Fields $\mathbb{Q}(\sqrt[3]{d}, \zeta_3)$ whose 3-class group is of type (9, 3) S. AOUISSI, M. C. ISMAILI, M. TALBI and A. AZIZI

Abstract: Let $\mathbf{k} = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$, with d a cube-free positive integer. Let $C_{\mathbf{k},3}$ be the 3-class group of \mathbf{k} . With the aid of genus theory, arithmetic properties of the pure cubic field $\mathbb{Q}(\sqrt[3]{d})$ and some results on the 3-class group $C_{\mathbf{k},3}$, we determine all integers d such that

 $C_{\mathbf{k},3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$

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1 Introduction

Let d be a cube-free positive integer, $\mathbf{k} = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$, and $C_{\mathbf{k},3}$ be the 3-class group of \mathbf{k} . A number of researchers have studied the 3-class group $C_{\mathbf{k},3}$ and the calculation of its rank. Calegari and Emerton [7, Lemma 5.11] proved that the rank of the 3-class group of $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, with a prime $p \equiv 1 \pmod{9}$, is equal to two if 9 divides the class number of $\mathbb{Q}(\sqrt[3]{p})$. The converse of the Calegari-Emerton result was proved by Frank Gerth III in [11, Theorem 1, p. 471].

The purpose of this paper is to classify all integers d for which $C_{k,3}$ is of type (9,3), i.e. $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. This investigation can be viewed as a continuation of the previous more general works [7, Lemma 5.11] and [11, Theorem 1, p. 471]. Effectively, we shall prove the following main theorem:

Theorem 1.1. Let $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field, where $d \ge 2$ is a cube-free integer, and let $k = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$ be its normal closure. Denote by u the index of the subgroup generated by the units of intermediate fields of the extension k/\mathbb{Q} in the group of units of k.

- 1) If the field k has a 3-class group of type (9,3), then $d = p^e$, where p is a prime number congruent to 1 (mod 9) and e = 1 or 2.
- 2) If p is a prime number congruent to 1 (mod 9), 9 divides the class number of Γ exactly, and u = 1, then the 3-class group of k is of type (9,3).

This result will be underpinned by numerical examples obtained with the computational number theory system PARI [17] in § 3. In section 2, where Theorem 1.1 is proved, we only state results that will be needed in this paper. More information on 3-class groups can be found in [10] and [9]. For the prime ideal factorization in the pure cubic field $\mathbb{Q}(\sqrt[3]{d})$, we refer the reader to the papers [8], [5], [4] and [16]. For the prime factorization rules of the third cyclotomic field $\mathbb{Q}(\zeta_3)$, we refer the reader to [13, Chap. 9, Sec. 1, Propositions 9.1.1–4, pp. 109-111].

Notations:

- The lower case letter p, respectively q, will denote a prime number congruent to 1, respectively -1, modulo 3;
- $\Gamma = \mathbb{Q}(\sqrt[3]{d})$: a pure cubic field, where $d \ge 2$ is a cube-free integer;
- $\mathbf{k}_0 = \mathbb{Q}(\zeta_3)$: the cyclotomic field, where $\zeta_3 = e^{2i\pi/3}$;
- $\mathbf{k} = \Gamma(\zeta_3)$: the normal closure of Γ ;
- Γ' and Γ'' : the two conjugate cubic fields of Γ , contained in k;
- u: the index of the subgroup E_0 generated by the units of intermediate fields of the extension k/\mathbb{Q} in the group of units of k;
- $\langle \tau \rangle = \text{Gal}(\mathbf{k}/\Gamma)$, such that $\tau^2 = id$, $\tau(\zeta_3) = \zeta_3^2$, and $\tau(\sqrt[3]{d}) = \sqrt[3]{d}$;
- $\langle \sigma \rangle = \text{Gal}(\mathbf{k}/\mathbf{k}_0)$, such that $\sigma^3 = id$, $\sigma(\zeta_3) = \zeta_3$, and $\sigma(\sqrt[3]{d}) = \zeta_3\sqrt[3]{d}$;
- $\lambda = 1 \zeta_3$ is a prime element above 3 of k_0 ;
- $q^* = 1$ or 0, according to whether ζ_3 is norm of an element of k or not;
- t: the number of prime ideals ramified in k/k_0 ;
- For an algebraic number field L:
 - $-\mathcal{O}_L, E_L$: the ring of integers of L, and the group of units of L;
 - $-C_{L,3}, h_L$: the 3-class group of L, and the class number of L;
 - $-L_3^{(1)}, L^*$: the Hilbert 3-class field of L, and the absolute genus field of L.

