

ARNOL'D'S TYPE THEOREM ON A NEIGHBORHOOD OF A CYCLE OF RATIONAL CURVES

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ABSTRACT. Arnol'd showed the uniqueness of the complex analytic structure of a small neighborhood of a non-singular elliptic curve embedded in a non-singular surface whose normal bundle satisfies Diophantine condition in the Picard variety. We show an analogue of this theorem for a neighborhood of a cycle of rational curves.

1. INTRODUCTION

Let C be a cycle of rational curves (i.e. a reduced singular complex curve with only nodes such that the dual graph is a cycle graph and each complement of the normalization is biholomorphic to the projective line \mathbb{P}^1) holomorphically embedded in a non-singular complex surface S . Assume that the normal line bundle $N_{C/S} := [C]|_C$ is topologically trivial, where $[C]$ is the holomorphic line bundle on S defined by the divisor C . Denote by $t(N_{C/S}) \in \mathbb{C}^*$ the complex number which corresponds to $N_{C/S}$ via the natural identification of Picard variety $\text{Pic}^0(C)$ of C with $H^1(C, \mathbb{C}^*) = \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ([U91, Lemma 1], see also §2.1.2). We show the following theorem on the uniqueness of the complex analytic structure of a small neighborhood of C under a Diophantine type condition for the normal bundle.

THEOREM 1.1. *Let C be a cycle of rational curves, and $i: C \rightarrow S$ and $i': C \rightarrow S'$ be holomorphic embeddings into non-singular complex surfaces S and S' respectively. Assume that $t(N_{i(C)/S}) = t(N_{i'(C)/S'}) = e^{2\pi\sqrt{-1}\theta}$ holds for a Diophantine irrational number $\theta \in \mathbb{R}$ (i.e. there exist positive constants α and A such that $|n \cdot \theta - m| \geq A \cdot n^{-\alpha}$ holds for any integer m and any positive integer n). Then there exists a biholomorphism $f: V \rightarrow V'$ between a neighborhood V of $i(C)$ in S and V' of $i'(C)$ in S' with $f|_{i(C)} = i' \circ i^{-1}$.*

Theorem 1.1 can be regarded as an analogue of Arnol'd's theorem [A], which states that the conclusion of the theorem holds for a non-singular elliptic C embedded in a non-singular surface S under the assumption that $N_{C/S}$ satisfies the Diophantine type condition in the Picard variety.

Note that, in our notation, C is a cycle of rational curves with only one irreducible component when C is a rational curve with a node. Neighborhoods of a rational curve with a node embedded in a surface was first investigated by Ueda in [U91] when $t(N_{C/S}) \in \mathbb{C}^* \setminus \text{U}(1)$, where $\text{U}(1) := \{t \in \mathbb{C}^* \mid |t| = 1\}$. In [K2], we slightly generalized his results to the case where, for example, C is a cycle of rational curves [K2, Theorem 1.6]. In that paper, we also treated the case where $t(N_{C/S}) \in \text{U}(1)$, which is the case that $N_{C/S}$ is a $\text{U}(1)$ -flat line bundle: i.e. $N_{C/S}$ admits a C^ω Hermitian metric with flat curvature. In

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this case, we showed the existence of a pseudoflat neighborhoods system of C under the assumption that $t(N_{C/S}) = e^{2\pi\sqrt{-1}\theta}$ holds for a Diophantine irrational number $\theta \in \mathbb{R}$ [K2, Theorem 1.4], which can be regarded as an analogue of Ueda's theorem for a non-singular compact curve embedded in a surface [U83, Theorem 3]. Here we remark that Theorem 1.1 is also regarded as an improved version of [K2, Theorem 1.4].

In the proof of Theorem 1.1, we compare the complex structure of a small neighborhood V of C with that of *the standard model* we describe in Example 2.1 or 2.4. We consider the cohomology class $\alpha = \alpha(C, V) \in H^1(V, \mathcal{O}_V)$ which corresponds to the difference of them. Then one can see that it is sufficient to show that the restriction $\alpha|_{V^*}$ of this class to a small neighborhood V^* of C in V is equal to zero in $H^1(V^*, \mathcal{O}_{V^*})$. Note that this class satisfies $\alpha|_C = 0 \in H^1(C, \mathcal{O}_C)$. Therefore the problem is reduced to showing the injectivity of the restriction morphism $\lim_{V^* \rightarrow} H^1(V^*, \mathcal{O}_{V^*}) \rightarrow H^1(C, \mathcal{O}_C)$, where V^* runs all the neighborhoods of C in V (Proposition 3.1). We show this by using a complex dynamical technique originated from [Sie], which is also used in the proofs of [U83, Theorem 3] and [K2, Theorem 1.4].

The main motivation of the present paper comes from [T] and [K3]. In [K3], as an application of Arnol'd's theorem [A], we constructed a K3 surface by holomorphically gluing two open complex surfaces obtained as the complements of tubular neighborhoods of non-singular elliptic curves embedded in the blow-ups of the projective planes at appropriate nine points. As described in [K3, §4.1.1], this construction can be regarded as a concrete description of a general fiber of a degeneration of K3 surfaces of type II. Theorem 1.1 can also be applied to nodal curves embedded in the blow-ups of the projective planes at appropriate nine points (Examples 2.2, 2.6, and 2.7). Toward a concrete description of a general fiber of a degeneration of K3 surfaces of type III, we will investigate these examples precisely.

