

# ON A QUESTION OF SCHMIDT AND SUMMERER CONCERNING 3-SYSTEMS

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ABSTRACT. Following a suggestion of W.M. Schmidt and L. Summerer, we construct a proper 3-system  $(P_1, P_2, P_3)$  with the property  $\overline{\varphi}_3 = 1$ . In fact, our method generalizes to provide  $n$ -systems with  $\overline{\varphi}_n = 1$ , for arbitrary  $n \geq 3$ . We visualize our constructions with graphics. We further present explicit examples of numbers  $\xi_1, \dots, \xi_{n-1}$  that induce the  $n$ -systems in question.

*Keywords:* parametric geometry of numbers, simultaneous approximation

Math Subject Classification 2010: 11J13, 11H06

## 1. PARAMETRIC DIOPHANTINE APPROXIMATION IN DIMENSION THREE

Let  $\xi_1, \xi_2$  be real numbers so that the set  $\{1, \xi_1, \xi_2\}$  is linearly independent over  $\mathbb{Q}$ . For  $q > 0$  a parameter, let  $K(q)$  be the box of points  $(z_0, z_1, z_2) \in \mathbb{R}^3$  that satisfy

$$|z_0| \leq e^{2q}, \quad |z_1| \leq e^{-q}, \quad |z_2| \leq e^{-q}.$$

Further let  $\Lambda$  be the lattice consisting of the points  $\{(x, \xi_1 x - y_1, \xi_2 x - y_2) : x, y_1, y_2 \in \mathbb{Z}\}$ . The successive minima  $\lambda_1(q), \lambda_2(q), \lambda_3(q)$  of  $K(q)$  with respect to  $\Lambda$  as functions of  $q$  contain the essential information on the simultaneous rational approximation to  $\xi_1, \xi_2$ . It is convenient to study the logarithms of the functions  $\lambda_j(q)$ , denoted by  $L_j(q) = \log \lambda_j(q)$  for  $j = 1, 2, 3$ . These functions have the nice property that their slopes are among  $\{-2, 1\}$ , and their sum is absolutely bounded uniformly in the parameter  $q$ . These properties motivated Schmidt and Summerer [7] to define so called 3-systems. A 3-system  $P = (P_1, P_2, P_3)$  is a triple of functions  $P_j : [0, \infty) \rightarrow \mathbb{R}$  with slopes among  $\{-2, 1\}$  with the properties that  $P_1(0) = P_2(0) = P_3(0) = 0$ ,  $P_1(q) \leq P_2(q) \leq P_3(q)$  and  $P_1(q) + P_2(q) + P_3(q) = 0$  for every  $q \geq 0$ . Hence, locally in a neighborhood of any  $q > 0$ , precisely one of the three functions decays while the other two rise, unless  $q$  is a switch point where some  $P_j$  are not differentiable (change slope). Moreover, for  $P$  to be a 3-system, it is additionally required that if at a switch point  $q$  some  $P_i$  changes from falling to rising and some other  $P_j$  from rising to falling, then  $i < j$  unless  $P_i(q) = P_j(q)$ . It has been shown in [7] that every function triple  $(L_1, L_2, L_3)$  as above, associated to some  $(\xi_1, \xi_2)$ , corresponds to a 3-system  $P$  up to a bounded amount, and conversely by Roy [2] that for any 3-system  $P$  there exist  $\xi_1, \xi_2$  satisfying the  $\mathbb{Q}$ -linear independence condition above and so that  $\sup_{q>0} \max_{j=1,2,3} |P_j(q) - L_j(q)| \ll 1$ . Roy's result employs a minor technical condition on the mesh of the system  $P$ , we do not rephrase it here. Both results [7, 2] are established in more generality.

