SOLUTIONS OF GROSS-PITAEVSKII EQUATION WITH PERIODIC POTENTIAL IN DIMENSION TWO.

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Abstract. Quasi-periodic solutions of a nonlinear polyharmonic equation for the case $4l > n+1$ in $\mathbb{R}^n$, $n > 1$, are studied. This includes Gross-Pitaevskii equation in dimension two ($l = 1, n = 2$). It is proven that there is an extensive "non-resonant" set $\mathcal{G} \subset \mathbb{R}^n$ such that for every $\vec{k} \in \mathcal{G}$ there is a solution asymptotically close to a plane wave $Ae^{i\langle \vec{k}, \vec{x}\rangle}$ as $|\vec{k}| \to \infty$, given A is sufficiently small.

1. Introduction

Let us consider a nonlinear polyharmonic equation with a periodic potential $V(\vec{x})$ and quasi-periodic boundary condition:

$$(-\Delta)^l u(\vec{x}) + V(\vec{x})u(\vec{x}) + \sigma|u(\vec{x})|^2 u(\vec{x}) = \lambda u(\vec{x}), \ \vec{x} \in [0, 2\pi]^n, \tag{1}$$

$$\begin{cases} u(x_1, \cdots, \underbrace{2\pi}_{s-th}, \cdots, x_n) = e^{2\pi i t_s} u(x_1, \cdots, \underbrace{0}_{s-th}, \cdots, x_n), \\ \frac{\partial}{\partial x_s} u(x_1, \cdots, \underbrace{2\pi}_{s-th}, \cdots, x_n) = e^{2\pi i t_s} \frac{\partial}{\partial x_s} u(x_1, \cdots, \underbrace{0}_{s-th}, \cdots, x_n), \\ \vdots \\ \frac{\partial^{2l-1}}{\partial x_s^{2l-1}} u(x_1, \cdots, \underbrace{2\pi}_{s-th}, \cdots, x_n) = e^{2\pi i t_s} \frac{\partial^{2l-1}}{\partial x_s^{2l-1}} u(x_1, \cdots, \underbrace{0}_{s-th}, \cdots, x_n), \\ s = 1, \cdots, n. \end{cases} \tag{2}$$

where l is an integer, $4l > n+1$, $\vec{t} = (t_1, \cdots, t_n) \in K = [0,1]^n$, σ is a real number and $V(\vec{x})$ is a trigonometric polynomial and $\int_Q V(\vec{x})d\vec{x} = 0$, $Q = [0, 2\pi]^n$ being the elementary cell of period 2π. More precisely,

$$V(\vec{x}) = \sum_{q \neq 0, |q| \le R_0} v_q e^{i\langle q, \vec{x}\rangle}, \tag{3}$$

v_q being Fourier coefficients.

When $l = 1$, $n = 1, 2, 3$, equation (1) is a famous Gross-Pitaevskii equation for Bose-Einstein condensate, see e.g. [5]. In physics papers, e.g. [3], [4], [6], [7], a big variety of numerical computations for Gross-Pitaevskii equation is made. However, they are restricted to the one dimensional case and there is a lack of theoretical considerations even for the case $n = 1$. In this paper we study the case $4l > n+1$ which includes $l = 1, n = 2$.

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The goal of this paper is to construct asymptotic formulas for $u(\vec{x})$ as $\lambda \to \infty$. We show that there is an extensive "non-resonant" set $\mathcal{G} \subset \mathbb{R}^n$ such that for every $\vec{k} \in \mathcal{G}$ there is a quasi-periodic solution of (1) close to a plane wave $Ae^{i\langle \vec{k},\vec{x}\rangle}$ with $\lambda = \lambda(\vec{k}, A)$ close to $|\vec{k}|^{2l} + \sigma|A|^2$ as $|\vec{k}| \to \infty$ (Theorem 3.11). We assume $A \in \mathbb{C}$ and $|A|$ is sufficiently small:

$$|\sigma||A|^2 < \lambda^{\gamma}, \quad \gamma < 2l - n. \tag{4}$$

Note that γ is any negative number for the Gross-Pitaevskii equation $l = 1, n = 2$. The quasi-momentum $\vec{t}$ in (1) is defined by the formula: $\vec{k} = \vec{t} + 2\pi j$, $j \in \mathbb{Z}^n$.

We show that the non-resonant set $\mathcal{G}$ has an asymptotically full measure in $\mathbb{R}^n$:

$$\lim_{R\to\infty} \frac{|\mathcal{G} \cap B_R|_n}{|B_R|_n} = 1, \tag{5}$$

where B_R is a ball of radius R in $\mathbb{R}^n$ and $|\cdot|_n$ is Lebesgue measure in $\mathbb{R}^n$.

Moreover, we investigate a set $\mathcal{D}(\lambda, A)$ of vectors $\vec{k} \in \mathcal{G}$, corresponding to a fixed sufficiently large λ and a fixed A. The set $\mathcal{D}(\lambda, A)$, defined as a level (isoenergetic) set for $\lambda(\vec{k}, A)$,

$$\mathcal{D}(\lambda, A) = \left\{ \vec{k} \in \mathcal{G} : \lambda(\vec{k}, A) = \lambda \right\}, \tag{6}$$

is proven to be a slightly distorted n-dimensional sphere with a finite number of holes (Theorem 3.13). For any sufficiently large λ, it can be described by the formula:

$$\mathcal{D}(\lambda, A) = \{\vec{k} : \vec{k} = \varkappa(\lambda, A, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}(\lambda)\}, \tag{7}$$

where $\mathcal{B}(\lambda)$ is a subset of the unit sphere S_{n-1}. The set $\mathcal{B}(\lambda)$ can be interpreted as a set of possible directions of propagation for almost plane waves. Set $\mathcal{B}(\lambda)$ has an asymptotically full measure on S_{n-1} as $\lambda \to \infty$:

$$|\mathcal{B}(\lambda)| =_{\lambda\to\infty} \omega_{n-1} + O\left(\lambda^{-\delta}\right), \quad \delta > 0, \tag{8}$$

here $|\cdot|$ is the standard surface measure on S_{n-1}, $\omega_{n-1} = |S_{n-1}|$. The value $\varkappa(\lambda, A, \vec{\nu})$ in (7) is the "radius" of $\mathcal{D}(\lambda, A)$ in a direction $\vec{\nu}$. The function $\varkappa(\lambda, A, \vec{\nu}) - (\lambda - \sigma|A|^2)^{1/2l}$ describes the deviation of $\mathcal{D}(\lambda, A)$ from the perfect sphere (circle) of the radius $(\lambda - \sigma|A|^2)^{1/2l}$ in $\mathbb{R}^n$. It is proven that the deviation is asymptotically small:

$$\varkappa(\lambda, A, \vec{\nu}) =_{\lambda\to\infty} \left(\lambda - \sigma|A|^2\right)^{1/2l} + O\left(\lambda^{-\gamma_1}\right), \quad \gamma_1 > 0. \tag{9}$$

