathbb{Z}_p -module $H^1(K_{\infty}/K, W)$. Take any $n \in \mathbb{Z}_{\geq 0}$. By the assumptions **(Abs)** and **(NT)**, the natural map $H^1(K_n/K, W[p^n]) \longrightarrow H^1(K_n/K, W)$ is injective. So, by the inflation-restriction exact sequence, we obtain an injection Ker res_{n,W} $\hookrightarrow H^1(K_{\infty}/K, W)$. Hence $\nu_{\rm im} := \nu$ satisfies the desired properties. \Box

Remark 2.7. Note that if the image $\rho_1(\operatorname{Gal}(K_1/K)) \subseteq \operatorname{GL}_d(\mathbb{F}_p)$ contains a non-trivial scalar matrix, then we can take $\nu_{\rm im} = 0$. Indeed, in such cases, similarly to [LW] §2 Lemma 3, we can show that $H^1(K_{\infty}/K, W) = 0$.

Example 2.8. Suppose $d = \dim_{\mathbb{Q}_p} V = 2$, and the following assumption (Full).

(Full) If p is odd, then the map $\rho_1: G_{K,\Sigma} \longrightarrow \operatorname{Aut}_{\mathbb{F}_p}(W[p]) \simeq \operatorname{GL}_2(\mathbb{F}_p)$ is surjective. If p = 2, then $\rho: G_{K,\Sigma} \longrightarrow \operatorname{Aut}_{\mathbb{Z}_2}(T) \simeq \operatorname{GL}_2(\mathbb{Z}_2)$ is surjective.

Then, we may take $\nu_{\rm im} = 0$ because of the following Claim 2.9. Note that in [SY1], [SY2] and [Hi], Sairaiji, Yamauchi and Hiranouchi assumed the hypothesis (Full). So, the constant $\nu_{\rm im}$ does not appear explicitly in these works.

Claim 2.9. If d = 2, and if T satisfies (Full), then we have $H^1(K_{\infty}/K, W) = 0$.

Proof of Claim 2.9. If p is odd, then the claim follows from similar arguments to those in the proof of [LW] §2 Lemma 3. So, we may suppose that p = 2. It suffices to show that $H^0(K, H^1(K_{n+1}/K_n, W[2])) = 0$ for each $n \in \mathbb{Z}_{\geq 0}$. Note that we have $G_{1,0} \simeq \operatorname{GL}_2(\mathbb{F}_2) \simeq \mathfrak{S}_3$. Let A be a unique normal subgroup of $G_{1,0}$ of order 3. Then, we have $W[2]^A = 0$, and $H^1(A, W[2]) = 0$. So, we have $H^1(K_1/K, W[2]) = 0$.

Take any $n \ge 1$, and let us show that

(2.1)
$$H^{0}(K, H^{1}(K_{n+1}/K_{n}, W[2])) = \operatorname{Hom}(G_{n+1,n}, W[2])^{G_{n+1,0}} = 0.$$

The map $\rho_{n+1}: G_{n+1,0} \longrightarrow \operatorname{GL}_2(\mathbb{Z}/2^{n+1}\mathbb{Z})$ induces an isomorphism from $G_{n+1,n}$ to a subgroup of $(1 + 2^n M_2(\mathbb{Z}_2))/((1 + 2^{n+1} M_2(\mathbb{Z}_2))) \simeq M_2(\mathbb{F}_2)$ preserving the conjugate action of $G_{n+1,0} \simeq \rho_{n+1}(G_{n+1,0})$, which factors through $G_{1,0} \simeq \operatorname{GL}_2(\mathbb{F}_p)$. By the

assumption (Full), we have $\operatorname{Gal}(K_{n+1}/K_n) \simeq M_2(\mathbb{F}_2)$. Note that the $\mathbb{F}_2[\operatorname{GL}(\mathbb{F}_2)]$ submodules of $M_2(\mathbb{F}_2)$ are $0 \subseteq \mathbb{F}_2 \subseteq \mathfrak{sl}(\mathbb{F}_2) \subseteq M_2(\mathbb{F}_2)$. So, we deduce that $M_2(\mathbb{F}_2)$ never has a quotient isomorphic to \mathbb{F}_2^2 . Hence the equality (2.1) holds.

Next, by using Galois cohomology classes contained in the Selmer group, we shall construct certain number fields. Let $n \in \mathbb{Z}_{\geq 0}$ be any element. Clearly, we have a natural isomorphism $H^1(K_{\Sigma}/K_n, W[p^n]) \simeq \operatorname{Hom}_{\operatorname{cont}}(G_n, W[p^n])$, where $\operatorname{Hom}_{\operatorname{cont}}(G_n, W[p^n])$ denotes the group consisting of continuous homomorphisms from G_n to $W[p^n]$. Since G_n is a normal subgroup of $G_{K,\Sigma} = G_0$, we can define a left action

 $G_{n,0} \times \operatorname{Hom}_{\operatorname{cont}}(G_n, W[p^n]) \longrightarrow \operatorname{Hom}_{\operatorname{cont}}(G_n, W[p^n]); \ (\sigma, f) \longmapsto \sigma * f$

of $G_{n,0}$ on $\operatorname{Hom}_{\operatorname{cont}}(G_n, W[p^n])$ by $(\sigma * f)(x) := \sigma(f(\widetilde{\sigma}^{-1}x\widetilde{\sigma}))$ for each $x \in G_n$, where $\widetilde{\sigma} \in G_0$ is a lift of σ . Note that he definition of $\sigma * f$ is independent of the choice of $\widetilde{\sigma}$. The following lemma is a key of the proof of Theorem 1.1.

Lemma 2.10. Take any $n \in \mathbb{Z}_{\geq 0}$. Let M be a \mathbb{Z}_p -submodule of

$$\mathcal{H}_n := H^0(K, H^1(K_{\Sigma}/K_n, W[p^n])) = \operatorname{Hom}_{\operatorname{cont}}(G_n, W[p^n])^{G_{n,0}}.$$

We define $K_n(M)$ to be the maximal subfield of K_{Σ} fixed by $\bigcap_{h \in M} \operatorname{Ker} h$. Then $K_n(M)/K_n$ is Galois, and $[K_n(M) : K_n] = p^{d\ell_{\mathbb{Z}_p}(M)}$. Moreover, the evaluation map $e_M \colon M \longrightarrow \operatorname{Hom}_{\mathbb{Z}_p[G_{n,0}]}(\operatorname{Gal}(K_n(M)/K_n), W[p^n])$ is an isomorphism of \mathbb{Z}_p -modules.