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For given  $\xi_1, \xi_2$  with induced functions  $L_j(q)$ , let  $\varphi_j(q) = L_j(q)/q$  and put

$$\underline{\varphi}_j = \liminf_{q \rightarrow \infty} \varphi_j(q), \quad \overline{\varphi}_j = \limsup_{q \rightarrow \infty} \varphi_j(q),$$

for  $j = 1, 2, 3$ . Since  $L_j$  have slopes  $-2$  and  $1$  only, it is clear that

$$(1) \quad -2 \leq \underline{\varphi}_j \leq \overline{\varphi}_j \leq 1, \quad j = 1, 2, 3.$$

By virtue of the results from [2, 7] quoted above, in the sequel we will identify the values  $\underline{\varphi}_j, \overline{\varphi}_j$  with quantities derived from an associated 3-system  $P$  via

$$(2) \quad \underline{\varphi}_j \longleftrightarrow \liminf_{q \rightarrow \infty} \frac{P_j(q)}{q}, \quad \overline{\varphi}_j \longleftrightarrow \limsup_{q \rightarrow \infty} \frac{P_j(q)}{q}, \quad j = 1, 2, 3,$$

and vice versa. M. Laurent [1] provided estimates for classical exponents of approximation related to any pair  $(\xi_1, \xi_2)$  that is  $\mathbb{Q}$ -linearly independent with  $\{1\}$ . As pointed out in [9] they translate into the language of the functions  $\varphi_j$  as

$$(3) \quad \begin{aligned} 0 &\leq \underline{\varphi}_3 \leq \overline{\varphi}_3 \leq 1, \\ \underline{\varphi}_3 + \underline{\varphi}_3 \overline{\varphi}_1 + \overline{\varphi}_1 &= 0, \end{aligned}$$

$$(4) \quad 2\underline{\varphi}_1 + \overline{\varphi}_3 \leq -\underline{\varphi}_3(3 + \underline{\varphi}_1 + 2\overline{\varphi}_3),$$

$$(5) \quad 2\overline{\varphi}_3 + \underline{\varphi}_1 \geq -\overline{\varphi}_1(3 + \overline{\varphi}_3 + 2\underline{\varphi}_1).$$

Schmidt and Summerer [9] recently provided additional information by including the second successive minimum in the picture.

**Theorem 1.1** (Schmidt/Summerer, 2017). *For any  $\xi_1, \xi_2$  with  $\{1, \xi_1, \xi_2\}$  linearly independent over  $\mathbb{Q}$ , if  $0 \leq \underline{\varphi}_3 < 1$ , additionally to the above relations we have*

$$(6) \quad \overline{\varphi}_2 \leq \overline{\Omega} := \frac{\overline{\varphi}_1 - \underline{\varphi}_1}{2 - \overline{\varphi}_1 - \overline{\varphi}_1 \underline{\varphi}_1},$$

and

$$(7) \quad \underline{\varphi}_2 \geq \underline{\Omega} := \frac{\underline{\varphi}_3 - \overline{\varphi}_3}{2 - \underline{\varphi}_3 - \overline{\varphi}_3 \underline{\varphi}_3}.$$

Moreover, these estimates are best possible in the sense that for given numbers  $\underline{\varphi}_1, \overline{\varphi}_1, \underline{\varphi}_3, \overline{\varphi}_3$  with  $0 \leq \underline{\varphi}_3 < 1$  and (3), (4), (5) there are  $\xi_1, \xi_2$  with  $\{1, \xi_1, \xi_2\}$  linearly independent over  $\mathbb{Q}$  for whose approximation constants we have  $\underline{\varphi}_2 = \underline{\Omega}$  and  $\overline{\varphi}_2 = \overline{\Omega}$ .

Schmidt and Summerer enclose a remark to Theorem 1.1 pointing out that in the case  $\underline{\varphi}_3 = 1$  excluded in its claim, we have  $\overline{\varphi}_2 = 1$  and  $\overline{\varphi}_1 = \underline{\varphi}_2 = -1/2$  (by mistake they denoted  $-1/3$  instead in [9]). However, there is a gap in Theorem 1.1 concerning the *existence* of graphs with the property  $\underline{\varphi}_3 = 1$ , and related real numbers  $\xi_1, \xi_2$ . In [9] they state "But one really should prove that  $\underline{\xi} = (\xi_1, \xi_2)$  with  $(1, \xi_1, \xi_2)$  linearly independent over  $\mathbb{Q}$  with  $\overline{\varphi}_3 = 1$  exist. We invite the reader to construct a proper 3-system  $P$  with this property." The main purpose of this paper is to provide the desired construction. Before we turn to constructing the 3-system, we point out that explicit examples of  $\mathbb{Q}$ -linearly independent  $\{1, \xi_1, \xi_2\}$  inducing  $\underline{\varphi}_3 = \overline{\varphi}_3 = 1$  can be derived from previous results of the author. Concretely [4, Corollary 2.11], upon putting  $k = n - 1 = 2$  and  $C = \infty$ , yields the following example.