To prove the results above, we consider the term $V + \sigma|u|^2$ in equation (1) as a periodic potential and formally change the nonlinear equation to a linear equation with an unknown potential $V(\vec{x}) + \sigma|u(\vec{x})|^2$:

$$(-\Delta)^l u(\vec{x}) + \big(V(\vec{x}) + \sigma|u(\vec{x})|^2\big)u(\vec{x}) = \lambda u(\vec{x}).$$

Further, we use known results for linear polyharmonic equations with periodic potentials. To start with, we consider a linear operator in $L^2(Q)$ described by the formula:

$$H(\vec{t}) = (-\Delta)^l + V, \tag{10}$$

and quasi-periodic boundary condition (2). The free operator $H_0(\vec{t})$, corresponding to $V=0$, has eigenfunctions given by:

$$\psi_j(\vec{x}) = e^{i\langle \vec{p}_j(\vec{t}),\vec{x}\rangle},\ \vec{p}_j(\vec{t}) := \vec{t} + 2\pi j,\ j \in \mathbb{Z}^n,\ \vec{t} \in K, \tag{11}$$

and the corresponding eigenvalues $p_j^{2l}(\vec{t}) := |\vec{p}_j(\vec{t})|^{2l}$. Perturbation theory for a linear operator $H(\vec{t})$ with a periodic potential V is developed in [1]. It is shown that at high energies, there is an extensive set of generalized eigenfunctions being close to plane waves. Below (See Theorem 2.2), we describe this result in details. Now, we define a map $\mathcal{M} : L^\infty(Q) \to L^\infty(Q)$ by the formula:

$$\mathcal{M}W(\vec{x}) = V(\vec{x}) + \sigma |u_{\tilde{W}}(\vec{x})|^2. \tag{12}$$

Here, $\tilde{W}$ is a shift of W by a constant such that $\int_Q \tilde{W}(\vec{x})d\vec{x} = 0$,

$$\tilde{W}(\vec{x}) = W(\vec{x}) - \frac{1}{(2\pi)^n}\int_Q W(\vec{x})d\vec{x}, \tag{13}$$

and $u_{\tilde{W}}$ is an eigenfunction of the linear operator $(-\Delta)^l + \tilde{W}$ with the boundary condition (2). Next, we consider a sequence $\{W_m\}_{m=0}^\infty$:

$$W_0 = V + \sigma|A|^2, \quad \mathcal{M}W_m = W_{m+1}. \tag{14}$$

Note that the sequence is well-defined by induction, since for each $m = 1, 2, 3, \ldots$ and $\vec{t}$ in a non-resonant set $\mathcal{G}$ described in Section 2, there is an eigenfunction $u_m(\vec{x})$ corresponding to the potential $\tilde{W}_m$:

$$H_m(\vec{t})u_m = \lambda_m u_m,$$
$$H_m(\vec{t})u_m := (-\Delta)^l u_m + \tilde{W}_m u_m,$$

λ_m, u_m being defined by formal series of the form (22)–(25), (33) with $\tilde{W}_m$ instead of V. Those series are proven to be convergent, thus justifying our construction. Next, we prove that the sequence $\{W_m\}_{m=0}^\infty$ is a Cauchy sequence of periodic functions in Q with respect to a norm

$$\|W\|_* = \sum_{q\in\mathbb{Z}^n} |w_q|, \tag{15}$$

w_q being Fourier coefficients of W. This implies that

$$W_m \to W \text{ with respect to the norm } \|\cdot\|_*,\ W \text{ is a periodic function.}$$

Further, we show that

$$u_m \to u_{\tilde{W}} \text{ in } L^\infty(Q), \qquad \lambda_m \to \lambda_{\tilde{W}} \text{ in } \mathbb{R},$$

where $u_{\tilde{W}}$, $\lambda_{\tilde{W}}$ correspond to the potential $\tilde{W}$ (via (22)–(25), (33) with $\tilde{W}$ instead of V). It follows from (12) and (14) that $\mathcal{M}W = W$ and, hence, $u := u_{\tilde{W}}$ solves the nonlinear equation (1) with quasi-periodic boundary condition (2).

Note, that the results of the paper can be easily generalized for the case of a sufficiently smooth potential $V(x)$. Generalization for the case $l = 1, n = 3$ (Gross-Pitaevskii equation in dimension three) is also possible. However, it requires more subtle considerations than here and will be done in a forthcoming paper.

The paper is organized as follows. In Section 2, we introduce results for the linear operator $(-\Delta)^l + V$ which include the perturbation formulas for an eigenvalue and its spectral projection. In Section 3, we prove existence of solutions of the equation (1) with boundary condition (2) and investigate their properties. Isoenergetic surfaces are also introduced and described there.

2. Linear Operator

Let us consider an operator

$$H = (-\Delta)^l + V, \tag{16}$$

in $L^2(\mathbb{R}^n)$, $4l > n+1$ and $n \geq 2$ where l is an integer and $V(\vec{x})$ is defined by (3). Since $V(\vec{x})$ is periodic with an elementary cell Q, the spectral study of (16) can be reduced to that of a family of Bloch operators $H(\vec{t})$ in $L^2(Q)$, $\vec{t} \in K$ (see formula (10) and quasi-periodic conditions (2)).

The free operator $H_0(\vec{t})$, corresponding to $V = 0$, has eigenfunctions given by (11) and the corresponding eigenvalue is $p_j^{2l}(\vec{t}) := |\vec{p}_j(\vec{t})|^{2l}$. Next, we describe an isoenergetic surface of $H_0(\vec{t})$ in K. To start with, we consider the sphere $S(k)$ of radius k centered at the origin in $\mathbb{R}^n$. For each $j \in \mathbb{Z}^n$ such that $j + K \cap S(k) \neq \emptyset$, $K := [0,1]^n$, we translate the corresponding piece of $S(k)$ into K, thus obtaining the sphere of radius k "packed" into K. We denote it by $S_0(k)$. Namely,

$$S_0(k) = \left\{ \vec{t} \in K : \text{ there is a } \ j \in \mathbb{Z}^n \text{ such that } p_j^{2l}(\vec{t}) = k^{2l} \right\}.$$

