# Fields Q(  $\frac{3}{4}$  $d, \zeta_3$ ) whose 3-class group is of type  $(9, 3)$ S. AOUISSI, M. C. ISMAILI, M. TALBI and A. AZIZI

**Abstract:** Let  $k = \mathbb{Q}(\sqrt[3]{})$  $d, \zeta_3$ , with d a cube-free positive integer. Let  $C_{k,3}$  be the 3-class group of k. With the aid of genus theory, arithmetic properties of the pure cubic field<br>  $\mathcal{D}(\sqrt[3]{d})$  and gave nearble an the 2 class weave  $G$  and determine all integers developed  $\mathbb{Q}(\sqrt[3]{d})$  and some results on the 3-class group  $C_{k,3}$ , we determine all integers d such that  $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$ 

Keywords: Pure cubic fields; 3-class groups; structure of groups. Mathematics Subject Classification 2010: 11R11, 11R16, 11R20, 11R27, 11R29, 11R37.

### 1 Introduction

Let *d* be a cube-free positive integer,  $k = \mathbb{Q}(\sqrt[3]{x})$  $d, \zeta_3$ , and  $C_{k,3}$  be the 3-class group of k. A number of researchers have studied the 3-class group  $C_{k,3}$  and the calculation of its rank. Calegari and Emerton [\[7,](#page-9-0) Lemma 5.11] proved that the rank of the 3-class group of  $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ , with a prime  $p \equiv 1 \pmod{9}$ , is equal to two if 9 divides the class number of  $\mathbb{Q}(\sqrt[3]{p})$ . The converse of the Calegari-Emerton result was proved by Frank Gerth III in [\[11,](#page-10-0) Theorem 1, p. 471].

The purpose of this paper is to classify all integers d for which  $C_{k,3}$  is of type  $(9,3)$ , i.e.  $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ . This investigation can be viewed as a continuation of the previous more general works [\[7,](#page-9-0) Lemma 5.11] and [\[11,](#page-10-0) Theorem 1, p. 471]. Effectively, we shall prove the following main theorem:

<span id="page-0-0"></span>**Theorem 1.1.** Let  $\Gamma = \mathbb{Q}(\sqrt[3]{d})$  be a pure cubic field, where  $d \geq 2$  is a cube-free integer, and let  $k = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$  be its normal closure. Denote by u the index of the subgroup generated by the units of intermediate fields of the extension  $k/\mathbb{Q}$  in the group of units of k.

- 1) If the field k has a 3-class group of type  $(9,3)$ , then  $d = p^e$ , where p is a prime number congruent to 1 (mod 9) and  $e = 1$  or 2.
- 2) If p is a prime number congruent to 1 (mod 9), 9 divides the class number of  $\Gamma$ exactly, and  $u = 1$ , then the 3-class group of k is of type  $(9, 3)$ .

This result will be underpinned by numerical examples obtained with the computational number theory system PARI [\[17\]](#page-10-1) in § [3.](#page-8-0) In section [2,](#page-1-0) where Theorem [1.1](#page-0-0) is proved, we only state results that will be needed in this paper. More information on 3-class groups can<br>be found in [10] and [0]. For the prime ideal fortesimiliar in the numeralis field  $\mathcal{N}^{(3/7)}$ be found in [\[10\]](#page-10-2) and [\[9\]](#page-10-3). For the prime ideal factorization in the pure cubic field  $\mathbb{Q}(\sqrt[3]{d})$ , we refer the reader to the papers [\[8\]](#page-10-4), [\[5\]](#page-9-1), [\[4\]](#page-9-2) and [\[16\]](#page-10-5). For the prime factorization rules of the third cyclotomic field  $\mathbb{Q}(\zeta_3)$ , we refer the reader to [\[13,](#page-10-6) Chap. 9, Sec. 1, Propositions 9.1.1–4, pp. 109-111].

#### Notations:

- The lower case letter  $p$ , respectively  $q$ , will denote a prime number congruent to 1, respectively  $-1$ , modulo 3;
- respectively  $-1$ , modulo 3;<br>  $\bullet \Gamma = \mathbb{Q}(\sqrt[3]{d})$ : a pure cubic field, where  $d \geq 2$  is a cube-free integer;
- $k_0 = \mathbb{Q}(\zeta_3)$ : the cyclotomic field, where  $\zeta_3 = e^{2i\pi/3}$ ;
- $k = \Gamma(\zeta_3)$ : the normal closure of Γ;
- $\Gamma'$  and  $\Gamma''$ : the two conjugate cubic fields of  $\Gamma$ , contained in k;
- u : the index of the subgroup  $E_0$  generated by the units of intermediate fields of the extension  $k/\mathbb{Q}$  in the group of units of k;
- extension  $\kappa/\psi$  in the group of units of  $\kappa$ ;<br>  $\bullet \langle \tau \rangle = \text{Gal}(\mathbf{k}/\Gamma)$ , such that  $\tau^2 = id$ ,  $\tau(\zeta_3) = \zeta_3^2$ , and  $\tau(\sqrt[3]{d}) = \sqrt[3]{d}$ ; √3
- $\langle \sigma \rangle$  = Gal (k/k<sub>0</sub>), such that  $\sigma^3 = id$ ,  $\sigma(\zeta_3) = \zeta_3$ , and  $\sigma(\zeta_3)$  $\zeta(d)=\zeta_3$ ;<br>∛d;
- $\lambda = 1 \zeta_3$  is a prime element above 3 of k<sub>0</sub>;
- $q^* = 1$  or 0, according to whether  $\zeta_3$  is norm of an element of k or not;
- t : the number of prime ideals ramified in  $k/k_0$ ;
- For an algebraic number field  $L$ :
	- $\mathcal{O}_L$ ,  $E_L$ : the ring of integers of L, and the group of units of L;
	- $C_{L,3}, h_L$ : the 3-class group of L, and the class number of L;
	- $L_3^{(1)}$  $L_3^{(1)}$ ,  $L^*$ : the Hilbert 3-class field of L, and the absolute genus field of L.