Proof. By definition, the extension $K_n(M)/K_n$ is clearly Galois, and e_M is a welldefined injective homomorphism. Let us show the rest of the assertion of Lemma 2.10 by induction on $\ell_{\mathbb{Z}_p}(M)$.

When $\ell_{\mathbb{Z}_p}(M) = 0$, the assertion of Lemma 2.10 is clear.

Let ℓ be a positive integer, and suppose that the assertion of Lemma 2.10 holds for any \mathbb{Z}_p -submodule M' of \mathcal{H}_n satisfying $\ell_{\mathbb{Z}_p}(M') < \ell$. Let M be any \mathbb{Z}_p -submodule of \mathcal{H}_n satisfying $\ell_{\mathbb{Z}_p}(M) = \ell$. Take a \mathbb{Z}_p -submodule M_0 of M such that $\ell_{\mathbb{Z}_p}(M_0) = \ell - 1$. By definition, we have $K_n(M_0) \subseteq K_n(M)$. Since e_{M_0} is an isomorphism by the hypothesis of induction, and since e_M is an injection, we deduce that

$$[(2.2) [K_n(M):K_n(M_0)] > 1.$$

For each \mathbb{Z}_p -submodule N of M, we put $\mathfrak{K}(N) := \bigcap_{h' \in N} \operatorname{Ker} h' = \operatorname{Gal}(K_{\Sigma}/K_n(N))$. Take an element $f \in M$ not contained in M_0 . Then, we have $\mathfrak{K}(M) = \operatorname{Ker} f \cap \mathfrak{K}(M_0)$. Note that the abelian group $\operatorname{Gal}(K_n(M)/K_n(M_0)) = \mathfrak{K}(M_0)/\mathfrak{K}(M)$ is annihilated by p. (Indeed, if there exists an element $\operatorname{Gal}(K_n(M)/K_n(M_0))$ which is not annihilated by p, then we obtain a sequence $M_0 \subset pM + M_0 \subset M$, which contradicts the fact that M/M_0 is a simple \mathbb{Z}_p -module.) So, the map f induces an injective $\mathbb{F}_p[G_{n,0}]$ -linear map from $\operatorname{Gal}(K_n(M)/K_n(M_0)) = \mathfrak{K}(M_0)/(\operatorname{Ker} f \cap \mathfrak{K}(M_0))$ into $(W[p^n])[p] = W[p]$. By the inequality (2.2) and the assumption (Abs), we deduce that

(2.3)
$$\operatorname{Gal}(K_n(M)/K_n(M_0)) \simeq W[p].$$

Since we have $[K_n(M_0):K_n] = p^{d(\ell-1)}$ by the induction hypothesis, we obtain

$$[K_n(M):K_n] = p^{d(\ell-1)} \cdot \#W[p] = p^{d\ell} = p^{d\ell_{\mathbb{Z}_p}(M)}$$

In order to complete the proof of Lemma 2.10, it suffices to prove that the map e_M is surjective. For each \mathbb{Z}_p -submodule N of M, we put

$$X(N) := \operatorname{Hom}_{\mathbb{Z}_p[G_n]} \left(\operatorname{Gal}(K_n(M)/K_n), W[p^n] \right).$$

Since e_M is injective, and since $\ell_{\mathbb{Z}_p}(M) = \ell$, it suffices to show that $\ell_{\mathbb{Z}_p}(X(M)) \leq \ell$. By the induction hypothesis, we have $\ell_{\mathbb{Z}_p}(X(M_0)) = \ell - 1$. Put

$$X(M; M_0) := \operatorname{Hom}_{\mathbb{Z}_p[G_{n,0}]} (\operatorname{Gal}(K_n(M)/K_n(M_0)), W[p^n]).$$

Since we have an exact sequence $0 \longrightarrow X(M; M_0) \longrightarrow X(M) \longrightarrow X(M_0)$, it suffices to show that the \mathbb{Z}_p -module $X(M; M_0)$ is simple. By (2.3), we obtain

$$X(M; M_0) \simeq \operatorname{Hom}_{\mathbb{Z}_p[G_{n,0}]}(W[p], W[p^n]) \simeq \operatorname{End}_{\mathbb{F}_p[G_{n,0}]}(W[p]).$$

Since the representation W[p] of G_n is absolutely irreducible over \mathbb{F}_p by the assumption **(Abs)**, we obtain $\operatorname{End}_{\mathbb{F}_p[G_{n,0}]}(W[p]) = \mathbb{F}_p$. Hence the \mathbb{Z}_p -module $X(M; M_0)$ is simple. This completes the proof of Lemma 2.10.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let us fix our notation. Again, let K, Σ and T be as in §1. Assume that T satisfies the condition **(Abs)** and **(NT)**. Take any local condition \mathcal{F} on (V, Σ) . Recall that we put $r_{\text{Sel}} = r_{\text{Sel}}(T, \mathcal{F}) = \text{corank}_{\mathbb{Z}_p} \text{Sel}_{\mathcal{F}}(K, W)$. For each $n \in \mathbb{Z}_{\geq 0}$, we denote by M_n the image of $\text{Sel}_{\mathcal{F}}(K, W[p^n]) \longrightarrow \text{Sel}_{\mathcal{F}}(K_n, W[p^n])$, and we put $L_n := K_n(M_n)$ in the sense of Lemma 2.10.

Proposition 3.1. Let ν_{im} be as in Lemma 2.6. For any $wn \in \mathbb{Z}_{\geq 0}$, the extension L_n/K_n is unramified outside Σ_{K_n} , and $\operatorname{ord}_p[L_n:K_n] \geq d(nr_{\text{Sel}} - \nu_{\text{im}})$.

Proof. Since every $f \in M_n$ is a map defined on $G_{K,\Sigma}$, we deduce that L_n is unramified outside Σ . By Lemma 2.6, the length of the \mathbb{Z}_p -module M_n is at least $nr_{\text{Sel}} - \nu_{\text{im}}$. So, Lemma 2.10 implies that $\operatorname{ord}_p[L_n:K_n] \geq d(nr_{\text{Sel}} - \nu_{\text{im}})$.