Obviously, operator $H_0(\vec{t})$ has an eigenvalue equal to k^{2l} if and only if $\vec{t} \in S_0(k)$. For this reason, $S_0(k)$ is called an isoenergetic surface of $H_0(\vec{t})$. When $\vec{t}$ is a point of self-intersection of $S_0(k)$, there exists $q \neq j$ such that

$$p_q^{2l}(\vec{t}) = p_j^{2l}(\vec{t}). \tag{17}$$

In other words, there is a non-simple eigenvalue of $H_0(\vec{t})$. We remove from the set $S_0(k)$ the $(k^{-n+1-\delta})$-neighborhoods of all self-intersections (17). We call the remaining set a non-resonant set and denote is by $\chi_0(k, \delta)$, The removed neighborhood of self-intersections is relatively small and, therefore, $\chi_0(k, \delta)$ has asymptotically full measure with respect to $S_0(k)$:

$$\frac{|\chi_0(k,\delta)|}{|S_0(k)|} = 1 + O(k^{-\delta/8}),$$

here and below $|\cdot|$ is Lebesgue measure of a surface in $\mathbb{R}^n$. It can be easily shown that for any $\vec{t} \in \chi_0(k, \delta)$, there is a unique $j \in \mathbb{Z}^n$ such that $p_j^{2l}(\vec{t}) = k^{2l}$ and

$$\min_{q \neq j} \left| p_q^{2l}(\vec{t}) - p_j^{2l}(\vec{t}) \right| > k^{2l-n-\delta}. \tag{18}$$

This means that the distance from $p_j^{2l}(\vec{t})$ to the nearest eigenvalue $p_q^{2l}(\vec{t})$, $q \neq j$ is greater than $k^{2l-n-\delta}$. If $2l > n$, then this distance is large and standard perturbation series can be constructed for $p_j^{2l}(\vec{t})$, $t \in \chi_0(k, \delta)$. However, the denseness of the eigenvalues increases infinitely when $k \to \infty$ and $2l < n$. Hence, eigenvalues of the free operator $H_0(\vec{t})$ strongly interact with each other when $2l < n$, the case $2l = n$

being intermediate. Nevertheless, the perturbation series for eigenvalues and their spectral projections were constructed in [1] for $4l > n+1$ when $\vec{t}$ belongs to a non-resonant set χ_1.

Lemma 2.1. *For any $0 < \beta < 1$, $0 < 2\delta < (n-1)(1-\beta)$ and sufficiently large $k > k_0(\beta,\delta)$, there is a non-resonant set $\chi_1(k,\beta,\delta)$ such that for any $t \in \chi_1(k,\beta,\delta)$ there is a unique $j \in \mathbb{Z}^n$: $p_j^{2l}(\vec{t}) = k^{2l}$ and if $\vec{t}$ is in the $(k^{-n+1-2\delta})$-neighborhood of $\chi_1(k,\beta,\delta)$ in K, then for $z \in C_0 = \{z \in \mathbb{C} : |z - k^{2l}| = k^{2l-n-\delta}\}$ we have*

$$\min_{i\in\mathbb{Z}^n} \left| p_i^{2l}(\vec{t}) - z \right| > k^{2l-n-\delta}, \tag{19}$$

$$200|p_i^{2l}(\vec{t}) - z||p_{i+q}^{2l}(\vec{t}) - z| > k^{2\gamma_2},\ i \in \mathbb{Z}^n,\ |q| < k^{\beta},\ q \neq 0, \tag{20}$$

here and below:

$$2\gamma_2 = 4l - n - 1 - \beta(n-1) - 2\delta > 0. \tag{21}$$

The non-resonant set $\chi_1(k,\beta,\delta)$ has an asymptotically full measure on $S_0(k)$:

$$\frac{s\Big(S_0(k) \setminus \chi_1(k,\beta,\delta)\Big)}{s\Big(S_0(k)\Big)} = O(k^{-\delta/8}).$$

Theorem 2.2. *Under the conditions of Lemma 2.1, there exists a single eigenvalue of the operator $H(\vec{t})$ in the interval $\varepsilon(k,\delta) \equiv (k^{2l} - k^{2l-n-\delta}, k^{2l} + k^{2l-n-\delta})$. It is given by the series:*

$$\lambda(\vec{t}) = p_j^{2l}(\vec{t}) + \sum_{r=2}^{\infty} g_r(k,t), \tag{22}$$

converging absolutely, where the index j is uniquely determined from the relation $p_j^{2l}(\vec{t}) \in \varepsilon(k,\delta)$ and

$$g_r(k,\vec{t}) = \frac{(-1)^r}{2\pi i r} \operatorname{Tr} \oint_{C_0} \left((H_0(\vec{t}) - z)^{-1} V\right)^r dz. \tag{23}$$

The spectral projection, corresponding to $\lambda(\vec{t})$ is given by the series:

$$E(t) = E_j + \sum_{r=1}^{\infty} G_r(k,t), \tag{24}$$

which converges in the trace class S_1 uniformly, where

$$G_r(k,\vec{t}) = \frac{(-1)^{r+1}}{2\pi i} \oint_{C_0} \left((H_0(\vec{t}) - z)^{-1} V\right)^r (H_0(\vec{t}) - z)^{-1} dz. \tag{25}$$

Moreover, for coefficients $g_r(k,\vec{t}), G_r(k,\vec{t})$, the following estimates hold:

$$|g_r(k,\vec{t})| < k^{2l-n-\delta} k^{-\gamma_2 r}, \tag{26}$$

$$\|G_r(k,\vec{t})\|_{S_1} \leq \hat{v} k^{-\gamma_2 r},\ \hat{v} = c R_0^n \max_{m\in\mathbb{Z}^n} |v_m|. \tag{27}$$

Remark 2.3. We use the following norm $\|T\|_1$ of an operator T in $l_2(\mathbb{Z}^2)$:

$$\|T\|_1 = \max_i \sum_p |T_{pi}|.$$

It can be easily seen from construction in [1] that estimates (27) hold with respect to this norm too.

Let us introduce the notations:

$$T(m) \equiv \frac{\partial^{|m|}}{\partial t_1^{m_1} \partial t_2^{m_2} ... \partial t_n^{m_n}}, \tag{28}$$

$$| \, m \, | \equiv m_1 + m_2 + ... + m_n, \;\; m! \equiv m_1! m_2! ... m_n!,$$

$$0 \leq | \, m \, | < \infty, \;\; T(0) f \equiv f.$$

The following theorem and corollary are proven in [1].