2 Fields $\mathbb{Q}(\sqrt[3]{d}, \zeta_3)$ whose 3-Class Group is of Type (9,3)

2.1 Preliminary results

In [15, Chap. 7, pp. 87–96], Ishida has explicitly given the genus field of any pure field. For the pure cubic field $\Gamma = \mathbb{Q}(\sqrt[3]{d})$, where d is a cube-free natural number, we have the following theorem.

Theorem 2.1. Let $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field, where $d \ge 2$ is a cube-free integer, and let p_1, \ldots, p_r be all prime divisors of d such that p_i is congruent to 1 (mod 3) for each $i \in \{1, 2, \ldots, r\}$. Let Γ^* be the absolute genus field of Γ , then

$$\Gamma^* = \prod_{i=1}^r M(p_i) \cdot \Gamma,$$

where $M(p_i)$ denotes the unique subfield of degree 3 of the cyclotomic field $\mathbb{Q}(\zeta_{p_i})$. The genus number of Γ is given by $g_{\Gamma} = 3^r$.

Remark 2.1. (1) If no prime $p \equiv 1 \pmod{3}$ divides d, i.e. r = 0, then $\Gamma^* = \Gamma$.

(2) For any value $r \ge 0$, Γ^* is contained in the Hilbert 3-class field $\Gamma_3^{(1)}$ of Γ .

(3) The cubic field M(p) is determined explicitly in [12, § 4, Proposition 1, p. 11].

Assuming that h_{Γ} is divisible exactly by 9, we can explicitly construct the absolute genus field Γ^* as follows:

Lemma 2.1. Let $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field, where $d \ge 2$ is a cube-free integer. If h_{Γ} is exactly divisible by 9, then there are at most two primes congruent to 1 (mod 3) dividing d.

Proof. If p_1, \ldots, p_r are all prime numbers congruent to 1 (mod 3) dividing d, then $3^r | h_{\Gamma}$. Therefore, if h_{Γ} is exactly divisible by 9, then $r \leq 2$. So there are two primes p_1 and p_2 dividing d such that $p_i \equiv 1 \pmod{3}$ for $i \in \{1, 2\}$, or there is only one prime $p \equiv 1 \pmod{3}$ with p|d, or there is no prime $p \equiv 1 \pmod{3}$ such that p|d.

Lemma 2.2. Let $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field, where $d \geq 2$ is a cube-free integer. If h_{Γ} is exactly divisible by 9 and if the integer d is divisible by two primes p_1 and p_2 such that $p_i \equiv 1 \pmod{3}$ for $i \in \{1, 2\}$, then $\Gamma^* = \Gamma_3^{(1)}$, $(\Gamma')^* = {\Gamma'_3}^{(1)}$ and $(\Gamma'')^* = {\Gamma''_3}^{(1)}$. Furthermore, $\mathbf{k} \cdot \Gamma_3^{(1)} = \mathbf{k} \cdot {\Gamma'_3}^{(1)} = \mathbf{k} \cdot {\Gamma''_3}^{(1)}$.

Proof. If h_{Γ} is exactly divisible by 9 and d is divisible by two prime numbers p_1 and p_2 which are congruent to 1 (mod 3), then $g_{\Gamma} = 9$ so $\Gamma^* = \Gamma_3^{(1)} = \Gamma \cdot M(p_1) \cdot M(p_2)$, where $M(p_1)$ (respectively $M(p_2)$) is the unique cubic subfield of $\mathbb{Q}(\zeta_{p_1})$ (respectively $\mathbb{Q}(\zeta_{p_2})$). The equations

$$\begin{cases} (\Gamma')^* = \Gamma' \cdot M(p_1) \cdot M(p_2) = {\Gamma'}_3^{(1)}, \\ (\Gamma'')^* = \Gamma'' \cdot M(p_1) \cdot M(p_2) = {\Gamma''}_3^{(1)} \end{cases}$$

can be deduced by the fact that in the general case we have

$$\begin{cases} (\Gamma')^* = \Gamma' \prod_{i=1}^r M(p_i), \\ (\Gamma'')^* = \Gamma'' \prod_{i=1}^r M(p_i), \end{cases}$$

where p_i , for each $1 \leq i \leq r$, is a prime divisor of d such that $p_i \equiv 1 \pmod{3}$. From the fact that h_{Γ} is exactly divisible by 9, we conclude that $h_{\Gamma'}$ is exactly divisible by 9 and $h_{\Gamma''}$ is exactly divisible by 9, because Γ , Γ' and Γ'' are isomorphic. Moreover,