Let $v \in \Sigma$ be any finite place. We denote by $P_{n,v}$ the set of all places of K_n above v. For each $w \in P_{n,v}$, we denote by $I_w(L_n/K_n)$ the inertia subgroup of $\operatorname{Gal}(L_n/K_n)$ at w. We define $I_{n,v}$ to be the subgroup of $\operatorname{Gal}(L_n/K_n)$ generated by $\bigcup_{w \in P_{n,v}} I_w(L_n/K_n)$. Recall that $r_v = r_v(T, \mathcal{F})$ denotes the corank of the \mathbb{Z}_p -module $\mathcal{H}_v := H^1_{\mathcal{F}}(K_v, W)/(H^1_{\mathcal{F}}(K_v, W) \cap H^1_{\operatorname{ur}}(K_v, W))$. Let \mathcal{K}/K_v be an algebraic extension. Put $W(\mathcal{K}) := H^0(\mathcal{K}, W)$. We define $W(\mathcal{K})_{\operatorname{div}}$ to be the maximal divisible \mathbb{Z}_p -submodule of $W(\mathcal{K})$. For each $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we define

$$\nu_{v,n} := \ell_{\mathbb{Z}_p} \left(H^0(K_v, W(K_v^{\mathrm{ur}})/W(K_v^{\mathrm{ur}})_{\mathrm{div}}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z} \right).$$

Note that $\{\nu_{v,n}\}_{n\geq 0}$ is a bounded increasing sequence, and for any sufficiently large m, we have $\nu_{v,m} = \nu_{v,\infty}$. Let us prove the following proposition.

Proposition 3.2. We have $\ell_{\mathbb{Z}_p}(I_{n,v}) \leq d(r_v n + \nu_{v,n})$ for any $v \in \Sigma$ and $n \in \mathbb{Z}_{\geq 0}$.

In order to show that Proposition 3.2, we need the following lemma.

Lemma 3.3. Let $n \in \mathbb{Z}_{\geq 0}$ be any element, and define

 $\widetilde{\mathcal{H}}_{v,n} := H^1_{\mathcal{F}}(K_v, W[p^n]) / (H^1_{\mathcal{F}}(K_v, W[p^n]) \cap H^1_{\mathrm{ur}}(K_v, W[p^n])).$

Then, we have $\ell_{\mathbb{Z}_p}(\widetilde{\mathcal{H}}_{v,n}) \leq r_v n + \nu_{v,n}$.

Proof of Lemma 3.3. Let \mathcal{K}/K_v be an algebraic extension, and

$$\iota_{\mathcal{K},n} \colon H^1(\mathcal{K}, W[p^n]) \longrightarrow H^1(\mathcal{K}, W)$$

the natural map. By the short exact sequence $0 \longrightarrow W[p^n] \longrightarrow W \longrightarrow W \longrightarrow 0$, we obtain isomorphism Ker $\iota_{\mathcal{K},n} \simeq W(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^N \mathbb{Z} \simeq (W(\mathcal{K})/W(\mathcal{K})_{\text{div}}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^N \mathbb{Z}$.

We put $\widetilde{Y}_n := H^1_{\mathcal{F}}(K_v, W[p^n]) \cap H^1_{\mathrm{ur}}(K_v, W[p^n])$, and $Y := H^1_{\mathcal{F}}(K_v, W) \cap H^1_{\mathrm{ur}}(K_v, W)$. By definition, it clearly holds that $\widetilde{Y}_n \subseteq \iota_{K_v,n}^{-1}(Y)$, and $H^1_{\mathcal{F}}(K_v, W[p^n])/\widetilde{Y}_n = \widetilde{\mathcal{H}}_{v,n}$. Moreover, we have an injection $H^1_{\mathcal{F}}(K_v, W[p^n])/\iota_{K_v,n}^{-1}(Y) \hookrightarrow \mathcal{H}_v[p^n]$. So, we obtain

$$\ell_{\mathbb{Z}_p}(\widetilde{\mathcal{H}}_{v,n}) \leq \ell_{\mathbb{Z}_p}(\mathcal{H}_v[p^n]) + \ell_{\mathbb{Z}_p}(\iota_{K_v,n}^{-1}(Y)/\widetilde{Y}_n).$$

On the one hand, since $H^1_{\mathcal{F}}(K_v, W)$ is a divisible \mathbb{Z}_p -module by definition, so is the quotient module \mathcal{H}_v . This implies that $\ell_{\mathbb{Z}_p}(\mathcal{H}_v[p^n]) = r_v n$. On the other hand, the restriction map $H^1(K_v, W[p^n]) \longrightarrow H^0(K_v, H^1(K_v^{\mathrm{ur}}, W[p^n]))$ induces an injection

$$\iota_{K_v,n}^{-1}(Y)/\tilde{Y}_n \hookrightarrow H^0(K_v, \operatorname{Ker} \iota_{K_v^{\operatorname{ur}},n}) \simeq H^0(K_v, W(K_v^{\operatorname{ur}})/W(K_v^{\operatorname{ur}})_{\operatorname{div}}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}.$$

So, we obtain $\ell_{\mathbb{Z}_p}(\iota_{K_{v,n}}^{-1}(Y)/\widetilde{Y}_n) \leq \nu_{v,n}$. Hence $\ell_{\mathbb{Z}_p}(\widetilde{\mathcal{H}}_{v,n}) \leq r_v n + \nu_{v,n}$. *Proof of Proposition 3.2.* Take any $v \in \Sigma$ and $n \in \mathbb{Z}_{\geq 0}$. Let $w \in P_{n,v}$. We define

oof of Proposition 3.2. Take any
$$v \in \Sigma$$
 and $n \in \mathbb{Z}_{\geq 0}$. Let $w \in P_{n,v}$. We define

$$\operatorname{res}_{I,w}$$
: $\operatorname{Hom}_{\mathbb{Z}_p}(\operatorname{Gal}(L_n/K_n), W[p^n]) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_p}(I_w(L_n/K_n), W[p^n])$

to be the restriction maps, and put $M_{n,w}^{\mathrm{ur}} := \mathrm{Ker}(\mathrm{res}_{I,w} \circ \mathrm{res}_{D,w}|_{M_n}) \subseteq M_n$. By definition, the extension $K_n(M_{n,w}^{\mathrm{ur}})/K_n$ is unramified at w. So, we obtain

(3.1)
$$I_w(L_n/K_n) \subseteq \operatorname{Gal}(L_n/K_n(M_{n,w}^{\mathrm{ur}})).$$