The organization of the paper is as follows. In §2, we correct some fundamental facts on cycles of rational curves. Here we also fix coordinates on a neighborhood of each irreducible component of a cycle of rational curves by using Grauert's theorem [G] intrinsically. In §3, we show the injectivity of the morphism $\lim_{V^* \rightarrow} H^1(V^*, \mathcal{O}_{V^*}) \rightarrow H^1(C, \mathcal{O}_C)$, where V^* runs all the neighborhoods of C . In §4, we prove Theorem 1.1. In §5, we investigate Examples 2.2, 2.6, and 2.7 precisely.

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2. PRELIMINARIES

In this section, we collect some fundamental facts and fix some notation on a cycle of rational curves.

Let C be a cycle of rational curves embedded in a non-singular surface S with Diophantine condition as in Theorem 1.1. Take an open covering $\{U_j\}_j$ of C , and a small neighborhood V_j of U_j in S with $V_j \cap C = U_j$ for each j . Denote by V the neighborhood $\bigcup_j V_j$ of C .

It follows from [K2, Proposition 2.5 (2)] that the pair (C, S) is of infinite type in the sense of [K2]. Therefore, from [K2, Theorem 1.4], we have that there exists a defining

function w_j of U_j in V_j for each j such that $w_j = t_{jk}w_k$ holds for some $t_{jk} \in \mathbb{U}(1)$ on each $V_{jk} := V_j \cap V_k$ when $\{U_j\}$ and $\{V_j\}$ are sufficiently fine.

2.1. Preliminaries on a rational curve with a node.

2.1.1. *Notation.* Let C be a rational curve with a node. In this case, we choose open coverings $\{U_j\}_j$ and $\{V_j\}_j$ such that the index set is $\{0, 1\}$ as follows: Let U_0 be a small neighborhood of the nodal point of C and U_1 be the regular part $C_{\text{reg}} := C \setminus \{\text{nodal point}\}$ of C . By taking V_j as a sufficiently small neighborhood of U_j , we may assume that $V_0 \cap V_1$ consists of two connected component V^+ and V^- . Let t_{\pm} be elements of $\mathbb{U}(1)$ such that

$$w_1 = \begin{cases} t_+ \cdot w_0 & (\text{on } V^+) \\ t_- \cdot w_0 & (\text{on } V^-). \end{cases}$$

Note that $t := t_+/t_- = t(N_{C/S}) \in \mathbb{U}(1) \subset \mathbb{C}^* = H^1(C, \mathbb{C}^*)$, see §2.1.2.

Let z be a non-homogeneous coordinate of the normalization $\tilde{C} \cong \mathbb{P}^1$ of C such that the preimage of the nodal point is $\{0, \infty\}$. As we will see in §2.2, we can extend the function $z|_{U_1}$ to V_1 , where we are identifying $\tilde{C} \setminus \{0, \infty\}$ with U_1 (see also [Siu]). The resulting holomorphic function on V_1 is also denoted by z . Take coordinates (x, y) of V_0 such that $x \cdot y$ is a defining function of U_0 in V_0 . These functions (x, y) will also be chosen by more careful argument in §2.2 in actual. Denote by U_0^+ the subset $\{(x, y) \in V_0 \mid y = 0\}$ and U_0^- the subset $\{(x, y) \in V_0 \mid x = 0\}$. We may assume that $U^+ := V^+ \cap U_0$ coincides with $U_0^+ \setminus \{\text{nodal point}\}$, and that $U^- := V^- \cap U_0$ coincides with $U_0^- \setminus \{\text{nodal point}\}$.

2.1.2. *Picard variety and some cohomologies.* Let $L \in \text{Pic}^0(C)$ be a topologically trivial line bundle on C . Then there is a uniquely determined complex constant $t = t(L) \in \mathbb{C}^*$ with

$$L = [\{(U^+, t), (U^-, 1)\}] \in \check{H}^1(\{U_j\}, \mathcal{O}_C^*) = H^1(C, \mathcal{O}_C^*)$$

where we are using the notation in the previous section (see the arguments around [U91, Lemma 1]). In particular, it is observed that L admits \mathbb{C}^* -flat structure: i.e. L admits a flat connection. From this fact, one have that $\text{Pic}^0(C)$ is naturally identified with $H^1(C, \mathbb{C}^*) = \mathbb{C}^*$.

When $t(L) \in \mathbb{U}(1) \setminus \{1\}$, L is a non-trivial $\mathbb{U}(1)$ -flat line bundle. In this case, one can obtain by considering the long exact sequence comes from $0 \rightarrow \mathcal{O}_C(L) \rightarrow i_*\mathcal{O}_{\tilde{C}}(i^*L) \rightarrow \mathcal{O}_{\{\text{the nodal point}\}} \rightarrow 0$ that $H^0(C, L) = H^1(C, L) = 0$, where $i: \tilde{C} \rightarrow C$ is the normalization (see [K2, p. 852]). By the same argument, one also have that $H^0(C, \mathcal{O}_C) \cong H^1(C, \mathcal{O}_C) \cong \mathbb{C}$.

2.1.3. *Standard model of a neighborhood of a rational curve with a node and some examples.* The following example can be regarded as the standard model of a neighborhood of a rational curve with a node.

EXAMPLE 2.1. Let \tilde{V} be a neighborhood of the zero section \tilde{C} of the line bundle $\pi: \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow \mathbb{P}^1$. Let S be a non-homogeneous coordinate of \mathbb{P}^1 . We also use the non-homogeneous $T := S^{-1}$ especially when we observe a neighborhood of $\{S = \infty\}$. Let ξ_0 and ξ_∞ be fiber coordinates of $\mathcal{O}_{\mathbb{P}^1}(-2)$ defined in a neighborhood of $\{S = 0\}$ and $\{T = 0\}$, respectively. We may assume that $\xi_\infty = \xi_0 \cdot S^2$ holds in the fibers over $\mathbb{P}^1 \setminus \{S = 0, \infty\}$.