$$\begin{cases} \mathbf{k} \cdot \Gamma_3^{(1)} = \mathbf{k} \cdot \Gamma \cdot M(p_1) \cdot M(p_2) = \mathbf{k} \cdot M(p_1) \cdot M(p_2), \\ \mathbf{k} \cdot \Gamma_3^{(1)} = \mathbf{k} \cdot \Gamma' \cdot M(p_1) \cdot M(p_2) = \mathbf{k} \cdot M(p_1) \cdot M(p_2), \\ \mathbf{k} \cdot \Gamma_3^{\prime\prime(1)} = \mathbf{k} \cdot \Gamma'' \cdot M(p_1) \cdot M(p_2) = \mathbf{k} \cdot M(p_1) \cdot M(p_2). \end{cases}$$

Hence, $\mathbf{k} \cdot \Gamma_3^{(1)} = \mathbf{k} \cdot {\Gamma'}_3^{(1)} = \mathbf{k} \cdot {\Gamma''}_3^{(1)}$.

Now, let u be the index of units defined in the above notations. We assume that h_{Γ} is exactly divisible by 9 and u = 1. From [5, § 14, Theorem 14.1, p. 232], we have $h_{\rm k} = \frac{u}{3} \cdot h_{\Gamma}^2$, whence $h_{\rm k}$ is exactly divisible by 27. The structure of the 3-class group $C_{\rm k,3}$ is described by the following Lemma:

Lemma 2.3. Let Γ be a pure cubic field, k its normal closure, and u be the index of units defined in the notations above, then

 $C_{\mathbf{k},3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \quad \Leftrightarrow \quad [C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z} \quad and \quad u=1].$

Lemma 2.3 will be underpinned in section 3 by numerical examples obtained with the computational number theory system PARI [17].

Proof. Assume that $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Let $h_{\Gamma,3}$ (respectively, $h_{k,3}$) be the 3-class number of Γ (respectively, k), then $h_{k,3} = 27$. According to [5, § 14, Theorem 14.1, p. 232], we get $27 = \frac{u}{3} \cdot h_{\Gamma,3}^2$ with $u \in \{1,3\}$, and thus u = 1, because otherwise 27 would be a square in \mathbb{N} , which is a contradiction. Thus $h_{\Gamma,3}^2 = 81$ and $h_{\Gamma,3} = 9$.

Let $C_{\mathbf{k},3}^+ = \{ \mathcal{A} \in C_{\mathbf{k},3} \mid \mathcal{A}^{\tau} = \mathcal{A} \}$ and $C_{\mathbf{k},3}^- = \{ \mathcal{A} \in C_{\mathbf{k},3} \mid \mathcal{A}^{\tau} = \mathcal{A}^{-1} \}$. According to [9, § 2, Lemmas 2.1 and 2.2, p. 53], we have $C_{\mathbf{k},3} \simeq C_{\mathbf{k},3}^+ \times C_{\mathbf{k},3}^-$ and $C_{\mathbf{k},3}^+ \simeq C_{\Gamma,3}$, hence $|C_{\mathbf{k},3}^-| = 3$. Since $C_{\mathbf{k},3}$ is of type (9, 3), we deduce that $C_{\mathbf{k},3}^-$ is a cyclic group of order 3 and $C_{\mathbf{k},3}^+$ is a cyclic group of order 9. Therefore, we have

$$u = 1$$
 and $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$.

Conversely, assume that u = 1 and $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$. By [5, § 14, Theorem 14.1, p. 232], we deduce that $|C_{k,3}| = \frac{1}{3} \cdot |C_{\Gamma,3}|^2$, and so $|C_{k,3}| = 27$. Furthermore, $|C_{k,3}^-| = 3$ and

$$C_{\mathbf{k},3} \simeq C_{\Gamma,3} \times C_{\mathbf{k},3}^{-} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

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2.2 Proof of Theorem 1.1

Let $\Gamma = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field, where $d \geq 2$ is a cube-free integer, $\mathbf{k} = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$ be its normal closure, and $C_{\mathbf{k},3}$ be the 3-class group of \mathbf{k} .