We fix an element $w_0 \in P_{n,v}$. Let $\sigma \in G_n$ be any element. Then, the diagram

commutes. So, for each $w \in P_{n,v}$, we have $M_{n,w}^{\text{ur}} = M_{n,w_0}^{\text{ur}}$. Hence by (3.1), we obtain $I_{n,v} \subseteq \operatorname{Gal}(L_n/K_n(M_{n,w_0}^{\mathrm{ur}}))$. In order to prove Proposition 3.2, it suffices to show that $\ell_{\mathbb{Z}_p}\left(\operatorname{Gal}(L_n/K_n(M_{n,w_0}^{\operatorname{ur}}))\right) \leq d(r_v n + \nu_{v,n}).$ (3.2)

By Lemma 2.10, we have $\ell_{\mathbb{Z}_p}(\operatorname{Gal}(L_n/K_n(M_{n,w_0}^{\operatorname{ur}}))) = d\ell_{\mathbb{Z}_p}(M_n/M_{n,w_0}^{\operatorname{ur}})$. Since the natural surjection $\operatorname{Sel}_{\mathcal{F}}(K, W[p^n]) \longrightarrow M_n/M_{n,w_0}^{\operatorname{ur}} \simeq (\operatorname{res}_{I,w} \circ \operatorname{res}_{D,w})(M_n)$ factors through the \mathbb{Z}_p -module $\mathcal{H}_{v,n}$ in Lemma 3.3, we obtain the inequality (3.2).

Proof of Theorem 1.1. Take any $n \in \mathbb{Z}_{\geq 0}$. Let I be the subgroup of $\operatorname{Gal}(L_n/K_n)$ generated by $\bigcup_{v \in \Sigma} I_{n,v}$. Then, the extension L_n^I/K_n is unramified at every finite place, and the degree $[L_n^I: K_n]$ is a power of p. So, by the global class field theory, we have

$$\operatorname{ord}_p(h_n) \ge \operatorname{ord}_p[L_n^I : K_n] \ge \operatorname{ord}_p[L_n : K_n] - \sum_{v \in \Sigma} \operatorname{ord}_p(\#I_{n,v})$$

For each $n \in \mathbb{Z}_{\geq 0}$, we put $\nu_{im,n} := \ell_{\mathbb{Z}_p}(H^1(K_n/K, W[p^n]))$. Then, by Lemma 2.6, the sequence $\{\nu_{im,n}\}_{n\geq 0}$ is bounded. By Proposition 3.1 and Proposition 3.2, we obtain

(3.3)
$$\operatorname{ord}_p(h_n) \ge d\left(r_{\operatorname{Sel}} \cdot n - \nu_{\operatorname{im},n}\right) - d\sum_{v \in \Sigma} \left(r_v n + \nu_{v,n}\right) \succ d\left(r_{\operatorname{Sel}} - \sum_{v \in \Sigma} r_v\right) n.$$

This completes the proof of Theorem 1.1.

This completes the proof of Theorem 1.1.

By the above arguments, in particular by the inequality (3.3), we have also obtained a bit stronger result than Theorem 1.1.

Theorem 3.4. For any $n \in \mathbb{Z}_{\geq 0}$, we have

$$\operatorname{ord}_p(h_n) \ge d\left(r_{\operatorname{Sel}} - \sum_{v \in \Sigma} r_v\right) n - d\nu_{\operatorname{im},n} - d\sum_{v \in \Sigma} \nu_{v,n}.$$

4. Application to abelian varieties

In this section, we apply Theorem 1.1 to the extension defined by an abelian variety A. In §4.1, we prove Corollary 1.3. Moreover, from the view point of Theorem 3.4, we compare our results (of stronger form) with earlier results in the cases when A is an elliptic curve. In §4.2, we study the cases when A is a Hilbert–Blumenthal or CM abelian variety, and prove Corollary 1.5.

4.1. General cases. Let A be an abelian variety over a number field K, and fix a prime number p such that A[p] becomes an absolutely irreducible representation of the absolute Galois group G_K of K over \mathbb{F}_p . We denote the dimension of A over Kby g. Let T_pA be the p-adic Tate module of A, namely $T_pA := \lim_{K \to a} A[p^n]$, and put $V_pA := T_pA \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Note that T_pA is a free \mathbb{Z}_p -module of rank 2g. Let $\Sigma(A)$ be the subset of P_K consisting of all places dividing $p\infty$ and all places where A has bad reduction. Then, the natural action $\rho_A^{(p)}$ of G_K is unramified outside $\Sigma(A)$. Let L/K be any finite extension, and $w \in P_K$ any finite place above a prime number

Let L/K be any finite extension, and $w \in P_K$ any finite place above a prime number ℓ . With the aid of the implicit function theorem (for instance [Se] PART II Chapter III §10.2 Theorem) and the Jacobian criterion (for instance [Li] Chapter 4 Theorem 2.19), the projectivity and smoothness of A implies that $A(L_w)$ is a g-dimensional compact abelian analytic group over L_w . So, we have

(4.1)
$$A(L_w) \simeq \mathbb{Z}_{\ell}^{g[L_w:\mathbb{Q}_{\ell}]} \oplus \text{ (a finite abelian group)}$$

(See Corollary 4 of Theorem 1 in [Se] PART II Chapter V §7.) Related to this fact, the following is known.

Proposition 4.1 ([BK] Example 3.11). For any finite extension field L of K, and any finite place $w \in P_L$, we have a natural isomorphism $H^1_f(L_w, A[p^{\infty}]) \simeq A(L_w) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p$. If w lies above p, then the corank of the \mathbb{Z}_p -module $H^1_f(L_w, A[p^{\infty}])$ is equal to $g[L_w : \mathbb{Q}_p]$. If w does not lie above p, then $H^1_f(L_w, A[p^{\infty}]) = 0$.

Proof of Corollary 1.3. We define the Tate-Shafarevich group $\operatorname{III}(A/K)$ to be the kernel of $H^1(K, A(\overline{K})) \longrightarrow \prod_{v \in P_K} H^1(K_v, A(\overline{K}_v))$. Then, we have a short exact sequence

$$(4.2) \qquad 0 \longrightarrow A(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p / \mathbb{Z}_p \longrightarrow \operatorname{Sel}_f(K, A[p^{\infty}]) \longrightarrow \operatorname{III}(A/K)[p^{\infty}] \longrightarrow 0.$$

(See §C.4 in [HS] Appendix C. Note that by Proposition 4.1, our $\operatorname{Sel}_f(K, A[p^{\infty}])$ is naturally isomorphic to $\varinjlim_n \operatorname{Sel}^{([p^n])}(A/K)$ in the sense of [HS] Appendix C, where $[p^n]: A \longrightarrow A$ denotes the multiplication-by- p^n isogeny.) So, we obtain

$$\operatorname{corank}_{\mathbb{Z}_p} \operatorname{Sel}_f(K, A[p^{\infty}]) \ge r_{\mathbb{Z}}(A) := \operatorname{rank}_{\mathbb{Z}} A(K).$$

Hence by Theorem 1.1 for $(T, \Sigma, \mathcal{F}) = (T_p A, \Sigma(A), f)$ and Proposition 4.1, we obtain

$$\operatorname{ord}_p(h_n(A;p)) \succ 2g\left(r_{\mathbb{Z}}(A) - g\sum_{v|p} [K_v : \mathbb{Q}_p]\right) n = 2g\left(r_{\mathbb{Z}}(A) - g[K : \mathbb{Q}]\right) n.$$

This completes the proof of Corollary 1.3.