**Theorem 1.2.** *Let*

$$(8) \quad \xi_1 = \sum_{k=1}^{\infty} 10^{-(2k-1)!}, \quad \xi_2 = \sum_{k=1}^{\infty} 10^{-(2k)!}.$$

*Then*

$$(9) \quad \underline{\varphi}_1 = -2, \quad \underline{\varphi}_2 = -\frac{1}{2}, \quad \underline{\varphi}_3 = 1,$$

$$(10) \quad \overline{\varphi}_1 = -\frac{1}{2}, \quad \overline{\varphi}_2 = 1, \quad \overline{\varphi}_3 = 1.$$

While the results in [4] are originally formulated in the language of another type of exponents, the two types of exponents determine each other via the identities of [6, Theorem 1.4], and we derive Theorem 1.2. We note that for the sole purpose of  $\overline{\varphi}_3 = 1$ , as desired in [9] and rephrased above, in fact any numbers  $\xi_1, \xi_2$  which are simultaneously approximable to any order by rational numbers can be chosen. In particular, one may choose the pair  $(\xi, \xi^2)$  with  $\xi$  any Liouville number, see [5, Theorem 3.1]. However, then we always have  $\underline{\varphi}_3 = 0$ . For Liouville's constant given as  $\xi = 10^{-1!} + 10^{-2!} + 10^{-3!} + \dots$ , by [5, Theorem 3.2] in place of (9), (10) we have

$$(11) \quad \underline{\varphi}_1 = -2, \quad \underline{\varphi}_2 = -\frac{1}{2}, \quad \underline{\varphi}_3 = 0$$

$$(12) \quad \overline{\varphi}_1 = 0, \quad \overline{\varphi}_2 = 1, \quad \overline{\varphi}_3 = 1.$$

Alternatively to the above examples, the pure existence of pairs  $(\xi_1, \xi_2)$  inducing  $\overline{\varphi}_3 = 1$  (or  $\underline{\varphi}_3 = 1$ ) also follows from Roy's results [2] and [3, Theorem 11.5] (the latter result, already quoted in [9], provides an explicit description of the spectrum of sextuples  $\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3, \overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3$  by a system of complicated inequalities). The main concern of the question of Schmidt and Summerer appears to be the construction of a suitable 3-system, carried out in Section 2.1 below.

## 2. CONSTRUCTION OF A 3-SYSTEM WITH $\overline{\varphi}_3 = 1$

We want to present an effective construction of a 3-system with (9), (10), in particular  $\overline{\varphi}_3 = 1$ . It resembles the combined graph  $(L_1, L_2, L_3)$  with respect to the pair  $(\xi_1, \xi_2)$  in (8), in an idealized form. In fact the resulting 3-system can be interpreted as the idealized extremal case of the regular graph defined in [8], for the parameter  $\rho = \infty$ . In Section 2.3 we will briefly sketch how to modify the method to obtain a graph with (11), (12) instead, and give generalizations to  $n$ -systems.

**2.1. The construction.** We construct the graphs piecewise as follows. Let

$$0 < l_0 < l_1 < l_2 < l_3 < \dots,$$

be a fast increasing lacunary sequence of real numbers with the property

$$(13) \quad \lim_{i \rightarrow \infty} \frac{l_{i+1}}{l_i} = \infty.$$

Let  $r_0 = 0$ . In the interval  $[r_0, l_0] = [0, l_0]$  let  $P_1$  decay with slope  $-2$  and  $P_2, P_3$  rise with slope 1, so that  $P_1(l_0) = -2l_0$  and  $P_2(l_0) = P_3(l_0) = l_0$ . Let  $w_0 = l_0$  for consistency with later notation. Let  $l_0$  be the first switch point where  $P_1$  starts to rise and  $P_2$  starts to decay. Then the graph of  $P_1$  will meet the graph of  $P_2$  at some