Fix a constant $0 < \varepsilon < 1$ and let us consider subsets

$$\widetilde{V}_0^+ := \{(S, \xi_0) \in \pi^{-1}(\mathbb{C}) \mid |S| < \varepsilon, |\xi_0| < \varepsilon\}$$

and

$$\widetilde{V}_0^- := \{(T, \xi_\infty) \in \pi^{-1}(\mathbb{P}^1 \setminus \{0\}) \mid |T| < \varepsilon, |\xi_\infty| < \varepsilon\}$$

of \widetilde{V} . By shrinking \widetilde{V} if necessary, we may assume that $\pi^{-1}(\{|S| < \varepsilon\}) \cap \widetilde{V} = \widetilde{V}_0^+$ and $\pi^{-1}(\{|T| < \varepsilon\}) \cap \widetilde{V} = \widetilde{V}_0^-$. Define a biholmorphism $F: \widetilde{V}_0^+ \rightarrow \widetilde{V}_0^-$ by $F^*(T, \xi_\infty) := (t \cdot \xi_0, S)$ (i.e. $F^*T := T \circ F := t \cdot \xi_0$ and $F^*\xi_\infty := \xi_\infty \circ F := S$), where $t \in \mathbb{U}(1)$ is a constant. Denote by $i: \widetilde{V} \rightarrow V$ the quotient by the relation induced by F . Then V is a non-singular surface and the compact analytic subset $C := i(\widetilde{C})$ is a rational curve with a node such that $t(N_{C/S}) = t$.

Next example is an analogue of Arnol'd–Ueda–Brunella's example [A] [U83] [B].

EXAMPLE 2.2. Take a plane cubic $C_0 \subset \mathbb{P}^2$ which admits only one nodal point, and nine points $Z \subset \{p_1, p_2, \dots, p_9\} \subset (C_0)_{\text{reg}}$, where $(C_0)_{\text{reg}}$ is the non-singular locus of C_0 . Denote by $\pi: S \rightarrow \mathbb{P}^2$ the blow-up at Z and by C the strict transform $(\pi^{-1})_*C_0$. Then it is known that, by taking a normalization $i: \mathbb{P}^1 \rightarrow C_0$ with $i^{-1}((C_0)_{\text{sing}}) = \{0, \infty\}$ appropriately ($(C_0)_{\text{sing}} := C_0 \setminus (C_0)_{\text{reg}}$), the complex constant $t = t(N_{C/S}) \in \mathbb{C}^*$ can be calculated by $t = \prod_{\nu=1}^9 i^{-1}(p_\nu) \in \mathbb{C}^* = H^1(C_0, \mathbb{C}^*)$, where we are identifying C_0 and C via π . Especially, each point of $\text{Pic}^0(C_0)$ is attained by choosing appropriate nine points configuration Z .

Finally, we give a counter example of Theorem 1.1 when $N_{C/S}$ does not satisfy Diophantine condition.

EXAMPLE 2.3. Let $\{(\widetilde{V}, \widetilde{V}_0^\pm, S, T, \xi_0, \xi_\infty)\}$ be those in Example 2.1. Denote by \widetilde{W}_0^+ the subset $\{(S, \xi_0) \in \pi^{-1}(\mathbb{C}) \cap \widetilde{V} \mid |S| < 1\}$ and by \widetilde{W}_0^- the subset $\{(T, \xi_\infty) \in \pi^{-1}(\mathbb{P}^1 \setminus \{0\}) \cap \widetilde{V} \mid |T| < 1\}$ of $\mathcal{O}_{\mathbb{P}^1}(-2)$. Note that $\widetilde{V}_0^+ \subset \widetilde{W}_0^+$ and $\widetilde{V}_0^- \subset \widetilde{W}_0^-$. For sufficiently small positive constant δ , set $\widetilde{W}_1 := \{(S, \xi_0) \in \widetilde{V} \mid 1/2 < |S| < 2, |\xi_0| < \delta\}$. We may assume that $\widetilde{V}_0^+ \cap \widetilde{W}_1 = \emptyset$ and $\widetilde{V}_0^- \cap \widetilde{W}_1 = \emptyset$ hold. Take a univalent holomorphic function φ defined on $\{w \in \mathbb{C} \mid |w| < \delta\}$ such that $\varphi(0) = 0$ and $\lambda := \varphi'(0) \in \mathbb{U}(1)$ hold. Denote by $\Phi_+: \widetilde{W}_1 \cap \widetilde{W}_0^+ \rightarrow \widetilde{W}_0^+$ the map defined by $(\Phi_+)^*(S, \xi_0) = (S, \varphi(S \cdot \xi_0) \cdot S^{-1})$ and by $\Phi_-: \widetilde{W}_1 \cap \widetilde{W}_0^- \rightarrow \widetilde{W}_0^-$ the natural injection. Define a surface W by $W := (\widetilde{W}_0^+ \amalg \widetilde{W}_1 \amalg \widetilde{W}_0^-) / \sim$, where \sim is the relation generated by