Remark 4.2. Here, we give remarks on the image of the modulo p representation $\rho_{A,1}^{(p)}$: $\operatorname{Gal}(K(A[p])/K) \longrightarrow \operatorname{Aut}(A[p])$. Let A be a principally polarized abelian variety of dimension g defined over K, and take an odd prime number p. Then, the image of ρ_1 can be regarded as a subgroup of $\operatorname{GSp}_{2g}(\mathbb{F}_p)$. Clearly if $\operatorname{Im} \rho_{A,1}^{(p)}$ contains $\operatorname{Sp}_{2g}(\mathbb{F}_p)$, then T_pA satisfies the conditions **(Abs)** and **(NT)**. Since p is odd, the non-trivial scalar -1 is contained in $\operatorname{Sp}_{2g}(\mathbb{F}_p)$. So, as noted in Remark 2.7, if $\operatorname{Im} \rho_{A,1}^{(p)}$ contains $\operatorname{Sp}_{2g}(\mathbb{F}_p)$, then we can take $\nu_{\mathrm{im}} = 0$, where ν_{im} denotes the error constant in Theorem 3.4 for $(T, \Sigma, \mathcal{F}) = (T_pA, \Sigma(A), f)$. It is proved by Banaszak, Gajda and Krasoń that $\operatorname{Im} \rho_{A,1}^{(p)}$ contains $\operatorname{Sp}_{2g}(\mathbb{F}_p)$ for sufficiently large p if A satisfies the following (i)–(iv).

- (i) The abelian variety A is simple.
- (ii) There is no endomorphism on A defined over \overline{K} except the multiplications by rational integers, namely $\operatorname{End}_{\overline{K}}(A) = \mathbb{Z}$.
- (iii) For any prime number ℓ , the Zariski closure of the image of the ℓ -adic representation $\rho_A^{(\ell)} \colon G_K \longrightarrow \operatorname{Aut}_{\mathbb{Z}_\ell}(V_\ell A) \simeq \operatorname{GL}_{2g}(\mathbb{Q}_\ell)$ is a connected algebraic group.
- (iv) The dimension g of A is odd.

See [BGK] Theorem 6.16 for the cases when $\operatorname{End}_{\overline{K}}(A) = \mathbb{Z}$. (Note that in [BGK], they proved more general results. For details, see loc. cit..)

Remark 4.3. Here, we shall describe the error terms $\nu_{v,n}$ in Theorem 3.4 for $T = T_p A$ in terminology related to the reduction of A at v. Let \mathcal{A} be the Néron model of Aover \mathcal{O}_K . Take any finite place $v \in P_K$, and denote by k_v the residue field of \mathcal{O}_{K_v} . We put $A_{0,v} := \mathcal{A} \otimes_{\mathcal{O}_K} k_v$, and define $A_{0,v}^0$ to be the identity component of $A_{0,v}$. Note that we have $H^0(K_v^{\mathrm{ur}}, A[p^{\infty}]) \simeq A_{0,v}(\overline{k_v})$. (See [ST] §1, Lemma 2.) By Chevalley decomposition, we have an exact sequence $0 \longrightarrow T_w \times U_w \longrightarrow A_{0,v}^0 \longrightarrow B_v \longrightarrow 0$ of group schemes over k, where T_w is a torus, U_w is a unipotent group, and B_v is an abelian variety. (For instance, see [Co] Theorem 1.1 and [Wa] Theorem 9.5.) In particular, if $U_w(\overline{k_v})[p^{\infty}] = 0$ (for instance, if v does not lie above p), then the divisible part of $A_{0,v}(\overline{k_v})[p^{\infty}]$ coincides with $A_{0,v}^0(\overline{k_v})[p^{\infty}]$, and hence

(4.3)
$$\nu_{v,n} = \ell_{\mathbb{Z}_p}(\pi_0(A_{0,v})(k_v)[p^{\infty}] \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}) = \ell_{\mathbb{Z}_p}(\pi_0(A_{0,v})(k_v)[p^n]).$$

where $\pi_0(A_{0,v})$ denotes the group of the connected components of $A_{0,v}$. (Note that the second equality holds since $\pi_0(A_{0,v})$ is finite.) We can compute the error factors $\nu_{v,n}$ explicitly if we know the structure of the reduction $A_{0,v}$ of A at each finite place v.

Example 4.4 (Error factors for elliptic curves). Now, we study the error factors $\nu_{v,n}$ in the setting of [SY2] and [Hi]. We set $K = \mathbb{Q}$, and let A be an elliptic curve with minimal discriminant Δ . Let p be a prime number satisfying (Full) in Example 2.8. We assume that p is odd. Moreover, we also assume the following hypothesis.

- If p = 3, then A does not have additive reduction at p.
- If A has additive reduction at p, then $A(\mathbb{Q}_p)[p] = 0$.
- If A has split multiplicative reduction at p, p does not divide $\operatorname{ord}_p(\Delta)$.

(These hypotheses are assumed in [SY2] for p > 2 and [Hi].) Note that $\Sigma(A)$ is the set of places dividing $\infty p\Delta$. In this situation, we have $U_p(\overline{\mathbb{F}}_p)[p^{\infty}] = 0$, where U_p is the unipotent part of $A_{0,p}^0$. So, by (4.3), we obtain $\nu_{p,n} = 0$ for each $n \in \mathbb{Z}_{\geq 0}$ since by [Si] CHAPTER IV §9 Tate's algorithm 9.4,

- if A has good reduction at p, then $A_{0,p}$ is connected;
- if A has split multiplicative reduction, then $\pi_0(A_{0,p})(\mathbb{F}_p) \simeq \mathbb{Z}/\operatorname{ord}_p(\Delta)\mathbb{Z};$

- if A has non-split multiplicative reduction, then $\pi_0(A_{0,p})(\mathbb{F}_p) \simeq 0$ or $\mathbb{Z}/2\mathbb{Z}$;
- if A has additive reduction, then the order of $\pi_0(A_{0,p})(\mathbb{F}_p)$ is prime to p.