#### <span id="page-1-0"></span>2 Fields Q( √3  $d, \zeta_3)$  whose 3-Class Group is of Type  $(9,3)$

### 2.1 Preliminary results

In [\[15,](#page-10-7) Chap. 7, pp. 87–96], Ishida has explicitly given the genus field of any pure field.<br>For the nume only field  $\Gamma$ ,  $\mathbb{Q}(\sqrt[3]{d})$  where d is a sube free network number, we have the For the pure cubic field  $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ , where d is a cube-free natural number, we have the following theorem.

**Theorem 2.1.** Let  $\Gamma = \mathbb{Q}(\sqrt[3]{d})$  be a pure cubic field, where  $d \geq 2$  is a cube-free integer, and let  $p_1, \ldots, p_r$  be all prime divisors of d such that  $p_i$  is congruent to 1 (mod 3) for each  $i \in \{1, 2, \ldots, r\}$ . Let  $\Gamma^*$  be the absolute genus field of  $\Gamma$ , then

$$
\Gamma^* = \prod_{i=1}^r M(p_i) \cdot \Gamma,
$$

where  $M(p_i)$  denotes the unique subfield of degree 3 of the cyclotomic field  $\mathbb{Q}(\zeta_{p_i})$ . The genus number of  $\Gamma$  is given by  $g_{\Gamma} = 3^r$ .

*Remark* 2.1. (1) If no prime  $p \equiv 1 \pmod{3}$  divides d, i.e.  $r = 0$ , then  $\Gamma^* = \Gamma$ .

(2) For any value  $r \geq 0$ ,  $\Gamma^*$  is contained in the Hilbert 3-class field  $\Gamma_3^{(1)}$  $_3^{(1)}$  of  $\Gamma$ .

(3) The cubic field  $M(p)$  is determined explicitly in [\[12,](#page-10-8) § 4, Proposition 1, p. 11].

Assuming that  $h_{\Gamma}$  is divisible exactly by 9, we can explicitly construct the absolute genus field  $\Gamma^*$  as follows:

<span id="page-2-0"></span>**Lemma 2.1.** Let  $\Gamma = \mathbb{Q}(\sqrt[3]{d})$  be a pure cubic field, where  $d \geq 2$  is a cube-free integer. If  $h_{\Gamma}$  is exactly divisible by 9, then there are at most two primes congruent to 1 (mod 3) dividing d.

*Proof.* If  $p_1, \ldots, p_r$  are all prime numbers congruent to 1 (mod 3) dividing d, then  $3^r | h_{\Gamma}$ . Therefore, if  $h_{\Gamma}$  is exactly divisible by 9, then  $r \leq 2$ . So there are two primes  $p_1$  and  $p_2$  dividing d such that  $p_i \equiv 1 \pmod{3}$  for  $i \in \{1,2\}$ , or there is only one prime  $p \equiv 1$ (mod 3) with  $p|d$ , or there is no prime  $p \equiv 1 \pmod{3}$  such that  $p|d$ .  $\Box$ 

<span id="page-2-1"></span>**Lemma 2.2.** Let  $\Gamma = \mathbb{Q}(\sqrt[3]{d})$  be a pure cubic field, where  $d \geq 2$  is a cube-free integer. If  $h_{\Gamma}$  is exactly divisible by 9 and if the integer d is divisible by two primes  $p_1$  and  $p_2$ such that  $p_i \equiv 1 \pmod{3}$  for  $i \in \{1,2\}$ , then  $\Gamma^* = \Gamma_3^{(1)}$ ,  $(\Gamma')^* = \Gamma_3'^{(1)}$  and  $(\Gamma'')^* = \Gamma_3''^{(1)}$ . Furthermore,  $k \cdot \Gamma_3^{(1)} = k \cdot \Gamma_3'^{(1)} = k \cdot \Gamma_3''^{(1)}$ .

*Proof.* If  $h_{\Gamma}$  is exactly divisible by 9 and d is divisible by two prime numbers  $p_1$  and  $p_2$ which are congruent to 1 (mod 3), then  $g_{\Gamma} = 9$  so  $\Gamma^* = \Gamma_3^{(1)} = \Gamma \cdot M(p_1) \cdot M(p_2)$ , where  $M(p_1)$  (respectively  $M(p_2)$ ) is the unique cubic subfield of  $\mathbb{Q}(\zeta_{p_1})$  (respectively  $\mathbb{Q}(\zeta_{p_2})$ ). The equations

$$
\begin{cases}\n(\Gamma')^* = \Gamma' \cdot M(p_1) \cdot M(p_2) = \Gamma'^{(1)}_3, \\
(\Gamma'')^* = \Gamma'' \cdot M(p_1) \cdot M(p_2) = \Gamma''^{(1)}_3\n\end{cases}
$$

can be deduced by the fact that in the general case we have

$$
\begin{cases}\n(\Gamma')^* = \Gamma' \prod_{i=1}^r M(p_i), \\
(\Gamma'')^* = \Gamma'' \prod_{i=1}^r M(p_i),\n\end{cases}
$$

where  $p_i$ , for each  $1 \leq i \leq r$ , is a prime divisor of d such that  $p_i \equiv 1 \pmod{3}$ . From the fact that  $h_{\Gamma}$  is exactly divisible by 9, we conclude that  $h_{\Gamma}$  is exactly divisible by 9 and  $h_{\Gamma}$ <sup>n</sup> is exactly divisible by 9, because  $\Gamma$ ,  $\Gamma'$  and  $\Gamma''$  are isomorphic. Moreover,

$$
\begin{cases}\nk \cdot \Gamma_3^{(1)} = k \cdot \Gamma \cdot M(p_1) \cdot M(p_2) = k \cdot M(p_1) \cdot M(p_2), \\
k \cdot \Gamma_3'^{(1)} = k \cdot \Gamma' \cdot M(p_1) \cdot M(p_2) = k \cdot M(p_1) \cdot M(p_2), \\
k \cdot \Gamma_3''^{(1)} = k \cdot \Gamma'' \cdot M(p_1) \cdot M(p_2) = k \cdot M(p_1) \cdot M(p_2).\n\end{cases}
$$