(1) Assume that the 3- class group $C_{k,3}$ is of type (9,3). We first write the integer d in the form given by equation (3.2) of [9, p. 55]:

$$d = 3^{e} p_{1}^{e_{1}} \dots p_{v}^{e_{v}} p_{v+1}^{e_{v+1}} \dots p_{w}^{e_{w}} q_{1}^{f_{1}} \dots q_{I}^{f_{I}} q_{I+1}^{f_{I+1}} \dots q_{J}^{f_{J}}, \tag{1}$$

where p_i and q_i are positive rational primes such that:

 $\begin{cases} p_i \equiv 1 \pmod{9}, & \text{for } 1 \leq i \leq v, \\ p_i \equiv 4 \text{ or } 7 \pmod{9}, & \text{for } v+1 \leq i \leq w, \\ q_i \equiv -1 \pmod{9}, & \text{for } 1 \leq i \leq I, \\ q_i \equiv 2 \text{ or } 5 \pmod{9}, & \text{for } 1 \leq i \leq J, \\ e_i = 1 \text{ or } 2, & \text{for } 1 \leq i \leq w, \\ f_i = 1 \text{ or } 2, & \text{for } 1 \leq i \leq J, \\ e = 0, 1 \text{ or } 2. \end{cases}$

Let $C_{k,3}^{(\sigma)}$ be the ambiguous ideal class group of k/k₀, where σ is a generator of Gal (k/k₀). It is known that $C_{k,3}^{(\sigma)}$ is an elementary abelian 3-group, because an ambiguous class C satisfies $\sigma(\mathcal{C}) = \mathcal{C}$, by definition, and therefore $\mathcal{C}^3 = \mathcal{C} \cdot \sigma(\mathcal{C}) \cdot \sigma^2(\mathcal{C}) = \mathcal{N}_{k/k_0}(\mathcal{C}) = 1$, since k_0 has class number 1.

The fact that the 3-class group $C_{k,3}$ is of type (9,3) implies that rank $C_{k,3}^{(\sigma)} = 1$. In fact, it is clear that if $C_{k,3}$ is of type (9,3), then rank $C_{k,3}^{(\sigma)} = 1$ or 2.

Let us assume that rank $C_{k,3}^{(\sigma)} = 2$. From [10, § 5, Theorem 5.3, pp. 97–98], we have

rank
$$C_{k,3} = 2t - s$$
,

where the integers t and s are defined in $[10, \S 5, \text{Theorem 5.3, pp. 97-98}]$ as follows:

- $t := \operatorname{rank} C_{\mathbf{k},3}^{(\sigma)},$
- s: the rank of the matrix $(\beta_{i,j})$ defined in [10, § 5, Theorem. 5.3, pp. 97–98].

Since $C_{k,3}$ is of type (9,3), then rank $C_{k,3} = 2$, and according to our hypothesis we have $t = \operatorname{rank} C_{k,3}^{(\sigma)} = 2$. So we get s = 2.

By [10, § 5, Theorem 5.3, pp. 97–98], the 3-class group $C_{k,3}$ is isomorphic to the direct product of an abelian 3-group of rank 2(t-s) and an elementary abelian 3-group of rank s. Here t = s = 2. Thus $C_{k,3}$ is isomorphic to the direct product of an abelian 3-group of rank 0 and an elementary abelian 3-group of rank 2, we get

$$C_{\mathbf{k},3} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z},$$

which contradicts the fact that $C_{k,3}$ is of type (9,3). We conclude rank $C_{k,3}^{(\sigma)} = 1$.

On the one hand, suppose that d is not divisible by any rational prime p such that $p \equiv 1 \pmod{3}$, i.e. w = 0 in Eq. (1) above. According to [9, § 5, Theorem 5.1, p. 61], this implies that $C_{k,3} \simeq C_{\Gamma,3} \times C_{\Gamma,3}$. Since the 3-group $C_{k,3}$ is of type (9,3), then $|C_{k,3}| = 3^3$, and by Lemma 2.3 we have $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$, so we get $|C_{\Gamma,3} \times C_{\Gamma,3}| = 3^4$, which is a contradiction. Hence, the integer d is divisible by at least one rational prime p such that $p \equiv 1 \pmod{3}$.

On the other hand, the fact that the 3-class group $C_{k,3}$ is of type (9,3) implies that $C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}$ according to Lemma 2.3. Since h_{Γ} is exactly divisible by 9, then according to Lemma 2.1, there are at most two primes congruent to 1 (mod 3) dividing d.