Let ℓ be a prime number distinct from p. Then, we have $U_{\ell}(\overline{\mathbb{F}}_{\ell})[p^{\infty}] = 0$. So, similarly to the above arguments, we obtain

(4.4)
$$\nu_{\ell,n} = \begin{cases} \min\{\operatorname{ord}_p(\operatorname{ord}_\ell(\Delta)), n\} \} & \left(\begin{array}{c} \text{if } A \text{ has split multiplicative} \\ \operatorname{reduction at } \ell \end{array} \right) \\ 0 & (\text{otherwise}). \end{cases}$$

Combining with Example 2.8, Theorem 3.4 implies that

$$\operatorname{ord}_p(h_n(A; p)) \ge 2 \left(r_{\mathbb{Z}}(A) - 1 \right) n - \sum_{p \neq \ell \mid \Delta} \nu_{\ell, n},$$

where $\nu_{\ell,n}$ is given by (4.4). This inequality coincides with that obtained in [SY1], [SY2] and [Hi] when p is odd. In [SY2], Sairaiji and Yamauchi also treat the cases when p = 2. Note that when p = 2, an inequality following from Theorem 3.4 is weaker than that obtained in [SY2]. Indeed, our $\nu_{p,n}$ is always non-negative by definition, but instead of it, in [SY2], they introduced a constant δ_2 related to the local behavior of A at p = 2 which may become a negative integer.

4.2. **RM and CM cases.** Here we shall prove Corollary 1.5. Let K/K^+ , p, π, A and $h_n(A; \pi)$ be as in Corollary 1.5. Take a subset $\Phi = \{\phi_1, \ldots, \phi_g\} \subseteq \text{Gal}(K/\mathbb{Q})$ such that we have an isomorphism

(4.5)
$$\operatorname{Lie}(A/K) \simeq \bigoplus_{i=1}^{g} (K, \phi_i)$$

of modules over the ring $K \otimes_{\mathbb{Z}} \operatorname{End}(A) = K \otimes_{\mathbb{Z}} \mathcal{O}_K = \prod_{\sigma \in \operatorname{Gal}(K/\mathbb{Q})} (K, \sigma)$. Note that $\phi_1|_{K^+}, \ldots, \phi_g|_{K^+}$ are distinct g elements of $\operatorname{Gal}(K^+/K)$.

Let us introduce notation related to the formal group law. Take any $\sigma \in \operatorname{Gal}(K/\mathbb{Q})$, and denote by $\sigma(\pi)$ the place of K corresponding to $\sigma(\pi)\mathcal{O}_K$ (by abuse of notation). We put $k_{\sigma(\pi)} := \mathcal{O}_K/\sigma(\pi)\mathcal{O}_K = \mathbb{F}_p$. We define $\mathcal{A}_{\sigma(\pi)}$ to be the Néron model of $A_{K_{\sigma(\pi)}} := A \otimes_K K_{\sigma(\pi)}$ over $\mathcal{O}_{K_{\sigma(\pi)}}$, and $\mathcal{O}_{\sigma(\pi),s}$ (resp. $\mathcal{O}_{\sigma(\pi),\eta}$) the origin of the special (resp. generic) fiber of $\mathcal{A}_{\sigma(\pi)}$. For each $\star \in \{s, \eta\}$, let $\mathfrak{m}_{\sigma(\pi),\star}$ be the maximal ideal of the local ring $\mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)},\star}$. Note that $\mathscr{O}_{\mathcal{A}_{\sigma(\pi)},s}$ is a regular local ring since A has good reduction at $\sigma(\pi)$. Let $s'_0 = p, s'_1, \ldots, s'_g \in \mathfrak{m}_{\sigma(\pi),s}$ be any regular system of parameters for the local ring $(\mathscr{O}_{\mathcal{A}_{\sigma(\pi)},\mathcal{O}_{\sigma(\pi),s}},\mathfrak{m}_{\sigma(\pi),s})$. Put $\mathfrak{n} := \mathscr{O}_{\mathcal{A}_{\sigma(\pi)},\mathcal{O}_{\sigma(\pi),s}} \cap \mathfrak{m}_{\sigma(\pi),\eta}$. Since we have the identity section $\operatorname{Spec} \mathcal{O}_{K_{\sigma(\pi)}} \longrightarrow \mathcal{A}_{\sigma(\pi)}$, it holds that $\mathscr{O}_{\mathcal{A}_{\sigma(\pi)},\mathcal{O}_{\sigma(\pi),s}}/\mathfrak{n} \simeq \mathcal{O}_{K_{\sigma(\pi)}}$. So, for each $i \in \mathbb{Z}$ with $1 \leq i \leq g$, we have a unique element $c_i \in \sigma(\pi) \mathcal{O}_{K_{\sigma(\pi)}}$ such that $s_i := s'_i - c_i \in \mathfrak{n}$. Since $\mathcal{O}_{K_{\sigma(\pi)}}$ is a DVR, and since we have

(4.6)
$$\mathfrak{n}/\mathfrak{n}^2 \otimes_{\mathcal{O}_{K_{\sigma(\pi)}}} k(\sigma(\pi)) \simeq \operatorname{coLie}(A_{0,\sigma(\pi)}/k(\sigma(\pi))),$$

(4.7)
$$\mathfrak{n}/\mathfrak{n}^2 \otimes_{\mathcal{O}_{K_{\sigma(\pi)}}} K_{\sigma(\pi)} \simeq \operatorname{coLie}(A_{K_{\sigma(\pi)}}/K_{\sigma(\pi)}),$$

the sequence s_1, \ldots, s_g forms a regular system of parameters for the regular local ring $(\mathcal{O}_{\mathcal{A}_{\sigma(\pi)}, \mathcal{O}_{\sigma(\pi), \eta}}, \mathfrak{m}_{\sigma(\pi), \eta})$, and it holds that

(4.8)
$$\mathfrak{n}/\mathfrak{n}^2 = \bigoplus_{i=1}^g \mathcal{O}_{K_{\sigma(\pi)}} \bar{s}_i,$$

where $A_{0,\sigma(\pi)}$ denotes the special fiber of $\mathcal{A}_{\sigma(\pi)}$, and \bar{s}_i denotes the image of s_i in $\mathfrak{n}/\mathfrak{n}^2$.