Hence,  $k \cdot \Gamma_3^{(1)} = k \cdot \Gamma_3'^{(1)} = k \cdot \Gamma_3''^{(1)}$ .

Now, let u be the index of units defined in the above notations. We assume that  $h_{\Gamma}$  is exactly divisible by 9 and  $u = 1$ . From [\[5,](#page-9-1) § 14, Theorem 14.1, p. 232], we have  $h_k =$ u 3  $\cdot h_{\Gamma}^2,$ whence  $h_k$  is exactly divisible by 27. The structure of the 3-class group  $C_{k,3}$  is described by the following Lemma:

<span id="page-3-0"></span>**Lemma 2.3.** Let  $\Gamma$  be a pure cubic field, k its normal closure, and u be the index of units defined in the notations above, then

 $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \quad \Leftrightarrow \quad [C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z} \quad and \quad u = 1].$ 

Lemma [2.3](#page-3-0) will be underpinned in section [3](#page-8-0) by numerical examples obtained with the computational number theory system PARI [\[17\]](#page-10-1).

*Proof.* Assume that  $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ . Let  $h_{\Gamma,3}$  (respectively,  $h_{k,3}$ ) be the 3-class number of Γ (respectively, k), then  $h_{k,3} = 27$ . According to [\[5,](#page-9-1) § 14, Theorem 14.1, p. 232], we get  $27 = \frac{u}{3} \cdot h_{\Gamma,3}^2$  with  $u \in \{1,3\}$ , and thus  $u = 1$ , because otherwise 27 would be a square in N, which is a contradiction. Thus  $h_{\Gamma,3}^2 = 81$  and  $h_{\Gamma,3} = 9$ .

Let  $C_{\mathbf{k},3}^+ = \{ \mathcal{A} \in C_{\mathbf{k},3} \mid \mathcal{A}^{\tau} = \mathcal{A} \}$  and  $C_{\mathbf{k},3}^- = \{ \mathcal{A} \in C_{\mathbf{k},3} \mid \mathcal{A}^{\tau} = \mathcal{A}^{-1} \}$ . According to [\[9,](#page-10-3) § 2, Lemmas 2.1 and 2.2, p. 53], we have  $C_{k,3} \simeq C_{k,3}^+ \times C_{k,3}^ K_{k,3}^-$  and  $C_{k,3}^+ \simeq C_{\Gamma,3}$ , hence  $|C_{k,3}^-|$  $\vert_{k,3}^{-} \vert = 3.$ Since  $C_{k,3}$  is of type (9,3), we deduce that  $C_{k}^ \overline{k}_{1,3}$  is a cyclic group of order 3 and  $C_{k,4}^+$  $_{\rm k,3}^{\rm +(}$  is a cyclic group of order 9. Therefore, we have

$$
u = 1
$$
 and  $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$ .

Conversely, assume that  $u = 1$  and  $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$ . By [\[5,](#page-9-1) § 14, Theorem 14.1, p. 232], we deduce that  $|C_{k,3}|=\frac{1}{3}$  $\frac{1}{3} \cdot |C_{\Gamma,3}|^2$ , and so  $|C_{\mathbf{k},3}| = 27$ . Furthermore,  $|C_{\mathbf{k},\mathbf{k}}$  $|\mathbf{k},3| = 3$  and

$$
C_{k,3} \simeq C_{\Gamma,3} \times C_{k,3}^- \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.
$$



### 2.2 Proof of Theorem [1.1](#page-0-0)

Let  $\Gamma = \mathbb{Q}(\sqrt[3]{d})$  be a pure cubic field, where  $d \geq 2$  is a cube-free integer,  $k = \mathbb{Q}(\sqrt[3]{d})$  $d, \zeta_3)$  be its normal closure, and  $C_{k,3}$  be the 3-class group of k.

(1) Assume that the 3- class group  $C_{k,3}$  is of type  $(9,3)$ . We first write the integer d in the form given by equation  $(3.2)$  of  $[9, p. 55]$  $[9, p. 55]$ :

<span id="page-3-1"></span>
$$
d = 3^{e} p_1^{e_1} \dots p_v^{e_v} p_{v+1}^{e_{v+1}} \dots p_w^{e_w} q_1^{f_1} \dots q_I^{f_I} q_{I+1}^{f_{I+1}} \dots q_J^{f_J}, \tag{1}
$$

where  $p_i$  and  $q_i$  are positive rational primes such that:

 $\sqrt{ }$  $\begin{array}{c} \end{array}$  $\begin{array}{c} \end{array}$  $p_i \equiv 1 \pmod{9}$ , for  $1 \le i \le v$ ,  $p_i \equiv 4 \text{ or } 7 \pmod{9}$ , for  $v + 1 \le i \le w$ ,  $q_i \equiv -1 \pmod{9}$ , for  $1 \le i \le I$ ,  $q_i \equiv 2 \text{ or } 5 \pmod{9}, \qquad \text{for} \quad I+1 \leq i \leq J,$  $e_i = 1$  or 2, for  $1 \leq i \leq w$ ,  $f_i = 1$  or 2, for  $1 \leq i \leq J$ ,  $e = 0, 1$  or 2.