point  $(r_1, P_1(r_1))$  with  $r_1 > l_0$ . We may assume  $l_1 > r_1$ . In the interval  $[r_1, l_1]$  we define  $P_1$  as decaying with slope  $-2$  again and the other two functions rising with slope  $1$ . Note that  $P_3(l_1) = l_1$  since it has not changed slope yet. Assume this construction of the graphs in  $[0, l_1]$  was step 0 of our construction. Now we carry out how to complete the process with identical steps  $1, 2, 3, \dots$  where in step  $i$  we define the graphs of  $P_1, P_2, P_3$  in the interval  $[l_i, l_{i+1}]$ . At position  $q = l_1$  we let  $P_1$  and  $P_3$  change slopes so that  $P_1$  rises with slope  $1$  and  $P_3$  decays with slope  $-2$ . The function  $P_2$  still rises with slope  $1$ . We keep these slopes until  $P_2$  meets  $P_3$  at position  $q = w_1$ . Then we let  $P_2$  decay with slope  $-2$  and the other functions rise with slope  $1$  until  $P_2$  meets  $P_1$  at some point  $(r_2, P_1(r_2))$ . We may assume  $l_2 > r_2$ . Then we let  $P_1$  decay with slope  $-2$  up to  $q = l_2$ , and the other two functions rise with slope  $1$  in this interval. This completes step 1. At  $q = l_2$  we let  $P_1$  again switch from decaying to rising and conversely for  $P_3$ , and so on. When we repeat the whole process ad infinitum, we claim that  $P_1, P_2, P_3$  represent the combined graph of a 3-system with the properties (9), (10). A sketch of such a 3-system in an initial interval is shown in Figure 1 below. For size reasons we used the slopes  $-1, 1/2$  instead of  $-2, 1$ , thereby sketching  $P_j(q)/2$  for  $j = 1, 2, 3$ .