$$\begin{cases} \widetilde{W}_0^+ \ni p \sim F(p) \in \widetilde{W}_0^- & \text{if } p \in \widetilde{V}_0^+ \\ \widetilde{W}_1 \ni p \sim \Phi_+(p) \in \widetilde{W}_0^+ & \text{if } p \in \{(S, \xi_0) \in \widetilde{W}_1 \mid \frac{1}{2} < |S| < 1\} \\ \widetilde{W}_1 \ni p \sim \Phi_-(p) \in \widetilde{W}_0^- & \text{if } p \in \{(S, \xi_0) \in \widetilde{W}_1 \mid 1 < |S| < 2\}, \end{cases}$$

where F is the one in Example 2.1 with $t = 1$. Denote by C the image of \widetilde{C} by the quotient map. Note that C is a compact leaf of the holomorphic foliation \mathcal{F} on W whose leaves are defined by

$$\begin{cases} \{S \cdot \xi_0 = \text{constant}\} & \text{(on } \widetilde{W}_0^+) \\ \{S \cdot \xi_0 = \text{constant}\} & \text{(on } \widetilde{W}_1) \\ \{T \cdot \xi_\infty = \text{constant}\} & \text{(on } \widetilde{W}_0^-). \end{cases}$$

Assume that φ is the one as in [U83, p. 606]. Then $t(N_{C/S}) = \varphi'(0)$ is a non-torsion element of $U(1)$, and any small neighborhood $W^* \subset W$ of C includes a compact leaf of \mathcal{F} which is biholomorphic to an elliptic curve and has no intersection with C . As it follows from the same argument as in [U83, §5.3] that there is no compact subvariety $W^* \setminus C$ for sufficiently small W^* if C admits pseudoflat neighborhoods system, we have that C does not admit a neighborhood as in Example 2.1 in this example.

2.2. Definition of the covering map $\tilde{V} \rightarrow V$ and outline of the proof of Theorem 1.1. Here we use the notation in §2.1.1. Take a copy \tilde{V}_1 of V_1 and two copies \tilde{V}_0^+ and \tilde{V}_0^- of V_0 . Denote by \tilde{V} the manifold constructed by patching \tilde{V}_0^+ , \tilde{V}_1 and \tilde{V}_0^- by considering the natural injections

$$\begin{array}{ccccc} & \tilde{V}_0^+ & & \tilde{V}_1 & & \tilde{V}_0^- \\ & \swarrow & & \swarrow & & \swarrow \\ & V^+ & & V^- & & \end{array}$$

of V^\pm . Note that \tilde{V} can be regarded as an open submanifold of the universal covering of V . Denote by $i: \tilde{V} \rightarrow V$ the natural map. In what follows, we regard \tilde{V}_0^\pm and \tilde{V}_1 as subsets of \tilde{V} . Then $i|_{\tilde{C}}$ is a normalization of C , where $\tilde{C} \subset i^{-1}(C)$ is the irreducible component which is compact. By identifying \tilde{C} and \mathbb{P}^1 , we may assume that the preimage of the nodal point is $\{0, \infty\}$. Denote by D_0 and D_∞ the other two irreducible components of $i^{-1}(C)$ which intersects \tilde{C} at 0 and ∞ , respectively. Define the defining function \tilde{w} of the divisor $i^*C = \tilde{C} + D_0 + D_\infty$ of \tilde{V} by

$$\tilde{w} := \begin{cases} (\tilde{V}_0^+ \rightarrow V_0)^*(t_+ \cdot w_0) & (\text{on } \tilde{V}_0^+) \\ (\tilde{V}_1 \rightarrow V_1)^*w_1 & (\text{on } \tilde{V}_1) \\ (\tilde{V}_0^- \rightarrow V_0)^*(t_- \cdot w_0) & (\text{on } \tilde{V}_0^-), \end{cases}$$

where $\tilde{V}_0^+ \rightarrow V_0$, $\tilde{V}_1 \rightarrow V_1$ and $\tilde{V}_0^- \rightarrow V_0$ be the natural biholomorphisms. By a simple argument, we have that $\deg N_{\tilde{C}/\tilde{V}} = -2$. Therefore, it follows from Grauert's theorem [G] that \tilde{V} can be holomorphically embedded in the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow \mathbb{P}^1$ by shrinking \tilde{V} (see also [CM, Theorem 2.5.2]). In what follows, we regard \tilde{V} as a subset of $\mathcal{O}_{\mathbb{P}^1}(-2)$ and identify \tilde{C} with the zero section via this embedding.

Take a strictly pseudoconvex neighborhood \mathcal{V} of \tilde{C} in \tilde{V} . It follows from Ohsawa's vanishing theorem [O, Theorem 4.5] that $H^1(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) = 0$. Thus we have that the line bundle on \tilde{V} corresponds to the divisor $D_0 - D_\infty$ is holomorphically trivial. Therefore there exists a holomorphic map $p: \mathcal{V} \rightarrow \mathbb{P}^1$ such that $p^*(\{0\} - \{\infty\}) = D_0 - D_\infty$ holds as divisors. By shrinking \tilde{V} so that $\tilde{V} \subset \mathcal{V}$, we may assume that the map p is defined on \tilde{V} .