Theorem 2.4. *Under the conditions of Theorem 2.2, the series (22), (24) can be differentiated with respect to $\vec{t}$ any number of times, and they retain their asymptotic character. Coefficients $g_r(k,\vec{t})$ and $G_r(k,\vec{t})$ satisfy the following estimates in the $(k^{-n+1-2\delta})$-neighborhood in $\mathbb{C}^n$ of the nonsingular set $\chi_1(k,\beta,\delta)$:*

$$| \, T(m) g_r(k,\vec{t}) \, | < m! k^{2l-n-\delta} (\hat{v} k^{-\gamma_2})^r k^{|m|(n-1+2\delta)}, \tag{29}$$

$$\|T(m) G_r(k,\vec{t})\|_1 < m! (\hat{v} k^{-\gamma_2})^r k^{|m|(n-1+2\delta)}. \tag{30}$$

Corollary 2.5. *There are the estimates for the perturbed eigenvalue and its spectral projection:*

$$| \, T(m)(\lambda(\vec{t}) - p_j^{2l}(\vec{t})) \, | < c m! k^{(n-1+2\delta)|m|} k^{2l-n-\delta-2\gamma_2}, \tag{31}$$

$$\|T(m)(E(\vec{t}) - E_j)\|_1 < c m! k^{(n-1+2\delta)|m|} k^{-\gamma_2}. \tag{32}$$

Corollary 2.6. *There is a one-dimensional space of Bloch eigenfunctions u_0 corresponding to the projection $E(t)$ given by* (24)*. They are given by the formula:*

$$u_0(\vec{x}) = A E(\vec{t}) e^{i\langle \vec{p}_j(\vec{t}), \vec{x}\rangle} = A \sum_{m \in \mathbb{Z}^n} E(\vec{t})_{mj} e^{i\langle \vec{p}_m(\vec{t}), \vec{x}\rangle} \tag{33}$$

$$= A e^{i\langle \vec{p}_j(\vec{t}), \vec{x}\rangle} \left(1 + \sum_{q \neq 0} \frac{v_q}{p_j^{2l}(\vec{t}) - p_{j+q}^{2l}(\vec{t})} e^{i\langle \vec{p}_q(\vec{0}), \vec{x}\rangle} + \cdots \right), \;\; j, q \in \mathbb{Z}^n, \;\; A \in \mathbb{C}.$$

Let $\tilde{\chi}_1(k,\beta,\delta) \subset S(k)$ be the image of $\chi_1(k,\beta,\delta) \subset S_0(k)$ on the sphere $S(k)$:

$$\tilde{\chi}_1(k,\beta,\delta) = \{\vec{p}_j(\vec{t}) \in S(k) : \; \vec{t} \in \chi_1(k,\beta,\delta)\}. \tag{34}$$

Note that $\tilde{\chi}_1(k,\beta,\delta)$ is well-defined, since $\chi_1(k,\beta,\delta)$ does not contain self intersections of $S_0(k)$. Let $\mathcal{B}(\lambda) \subset S_{n-1}$ be the set of directions corresponding to the nonsingular set $\tilde{\chi}_1(k,\beta,\delta)$:

$$\mathcal{B}(\lambda) = \left\{ \vec{\nu} \in S_{n-1} : k\vec{\nu} \in \tilde{\chi}_1(k,\beta,\delta) \right\}, \;\; k^{2l} = \lambda. \tag{35}$$

The set $\mathcal{B}(\lambda)$ can be interpreted as a set of possible directions of propagation for almost plane waves (33). We define the non-resonance set $\mathcal{G} \subset \mathbb{R}^n$ as the union of all $\tilde{\chi}_1(k, \beta, \delta)$:

$$\mathcal{G} = \bigcup_{k > k_0(\beta,\delta)} \tilde{\chi}_1(k, \beta, \delta) \tag{36}$$

Further we denote vectors of $\mathcal{G}$ by $\vec{k}$. Formulas (35), (36) yield:

$$\mathcal{G} = \big\{\vec{k} = k\vec{\nu} : \vec{\nu} \in \mathcal{B}(k^{2l}),\ k > k_0(\beta, \delta)\big\}. \tag{37}$$

Since any vector $\vec{k}$ can be written as $\vec{k} = \vec{p}_j(t)$ in a unique way, formula (36) yields:

$$\mathcal{G} = \big\{\vec{p}_j(\vec{t}) :\ \vec{t} \in \chi_1(k, \beta, \delta),\ \text{where } k = p_j(\vec{t}),\ k > k_0(\beta, \delta)\big\}. \tag{38}$$

Let $\lambda(\vec{k})$ be defined by (22), where $\vec{k} = \vec{p}_j(\vec{t})$.

Next, we describe isoenergetic surfaces for the operator (16). The set $\mathcal{D}(\lambda)$, defined as a level (isoenergetic) set for $\lambda(\vec{k})$,

$$\mathcal{D}(\lambda) = \left\{\vec{k} \in \mathcal{G} : \lambda(\vec{k}) = \lambda\right\}. \tag{39}$$

Lemma 2.7. *For any sufficiently large λ, $\lambda > k_0(\beta, \delta)^{2l}$, and for every $\vec{\nu} \in \mathcal{B}(\lambda)$, there is a unique $\varkappa = \varkappa(\lambda, \vec{\nu})$ in the interval*

$$I := [k - k^{-n+1-2\delta}, k + k^{-n+1-2\delta}], \quad k^{2l} = \lambda,$$

such that

$$\lambda(\varkappa\vec{\nu}) = \lambda. \tag{40}$$

Furthermore, $|\varkappa - k| \leq ck^{2l-n-\delta-2\gamma_2-2l+1} = ck^{-\gamma_1}$, $\gamma_1 = 4l - 2 - \beta(n-1) - \delta > 0$.

The Lemma easily follows from (31) for $|m| = 1$.

Lemma 2.8. (1) *For any sufficiently large λ, $\lambda > k_0(\beta, \delta)^{2l}$, the set $\mathcal{D}(\lambda)$, defined by (39) is a distorted sphere with holes; it is described by the formula:*

$$\mathcal{D}(\lambda) = \{\vec{k} : \vec{k} = \varkappa(\lambda, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}(\lambda)\}, \tag{41}$$

where $\varkappa(\lambda, \vec{\nu}) = k + h(\lambda, \vec{\nu})$ and $h(\lambda, \vec{\nu})$ obeys the inequalities:

$$|h| < ck^{-\gamma_1}, \quad |\nabla_{\vec{\nu}} h| < ck^{-\gamma_1+n-1+2\delta} = ck^{-2\gamma_2+\delta}. \tag{42}$$

(2) *The measure of $\mathcal{B}(\lambda) \subset S_{n-1}$ satisfies the estimate (8).*
(3) *The surface $\mathcal{D}(\lambda)$ has the measure that is asymptotically close to that of the whole sphere of the radius k in the sense that*

$$\big|\mathcal{D}(\lambda)\big| \underset{\lambda\to\infty}{=} \omega_{n-1}k^{n-1}\big(1 + O(k^{-\delta})\big), \quad \lambda = k^{2l}. \tag{43}$$

The proof is based on Implicit Function Theorem.