Let  $C_{\rm k.3}^{(\sigma)}$  $\kappa_{k,3}^{(\sigma)}$  be the ambiguous ideal class group of k/k<sub>0</sub>, where  $\sigma$  is a generator of Gal (k/k<sub>0</sub>). It is known that  $C_{k,3}^{(\sigma)}$  $\mathcal{L}_{k,3}^{(\sigma)}$  is an elementary abelian 3-group, because an ambiguous class C satisfies  $\sigma(\mathcal{C}) = \mathcal{C}$ , by definition, and therefore  $\mathcal{C}^3 = \mathcal{C} \cdot \sigma(\mathcal{C}) \cdot \sigma^2(\mathcal{C}) = \mathcal{N}_{k/k_0}(\mathcal{C}) = 1$ , since  $k_0$  has class number 1.

The fact that the 3-class group  $C_{k,3}$  is of type  $(9,3)$  implies that rank  $C_{k,3}^{(\sigma)} = 1$ . In fact, it is clear that if  $C_{k,3}$  is of type  $(9,3)$ , then rank  $C_{k,3}^{(\sigma)} = 1$  or 2.

Let us assume that rank  $C_{k,3}^{(\sigma)} = 2$ . From [\[10,](#page-10-2) § 5, Theorem 5.3, pp. 97–98], we have

$$
rank C_{k,3} = 2t - s,
$$

where the integers t and s are defined in  $[10, \S 5,$  $[10, \S 5,$  Theorem 5.3, pp. 97–98] as follows:

- $t := \text{rank } C_{k,3}^{(\sigma)}$ ι(σ)<br>k,3,
- s : the rank of the matrix  $(\beta_{i,j})$  defined in [\[10,](#page-10-2) § 5, Theorem. 5.3, pp. 97–98].

Since  $C_{k,3}$  is of type (9,3), then rank  $C_{k,3} = 2$ , and according to our hypothesis we have  $t = \text{rank } C_{k,3}^{(\sigma)} = 2.$  So we get  $s = 2$ .

By  $[10, \S 5]$  $[10, \S 5]$ , Theorem 5.3, pp. 97–98, the 3-class group  $C_{k,3}$  is isomorphic to the direct product of an abelian 3-group of rank  $2(t - s)$  and an elementary abelian 3-group of rank s. Here  $t = s = 2$ . Thus  $C_{k,3}$  is isomorphic to the direct product of an abelian 3-group of rank 0 and an elementary abelian 3-group of rank 2, we get

$$
C_{k,3} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},
$$

which contradicts the fact that  $C_{k,3}$  is of type (9,3). We conclude rank  $C_{k,3}^{(\sigma)} = 1$ .

On the one hand, suppose that d is not divisible by any rational prime p such that  $p \equiv 1$ (mod 3), i.e.  $w = 0$  in Eq. [\(1\)](#page-3-1) above. According to [\[9,](#page-10-3) § 5, Theorem 5.1, p. 61], this implies that  $C_{k,3} \simeq C_{\Gamma,3} \times C_{\Gamma,3}$ . Since the 3-group  $C_{k,3}$  is of type  $(9,3)$ , then  $|C_{k,3}| = 3^3$ , and by Lemma [2.3](#page-3-0) we have  $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$ , so we get  $|C_{\Gamma,3} \times C_{\Gamma,3}| = 3^4$ , which is a contradiction. Hence, the integer d is divisible by at least one rational prime p such that  $p \equiv 1 \pmod{3}$ .

On the other hand, the fact that the 3-class group  $C_{k,3}$  is of type  $(9,3)$  implies that  $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$  according to Lemma [2.3.](#page-3-0) Since  $h_{\Gamma}$  is exactly divisible by 9, then according to Lemma [2.1,](#page-2-0) there are at most two primes congruent to 1 (mod 3) dividing  $d$ .

Now, assume that there are exactly two different primes  $p_1$  and  $p_2$  dividing d such that  $p_1 \equiv p_2 \equiv 1 \pmod{3}$ , then, according to Lemma [2.2,](#page-2-1) we get:

$$
\Gamma^* = \Gamma_3^{(1)} = \Gamma \cdot M(p_1) \cdot M(p_2),
$$

where  $M(p_1)$  (respectively,  $M(p_2)$ ) is the unique subfield of degree 3 of  $\mathbb{Q}(\zeta_{p_1})$  (respectively,  $\mathbb{Q}(\zeta_{p_2})$ ). If  $M(p_1) \neq M(p_2)$ , then  $\Gamma \cdot M(p_1)$  and  $\Gamma \cdot M(p_2)$  are two different subfields of  $\Gamma_3^{(1)}$ 3 over  $\Gamma$  of degree 3. According to class field theory, we have

$$
\operatorname{Gal}(\Gamma_3^{(1)}/\Gamma) \cong C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}.
$$

Since  $Gal(\Gamma_3^{(1)}/\Gamma)$  is a cyclic 3-group, there exist only one sub-group of  $Gal(\Gamma_3^{(1)}/\Gamma)$  of order 3. By the Galois correspondence, there exist a unique sub-field of  $\Gamma_3^{(1)}$  $_3^{(1)}$  over  $\Gamma$  of degree 3. We conclude that  $\Gamma \cdot M(p_1) = \Gamma \cdot M(p_2)$ , which is a contradiction.