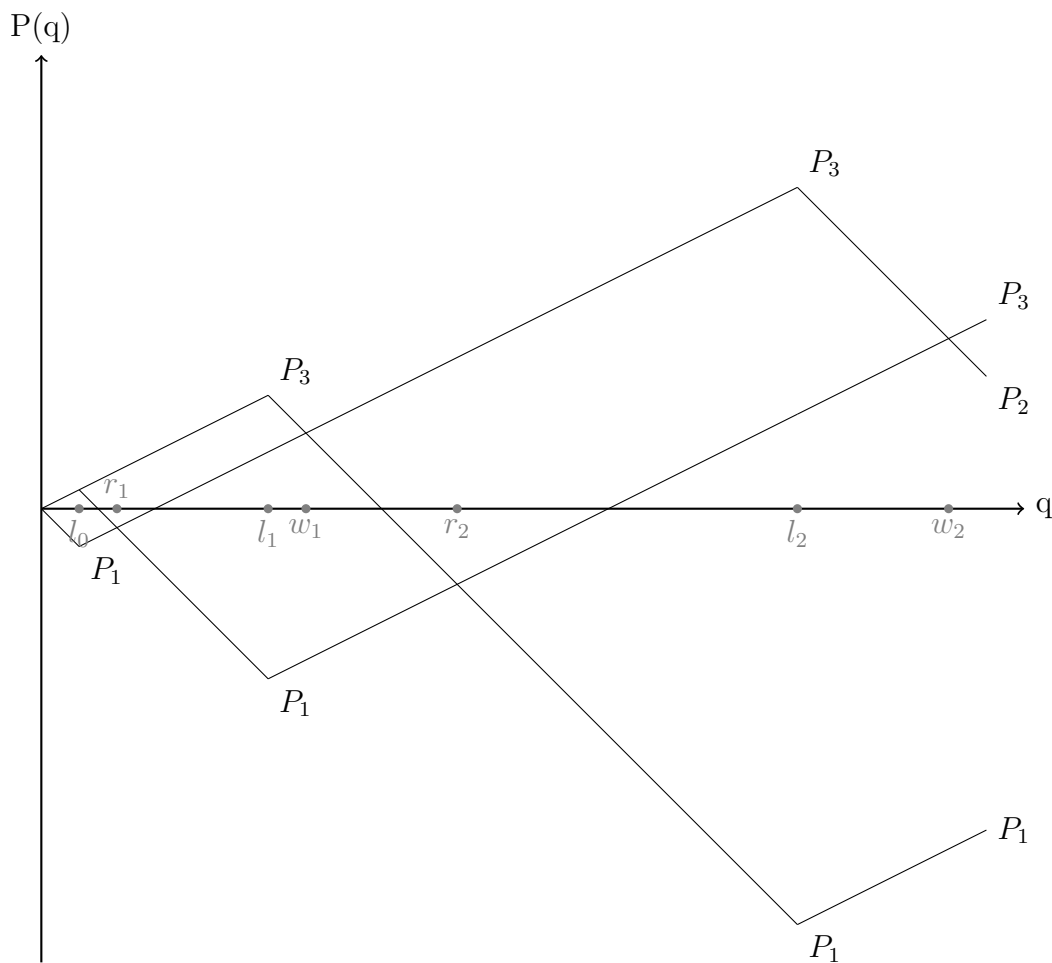


Figure 1: Visualization of case  $\bar{\varphi}_3 = 1$ , slopes scaled by factor  $1/2$

**2.2. The proof.** Keep in mind for the following that the switching positions in our construction are ordered

$$0 = r_0 < l_0 = w_0 < r_1 < l_1 < w_1 < r_2 < l_2 < w_2 < \dots,$$

and also the identification (2). First it is clear that the process yields the combined graph of a 3-system  $P$ . Indeed, by construction there is always precisely one  $P_j$  decaying, there are infinitely many positions where  $P_1 = P_2$  and  $P_2 = P_3$  respectively hold, and the switches occur in a way that respects the additional 3-system condition on a local maximum having higher index than a local minimum at switch points mentioned in the introduction. To obtain (9), (10), we first look at positions  $q = l_i$  and claim that

$$(14) \quad \lim_{i \rightarrow \infty} \frac{P_1(l_i)}{l_i} = -2, \quad \lim_{i \rightarrow \infty} \frac{P_2(l_i)}{l_i} = \lim_{i \rightarrow \infty} \frac{P_3(l_i)}{l_i} = 1.$$

By the identification (2) and by (1) this implies  $\underline{\varphi}_1 = -2$  and  $\overline{\varphi}_2 = \overline{\varphi}_3 = 1$ . By construction  $P_1$  decays with slope  $-2$  in intervals of the form  $I_t := [r_t, l_t]$  for  $t \geq 0$  and rises in intervals  $J_t := [l_{t-1}, r_t]$  for  $t \geq 1$ . We next check that

$$(15) \quad r_t < 2l_{t-1}, \quad t \geq 1.$$

We trivially have  $P_3(l_{t-1}) - P_1(l_{t-1}) \leq l_{t-1} - (-2l_{t-1}) = 3l_{t-1}$ . On the other hand, since  $P_1$  decays in  $J_t$  with slope  $-2$  whereas  $P_3$  rises with slope  $1$ , the function  $P_3 - P_1$  has slope  $3$  in  $J_t$  so that they must meet within distance  $3l_{t-1}/3 = l_{t-1}$  in the first coordinate on the right from  $l_{t-1}$ . This intersection point has first coordinate  $r_t$ , and we deduce (15).

The estimate (15) and the assumption (13) clearly imply that the sums of the lengths of the intervals  $I_t$  over  $t = 1, 2, \dots, i$  exceeds the according sums of the intervals  $J_t$  by any given factor  $\rho > 0$  for large enough  $i$ , i.e.

$$\sum_{t=0}^i |I_t| > \rho \sum_{t=1}^i |J_t|, \quad i \geq i_0(\rho).$$

Thus since

$$l_i = \sum_{t=0}^i |I_t| + \sum_{t=1}^i |J_t|$$

and

$$P_1(l_i) = -2 \sum_{t=0}^i |I_t| + \sum_{t=1}^i |J_t|,$$

indeed for sufficiently large  $i$  we have

$$\frac{P_1(l_i)}{l_i} < -\frac{2 + \rho^{-1}}{1 + \rho^{-1}}.$$

As we can choose  $\rho$  arbitrarily large indeed  $\lim_{i \rightarrow \infty} P_1(l_i)/l_i = -2$ , hence  $\underline{\varphi}_1 = -2$  by (1), (2). Since  $P_2$  and  $P_3$  rise with slope  $1$  in any  $I_t$  we infer the remaining claims of (14) by a very similar argument, or directly by using the bounded sum property at  $q = l_i$ .