3. Proof of The Main Result

First, we prove that $\{W_m\}_{m=0}^{\infty}$ in (14) is a Cauchy sequence with respect to the norm defined by (15). Further we need the following obvious properties of norm $\|\cdot\|_*$:

$$\|f\|_* = \|\bar{f}\|_*, \quad \|\Re(f)\|_* \le \|f\|_*, \quad \|\Im(f)\|_* \le \|f\|_*, \quad \|fg\|_* \le \|f\|_*\|g\|_*. \tag{44}$$

where $\Re(f)$ and $\Im(f)$ are real and imaginary part for f, respectively.

We define the value $k_1 = k_1(\|V\|_*, \delta, \beta)$ as

$$k_1(\|V\|_*, \delta, \beta) = \max\Big\{ (16\|V\|_*)^{1/\gamma_2}, \ k_0(\beta,\delta)\Big\}, \tag{45}$$

$\gamma_2 > 0$ being defined by (21) and $k_0(\beta,\delta)$ being as in Corollary 2.1.

Lemma 3.1. *The following inequalities hold for any $m = 1, 2, \dots$:*

$$\|\tilde{W}_m - V\|_* \le 8|\sigma||A|^2\|V\|_* k^{-\gamma_2}, \tag{46}$$

$$\|W_m - W_{m-1}\|_* \le 4|\sigma||A|^2\|V\|_* k^{-\gamma_2}(|\sigma||A|^2 k^{-\gamma_0})^{m-1}, \tag{47}$$

$$\|E_m(\vec{t}) - E_{m-1}(\vec{t})\|_1 \le 8|\sigma||A|^2\|V\|_* k^{-(2l-n-\delta)-\gamma_2}(|\sigma||A|^2 k^{-\gamma_0})^{m-1}, \tag{48}$$

where $\gamma_0 = 2l - n - 2\delta$, $\delta > 0$, and $|\sigma||A|^2 < k^{\gamma_0 - \delta}$, k being sufficiently large $k > k_1(\|V\|_*, \delta, \beta)$.

Corollary 3.2. *There is a periodic function W such that W_m converges to W with respect to the norm $\|\cdot\|_*$:*

$$\|W - W_m\|_* \le 8|\sigma||A|^2\|V\|_* k^{-\gamma_2}(|\sigma||A|^2 k^{-\gamma_0})^m. \tag{49}$$

Proof of Lemma 3.1. Let us consider the function (33) written in the form

$$u_0(\vec{x}) = \psi_0(\vec{x}) e^{i\langle \vec{p}_j(\vec{t}), \vec{x}\rangle}, \tag{50}$$

where

$$\psi_0(\vec{x}) = A \sum_{q\in\mathbb{Z}^n} E(\vec{t})_{j+q,j} e^{i\langle \vec{p}_q(\vec{0}), \vec{x}\rangle}, \tag{51}$$

is called the periodic part of u_0.

First, we prove (47) for $m = 1$. It follows from (12), (14) and (44) that

$$\begin{aligned}\big\|W_1 - W_0\big\|_* =& |\sigma|\big\||u_0|^2 - |A|^2\big\|_* = |\sigma|\big\||\psi_0|^2 - |A|^2\big\|_* \\ \le& |\sigma|\big\||\psi_0|^2 - |A|^2 + 2i\Im(\bar{A}\psi_0)\big\|_* = |\sigma|\big\|(\psi_0 - A)(\bar{\psi}_0 + \bar{A})\big\|_* \\ \le& |\sigma|\big\|\psi_0 - A\big\|_*\big\|\bar{\psi}_0 + \bar{A}\big\|_*. \end{aligned} \tag{52}$$

Let us consider

$$B_0(z) = (H_0(\vec{t}) - z)^{-\frac{1}{2}} V (H_0(\vec{t}) - z)^{-\frac{1}{2}}. \tag{53}$$

Then it follows from Lemma 2.1 that:

$$\max_{z\in C_0}\Big\|(H_0(\vec{t}) - z)^{-1}\Big\|_1 < k^{-2l+n+\delta}. \tag{54}$$

$$\max_{z\in C_0}\|B_0(z)\|_1 < \|V\|_* k^{-\gamma_2}, \tag{55}$$

γ_2 being defined by (21). By (25) and (53),

$$G_r(k,\vec{t}) = \frac{(-1)^{r+1}}{2\pi i}\oint_{C_0}(H_0(\vec{t})-z)^{-\frac{1}{2}}B_0(z)^r(H_0(\vec{t})-z)^{-\frac{1}{2}}dz. \tag{56}$$

It is easy to see that

$$\|G_r(k,\vec{t})\|_1 < \|V\|_*^r k^{-\gamma_2 r}. \tag{57}$$

Next, by (51), (25) and (27),

$$\begin{aligned}\|\psi_0 - A\|_* \leq & \Big|AE(\vec{t})_{jj} - A\Big| + |A|\sum_{q\in\mathbb{Z}^n\setminus\{0\}}\Big|E(\vec{t})_{j+q,j}\Big| \\ \leq & |A|\sum_{r=1}^{\infty}\|G_r(k,\vec{t})\|_1 \leq \|V\|_*|A|k^{-\gamma_2}(1+o(1)).\end{aligned} \tag{58}$$

It follows:

$$\|\psi_0\|_* = \|\bar{\psi}_0\|_* \leq |A| + O\big(|A|k^{-\gamma_2}\big). \tag{59}$$

Using (52), (58) and (59), we get

$$\|W_1 - W_0\|_* \leq 4|\sigma||A|^2\|V\|_* k^{-\gamma_2}.$$

Since $\|\tilde{W}_1 - V\|_* = \|\tilde{W}_1 - \tilde{W}_0\|_* \leq \|W_1 - W_0\|_*$, we have:

$$\|\tilde{W}_1 - V\|_* \leq 4|\sigma||A|^2\|V\|_* k^{-\gamma_2}. \tag{60}$$

Now, we use mathematical induction to show simultaneously,

$$\|\tilde{W}_m - V\|_* \leq 8|\sigma||A|^2\|V\|_* k^{-\gamma_2}, \tag{61}$$

$$\|W_m - W_{m-1}\|_* \leq 4|\sigma||A|^2\|V\|_* k^{-\gamma_2}(|\sigma||A|^2k^{-\gamma_0})^{m-1}. \tag{62}$$

Suppose that for all $1 \leq s \leq m-1$,

$$\|\tilde{W}_s - V\|_* \leq 8|\sigma||A|^2\|V\|_* k^{-\gamma_2}, \tag{63}$$

$$\|W_s - W_{s-1}\|_* \leq 4|\sigma||A|^2\|V\|_* k^{-\gamma_2}(|\sigma||A|^2k^{-\gamma_0})^{s-1}. \tag{64}$$