Let $S = T^{-1}$ be non-homogeneous coordinate of \mathbb{P}^1 . Denote also by S and T the meromorphic functions p^*S and p^*T on \tilde{V} , respectively. Then we have that $D_0 = \{S = 0\} = \{T = \infty\}$ and $D_\infty = \{S = \infty\} = \{T = 0\}$. Setting $\xi_0 := \tilde{w} \cdot S^{-1}$ on a neighborhood of D_0 and $\xi_\infty := \tilde{w} \cdot T^{-1}$ on a neighborhood of D_∞ , we regard (S, ξ_0) and (T, ξ_∞) as coordinates of a neighborhood of D_0 and D_∞ , respectively. Denote by $F: \tilde{V}_0^+ \rightarrow \tilde{V}_0^-$ the biholomorphism such that $i^{-1}(i(p)) = \{p, F(p)\}$ for each $p \in \tilde{V}_0^+$. As it hold that

$F^*\tilde{w} = t \cdot \tilde{w}$, $F^*D_\infty = \tilde{C} \cap \tilde{V}_0^+$ and that $F^*(\tilde{C} \cap \tilde{V}_0^-) = D_0$, we have that

$$F^*(T, \xi_\infty) = \left(\frac{t \cdot \xi_0}{G(S, \xi_0)}, G(S, \xi_0) \cdot S \right)$$

holds for a nowhere vanishing holomorphic function G defined on \tilde{V}_0^+ , where $t = t_+/t_-$. By changing the scaling of \tilde{w} , we may assume that $G(0, 0) = 1$. In §4, we will prove Theorem 1.1 by showing that one may assume that $G \equiv 1$ by changing coordinate functions appropriately.

2.3. Preliminaries on a cycle of rational curves in general. Let C be a cycle of rational curves in general. Denote by $n = n(C)$ the number of the irreducible components of C . Here we treat the case where $n \geq 2$. Denote by $\{C_{(\nu)}\}_{\nu=1}^n$ the set of all irreducible components of C . We sometimes use the notation $C_{(0)} := C_{(n)}$. We may assume that $C_{(\nu)} \cap C_{(\mu)} \neq \emptyset$ if and only if $\nu - \mu = \pm 1$ modulo n . It holds that $H^1(C, \mathcal{O}_C) = \mathbb{C}$ also in this case, since $H^1(C, i_*\mathcal{O}_{\tilde{C}}) = H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) = 0$ follows from the same exact sequence as we considered in §2.1.2, where $i: \tilde{C} \rightarrow C$ is the normalization (Note that the higher direct images vanish for i , since i is a finite morphism). Thus $\text{Pic}^0(C)$ is naturally identified with $H^1(C, \mathbb{C}^*) = \mathbb{C}^*$ also in this case.

The following example can be regarded as the standard model of a neighborhood of C .

EXAMPLE 2.4. Let $\{(\tilde{V}_{(\nu)}, \tilde{C}_{(\nu)}, \tilde{V}_{0(\nu)}^\pm, S_{(\nu)}, T_{(\nu)}, \xi_{0(\nu)}, \xi_{\infty(\nu)})\}_{\nu=1}^n$ be n -copies of $(\tilde{V}, \tilde{C}, \tilde{V}_0^\pm, S, T, \xi_0, \xi_\infty)$ in Example 2.1. Define a biholomorphism $F_{\nu+1, \nu}: \tilde{V}_{0(\nu+1)}^+ \rightarrow \tilde{V}_{0(\nu)}^-$ by $(F_{\nu+1, \nu})^*(T_{(\nu)}, \xi_{\infty(\nu)}) = (t_{\nu+1, \nu} \cdot \xi_{0(\nu)}, S_{(\nu)})$ for $\nu = 0, 1, \dots, n-1$, where $t_{\nu+1, \nu} \in U(1)$ and

$$(\tilde{V}_{(0)}, \tilde{C}_{(0)}, \tilde{V}_{0(0)}^\pm, S_{(0)}, T_{(0)}, \xi_{0(0)}, \xi_{\infty(0)}) := (\tilde{V}_{(n)}, \tilde{C}_{(n)}, \tilde{V}_{0(n)}^\pm, S_{(n)}, T_{(n)}, \xi_{0(n)}, \xi_{\infty(n)}).$$

Let $i: \tilde{V} := \coprod_{\nu=1}^n \tilde{V}_\nu \rightarrow V$ be the quotient by the relation induced by $F_{\nu+1, \nu}$'s. Denote by C the image $i(\tilde{C})$, where $\tilde{C} := \coprod_{\nu=1}^n \tilde{C}_{(\nu)}$. Then C is a cycle of n rational curves embedded in V with $t(N_{C/V}) = \prod_{\nu=0}^{n-1} t_{\nu+1, \nu}$.

REMARK 2.5. It follows from the same argument as in §2.2 that one can construct a finite covering map $i: \tilde{V} \rightarrow V$ as in Example 2.4 for a small neighborhood V of C also in the case where C consists of n irreducible components ($n \geq 2$). In this case, \tilde{V} is the disjoint union of a neighborhood $\tilde{V}_{(\nu)}$ of each irreducible component $C_{(\nu)}$ of C with the same local coordinates $(S_{(\nu)}, T_{(\nu)}, \xi_{0(\nu)}, \xi_{\infty(\nu)})$ as in Example 2.4 (Here we use Grauert's theorem [G] again). In general, the gluing morphism $F_{\nu+1, \nu}: \tilde{V}_{0(\nu+1)}^+ \rightarrow \tilde{V}_{0(\nu)}^-$ needs not to coincide with the one in Example 2.4. From the same argument as in §2.2 by using [K2, Theorem 1.4], it follows that, by choosing $S_{(\nu)}, T_{(\nu)}, \xi_{0(\nu)}$ and $\xi_{\infty(\nu)}$ suitably, we may assume that

$$(F_{\nu+1, \nu})^*(T_{(\nu)}, \xi_{\infty(\nu)}) = \left(\frac{t_{\nu+1, \nu} \cdot \xi_{0(\nu)}}{G_{\nu+1, \nu}(S_{(\nu)}, \xi_{0(\nu)})}, G_{\nu+1, \nu}(S_{(\nu)}, \xi_{0(\nu)}) \cdot S_{(\nu)} \right)$$

holds for a nowhere vanishing holomorphic function $G_{\nu+1, \nu}$ defined on $\tilde{V}_{0(\nu+1)}^+$ and a constant $t_{\nu+1, \nu} \in U(1)$. In §4, we will prove Theorem 1.1 by showing that one may assume that $G_{\nu+1, \nu} \equiv 1$ by changing the coordinate functions appropriately.