We denote by $\widehat{\mathcal{A}}_{\sigma(\pi)} = \operatorname{Spf} \mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}}$ the completion of $\mathcal{A}_{\sigma(\pi)}$ along $\mathcal{O}_{\sigma(\pi),s}$, and by $\widehat{\mathfrak{m}}_{\sigma(\pi)}$ the maximal ideal of $\mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}}$. Note that $s_0 := p, s_1, \ldots, s_g$ forms a regular system of parameters for the complete regular local ring $(\mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}}, \widehat{\mathfrak{m}}_{\sigma(\pi)})$ We also note that p splits completely (unramified in particular) in K/\mathbb{Q} by our assumption. So, we have $\mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}} = \mathcal{O}_{K_{\sigma(\pi)}}[[s_1, \ldots, s_g]]$. (See [Ma] Theorem 29.7.) For each $i \in \mathbb{Z}$ with $1 \leq i \leq g$, we define a formal power series $\mathscr{F}_{A,\sigma(\pi),i} \in \mathcal{O}_{K_{\sigma(\pi)}}[[x_1, \ldots, x_g, y_1, \ldots, y_g]]$ by

$$\mathscr{F}_{A,\sigma(\pi),i} := \operatorname{add}_{A,\sigma(\pi)}^{\sharp}(s_i),$$

where $\operatorname{add}_{A,\sigma(\pi)}^{\sharp} \colon \mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}} \longrightarrow \mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}} \widehat{\otimes}_{\mathcal{O}_{K_{\sigma(\pi)}}} \mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}} = \mathcal{O}_{K_{\sigma(\pi)}}[[x_1,\ldots,x_g,y_1,\ldots,y_g]]$ is the ring homomorphism corresponding to the group structure of the formal group scheme $\widehat{\mathcal{A}}_{\sigma(\pi)}$. Note that since s_1,\ldots,s_g forms a regular system of parameters for the regular local ring $\mathscr{O}_{\mathcal{A}_{\sigma(\pi)},\mathcal{O}_{\sigma(\pi),\eta}}$, the correction $\mathscr{F}_{A,\sigma(\pi)} = (\mathscr{F}_{A,\sigma(\pi),i})_{i=1}^g$ is a gdimensional commutative formal group law over $K_{\sigma(\pi)}$, and hence that over $\mathcal{O}_{K_{\sigma(\pi)}}$. (For details, see Lemma C.2.1 in Appendix C of [HS].) Let $\alpha \in \mathcal{O}_K$ be any element, and $[\alpha]_{\mathcal{A},\sigma(\pi)}^{\sharp} \colon \mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}} \longrightarrow \mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}}$ the ring homomorphism corresponding to the multiplication-by- α endomorphism on the formal group scheme $\widehat{\mathcal{A}}_{\sigma(\pi)}$. For each $i \in \mathbb{Z}$ with $1 \leq i \leq g$, we put

$$[\alpha]_{A,\sigma(\pi),i}(s_1,\ldots,s_g) := [\alpha]_{A,\sigma(\pi)}^{\sharp}(s_i) \in \mathscr{O}_{\widehat{\mathcal{A}}_{\sigma(\pi)}} = \mathcal{O}_{K_{\sigma(\pi)}}[[s_1,\ldots,s_g]].$$

Lemma 4.5. There exists a regular system of parameters $s_0 = p, s_1, \ldots, s_g \in \mathfrak{m}_{\sigma(\pi),s}$ such that for any $i \in \mathbb{Z}$ with $1 \leq i \leq g$ and any $\alpha \in \mathcal{O}_K$, it holds that $s_i \in \mathfrak{n}$, and

$$[\alpha]_{A,\sigma(\pi),i}(s_1,\ldots,s_g) \equiv \phi_i(\alpha)s_i \mod \mathfrak{n}^2.$$

Proof. Since we have a direct product decomposition

$$\mathcal{O}_{K_{\sigma(\pi)}} \otimes_{\mathbb{Z}} \mathcal{O}_{K} = \mathbb{Z}_{p} \otimes_{\mathbb{Z}} \mathcal{O}_{K} = \varprojlim_{n} \mathcal{O}_{K}/p^{n} \mathcal{O}_{K} = \prod_{\tau \in \operatorname{Gal}(K/\mathbb{Q})} \mathcal{O}_{K_{\tau(\pi)}}$$

by Chinese remainder theorem, and by (4.5), (4.7) and (4.8), we can take an $\mathcal{O}_{K_{\sigma(\pi)}}$ basis $\bar{s}_1, \dots, \bar{s}_g$ of $\mathfrak{n}/\mathfrak{n}^2$ such that for each $i \in \mathbb{Z}$ with $1 \leq i \leq g$ and each $\alpha \in \mathcal{O}_K$, the element \bar{s}_i is an eigenvector of the multiplication-by- α map $[\alpha]$ attached to the eigenvalue $\phi_i(\alpha)$. We take any lift $s_1, \dots, s_g \in \mathfrak{n}$ of $\bar{s}_1, \dots, \bar{s}_g$. Then the sequence $s_0 = p, s_1, \dots, s_g$ is the one as desired.

From now on, let the parameters s_1, \ldots, s_g be as in 4.5. We define

$$\mathscr{F}_{A,\sigma(\pi)}(\sigma(\pi)\mathcal{O}_{K_{\sigma(\pi)}}) = \left((\sigma(\pi)\mathcal{O}_{K_{\sigma(\pi)}})^{g}, \mathscr{F}_{A,\sigma(\pi)} \right)$$

to be the set $(\sigma(\pi)\mathcal{O}_{K_{\sigma(\pi)}})^g$ equipped with a group structure defined by the formal group law $\mathscr{F}_{A,\sigma(\pi)}$. Note that by the scalar multiplication defined by the collection $([\alpha]_{A,\sigma(\pi),i})_{i=1}^g$ of the power series, we can regard $\mathscr{F}_{A,\sigma(\pi)}(\sigma(\pi)\mathcal{O}_{K_{\sigma(\pi)}})$ as an \mathcal{O}_K -module. By Lemma 4.5, the following holds.