Next we show

$$(16) \quad \lim_{q \rightarrow \infty} \frac{P_3(q)}{q} = 1.$$

By construction  $P_3$  has local minima precisely at positions  $w_i$  and it rises with slope 1 everywhere outside of the intervals  $[l_i, w_i]$ , in which it decays with slope  $-2$ . In view of (1) it suffices to check that

$$(17) \quad \liminf_{i \rightarrow \infty} \frac{P_3(w_i)}{w_i} \geq 1.$$

By construction

$$P_3(w_i) = l_0 - 2 \sum_{j=0}^i (w_j - l_j) + \sum_{j=0}^{i-1} (l_{j+1} - w_j).$$

Hence, in view of (13), to verify (17) it suffices to check

$$(18) \quad \lim_{i \rightarrow \infty} \frac{w_i}{l_i} = 1.$$

Now by construction in the interval  $[l_i, w_i]$  the function  $P_2$  rises with slope 1 whereas  $P_3$  decays with slope  $-2$ , hence  $w_i = l_i + u_i$  with  $u_i$  defined implicitly by the identity  $P_3(l_i) - 2u_i = P_2(l_i) + u_i$ , that is  $w_i = l_i + (P_3(l_i) - P_2(l_i))/3$ . On the other hand, by (14) we have  $P_2(l_i) = l_i(1 + o(1))$  and  $P_3(l_i) = l_i(1 + o(1))$ , hence inserting we derive  $w_i = l_i(1 + o(1))$  as  $i \rightarrow \infty$ , as desired. Thus (16) is shown.

Finally we show that

$$(19) \quad \lim_{i \rightarrow \infty} \frac{P_1(r_i)}{r_i} = \lim_{i \rightarrow \infty} \frac{P_2(r_i)}{r_i} = -\frac{1}{2}.$$

Since by construction the local maxima of  $P_1$  and the local minima of  $P_2$  both are attained precisely at the positions  $r_i$ , the remaining identities from (9) and (10) are implied. Let  $K_t = [w_{t-1}, r_t]$ , so that  $K_t \subseteq J_t$  and by (13), (18) the complement  $J_t \setminus K_t$  is small compared to  $J_t$ . In  $K_t$ , the function  $P_1$  rises with slope 1 whereas  $P_2$  decays with slope  $-2$ . Moreover, by (14) and (18) and since the slopes are bounded

$$\lim_{i \rightarrow \infty} \frac{P_1(w_i)}{w_i} = -2, \quad \lim_{i \rightarrow \infty} \frac{P_2(w_i)}{w_i} = 1.$$

Combining these two facts and by definition of  $r_i$ , for large  $i$  we readily conclude  $r_i = w_{i-1}(2 - o(1))$  and thus the asymptotic value at  $r_i$  is  $P_1(r_i) = P_1(w_{i-1}) + r_i - w_{i-1} = w_{i-1}(-1 + o(1))$ , hence indeed  $P_1(r_i)/r_i = -1/2 + o(1)$  for large  $i$ . Thus (19) holds and the proof is finished.

**2.3. Generalizations and variations.** A similar construction as in Section 2.1 can be done in arbitrary dimension  $n$ , where the slopes of the  $P_j$  are among  $\{-n+1, 1\}$ . Instead of one sequence  $(w_i)_{i \geq 0}$  with  $l_i < w_i < r_{i+1}$ , we obtain  $n-2$  sequences  $(w_i^h)_{i \geq 0}$ ,  $1 \leq h \leq n-2$ , induced by positions where  $P_{n-h+1}$  meets  $P_{n-h}$ , ordered  $l_i < w_i^1 < w_i^2 < \dots < w_i^{n-2} < r_{i+1}$ . We derive  $n$ -systems  $P = (P_1, \dots, P_n)$  whose approximation constants (via identification (2)) satisfy

$$\underline{\varphi}_1 = -n + 1, \quad \underline{\varphi}_2 = \frac{2-n}{2}, \quad \underline{\varphi}_j = 1, \quad 3 \leq j \leq n,$$

and

$$\bar{\varphi}_1 = \frac{2-n}{2}, \quad \bar{\varphi}_j = 1, \quad 2 \leq j \leq n.$$