Let, by analogy with (33),

$$u_s(\vec{x}) := A\sum_{m\in\mathbb{Z}^n}E_s(\vec{t})_{m,j}e^{i\langle\vec{p}_m(\vec{t}),\vec{x}\rangle}, \tag{65}$$

where $E_s(\vec{t})$ is the spectral projection (24) with the potential $\tilde{W}_s$. Obviously,

$$u_s(\vec{x}) = \psi_s(\vec{x})e^{i\langle\vec{p}_j(\vec{t}),\vec{x}\rangle}, \tag{66}$$

where the function,

$$\psi_s(\vec{x}) = A\sum_{q\in\mathbb{Z}^n}E_s(\vec{t})_{j+q,j}e^{i\langle\vec{p}_q(\vec{0}),\vec{x}\rangle}, \tag{67}$$

is the periodic part of u_s. Clearly,

$$\|\psi_s\|_* \leq |A|\|E_s(\vec{t})\|_1. \tag{68}$$

Let

$$B_s(z) = (H_0(\vec{t})-z)^{-\frac{1}{2}}\tilde{W}_s(H_0(\vec{t})-z)^{-\frac{1}{2}}. \tag{69}$$

Using (63), (55) and (19), we easily obtain:

$$\|B_s(z)\|_1 \leq 8|\sigma||A|^2\|V\|_* k^{-2l+n+\delta-\gamma_2} + \|V\|_* k^{-\gamma_2} \leq 2\|V\|_* k^{-\gamma_2}, \quad z \in C_0, \tag{70}$$

for any $1 \leq s \leq m-1$. It is easy to see now that

$$\|G_{s,r}(k,\vec{t})\|_1 \leq \left(4\|V\|_* k^{-\gamma_2}\right)^r, \quad 1 \leq s \leq m-1, \tag{71}$$

here $G_{s,r}(k,\vec{t})$ is given by (25) with $\tilde{W}_s$ instead of V. It follows:

$$\begin{aligned}\|E_s(\vec{t})\|_1 \leq& 1 + \sum_{r=1}^{\infty} \|G_{s,r}(k,\vec{t})\|_1 \\ \leq& 1 + 8\|V\|_* k^{-\gamma_2} \leq 2, \quad 1 \leq s \leq m-1.\end{aligned} \tag{72}$$

Next, we note that

$$\begin{aligned}&\max_{z\in C_0} \|B_{m-1}^r(z) - B_{m-2}^r(z)\|_1 \\ \leq& \max_{z\in C_0} \|B_{m-1}(z) - B_{m-2}(z)\|_1 \left(\|B_{m-1}(z)\|_1 + \|B_{m-2}(z)\|_1\right)^{r-1} \\ \leq& k^{-(2l-n-\delta)} \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_* \Big(4\|V\|_* k^{-\gamma_2}\Big)^{r-1}.\end{aligned} \tag{73}$$

Hence,

$$\|G_{m-1,r}(k,\vec{t}) - G_{m-2,r}(k,\vec{t})\|_1 \leq k^{-(2l-n-\delta)} \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_* \Big(4\|V\|_* k^{-\gamma_2}\Big)^{r-1}. \tag{74}$$

Estimate (74) yields:

$$\begin{aligned}\|E_{m-1}(\vec{t}) - E_{m-2}(\vec{t})\|_1 \leq& \sum_{r=1}^{\infty} \|G_{m-1,r}(k,\vec{t}) - G_{m-2,r}(k,\vec{t})\|_1 \\ \leq& 2k^{-(2l-n-\delta)} \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_*.\end{aligned} \tag{75}$$

Next, considering as in (52), we obtain:

$$\big\|W_m - W_{m-1}\big\|_* \leq |\sigma| \big\|\psi_{m-1} - \psi_{m-2}\big\|_* \big\|\bar{\psi}_{m-1} + \bar{\psi}_{m-2}\big\|_*, \tag{76}$$

and, hence, by (67),

$$\|W_m - W_{m-1}\|_* \leq |\sigma||A|^2 \|E_{m-1}(\vec{t}) - E_{m-2}(\vec{t})\|_1 \left(\|E_{m-1}(\vec{t})\|_1 + \|E_{m-2}(\vec{t})\|_1\right). \tag{77}$$

Using (72) and (75), we obtain

$$\|W_m - W_{m-1}\|_* \leq 8|\sigma||A|^2 k^{-(2l-n-\delta)} \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_*. \tag{78}$$

Considering $\|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_* \leq \|W_{m-1} - W_{m-2}\|_*$ and using (64) for $s = m-1$, we arrive at the estimate:

$$\begin{aligned}\|W_m - W_{m-1}\|_* \leq& 8|\sigma||A|^2 k^{-(2l-n-\delta)} 4|\sigma||A|^2\|V\|_* k^{-\gamma_2} \left(|\sigma||A|^2 k^{-\gamma_0}\right)^{m-2} \\ \leq& 4|\sigma||A|^2\|V\|_* k^{-\gamma_2} \left(|\sigma||A|^2 k^{-\gamma_0}\right)^{m-1},\end{aligned} \tag{79}$$

when $k > k_1(\|V\|_*, \delta, \beta)$. Further, (79) and (60) enable the estimate

$$\begin{aligned}\|\tilde{W}_m - V\|_* \leq& \|\tilde{W}_m - \tilde{W}_{m-1}\|_* + \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_* + \cdots + \|\tilde{W}_1 - V\|_* \\ \leq& 8|\sigma||A|^2\|V\|_* k^{-\gamma_2},\end{aligned}$$

which completes the proof of (46) and (47). Using (75), we obtain (48). ∎

Lemma 3.3. *Suppose $\vec{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in K of the non-resonant set $\chi_1(k,\beta,\delta)$. Then for every sufficiently large $k > k_1(\|V\|_*,\delta,\beta)$ and every $A \in \mathbb{C} : |\sigma||A|^2 < k^{\gamma_0-\delta}$, the sequence $E_m(\vec{t})$ converges with respect to $\|\cdot\|_1$ to a one-dimensional spectral projection $E_{\tilde{W}}(\vec{t})$ of $H_0(t)+\tilde{W}$:*

$$\|E_m(\vec{t}) - E_{\tilde{W}}(\vec{t})\|_1 \leq 8\|V\|_* k^{-\gamma_2}(|\sigma||A|^2k^{-\gamma_0})^{m+1}. \tag{80}$$

The projection $E_{\tilde{W}}(\vec{t})$ is given by the series (24), (25) with $\tilde{W}$ instead of V. The series converges with respect to $\|\cdot\|_1$:

$$\|G_r(k,\vec{t})\|_1 \leq \left(2\|V\|_* k^{-\gamma_2}\right)^r \tag{81}$$

Proof. Let $B(z)$ be given by (69) with $\tilde{W}$ instead of $\tilde{W}_s$. Obviously, $B(z)$ is the limit of $B_m(z)$ in $\|\cdot\|_1$-norm. The estimate (70) yields:

$$\|B(z)\|_1 \leq 2\|V\|_* k^{-\gamma_2}, \quad z \in C_0. \tag{82}$$

It follows that the perturbation series for the resolvent of $H_0(\vec{t})+W$ converges with respect to $\|\cdot\|_1$ norm on C_0. Integrating the series of z we obtain that $E(t)$ admits the expansion (24), (25) and (81) holds. Obviously, G_r corresponding to $\tilde{W}$ is the limit of $G_{m,r}$ in $\|\cdot\|_1$ norm. Summing the estimates (75), we obtain (80).