Corollary 4.6. Let $\sigma \in \text{Gal}(K/\mathbb{Q})$ any element. Then, we have

$$\mathscr{F}_{A,\sigma(\pi)}\left(\sigma(\pi)\mathcal{O}_{K_{\sigma(\pi)}}\right)\otimes_{\mathcal{O}_{K}} \lim_{n \to 0} \pi^{-n}\mathcal{O}_{K}/\mathcal{O}_{K} \simeq \begin{cases} \mathbb{Q}_{p}/\mathbb{Z}_{p} & (if \ \sigma \in \Phi), \\ 0 & (if \ \sigma \notin \Phi). \end{cases}$$

Let $A_{0,\sigma(\pi)}$ be the special fiber of $\mathcal{A}_{\sigma(\pi)}$. Since A has good reduction at $\sigma(\pi)$, we can define the reduction map $\operatorname{red}_{A,\sigma(\pi)} : A(K_{\sigma(\pi)}) \longrightarrow A_{0,v}(k_{\sigma(\pi)})$.

Lemma 4.7 (For instance, see [HS] Theorem C.2.6). For any finite place $v \in P_K$, the \mathcal{O}_K -module Ker red_{A,\sigma(\pi)} is isomorphic to $\mathscr{F}_{A,\sigma(\pi)}(\pi_v \mathcal{O}_{K_{\sigma(\pi)}})$.

Remark 4.8. Note that [HS] Theorem C.2.6 says that Ker $\operatorname{red}_{A,\sigma(\pi)}$ is isomorphic to $\mathscr{F}_A(\pi_v \mathcal{O}_{K_v})$ only as a group, but it is easy to verify that the group isomorphism constructed in the proof of [HS] Theorem C.2.6 preserves the scalar action of \mathcal{O}_K .

We define $T_{\pi}A := \lim_{n \to \infty} A[\pi^n]$. Let $\Sigma(A)$ be a subset of P_K consisting of all places dividing $p\infty$ and all places where A has bad reduction. Since $T_{\pi}A$ is regarded as a \mathbb{Z}_p submodule of T_pA , the action of $\operatorname{Gal}(\overline{K}/K)$ on $T_{\pi}A$ is unramified outside $\Sigma(A)$. Since p splits completely in K, the \mathbb{Z}_p -module $T_{\pi}A$ is free of rank $2 \dim / [K : \mathbb{Q}] = 2/[K : K^+]$. Note that $T_{\pi}A$ satisfies (Abs) and (NT) by our assumption.

Take a finite place $v \in P_K$ above a prime number ℓ . Since $H^1(K_v, A[\pi^{\infty}])$ is a direct summand of $H^1(K_v, A[p^{\infty}])$ consisting of elements annihilated by π^n for some $n \in \mathbb{Z}_{\geq 0}$, Proposition 4.1 implies that

$$H^1_f(K_v, A[\pi^{\infty}]) = H^1_f(K_v, A[p^{\infty}]) \cap H^1(K_v, A[\pi^{\infty}]) = H^1_f(L_v, A[p^{\infty}])[\pi^{\infty}]$$
$$\simeq (A(K_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)[\pi^{\infty}] \simeq A(K_v) \otimes_{\mathcal{O}_K} K_\pi/\mathcal{O}_{K_\pi}.$$

By isomorphisms (4.1) (for $v \nmid p$), Corollary 4.6 and Lemma 4.7, the following holds.

Proposition 4.9. Let $v \in P_K$ be a finite place. Then, we have

$$H^1_f(K_v, A[\pi^{\infty}]) = \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p & (if \ v = \sigma(\pi) \ for \ some \ \sigma \in \Phi), \\ 0 & (otherwise). \end{cases}$$

Proof of Corollary 1.5. Note that the Selmer group $\operatorname{Sel}_f(K, A[\pi^{\infty}])$ is a direct summand of the \mathbb{Z}_p -module $\operatorname{Sel}_f(K, A[p^{\infty}])$ consisting of elements annihilated by π^n for some $n \in \mathbb{Z}_{\geq 0}$. So, we have $\operatorname{Sel}_f(K, A[\pi^{\infty}]) \simeq \operatorname{Sel}_f(K, A[p^{\infty}]) \otimes_{\mathcal{O}_K} K_{\pi}/\mathcal{O}_{K_{\pi}}$. Combining with the short exact sequence (4.2), this implies that $\operatorname{Sel}_f(K, A[\pi^{\infty}])$ has a \mathbb{Z}_p -submodule isomorphic to $A(K) \otimes_{\mathcal{O}_K} K_{\pi}/\mathcal{O}_{K_{\pi}}$. Thus it holds that

$$\operatorname{corank}_{\mathbb{Z}_p} \operatorname{Sel}_f(K, A[\pi^{\infty}]) \ge r_{\mathcal{O}_K}(A) := \dim_K(A(K) \otimes_{\mathcal{O}_K} K).$$

Hence by Theorem 1.1 for $(T, \Sigma, \mathcal{F}) = (T_{\pi}A, \Sigma(A), f)$ and Proposition 4.9, we obtain the assertion of Corollary 1.5, that is, $h_n(A; \pi) \succ 2[K:K^+]^{-1}(r_{\mathcal{O}_K}(A) - g)n$.

Remark 4.10. Here, let p be an odd prime number, and consider the image of $G_{1,0}$; = $\operatorname{Gal}(K_1/K)$ in $\operatorname{Aut}_{\mathbb{F}_p}(A[\pi])$. First, let K be a CM field. By Banaszak–Gajda–Krasoń's work ([BGK] Theorem 6.16 for the cases then $\operatorname{End}_{\overline{K}}(A) = \mathcal{O}_K$) it is known that the image of $G_{1,0} := \operatorname{Gal}(K_1/K)$ in $\operatorname{Aut}_{\mathbb{F}_p}(A[\pi]) = \operatorname{GL}_2(\mathbb{F}_p)$ contains $\operatorname{SL}_2(\mathbb{F}_p)$ if A is principally polarized, and if A satisfies (i) and (iii) in Remark 4.2. So, if A satisfies the conditions (i) and (iii) in Remark 4.2, then $T_{\pi}A$ satisfies (Abs) and (NT), and we can take $\nu_{\rm im} = 0$ by Remark 2.7.

Next, let K be a CM field. Then $T_{\pi}A$ obviously satisfies the condition (Abs) since $\dim_{\mathbb{F}_p} A[\pi] = 1$. Moreover, if K is a CM field, then $G_{1,0} = \operatorname{Gal}(K_1/K) \simeq \mathbb{F}_p^{\times}$. (See, for instance [Ro] Proposition 3.1.) So, in this situation, the π -adic Tate module $T_{\pi}A$ also satisfies the condition (NT), and we can take $\nu_{\rm im} = 0$ by Remark 2.7.

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