EXAMPLE 2.6. Fix a plane cubic $C_0 \subset \mathbb{P}^2$ which admits only nodal singularities and consists of two irreducible components, say $C_0^{(1)}$ and $C_0^{(2)}$. One may assume that $C_0^{(\nu)}$ is of degree ν for $\nu = 1, 2$. Take three points $\{p_1, p_2, p_3\} \subset C_0^{(1)} \cap (C_0)_{\text{reg}}$ and six points $\{p_4, p_5, \dots, p_9\} \subset C_0^{(2)} \cap (C_0)_{\text{reg}}$. Denote by $\pi: S \rightarrow \mathbb{P}^2$ the blow-up at $Z := \{p_1, p_2, \dots, p_9\}$ and by C the strict transform $(\pi^{-1})_* C_0$. Then it is known that, by taking a normalization $i: \mathbb{P}^1 \amalg \mathbb{P}^1 \rightarrow C_0$ with $i^{-1}((C_0)_{\text{sing}}) = \{0, \infty\}$ appropriately, the complex constant $t = t(N_{C/S}) \in \mathbb{C}^*$ can be calculated by $t = \prod_{\nu=1}^9 i^{-1}(p_\nu) \in \mathbb{C}^* = H^1(C_0, \mathbb{C}^*)$, where we are identifying C_0 and C via π .

EXAMPLE 2.7. Fix a plane cubic $C_0 \subset \mathbb{P}^2$ which admits only nodal singularities and consists of three irreducible components, say $C_0^{(1)}$, $C_0^{(2)}$ and $C_0^{(3)}$. Each $C_0^{(\nu)}$ is a line for $\nu = 1, 2, 3$. Take three points $\{p_1, p_2, p_3\} \subset C_0^{(1)} \cap (C_0)_{\text{reg}}$, $\{p_4, p_5, p_6\} \subset C_0^{(2)} \cap (C_0)_{\text{reg}}$, and $\{p_7, p_8, p_9\} \subset C_0^{(3)} \cap (C_0)_{\text{reg}}$. Denote by $\pi: S \rightarrow \mathbb{P}^2$ the blow-up at $Z := \{p_1, p_2, \dots, p_9\}$ and by C the strict transform $(\pi^{-1})_* C_0$. Then it is known that, by taking a normalization $i: \mathbb{P}^1 \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 \rightarrow C_0$ with $i^{-1}((C_0)_{\text{sing}}) = \{0, \infty\}$ appropriately, the complex constant $t = t(N_{C/S}) \in \mathbb{C}^*$ can be calculated by $t = \prod_{\nu=1}^9 i^{-1}(p_\nu) \in \mathbb{C}^* = H^1(C_0, \mathbb{C}^*)$, where we are identifying C_0 and C via π .

Note that each point of $\text{Pic}^0(C_0)$ is attained by choosing appropriate nine points configuration Z in Examples 2.6 and 2.7 (as in Example 2.2).

3. INJECTIVITY OF THE RESTRICTION $\lim_{V^* \rightarrow} H^1(V^*, \mathcal{O}_{V^*}) \rightarrow H^1(C, \mathcal{O}_C)$

We will show Theorem 1.1 in §4 by the strategy as we mentioned in §2.2 and Remark 2.5. When C is a rational curve with a node, for example, we will choose suitable coordinates of \tilde{V} so that $G \equiv 1$ holds. Consider the composition g of the natural biholomorphism $V_0 \rightarrow \tilde{V}_0^+$ and the branch of $\frac{1}{2\pi\sqrt{-1}} \log G$ such that $g(0, 0) = 0$. By the arguments we will explain the details in §4, the problem can be reduced to showing that the cohomology class $\alpha := [\{(V^+, -g|_{V^+}), (V^-, 0)\}] \in \check{H}^1(\{V_j\}, \mathcal{O}_V)$ is trivial. As it is easily observed that $\alpha|_C = 0 \in H^1(C, \mathcal{O}_C)$ (see the proof of “Proposition 3.3 \Rightarrow Proposition 3.2” below), it is sufficient to show the injectivity of the restriction $H^1(V, \mathcal{O}_V) \rightarrow H^1(C, \mathcal{O}_C)$ by shrinking V in a suitable sense. For such a purpose, we will show the following:

PROPOSITION 3.1. *Let C be a cycle of a curve embedded in non-singular surface V such that the normal bundle $N_{C/V}$ is topologically trivial and satisfies Diophantine condition as in Theorem 1.1. For any element α of the kernel of the restriction $H^1(V, \mathcal{O}_V) \rightarrow H^1(C, \mathcal{O}_C)$, there exists a neighborhood V^* of C such that $\alpha|_{V^*} = 0 \in H^1(V^*, \mathcal{O}_{V^*})$.*

3.1. Proof of Proposition 3.1 when C is a rational curve with a node.

3.1.1. *Notation and statement in this case.* Assume that C is a rational curve with a node. Then we can use the notation as in §2.2. By a simple argument, Proposition 3.1 can be reworded as follows:

PROPOSITION 3.2. *Let F_+ and F_- be holomorphic functions defined on V^+ and V^- , respectively, such that $\{(U^\pm, F_\pm|_{U^\pm})\}$ extends to a holomorphic function defined on U_0 . Then there exists a neighborhood V^* of C such that the class*

$$\alpha := [\{(V^* \cap V^+, F_+|_{V^* \cap V^+}), (V^* \cap V^-, F_-|_{V^* \cap V^-})\}] \in \check{H}^1(\{V^* \cap V_j\}, \mathcal{O}_{V^*})$$

is trivial.