Again this resembles the special case  $\rho = \infty$  of the regular graph [8] in dimension  $n$ , and suitable numbers  $(\xi_1, \dots, \xi_{n-1})$  inducing these approximation constants arise from [4, Corollary 2.11] upon taking  $k = n - 1, C = \infty$ , a particular choice is

$$\xi_j = \sum_{k=0}^{\infty} 10^{-(k(n-1)+j)!}, \quad 1 \leq j \leq n - 1.$$

Finally, we sketch the construction of a 3-system  $P$  with the properties (11), (12) in place of (9), (10). We have to alternate between the construction of Section 2.1 and another type of intermediate construction. Take  $i$  a large integer and follow the construction from Section 2.1 up to  $q = q_0 =: l_i$ . Recall  $P_1(l_i) \approx -2l_i$  and  $P_j(l_i) \approx l_i$  for  $j = 2, 3$  by (14). Then we make the first intermediate construction. Starting from  $q_0$ , let  $P_1$  rise with slope 1 and  $P_2, P_3$  decay with slope roughly  $-1/2$  in not too short intervals. The latter can be easily realized by changing the slopes of  $P_2, P_3$  rapidly so that there are many positions  $q$  with equality  $P_2(q) = P_3(q)$ . One may take these equality positions an arithmetic sequence  $b_0, b_1 = b_0 + D, b_2 = b_0 + 2D, \dots, b_h = b_0 + hD$  with some  $b_0 \geq q_0, h \geq 0$  and some small increment  $D > 0$ , in the following way. Fix  $D > 0$  small. Let  $(b_0, P_2(b_0))$  be the intersection point of the line passing through  $(q_0, P_2(q_0))$  with slope 1 (graph of  $P_2$ ) and the line passing through  $(q_0, P_3(q_0))$  with slope  $-2$  (graph of  $P_3$ ), corresponding to  $w_i$  in Section 2.1. In  $[b_0, b_0 + D/2]$ , let  $P_2$  decay with slope  $-2$  and  $P_3$  rise with slope 1. Then at  $q_0 + D/2$  interchange the slopes, such that at  $b_1 = b_0 + D$  we have  $P_2(b_1) = P_3(b_1) = P_2(b_0) - D/2$ . We repeat this procedure and stop at the largest index  $h$  so that the resulting graphs of  $P_2, P_3$  remain positive on  $[0, b_h]$ . For simplicity let  $\tilde{q} := b_h$ . Notice that  $P_j(b_l) - P_j(b_0) = -(b_l - b_0)/2 = -lD/2$  for  $l = 0, 1, \dots, h$ . Therefore, by (14) and since  $D$  is small, it is easy to see that  $|P_j(\tilde{q})|$  are all small for  $j = 1, 2, 3$ . Now starting at  $\tilde{q}$ , let  $P_1, P_3$  rise with slope 1 and  $P_2$  decay with slope  $-2$  until the graphs of  $P_1$  and  $P_2$  meet at some position  $q_1$ . Since  $|P_j(\tilde{q})|$  are all small, the expressions  $q_1 - \tilde{q}$  and  $|P_j(q_1)|$  for  $j = 1, 2, 3$ , are small (like  $o(q_1)$ ) as well. This ends the first intermediate construction, illustrated in Figure 2 below (again slopes are scaled with factor  $1/2$ ). Now we essentially apply the initial construction (step 0) from Section 2.1 from the interval  $[0, l_1]$  again, starting from  $q = q_1$  instead of  $q = 0$ . Let us denote by  $q_2$  the right endpoint in this construction, that is the value corresponding to  $l_1$  from Section 2.1. Notice that  $P_1$  has a local minimum inside the interval  $[q_1, q_2]$ , corresponding to  $l_0$  from Section 2.1, and another one at the right endpoint  $q_2$ . Since  $|P_j(q_1)|$  are small for  $j = 1, 2, 3$ , the  $P_j$  indeed behave in  $[q_1, q_2]$  essentially like they do in the construction of Section 2.1 in the interval  $[0, l_1]$  (see Figure 1). In particular, as for  $q = q_0$ , at  $q = q_2$  again we have  $P_1(q_2) \approx -2q_2$  and  $P_j(q_2) \approx q_2$  for  $j = 2, 3$ . Hence at this point we again switch to the intermediate construction to define the  $P_j$  in some interval  $[q_2, q_3]$ . We repeat this iterative process of constructing  $P$  in  $[q_{2k}, q_{2k+1}]$  and then in  $[q_{2k+1}, q_{2k+2}]$ , for all  $k \geq 1$ . It can be checked that the resulting combined graph satisfies (11), (12). Notice hereby that the condition  $\underline{\varphi}_2 = -1/2$  forced us to copy the behavior of the  $P_j$  on  $[0, l_1]$ , and not only on  $[0, l_0]$ , in intervals  $[q_{2k+1}, q_{2k+2}]$ . The procedure can again be generalized to dimension  $n$  to provide  $n$ -systems with the properties