Now, assume that there are exactly two different primes p_1 and p_2 dividing d such that $p_1 \equiv p_2 \equiv 1 \pmod{3}$, then, according to Lemma 2.2, we get:

$$\Gamma^* = \Gamma_3^{(1)} = \Gamma \cdot M(p_1) \cdot M(p_2),$$

where $M(p_1)$ (respectively, $M(p_2)$) is the unique subfield of degree 3 of $\mathbb{Q}(\zeta_{p_1})$ (respectively, $\mathbb{Q}(\zeta_{p_2})$). If $M(p_1) \neq M(p_2)$, then $\Gamma \cdot M(p_1)$ and $\Gamma \cdot M(p_2)$ are two different subfields of $\Gamma_3^{(1)}$ over Γ of degree 3. According to class field theory, we have

$$\operatorname{Gal}(\Gamma_3^{(1)}/\Gamma) \cong C_{\Gamma,3} \simeq \mathbb{Z}/9\mathbb{Z}.$$

Since $\operatorname{Gal}(\Gamma_3^{(1)}/\Gamma)$ is a cyclic 3-group, there exist only one sub-group of $\operatorname{Gal}(\Gamma_3^{(1)}/\Gamma)$ of order 3. By the Galois correspondence, there exist a unique sub-field of $\Gamma_3^{(1)}$ over Γ of degree 3. We conclude that $\Gamma \cdot M(p_1) = \Gamma \cdot M(p_2)$, which is a contradiction.

Thus, $M(p_1) = M(p_2)$, and then $p_1 = p_2$, which contradicts the fact that p_1 and p_2 are two different primes. Hence, there is exactly one prime congruent to 1 (mod 3) which divides d. Thus the integer d can be written in the following form:

$$d = 3^{e} p_{1}^{e_{1}} q_{1}^{f_{1}} \dots q_{I}^{f_{I}} q_{I+1}^{f_{I+1}} \dots q_{J}^{f_{J}},$$

with $p_1 \equiv -q_i \equiv 1 \pmod{3}$, where p_1 , e, e_1 , q_i and f_i (for $1 \leq i \leq J$) are defined in Eq. (1).

Next, since rank $C_{k,3}^{(\sigma)} = 1$, then according to [9, § 3, Lemma 3.1, p. 55], there are three possible cases as follows:

- Case 1 : 2w + J = 1,
- Case 2 : 2w + J = 2,
- Case 3 : 2w + J = 3,

where w and J are the integers defined in Eq. (1) above. We will successively treat these cases as follows:

• Case 1: We have 2w + J = 1, then w = 0 and J = 1. This case is impossible, because we have shown above that the integer d is divisible by exactly one prime number congruent to 1 (mod 3) and thus w = 1.

• Case 2: We have 2w + J = 2, and as in Case 1, we necessarily have w = 1 and J = 0, which implies that $d = 3^e p_1^{e_1}$, where p_1 is a prime number such that $p_1 \equiv 1 \pmod{3}$, $e \in \{0, 1, 2\}$ and $e_1 \in \{1, 2\}$. Then,

- If $d \equiv \pm 1 \pmod{9}$, then we necessarily have e = 0. Assume that $p_1 \equiv 4 \text{ or } 7 \pmod{9}$, then $d \not\equiv \pm 1 \pmod{9}$ which is an absurd. So we necessarily have $p_1 \equiv 1 \pmod{9}$. Thus the integer d will be written in the form $d = p_1^{e_1}$, where $p_1 \equiv 1 \pmod{9}$ and $e_1 \in \{1, 2\}$.
- If $d \not\equiv \pm 1 \pmod{9}$:
 - * If $e \neq 0$, the integer d is written as $d = 3^e p_1^{e_1}$, where $p_1 \equiv 1 \pmod{3}$ and $e, e_1 \in \{1, 2\}$.
 - * If e = 0, then *d* is written as $d = p_1^{e_1}$ with $p_1 \equiv 4$ or 7 (mod 9) and $e_1 \in \{1, 2\}$.

• Case 3: We have 2w + J = 3, then we necessarily get w = 1 and J = 1, because $w \neq 0$. Thus $d = 3^e p_1^{e_1} q_1^{f_1}$, where $p_1 \equiv 1 \pmod{3}$, $q_1 \equiv -1 \pmod{3}$, $e \in \{0, 1, 2\}$ and $e_1, f_1 \in \{1, 2\}$. Then:

- If $d \equiv \pm 1 \pmod{9}$, we necessarily have e = 0. If p_1 or $-q_1 \not\equiv 1 \pmod{9}$, then $d \not\equiv \pm 1 \pmod{9}$ which is an absurd. It remain only the case where $p_1 \equiv -q_1 \equiv 1 \pmod{9}$. Then the integer d will be written in the form $d = p_1^{e_1} q_1^{f_1}$, where $p_1 \equiv -q_1 \equiv 1 \pmod{9}$ and $e_1, f_1 \in \{1, 2\}$. - If $d \not\equiv \pm 1 \pmod{9}$:

According to [10, § 5, p. 92], the rank of $C_{\mathbf{k},3}^{(\sigma)}$ is specified as follows:

rank
$$C_{\mathbf{k},3}^{(\sigma)} = t - 2 + q^*$$
,

where t and q^* are defined in the notations.