As will be proven immediately after, Proposition 3.2 follows from:

PROPOSITION 3.3. *Let F_+ and F_- be holomorphic functions defined on V^+ and V^- , respectively, such that $F_{\pm}|_{U^{\pm}} \equiv 0$. Then there exists a neighborhood $V^* \subset V$ of C such that the Čech cohomology class*

$$\{[(V^* \cap V^+, F_+|_{V^* \cap V^+}), (V^* \cap V^-, F_-|_{V^* \cap V^-})]\} \in \check{H}^1(\{V^* \cap V_j\}, \mathcal{O}_{V^*}(-C))$$

is trivial.

Proof of “Proposition 3.3 \Rightarrow Proposition 3.2”. Denote by g_0 the extension of $\{(U^{\pm}, F_{\pm}|_{U^{\pm}})\}$ to U_0 . As V_0 is covered by a Stein neighborhood of U_0 , we obtain a holomorphic function G_0 on V_0 such that $G_0|_{U_0} = g_0$. By using a function G_1 on V_1 defined by $G_1 \equiv 0$, consider

$$\beta := \{[(V^+, (G_0 - G_1)|_{V^+}), (V^-, (G_0 - G_1)|_{V^-})]\} \in \check{H}^1(\{V_j\}, \mathcal{O}_V).$$

Then it follows from Proposition 3.3 that the class $(\alpha - \beta) \in \check{H}^1(\{V^* \cap V_j\}, \mathcal{O}_{V^*}(-C))$ is trivial for a neighborhood V^* of C , which proves Proposition 3.2. \square

Here we first give some notation which will be used in the proof of Proposition 3.3. Let V_0^* be a small neighborhood of the nodal point in V_0 . Denote by x the holomorphic function obtained by pulling back the function S by the natural biholomorphism $V_0 \rightarrow \tilde{V}_0^+$, and by y the one obtained by pulling back the function T by the natural biholomorphism $V_0 \rightarrow \tilde{V}_0^-$. We regard (x, y) as coordinates of a neighborhood of $\overline{V_0^*}$. Note that $x \cdot y$ is a local defining function of C in this locus. For sufficiently small positive constants ε and δ , we may assume that $V_0^* := \{(x, y) \in V_0 \mid \max\{|x|, |y|\} < 2\varepsilon, |w_0| < \delta\}$. Denote by U_0^* the subset $V_0^* \cap C$: i.e. $U_0^* = \{(x, y) \in V_0^* \mid |x| < 2\varepsilon, y = 0\} \cup \{(x, y) \in V_0^* \mid x = 0, |y| < 2\varepsilon\}$. In what follows we always assume that ε and δ are sufficiently small so that V_0^* is a relatively compact subset of V_0 .

Next we give a definition of a relatively compact subset V_1^* of V_1 . Denote by z the holomorphic function obtained by pulling back the function S by the natural biholomorphism $V_1 \rightarrow \tilde{V}_1$. Denote by V_1^* the subset $\{(z, w_1) \in V_1 \mid \varepsilon < |z| < 1/\varepsilon, |w_1| < \delta\}$, where we are regarding (z, w_1) as coordinates of this locus. Let U_1^* be the subset of U_1 defined by $U_1^* := V_1^* \cap C$: i.e. $U_1^* = \{(z, w_1) \in V_1^* \mid \varepsilon < |z| < 1/\varepsilon, w_1 = 0\}$. Set $U_+^* := U_0^+ \cap U_1^* = \{(x, y) \in V_0^* \mid \varepsilon < |x| < 2\varepsilon, y = 0\}$ and $U_-^* := U_0^- \cap U_1^* = \{(x, y) \in V_0^* \mid x = 0, \varepsilon < |y| < 2\varepsilon\}$. Denote by V_{\pm}^* the connected components of $V_0^* \cap V_1^*$ which includes U_{\pm}^* respectively, and by V^* the subset $V_0^* \cup V_1^* = i(\{|w| < \delta\})$.

In what follows, we fix ε and do not vary this value any more, whereas we will shrink δ as necessary.

3.1.2. Outline of the proof of Proposition 3.3. We will construct holomorphic functions F_j on V_j^* for each $j = 0, 1$ such that $F_0|_{V_{\pm}^*} - F_1|_{V_{\pm}^*} = F_{\pm}|_{V_{\pm}^*}$ holds on each V_{\pm}^* by shrinking δ . Actually, it is sufficient to construct such $\{(V_j^*, F_j)\}$, since we can construct $\widehat{F}_j: V^* \cap V_j \rightarrow \mathbb{C}$ such that $\delta\{(V^* \cap V_j, \widehat{F}_j)\} = \{(V^* \cap V^{\pm}, F_{\pm}|_{V^* \cap V^{\pm}})\}$ from them as follows: Set $\widehat{F}_j(p) := F_j(p)$ for $p \in V_j^*$. For $p \in (V^* \cap V_j) \setminus V_j^*$, set

$$\widehat{F}_j(p) := \begin{cases} F_{1-j}(p) + (-1)^j \cdot F_+(p) & (\text{if } p \in V^+) \\ F_{1-j}(p) + (-1)^j \cdot F_-(p) & (\text{if } p \in V^-). \end{cases}$$

Note that $p \in V_{1-j}^*$, and thus it holds that $p \in V_j \cap V_{1-j}^* \subset V_0 \cap V_1 = V^+ \cup V^-$ in this case.