∎

Definition 3.4. Let $u(\vec{x})$ be defined as in Corollary 2.6 for the potential $\tilde{W}(\vec{x})$. Let $\psi(\vec{x})$ be the periodic part of $u(\vec{x})$.

The next lemma follows from the estimate (80).

Lemma 3.5. *Suppose $\vec{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in K of the non-resonant set $\chi_1(k,\beta,\delta)$. Then for every sufficiently large $k > k_1(\|V\|_*,\delta,\beta)$ and every $A \in \mathbb{C} : |\sigma||A|^2 < k^{\gamma_0-\delta}$, the sequence $\psi_m(\vec{x})$ converges to the function $\psi(\vec{x})$ with respect to $\|\cdot\|_*$:*

$$\|\psi_m - \psi\|_* \leq 8|A|\|V\|_* k^{-\gamma_2}(|\sigma||A|^2k^{-\gamma_0})^{m+1}. \tag{83}$$

Corollary 3.6. *The sequence u_m converges to u in $L^\infty(Q)$.*

Corollary 3.7.

$$\mathcal{M}W = W.$$

Proof of Corollary 3.7. Considering as in (76), we obtain:

$$\left\|\mathcal{M}W_m - \mathcal{M}W\right\|_* \leq |\sigma|\left\|\psi_m - \psi\right\|_*\left\|\bar{\psi}_m + \bar{\psi}\right\|_*, \tag{84}$$

It immediately follows from Lemma 3.3 that $\mathcal{M}W_m \to \mathcal{M}W$ with respect to $\|\cdot\|_*$. Now, by (14) and (84), we have $\mathcal{M}W = W$. ∎

Let $\lambda_m(\vec{t})$, $\lambda_{\tilde{W}}(\vec{t})$ be the eigenvalues (22) corresponding to $\tilde{W}_m$ and $\tilde{W}$, respectively.

Lemma 3.8. *Under conditions of Lemma 3.3 the sequence $\lambda_m(\vec{t})$ converges to $\lambda_{\tilde{W}}(\vec{t})$ being given by (22) and*

$$|g_r(k,\vec{t})| < r^{-1}k^{2l-n-\delta}\left(4\|V\|_*k^{-\gamma_2}\right)^r, \tag{85}$$

where g_r is given by (23) with $\tilde{W}$ instead of V.

Proof. By perturbation theory, $\lambda_{\tilde{W}}(\vec{t})$ is the limit of $\lambda_m(\vec{t})$ as $m\to\infty$. Let us show that the series (22) converges. Let us consider two projections $E_0=E_j$, $E_1=I-E_j$, here E_j is the spectral projection of H_0, see (25). Note that

$$\oint_{C_0}(E_1B(z)E_1)^r\,dz=0,\quad r=1,2,...,$$

since the integrand is holomorphic inside C_0. Hence,

$$\oint_{C_0}B(z)^r dz=\oint_{C_0}\left(B(z)^r-(E_1B(z)E_1)^r\right)dz=$$
$$\sum_{i_1,\dots,i_{r+1}=0,1,\exists s:i_s=0}\oint_{C_0}E_{i_1}B(z)E_{i_2}B(z)....E_{i_r}B(z)E_{i_{r+1}}dz.$$

Obviously, $E_{i_1}B(z)_{i_2}B(z)....E_{i_r}B(z)E_{i_{r+1}}$ is in the trace class S_1 if at least one index i_s, $1\le s\le r+1$ is zero, since $E_0\in S_1$. Notice that for the adjoint operator B^* we have $B^*(z)=B(\bar{z})$. It follows:

$$\|E_{i_1}B(z)E_{i_2}B(z)....E_{i_r}B(z)E_{i_{r+1}}\|_{S_1}\le\|B\|^r\le\|B^*\|_1^{r/2}\|B\|_1^{r/2}<\left(2\|V\|_*k^{-\gamma_2}\right)^r.$$

Now, we easily obtain (85). ∎

Considering as in the proof of Theorem 2.4, one can prove an analogous theorem:

Theorem 3.9. *Under the conditions of Lemma 3.3 the series (22), (24) for the potential $\tilde{W}$ can be differentiated with respect to $\vec{t}$ any number of times, and they retain their asymptotic character. Coefficients $g_r(k,\vec{t})$ and $G_r(k,\vec{t})$ satisfy the following estimates in the $(k^{-n+1-2\delta})$-neighborhood in $\mathbb{C}^n$ of the nonsingular set $\chi_1(k,\beta,\delta)$:*

$$|\,T(m)g_r(k,\vec{t})\,|<m!k^{2l-n-\delta}\left(4\|V\|_*k^{-\gamma_2}\right)^r k^{|m|(n-1+2\delta)}, \tag{86}$$

$$\|T(m)G_r(k,\vec{t})\|_1<m!(2\|V\|_*k^{-\gamma_2})^r k^{|m|(n-1+2\delta)}. \tag{87}$$

Corollary 3.10. *There are the estimates for the perturbed eigenvalue and its spectral projection:*

$$\left|T(m)\left(\lambda_{\tilde{W}}(\vec{t})-p_j^{2l}(\vec{t})\right)\right|<C(\|V\|_*)m!k^{(n-1+2\delta)|m|}k^{2l-n-\delta-2\gamma_2}, \tag{88}$$

$$\|T(m)(E_{\tilde{W}}(\vec{t})-E_j)\|_1<C(\|V\|_*)m!k^{(n-1+2\delta)|m|}k^{-\gamma_2}. \tag{89}$$

In particular,

$$\left|\lambda_{\tilde{W}}(\vec{t})-p_j^{2l}(\vec{t})\right|<C(\|V\|_*)k^{2l-n-\delta-2\gamma_2}, \tag{90}$$

$$\|E_{\tilde{W}}(\vec{t})-E_j\|_1<C(\|V\|_*)k^{-\gamma_2}, \tag{91}$$

$$\left|\nabla\lambda_{\tilde{W}}(\vec{t})-2l\vec{p}_j(\vec{t})p_j^{2l-2}(\vec{t})\right|<C(\|V\|_*)k^{2l-1-2\gamma_2+\delta}. \tag{92}$$

We have the following main result for the nonlinear polyharmonic equation with quasi-periodic condition.