Thus,  $M(p_1) = M(p_2)$ , and then  $p_1 = p_2$ , which contradicts the fact that  $p_1$  and  $p_2$ are two different primes. Hence, there is exactly one prime congruent to 1 (mod 3) which divides d. Thus the integer d can be written in the following form:

$$
d=3^{e}p_1^{e_1}q_1^{f_1}\ldots q_I^{f_I}q_{I+1}^{f_{I+1}}\ldots q_J^{f_J},
$$

with  $p_1 \equiv -q_i \equiv 1 \pmod{3}$ , where  $p_1, e, e_1, q_i$  and  $f_i$  (for  $1 \leq i \leq J$ ) are defined in Eq.  $(1).$  $(1).$ 

Next, since rank  $C_{k,3}^{(\sigma)} = 1$ , then according to [\[9,](#page-10-3) § 3, Lemma 3.1, p. 55], there are three possible cases as follows:

- Case  $1: 2w + J = 1$ ,
- Case 2 :  $2w + J = 2$ ,
- Case  $3: 2w + J = 3$ ,

where w and J are the integers defined in Eq.  $(1)$  above. We will successively treat these cases as follows:

• Case 1: We have  $2w + J = 1$ , then  $w = 0$  and  $J = 1$ . This case is impossible, because we have shown above that the integer  $d$  is divisible by exactly one prime number congruent to 1 (mod 3) and thus  $w = 1$ .

• Case 2: We have  $2w + J = 2$ , and as in Case 1, we necessarily have  $w = 1$  and  $J = 0$ , which implies that  $d = 3^e p_1^{e_1}$ , where  $p_1$  is a prime number such that  $p_1 \equiv 1 \pmod{3}$ ,  $e \in \{0, 1, 2\}$  and  $e_1 \in \{1, 2\}$ . Then,

 $-$  If  $d \equiv \pm 1 \pmod{9}$ , then we necessarily have  $e = 0$ .

Assume that  $p_1 \equiv 4$  or 7 (mod 9), then  $d \not\equiv \pm 1 \pmod{9}$  which is an absurd. So we necessarily have  $p_1 \equiv 1 \pmod{9}$ . Thus the integer d will be written in the form  $d = p_1^{e_1}$ , where  $p_1 \equiv 1 \pmod{9}$  and  $e_1 \in \{1, 2\}$ .

- $-$  If  $d \not\equiv \pm 1 \pmod{9}$ :
	- ∗ If  $e \neq 0$ , the integer d is written as  $d = 3^e p_1^{e_1}$ , where  $p_1 \equiv 1 \pmod{3}$  and  $e, e_1 \in \{1, 2\}.$

∗ If  $e = 0$ , then d is written as  $d = p_1^{e_1}$  with  $p_1 \equiv 4$  or 7 (mod 9) and  $e_1 \in \{1, 2\}$ .

• Case 3: We have  $2w + J = 3$ , then we necessarily get  $w = 1$  and  $J = 1$ , because  $w \neq 0$ . Thus  $d = 3^e p_1^{e_1} q_1^{f_1}$ , where  $p_1 \equiv 1 \pmod{3}$ ,  $q_1 \equiv -1 \pmod{3}$ ,  $e \in \{0, 1, 2\}$  and  $e_1, f_1 \in \{1, 2\}$ . Then:

− If  $d \equiv \pm 1 \pmod{9}$ , we necessarily have  $e = 0$ . If  $p_1$  or  $-q_1 \not\equiv 1 \pmod{9}$ , then  $d \not\equiv \pm 1$ (mod 9) which is an absurd. It remain only the case where  $p_1 \equiv -q_1 \equiv 1 \pmod{9}$ . Then the integer d will be written in the form  $d = p_1^{e_1} q_1^{f_1}$ , where  $p_1 \equiv -q_1 \equiv 1$ (mod 9) and  $e_1, f_1 \in \{1, 2\}.$ 

 $-$  If  $d \not\equiv \pm 1 \pmod{9}$ :

According to [\[10,](#page-10-2) § 5, p. 92], the rank of  $C_{k,3}^{(\sigma)}$  $\kappa^{(\sigma)}_{k,3}$  is specified as follows:

rank 
$$
C_{k,3}^{(\sigma)} = t - 2 + q^*
$$
,

where  $t$  and  $q^*$  are defined in the notations.

On the one hand, the fact that rank  $C_{k,3}^{(\sigma)} = 1$  imply that  $t = 2$  or 3 according to whether  $\zeta_3$  is norm of an element of k or not.

On the other hand, we have  $d = 3^e p_1^{e_1} q_1^{f_1}$  with  $p_1 \equiv 1 \pmod{3}$  and  $q_1 \equiv -1 \pmod{3}$ . By [\[13,](#page-10-6) Chap. 9, Sec. 1, Proposition 9.1.4, p.110] we have  $q_1$  is inert in  $k_0$ , and by [\[8,](#page-10-4) Sec 4, pp. 51-54] we have  $q_1$  is ramifed in  $\Gamma = \mathbf{Q}(\sqrt[3]{d})$ . Since  $d \not\equiv \pm 1 \pmod{9}$ , then 3 is ramifed in  $\Gamma$  by [\[8,](#page-10-4) Sec 4, pp. 51-54], and  $3\mathcal{O}_{k_0} = (\lambda)^2$  where  $\lambda = 1 - \zeta_3$ . Since  $p_1 \equiv 1 \pmod{3}$ , then by [\[13,](#page-10-6) Chap. 9, Sec. 1, Proposition 9.1.4, p.110] we have  $p_1 = \pi_1 \pi_2$  where  $\pi_1$  and  $\pi_2$  are two primes of  $k_0$  such that  $\pi_2 = \pi_1^{\tau}$ , the prime  $p_1$ is ramified in Γ, then  $\pi_1$  and  $\pi_2$  are ramified in k. Hence, the number of prime ideals which are ramified in  $k/k_0$  is  $t = 4$ , which contradicts the fact that  $t = 2$  or 3.