On the one hand, the fact that rank $C_{k,3}^{(\sigma)} = 1$ imply that t = 2 or 3 according to whether ζ_3 is norm of an element of k or not.

On the other hand, we have $d = 3^e p_1^{e_1} q_1^{f_1}$ with $p_1 \equiv 1 \pmod{3}$ and $q_1 \equiv -1 \pmod{3}$. By [13, Chap. 9, Sec. 1, Proposition 9.1.4, p.110] we have q_1 is inert in k_0 , and by [8, Sec 4, pp. 51-54] we have q_1 is ramifed in $\Gamma = \mathbf{Q}(\sqrt[3]{d})$. Since $d \not\equiv \pm 1 \pmod{9}$, then 3 is ramifed in Γ by [8, Sec 4, pp. 51-54], and $3\mathcal{O}_{k_0} = (\lambda)^2$ where $\lambda = 1 - \zeta_3$. Since $p_1 \equiv 1 \pmod{3}$, then by [13, Chap. 9, Sec. 1, Proposition 9.1.4, p.110] we have $p_1 = \pi_1 \pi_2$ where π_1 and π_2 are two primes of k_0 such that $\pi_2 = \pi_1^{\tau}$, the prime p_1 is ramifed in Γ , then π_1 and π_2 are ramified in k. Hence, the number of prime ideals which are ramified in k/k_0 is t = 4, which contradicts the fact that t = 2 or 3.

We summarize all forms of integer d in the three cases 1, 2 and 3 as follows:

$$d = \begin{cases} p_1^{e_1} & \text{with } p_1 \equiv 1 \pmod{9}, \\ p_1^{e_1} & \text{with } p_1 \equiv 4 \text{ or } 7 \pmod{9}, \\ 3^e p_1^{e_1} \not\equiv \pm 1 \pmod{9} & \text{with } p_1 \equiv 1 \pmod{9}, \\ 3^e p_1^{e_1} \not\equiv \pm 1 \pmod{9} & \text{with } p_1 \equiv 4 \text{ or } 7 \pmod{9}, \\ p_1^{e_1} q_1^{f_1} \equiv \pm 1 \pmod{9} & \text{with } p_1 \equiv -q_1 \equiv 1 \pmod{9}, \end{cases}$$

where $e, e_1, f_1 \in \{1, 2\}$.

Our next goal is to show that the only possible form of the integer d is the first form $d = p_1^{e_1}$, where $p_1 \equiv 1 \pmod{9}$ and $e_1 = 1$ or 2.

- Case where $d = p_1^{e_1}$, with $p_1 \equiv 4$ or 7 (mod 9):
 - If $\left(\frac{3}{p}\right)_3 \neq 1$, then according to [3, § 1, Conjecture 1.1, p. 1], we have $C_{k,3} \simeq \mathbb{Z}/3\mathbb{Z}$, which contradict the fact that $C_{k,3}$ is of type (9,3). We note that in this case, the fields Γ and k are of Type III, respectively, α , in the terminology of [5, § 15, Theorem 15.6, pp. 235-236], respectively, [1, § 2.1, Theorem 2.1, p. 4].
 - If $\left(\frac{3}{p}\right)_3 = 1$, then according to [3, § 1, Conjecture 1.1, p. 1], we have $C_{k,3} \simeq$ $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, which is impossible. We note that in this case, the fields Γ and k are of Type I, respectively, β , in the terminology of [5, § 15, Theorem 15.6, pp. 235-236], respectively, $[1, \S 2.1, \text{Theorem } 2.1, \text{ p. } 4]$.