Theorem 3.11. *Suppose $\vec{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in K of the non-resonant set $\chi_1(k,\beta,\delta)$, $k > k_1(\|V\|_*,\delta,\beta)$ and $A \in \mathbb{C} : |\sigma||A|^2 < k^{\gamma_0-\delta}$. Then, there is a function $u(\vec{x})$, depending on $\vec{t}$ as a parameter, and a real value $\lambda(\vec{t})$, satisfying the equation*

$$(-\Delta)^l u(\vec{x}) + V(\vec{x})u(\vec{x}) + \sigma|u(\vec{x})|^2 u(\vec{x}) = \lambda u(\vec{x}),\ \vec{x} \in Q, \tag{93}$$

and the quasi-periodic boundary condition (2). The following formulas hold:

$$u(\vec{x}) = Ae^{i\langle \vec{p}_j(\vec{t}),\vec{x}\rangle}\left(1 + \tilde{u}(\vec{x})\right), \tag{94}$$

$$\lambda(\vec{t}) = p_j^{2l}(\vec{t}) + \sigma|A|^2 + O\left(\left(k^{2l-n-\delta} + \sigma|A|^2\right)k^{-2\gamma_2}\right), \tag{95}$$

where $\tilde{u}(\vec{x})$ is periodic and

$$\|\tilde{u}\|_* \leq k^{-\gamma_2}, \quad \gamma_2 > 0 \ \text{ is defined by } (21). \tag{96}$$

Proof. Let us consider the function u given by Definition 3.4 and the value $\lambda_{\tilde{W}}(\vec{t})$. They solve the equation

$$(-\Delta)^l u(\vec{x}) + \tilde{W}(\vec{x})u(\vec{x}) = \lambda_{\tilde{W}}(\vec{t})u(\vec{x}), \quad \vec{x} \in Q, \tag{97}$$

and u satisfies the quasi-boundary condition (2). By Corollary 3.7, we have

$$W(\vec{x}) = \mathcal{M}W(\vec{x}) = V(\vec{x}) + \sigma|u(\vec{x})|^2.$$

Hence,

$$\tilde{W}(\vec{x}) = W(\vec{x}) - \frac{1}{(2\pi)^n}\int_Q W(\vec{x})d\vec{x} = V(\vec{x}) + \sigma|u(\vec{x})|^2 - \sigma\|u\|^2_{L^2(Q)}.$$

Substituting the last expression into (97), we obtain that $u(\vec{x})$ satisfies (93) with

$$\lambda(\vec{t}) = \lambda_{\tilde{W}}(\vec{t}) + \sigma\|u\|^2_{L^2(Q)} = \lambda_{\tilde{W}}(\vec{t}) + \sigma|A|^2\sum_{q\in Z^n}\left|\left(E_{\tilde{W}}\right)_{qj}\right|^2 = \lambda_{\tilde{W}}(\vec{t}) + \sigma|A|^2\left(E_{\tilde{W}}\right)_{jj}. \tag{98}$$

Note that $(G_1)_{jj} = 0$ and, therefore, $\left(E_{\tilde{W}}\right)_{jj} = 1 + O(k^{-2\gamma_2})$. Further, by the definition of $u(\vec{x})$, we have

$$u(\vec{x}) := Ae^{i\langle \vec{p}_j(\vec{t}),\vec{x}\rangle}\sum_{q\in\mathbb{Z}^n}\left(E_{\tilde{W}}\right)_{q+j,j}e^{i\langle p_q(0),\vec{x}\rangle}. \tag{99}$$

Using formulas (98) and (99) and estimates (90) and (91), we obtain (94) and (96), respectively. ∎

Lemma 3.12. *For any sufficiently large λ, every $A \in \mathbb{C} : |\sigma||A|^2 < k^{\gamma_0-\delta}$, $\lambda = k^{2l}$ and for every $\vec{\nu} \in \mathcal{B}(\lambda)$, there is a unique $\varkappa = \varkappa(\lambda, A, \vec{\nu})$ in the interval*

$$I := [k - k^{-n+1-2\delta}, k + k^{-n+1+2\delta}],$$

such that

$$\lambda(\varkappa\vec{\nu}, A) = \lambda. \tag{100}$$

Furthermore,

$$|\varkappa(\lambda, A, \vec{\nu}) - \tilde{k}| \leq C(||V||_*) \left(k^{2l-n-\delta} + |\sigma||A|^2\right) k^{-2l+1-2\gamma_2}, \quad \tilde{k} = (\lambda - \sigma|A|^2)^{1/2l}. \tag{101}$$

Proof. Taking into account (35) and using formulas (88), (92) and Implicit Function Theorem, we prove the lemma. The proof is completely analogous to that for a linear case. ∎

Theorem 3.13. (1) *For any sufficiently large λ and every $A \in \mathbb{C} : |\sigma||A|^2 < k^{\gamma_0-\delta}$, the set $\mathcal{D}(\lambda, A)$, defined by (6) is a distorted sphere with holes; it can be described by the formula*

$$\mathcal{D}(\lambda, A) = \{\vec{k} : \vec{k} = \varkappa(\lambda, A, \vec{\nu})\vec{\nu}, \ \vec{\nu} \in \mathcal{B}(\lambda)\}, \tag{102}$$

where $\varkappa(\lambda, A, \vec{\nu}) = \tilde{k} + h(\lambda, A, \vec{\nu})$ and $h(\lambda, A, \vec{\nu})$ obeys the inequalities

$$|h| < C(||V||_*) \left(k^{2l-n-\delta} + |\sigma||A|^2\right) k^{-2l+1-2\gamma_2} < C(||V||_*) k^{-\gamma_1}, \tag{103}$$

with $\gamma_1 = 4l - 2 - \beta(n-1) - \delta > 0$,

$$|\nabla_{\vec{\nu}} h| < C(||V||_*) k^{-\gamma_1+n-1+2\delta} = C(||V||_*) k^{-2\gamma_2+\delta}. \tag{104}$$

(2) *The measure of $\mathcal{B}(\lambda) \subset S_{n-1}$ satisfies the estimate*

$$L\left(\mathcal{B}\right) = \omega_{n-1}(1 + O(k^{-\delta})). \tag{105}$$

(3) *The surface $\mathcal{D}(\lambda, A)$ has the measure that is asymptotically close to that of the whole sphere of the radius k in the sense that*

$$\left|\mathcal{D}(\lambda, A)\right| \underset{\lambda \to \infty}{=} \omega_{n-1} k^{n-1}\big(1 + O(k^{-\delta})\big). \tag{106}$$

Proof. The proof is based on Implicit Function Theorem. It is completely analogous to Lemma 2.11 in [1]. ∎

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