We summarize all forms of integer  $d$  in the three cases 1, 2 and 3 as follows:

$$
d = \begin{cases} p_1^{e_1} & \text{with } p_1 \equiv 1 \pmod{9}, \\ p_1^{e_1} & \text{with } p_1 \equiv 4 \text{ or } 7 \pmod{9}, \\ 3^e p_1^{e_1} \not\equiv \pm 1 \pmod{9} & \text{with } p_1 \equiv 1 \pmod{9}, \\ 3^e p_1^{e_1} \not\equiv \pm 1 \pmod{9} & \text{with } p_1 \equiv 4 \text{ or } 7 \pmod{9}, \\ p_1^{e_1} q_1^{f_1} \equiv \pm 1 \pmod{9} & \text{with } p_1 \equiv -q_1 \equiv 1 \pmod{9}, \end{cases}
$$

where  $e, e_1, f_1 \in \{1, 2\}.$ 

Our next goal is to show that the only possible form of the integer  $d$  is the first form  $d = p_1^{e_1}$ , where  $p_1 \equiv 1 \pmod{9}$  and  $e_1 = 1$  or 2.

- Case where  $d = p_1^{e_1}$ , with  $p_1 \equiv 4$  or 7 (mod 9):
	- $-$  If  $\left(\frac{3}{n}\right)$  $\left(\frac{3}{p}\right)$  $\frac{3}{3} \neq 1$ , then according to [\[3,](#page-9-3) § 1, Conjecture 1.1, p. 1], we have  $C_{k,3} \simeq \mathbb{Z}/3\mathbb{Z}$ , which contradict the fact that  $C_{k,3}$  is of type  $(9,3)$ . We note that in this case, the fields Γ and k are of Type III, respectively,  $\alpha$ , in the terminology of [\[5,](#page-9-1) § 15, Theorem 15.6, pp. 235-236], respectively, [\[1,](#page-9-4) § 2.1, Theorem 2.1, p. 4].
	- $-$  If  $\left(\frac{3}{n}\right)$  $\left(\frac{3}{p}\right)$  $S_3 = 1$ , then according to [\[3,](#page-9-3) § 1, Conjecture 1.1, p. 1], we have  $C_{k,3} \simeq$  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ , which is impossible. We note that in this case, the fields  $\Gamma$  and k are of Type I, respectively,  $\beta$ , in the terminology of [\[5,](#page-9-1) § 15, Theorem 15.6, pp. 235-236], respectively,  $[1, § 2.1, Theorem 2.1, p. 4]$  $[1, § 2.1, Theorem 2.1, p. 4]$ .

For this case, we see that in [\[5,](#page-9-1) § 17, Numerical Data, p. 238], and also in the tables of [\[6\]](#page-9-5) which give the class number of a pure cubic field, the prime numbers  $p = 61, 67, 103,$  and 151, which are all congruous to 4 or 7 (mod 9), verify the following properties:

- (i) 3 is a residue cubic modulo  $p$ ;
- (ii) 3 divide exactly the class number of  $\Gamma$ ;
- (iii)  $u=3$ ;
- (iv)  $C_{\Gamma,3} \simeq \mathbb{Z}/3\mathbb{Z}$ , and  $C_{\text{k,3}} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ .

• Case where  $d = 3^e p_1^{e_1} \not\equiv \pm 1 \pmod{9}$ , with  $p_1 \equiv 1 \pmod{9}$ :

Here  $e, e_1 \in \{1, 2\}$ . As  $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{a^2b})$ , we can choose  $e_1 = 1$ , i.e.  $d = 3^e p_1$  with  $e \in \{1, 2\}$ . On the one hand, the fact that  $p_1 \equiv 1 \pmod{3}$  implies by [\[13,](#page-10-6) Chap. 9, Sec. 1, Proposition 9.1.4, p.110 that  $p_1 = \pi_1 \pi_2$  with  $\pi_1^{\tau} = \pi_2$  and  $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{k_0}}$ , the prime  $p_1$  is totally ramified in Γ, then  $\pi_1$  and  $\pi_2$  are totally ramified in k and we have  $\pi_1 \mathcal{O}_k = \mathcal{P}_1^3$  and  $\pi_2 \mathcal{O}_k = \mathcal{P}_2^3$ , where  $\mathcal{P}_1, \mathcal{P}_2$  are two prime ideals of k.

We know that  $3\mathcal{O}_{k_0} = \lambda^2 \mathcal{O}_{k_0}$ , with  $\lambda = 1 - \zeta_3$  a prime element of  $k_0$ . Since  $d \not\equiv \pm 1$ (mod 9), then 3 is totally ramified in Γ, and then  $\lambda$  is ramified in k/k<sub>0</sub>. Hence, the number of ideals which are ramified in  $k/k_0$  is  $t = 3$ .

On the other hand, from [\[13,](#page-10-6) Chap. 9, Sec. 1, Proposition 9.1.4, p.110] we have On the other hand, from [13, Chap. 3, Sec. 1, 1 roposition 3.1.4, p.110] we have<br> $3 = -\zeta_3^2 \lambda^2$ , then  $k = k_0(\sqrt[3]{x})$  with  $x = \zeta_3^2 \lambda^2 \pi_1 \pi_2$ . The primes  $\pi_1$  and  $\pi_2$  are congruent to 1 (mod  $\lambda^3$ ) because  $p_1 \equiv 1 \pmod{9}$ , then according to [\[1,](#page-9-4) § 3, Lemma 3.3, p. 17] we have  $\zeta_3$  is a norm of an element of  $k \setminus \{0\}$ , so  $q^* = 1$ . We conclude according to [\[10,](#page-10-2) § 5, p. 92] that rank  $C_{k,3}^{(\sigma)} = 2$  which is impossible.