For this case, we see that in [5, § 17, Numerical Data, p. 238], and also in the tables of [6] which give the class number of a pure cubic field, the prime numbers p = 61, 67, 103, and 151, which are all congruous to 4 or 7 (mod 9), verify the following properties:

- (i) 3 is a residue cubic modulo p;
- (ii) 3 divide exactly the class number of Γ ;
- (iii) u = 3;
- (iv) $C_{\Gamma,3} \simeq \mathbb{Z}/3\mathbb{Z}$, and $C_{k,3} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

• Case where $d = 3^e p_1^{e_1} \not\equiv \pm 1 \pmod{9}$, with $p_1 \equiv 1 \pmod{9}$:

Here $e, e_1 \in \{1, 2\}$. As $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{a^2b})$, we can choose $e_1 = 1$, i.e. $d = 3^e p_1$ with $e \in \{1, 2\}$. On the one hand, the fact that $p_1 \equiv 1 \pmod{3}$ implies by [13, Chap. 9, Sec. 1, Proposition 9.1.4, p.110] that $p_1 = \pi_1 \pi_2$ with $\pi_1^{\tau} = \pi_2$ and $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{k_0}}$, the prime p_1 is totally ramified in Γ , then π_1 and π_2 are totally ramified in k and we have $\pi_1 \mathcal{O}_k = \mathcal{P}_1^3$ and $\pi_2 \mathcal{O}_k = \mathcal{P}_2^3$, where $\mathcal{P}_1, \mathcal{P}_2$ are two prime ideals of k.

We know that $3\mathcal{O}_{k_0} = \lambda^2 \mathcal{O}_{k_0}$, with $\lambda = 1 - \zeta_3$ a prime element of k_0 . Since $d \neq \pm 1 \pmod{9}$, then 3 is totally ramified in Γ , and then λ is ramified in k/k_0 . Hence, the number of ideals which are ramified in k/k_0 is t = 3.

On the other hand, from [13, Chap. 9, Sec. 1, Proposition 9.1.4, p.110] we have $3 = -\zeta_3^2 \lambda^2$, then $\mathbf{k} = \mathbf{k}_0(\sqrt[3]{x})$ with $x = \zeta_3^2 \lambda^2 \pi_1 \pi_2$. The primes π_1 and π_2 are congruent to 1 (mod λ^3) because $p_1 \equiv 1 \pmod{9}$, then according to [1, § 3, Lemma 3.3, p. 17] we have ζ_3 is a norm of an element of $\mathbf{k} \setminus \{0\}$, so $q^* = 1$. We conclude according to [10, § 5, p. 92] that rank $C_{\mathbf{k},3}^{(\sigma)} = 2$ which is impossible.

• Case where $d = 3^e p_1^{e_1} \not\equiv \pm 1 \pmod{9}$, with $p_1 \equiv 4 \text{ or } 7 \pmod{9}$:

Here $e, e_1 \in \{1, 2\}$. So we can choose e = 1, then $d = 3p_1^{e_1}$ with $e_1 \in \{1, 2\}$. As above we get π_1 and π_2 , and λ are ramified in k/k₀.

Put $p\mathcal{O}_{\Gamma} = \mathcal{P}^3$, $\pi_1\mathcal{O}_k = \mathcal{P}_1^3$, $\pi_2\mathcal{O}_k = \mathcal{P}_2^3$ and $\lambda\mathcal{O}_k = I^3$, where \mathcal{P} is a prime ideal of Γ , and $\mathcal{P}_1, \mathcal{P}_2$ and I are prime ideals of k. According to [14, § 3.2, Theorem 3.5, pp 36-39] we get $C_{k,3}$ is cyclic of order 3 which contradict the fact that $C_{k,3}$ is of type (9,3).

• Case where $d = p_1^{e_1} q_1^{f_1} \equiv \pm 1 \pmod{9}$, with $p_1 \equiv -q_1 \equiv 1 \pmod{9}$:

Since $d \equiv \pm 1 \pmod{9}$, then according to [8, Sec. 4, pp. 51-54] we have 3 is not ramified in the field Γ , so $\lambda = 1 - \zeta_3$ is not ramified in k/k₀. As $p_1 \equiv 1 \pmod{3}$, then by [13, Chap. 9, Sec. 1, Proposition 9.1.4, p.110] $p_1 = \pi_1 \pi_2$ with $\pi_1^{\tau} = \pi_2$ and $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{k_0}}$, so π_1 and π_2 are totally ramified in k. Since $q_1 \equiv -1 \pmod{3}$, then q_1 is inert in k₀. Hence, the primes ramifies in k/k₀ are π_1, π_2 and q_1 . Put $x = \pi_1^{e_1} \pi_2^{e_1} \pi^{f_1}$, where $-q_1 = \pi$ is a prime number of k₀, then we have $\mathbf{k} = \mathbf{k}_0(\sqrt[3]{x})$. The fact that $p_1 \equiv -q_1 \equiv 1 \pmod{9}$ imply that $\pi_1 \equiv \pi_2 \equiv \pi \equiv 1 \pmod{\lambda^3}$, then by [1, § 3, Lemma 3.3, p. 17], ζ_3 is a norm of an element of $\mathbf{k} \setminus \{0\}$, so $q^* = 1$. Then by [10, § 5, p. 92] we get rank $C_{\mathbf{k},3}^{(\sigma)} = t - 2 + q^* = 2$ which is impossible.