$$\underline{\varphi}_1 = -n + 1, \quad \underline{\varphi}_2 = \frac{2 - n}{2}, \quad \underline{\varphi}_j = 0, \quad 3 \leq j \leq n,$$

and

$$\bar{\varphi}_1 = 0, \quad \bar{\varphi}_j = 1, \quad 2 \leq j \leq n.$$

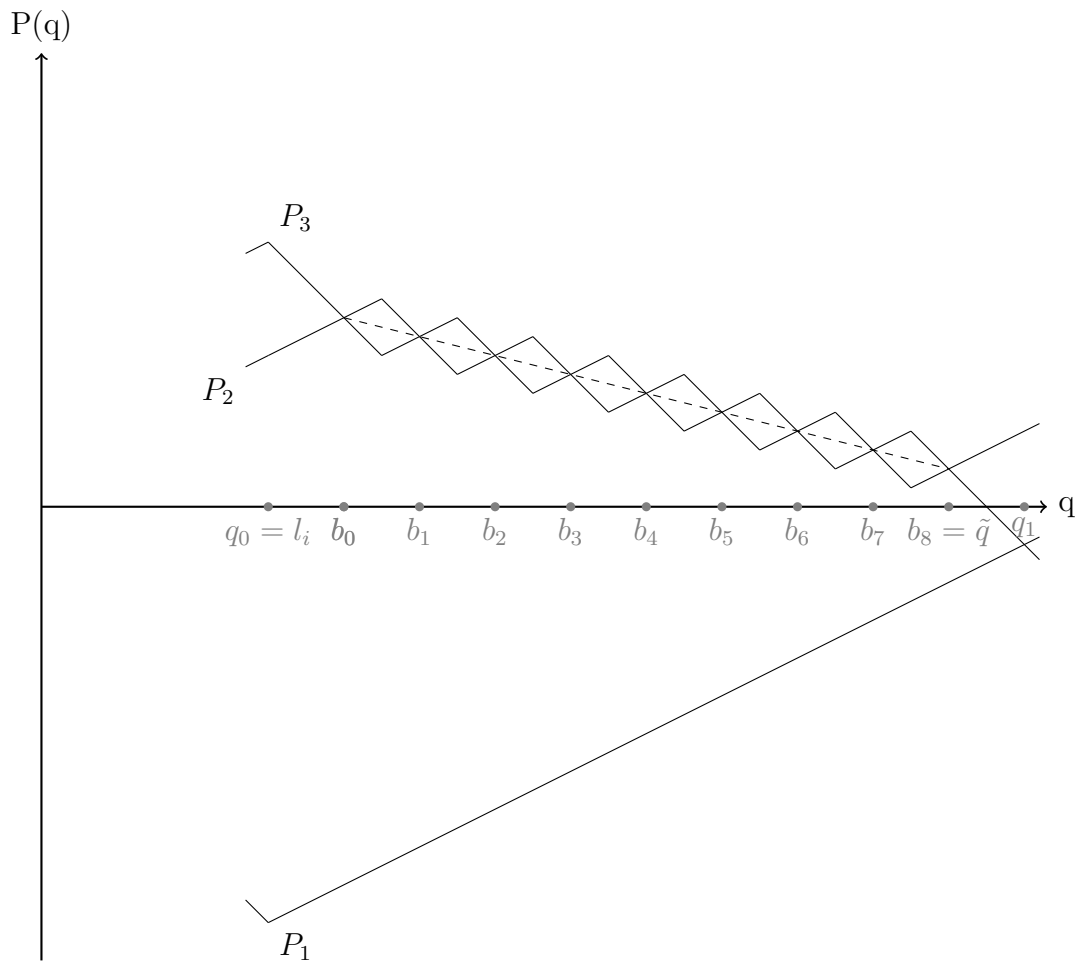


Figure 2: Intermediate construction in  $[q_0, q_1]$ , slopes scaled by factor  $1/2$

*The author thanks the referee for the careful reading and for pointing out some inaccuracies!*

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