• Case where  $d = 3^e p_1^{e_1} \not\equiv \pm 1 \pmod{9}$ , with  $p_1 \equiv 4$  or 7 (mod 9):

Here  $e, e_1 \in \{1, 2\}$ . So we can choose  $e = 1$ , then  $d = 3p_1^{e_1}$  with  $e_1 \in \{1, 2\}$ . As above we get  $\pi_1$  and  $\pi_2$ , and  $\lambda$  are ramified in k/k<sub>0</sub>.

Put  $p\mathcal{O}_{\Gamma} = \mathcal{P}^3$ ,  $\pi_1\mathcal{O}_k = \mathcal{P}_1^3$ ,  $\pi_2\mathcal{O}_k = \mathcal{P}_2^3$  and  $\lambda\mathcal{O}_k = I^3$ , where  $\mathcal P$  is a prime ideal of  $\Gamma$ , and  $\mathcal{P}_1,\mathcal{P}_2$  and I are prime ideals of k. According to [\[14,](#page-10-9) § 3.2, Theorem 3.5, pp 36-39] we get  $C_{k,3}$  is cyclic of order 3 which contradict the fact that  $C_{k,3}$  is of type  $(9,3)$ .

• Case where  $d = p_1^{e_1} q_1^{f_1} \equiv \pm 1 \pmod{9}$ , with  $p_1 \equiv -q_1 \equiv 1 \pmod{9}$ :

Since  $d \equiv \pm 1 \pmod{9}$ , then according to [\[8,](#page-10-4) Sec. 4, pp. 51-54] we have 3 is not ramified in the field  $\Gamma$ , so  $\lambda = 1 - \zeta_3$  is not ramified in k/k<sub>0</sub>. As  $p_1 \equiv 1 \pmod{3}$ , then by [\[13,](#page-10-6) Chap. 9, Sec. 1, Proposition 9.1.4, p.110]  $p_1 = \pi_1 \pi_2$  with  $\pi_1^{\tau} = \pi_2$  and  $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{k_0}}$ , so  $\pi_1$  and  $\pi_2$  are totally ramified in k. Since  $q_1 \equiv -1 \pmod{3}$ , then  $q_1$  is inert in k<sub>0</sub>. Hence, the primes ramifies in  $k/k_0$  are  $\pi_1, \pi_2$  and  $q_1$ . Put  $x = \pi_1^{e_1} \pi_2^{e_1} \pi_1^{f_1}$ , where  $-q_1 = \pi$  is a prime number of k<sub>0</sub>, then we have k = k<sub>0</sub>( $\sqrt[3]{x}$ ). The fact that  $p_1 \equiv -q_1 \equiv 1 \pmod{9}$  imply that  $\pi_1 \equiv \pi_2 \equiv \pi \equiv 1 \pmod{\lambda^3}$ , then by [\[1,](#page-9-4) § 3, Lemma 3.3, p. 17],  $\zeta_3$  is a norm of an element of  $k \setminus \{0\}$ , so  $q^* = 1$ . Then by [\[10,](#page-10-2) § 5, p. 92] we get rank  $C_{k,3}^{(\sigma)} = t - 2 + q^* = 2$  which is impossible.

Finally, we have shown that if the 3-class group  $C_{k,3}$  is of type  $(9,3)$ , then  $d = p^e$ , where p is a prime number such that  $p \equiv 1 \pmod{9}$  and  $e = 1$  or 2. We can see that this result is compatible with the first form of the integer d in  $\left[1, \S 1, \text{Theorem 1.1, p. 2}\right]$  $\left[1, \S 1, \text{Theorem 1.1, p. 2}\right]$  $\left[1, \S 1, \text{Theorem 1.1, p. 2}\right]$ .

(2) Suppose that  $d = p^e$ , with p is a prime number congruent to 1 (mod 9). Here  $e \in \{1, 2\}$ , (2) suppose that  $a = p$ , with p is a prime number congruent to 1 (mod 3). Here  $c \in \{1, 2\}$ ,<br>since  $\mathbb{Q}(\sqrt[3]{p}) = \mathbb{Q}(\sqrt[3]{p^2})$  we can choose  $e = 1$ . From [\[5,](#page-9-1) § 14, Theorem 14.1, p. 232], we have  $h_{k} = \frac{u}{3}$  $\frac{u}{3} \cdot h_{\Gamma}^2$ , the fact that  $u = 1$  and that  $h_{\Gamma}$  is exactly divisible by 9 implies that  $h_{\mathbf{k}}$ 

is exactly divisible by 27. Since  $h_{\Gamma}$  is exactly divisible by 9, then by [\[7,](#page-9-0) Lemma 5.11] we have rank  $C_{k,3} = 2$ . We conclude that  $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ .

# <span id="page-8-0"></span>3 Numerical Examples

Let  $\Gamma = \mathbb{Q}(\sqrt[3]{ab^2})$  be a pure cubic field, where a and b are coprime square-free integers. We point out that  $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{a^2b})$ . Assume that  $k = \mathbb{Q}(\sqrt[3]{ab^2}, \zeta_3)$  and  $C_{k,3}$  is of type (9, 3). Using the system Pari [\[17\]](#page-10-1), we compute class groups for  $b = 1$  and a prime  $a = p \equiv 1$ (mod 9). The following table illustrates our main result Theorem [1.1](#page-0-0) and Lemma [2.3.](#page-3-0) Here we denote by:

> $C_{\Gamma,3}$  (respectively,  $C_{\mathbf{k},3}$ ) : the 3-class group of  $\Gamma$  (respectively, k);  $h_{\Gamma,3}$  (respectively,  $h_{k,3}$ ) : the 3-class number of  $\Gamma$  (respectively, k).