Finally, we have shown that if the 3-class group $C_{k,3}$ is of type (9,3), then $d = p^e$, where p is a prime number such that $p \equiv 1 \pmod{9}$ and e = 1 or 2. We can see that this result is compatible with the first form of the integer d in [1, § 1, Theorem 1.1, p. 2].

(2) Suppose that $d = p^e$, with p is a prime number congruent to 1 (mod 9). Here $e \in \{1, 2\}$, since $\mathbb{Q}(\sqrt[3]{p}) = \mathbb{Q}(\sqrt[3]{p^2})$ we can choose e = 1. From [5, § 14, Theorem 14.1, p. 232], we have $h_k = \frac{u}{3} \cdot h_{\Gamma}^2$, the fact that u = 1 and that h_{Γ} is exactly divisible by 9 implies that h_k

is exactly divisible by 27. Since h_{Γ} is exactly divisible by 9, then by [7, Lemma 5.11] we have rank $C_{k,3} = 2$. We conclude that $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

3 Numerical Examples

Let $\Gamma = \mathbb{Q}(\sqrt[3]{ab^2})$ be a pure cubic field, where a and b are coprime square-free integers. We point out that $\mathbb{Q}(\sqrt[3]{ab^2}) = \mathbb{Q}(\sqrt[3]{a^2b})$. Assume that $k = \mathbb{Q}(\sqrt[3]{ab^2}, \zeta_3)$ and $C_{k,3}$ is of type (9,3). Using the system Pari [17], we compute class groups for b = 1 and a prime $a = p \equiv 1 \pmod{9}$. The following table illustrates our main result Theorem 1.1 and Lemma 2.3. Here we denote by:

> $C_{\Gamma,3}$ (respectively, $C_{k,3}$) : the 3-class group of Γ (respectively, k); $h_{\Gamma,3}$ (respectively, $h_{k,3}$) : the 3-class number of Γ (respectively, k).

Table : Some fields $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ whose 3-class group is of type (9, 3).

| p | p^2 | $p \pmod{9}$ | $h_{\Gamma,3}$ | $h_{\mathrm{k},3}$ | u | $C_{\Gamma,3}$ | $C_{\rm k,3}$ |
|------|----------|--------------|----------------|--------------------|---|----------------|---------------|
| 199 | 39601 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 271 | 73441 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 487 | 237169 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 523 | 273529 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 1297 | 1682209 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 1621 | 2627641 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 1693 | 2866249 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 1747 | 3052009 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 1999 | 3996001 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 2017 | 4068289 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 2143 | 4592449 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 2377 | 5650129 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 2467 | 6086089 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 2593 | 6723649 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 2917 | 8508889 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 3511 | 12327121 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 3673 | 13490929 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 3727 | 13890529 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 3907 | 15264649 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 4159 | 17297281 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 4519 | 20421361 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 4591 | 21077281 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 4789 | 22934521 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 4933 | 24334489 | 1 | 9 | 27 | 1 | [9] | [9, 3] |

| p | p^2 | $p \pmod{9}$ | $h_{\Gamma,3}$ | $h_{\mathrm{k},3}$ | u | $C_{\Gamma,3}$ | $C_{\mathrm{k},3}$ |
|------|----------|--------------|----------------|--------------------|---|----------------|--------------------|
| 4951 | 24512401 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 5059 | 25593481 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 5077 | 25775929 | 1 | 9 | 27 | 1 | [9] | [9, 3] |
| 5347 | 28590409 | 1 | 9 | 27 | 1 | [9] | [9, 3] |

Remark 3.1. Let p is a prime number such that $p \equiv 1 \pmod{9}$. Let $\Gamma = \mathbb{Q}(\sqrt[3]{p})$, $\mathbf{k} = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ be the normal closure of the pure cubic field Γ and $C_{\mathbf{k},3}$ be the 3-part of the class group of \mathbf{k} . If 9 divide exactly the class number of $\mathbb{Q}(\sqrt[3]{p})$ and u = 1, then according to Theorem 1.1, the 3-class group of $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ is of type (9,3). Furthermore, if 3 is not residue cubic modulo p, then a generators of 3-class group of $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ can be deduced by [2, § 3, Theorem 3.2, p. 10].

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