Table : Some fields  $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$  whose 3-class group is of type  $(9, 3)$ .



р	$n^{\omega}$	9 mod	$h_{\Gamma,3}$	$n_{k,3}$	$\boldsymbol{\mathit{u}}$	$\Gamma$ .3	k.3
495	24512401			27		$\left[9\right]$	3 9,
5059	25593481			27		$\left[9\right]$	$\mathbf{Q}$ Ί9, ◡
5077	25775929		g	27		$\left[9\right]$	3 9.
5347	28590409		y	27		$\left[9\right]$	$\mathbf{Q}^{\prime}$ 9. IJ

Remark 3.1. Let p is a prime number such that  $p \equiv 1 \pmod{9}$ . Let  $\Gamma = \mathbb{Q}(\sqrt[3]{p})$ , k =  $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$  be the normal closure of the pure cubic field  $\Gamma$  and  $C_{k,3}$  be the 3-part of the  $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$  $\mathcal{L}(\sqrt{p}, \varsigma_3)$  be the normal closure of the pure cubic field 1 and  $\mathcal{O}_{k,3}$  be the 3-part of the class group of k. If 9 divide exactly the class number of  $\mathbb{Q}(\sqrt[n]{p})$  and  $u = 1$ , then according to Theorem 1.1, the 3-class group of  $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$  is of type  $(9, 3)$ . Furthermore, if 3 is not residue cubic modulo p, then a generators of 3-class group of  $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$  can be deduced by residue cubic modulo p, then a generators of 3-class group of  $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$  can be deduced by [\[2,](#page-9-6) § 3, Theorem 3.2, p. 10].

## 4 Acknowledgements

The authors would like to thank Professors Daniel C. Mayer and Mohammed Talbi who were of a great help concerning correcting the spelling mistakes that gave more value to the work.

## References

- <span id="page-9-4"></span>[1] S. Aouissi, D. C. Mayer, M. C. Ismaili, M. Talbi, and A. Azizi, 3-rank of ambiguous class groups in cubic Kummer extensions, arXiv:1804.00767v4.
- <span id="page-9-6"></span>[2] S. Aouissi, M. C. Ismaili, M. Talbi and A. Azizi. The generators of 3-class group of some fields of degree 6 over Q, arXiv:1804.00692.
- <span id="page-9-3"></span>[3] S. Aouissi, M. Talbi, M. C. Ismaili and A. Azizi, On a conjecture of lemmermeyer, arXiv:1810.07172v3.
- <span id="page-9-2"></span>[4] P. Barrucand and H. Cohn, A rational genus, class number divisibility, and unit theory for pure cubic fields, J. Number Theory 2 (1970), 7-21.
- <span id="page-9-1"></span>[5] P. Barrucand and H. Cohn, Remarks on principal factors in a relative cubic field, J. Number Theory 3 (1971), 226-239.
- <span id="page-9-5"></span>[6] B. D. Beach, H. C. Williams, C. R. Zarnke Some computer results on units in quadratic and cubic fields, Proceeding of the twenty-fifth summer meeting of the Canadian Mathematical Congress (Lake Head University, Thunder Bay, 1971), 609-648.
- <span id="page-9-0"></span>[7] F. Calegari and M. Emerton, On the ramification of Hecke algebras at Eisenstein primes, Invent. Math. 160 (2005), 97-144.
- <span id="page-10-4"></span>[8] R. Dedekind, Über die Anzahl der Idealklassen in reinen kubischen Zahlkörpern. J. Reine Angew. Math. 121 (1900), 40-123.
- <span id="page-10-3"></span>[9] F. Gerth III, On 3-class groups of pure cubic fields, J. Reine Angew. Math. 278/279 (1975), 52-62.
- <span id="page-10-2"></span>[10] F. Gerth III, On 3-class groups of cyclic cubic extensions of certain number fields, J. Number Theory 8 (1976), 84-94.
- <span id="page-10-0"></span>[11] F. Gerth III, On 3-class groups of certain pure cubic fields, Aust. Math. Soc. 72(3) (2005) 471-476.
- <span id="page-10-8"></span>[12] T. Honda, Pure cubic fields whose class numbers are multiples of three, J. Number Theory 3 (1971), 7-12.
- <span id="page-10-6"></span>[13] K. Ireland and M. Rosen, A classical introduction to modern number theory, Chapter 9: Cubic and biquadratic reciprocity, 2nd edn (Springer, New Yark, 1992) 108-111.
- <span id="page-10-9"></span>[14] M. C. Ismaili, Sur la capitulation des 3-classes d'idéaux de la clôture normale d'un corps cubique pur, Thèse de doctorat, University Laval, Québec (1992).
- <span id="page-10-7"></span>[15] M. Ishida, The genus fields of algebraic number fields, Chapter 7: The genus fields of pure number fields, Lecture Notes in Mathematics. 555 (Springer-Verlag, 1976) 87-93.
- <span id="page-10-5"></span>[16] A. Markoff, Sur les nombres entiers dépendants d'une racine cubique d'un nombre entier ordinaire, Mem. Acad. Imp. Sci. St. Petersbourg VIII 38 (1892) 1-37.
- <span id="page-10-1"></span>[17] The PARI Group, PARI/GP, Version 2.9.4, Bordeaux, 2017, http://pari.math.ubordeaux.fr.

Abdelmalek AZIZI, Moulay Chrif ISMAILI and Siham AOUISSI Department of Mathematics and Computer Sciences, Mohammed first University, 60000 Oujda - Morocco, abdelmalekazizi@yahoo.fr, mcismaili@yahoo.fr, aouissi.siham@gmail.com.

Mohamed TALBI Regional Center of Professions of Education and Training, 60000 Oujda - Morocco, ksirat1971@gmail.com.