3.1.3. *Proof of Proposition 3.3 (Step 1: Construction of F_j 's as formal power series).* In this step, we will construct F_j 's in the form of

$$F_0(x, y) = \sum_{\nu=1}^{\infty} a_{0,\nu}(x, y) \cdot w_0^\nu$$

and

$$F_1(z, w_1) = \sum_{\nu=1}^{\infty} a_{1,\nu}(z) \cdot w_1^\nu$$

formally. Here $a_{1,\nu}$ is a function defined on U_1^* , which is also be regarded as a function on V_1^* by pulling back the natural projection $(z, w_1) \mapsto z$. Similarly, $a_{0,\nu}$ is a function defined on U_0^* with

$$a_{0,\nu} = \begin{cases} p_\nu + r_\nu & (\text{if } p \in U^+ \cap U_0^*) \\ q_\nu + r_\nu & (\text{if } p \in U^- \cap U_0^*), \end{cases}$$

where $p_\nu(x)$ is a holomorphic function on U_+^* with $p_\nu(0) = 0$, $q_\nu(y)$ is a holomorphic function on U_-^* with $q_\nu(0) = 0$, and $r_\nu \in \mathbb{C}$ is a constant. We also regard $a_{0,\nu}$ as a function defined on V_0^* by setting $a_{0,\nu}(x, y) := p_\nu(x) + q_\nu(y) + r_\nu$, where p_ν and q_ν are extended by considering the pull-back by the projection $(x, y) \mapsto x$ and $(x, y) \mapsto y$, respectively. Denote by

$$F_\pm(z, w_1) = \sum_{\nu=1}^{\infty} b_{\pm,\nu}(z) \cdot w_1^\nu$$

the expansion of F_\pm by w_1 on V_\pm^* .

First, let us construct $\{a_{j,1}\}_{j=0,1}$. As $N_{C/S}$ is non-torsion, it holds that $\check{H}^1(\{U_j\}, N_{C/S}^{-1}) = 0$ (see §2.1.2). Therefore, by considering the 1-cocycle $[\{(U_\pm^*, b_{\pm,1})\}] \in \check{H}^1(\{U_j^*\}, N_{C/S}^{-1})$, one can take $\{a_{j,1}\}$ such that

$$\begin{cases} t_+^{-1} a_{0,1}(z) - a_{1,1}(z) = b_{+,1}(z) & (\text{on } U_+^*) \\ t_-^{-1} a_{0,1}(z) - a_{1,1}(z) = b_{-,1}(z) & (\text{on } U_-^*). \end{cases}$$

Note that such $\{a_{j,1}\}$ is unique since $H^1(C, N_{C/S}^{-1}) = 0$. By letting r_1 be that value of $a_{0,1}$ at the nodal point, p_1 and q_1 are uniquely determined. We here remark that, for any choice of the other coefficients $\{a_{j,\nu}\}_{j=0,1,\nu \geq 2}$, we have that

$$F_0 - F_1 = \begin{cases} F_+ + O(w_1^2) & (\text{on } V_+^*) \\ F_- + O(w_1^2) & (\text{on } V_-^*) \end{cases}$$

holds as $w_1 \rightarrow 0$.

Next, we construct $\{a_{j,n+1}\}$ by assuming that $\{a_{j,\nu}\}_{j=0,1,\nu \leq n}$ is determined so that the following inductive assumption holds: for any choice of $\{a_{j,\nu}\}_{j=0,1,\nu \geq n+1}$,

$$F_0 - F_1 = \begin{cases} F_+ + O(w_1^{n+1}) & (\text{on } V_+^*) \\ F_- + O(w_1^{n+1}) & (\text{on } V_-^*) \end{cases}$$

holds as $w_1 \rightarrow 0$. In what follows, we regard $\{a_{j,\nu}\}_{j=0,1,\nu \geq n+1}$ as unknown functions. Denote by

$$p_\nu(x(z, w_1)) = \begin{cases} p_\nu(x(z)) & (\text{on } V_+^*) \\ \sum_{\lambda=1}^{\infty} P_{\nu,\lambda}^-(z) \cdot w_1^\lambda & (\text{on } V_-^*) \end{cases}$$

and

$$q_\nu(y(z, w_1)) = \begin{cases} \sum_{\lambda=1}^{\infty} Q_{\nu,\lambda}^+(z) \cdot w_1^\lambda & (\text{on } V_+^*) \\ q_\nu(y(z)) & (\text{on } V_-^*) \end{cases}$$

the expansion of p_ν and q_ν by w_1 respectively (Note that $x = x(z, w_1)$ and $y = y(z, w_1)$ do not depend on w_1 on V_+^* and V_-^* , respectively, in our coordinates, and that $q_\nu|_{U^+} \equiv q_\nu(0) = 0$ and $p_\nu|_{U^-} \equiv p_\nu(0) = 0$).

On V_+ , one can expand $F_0|_{V_+^*}$ as follows:

$$F_0|_{V_+^*} = \sum_{\nu=1}^{\infty} a_{0,\nu}(x, y) \cdot w_0^\nu = \sum_{\nu=1}^{\infty} t_+^{-\nu} \cdot \left(p_\nu(x(z)) + \sum_{\lambda=1}^{\infty} Q_{\nu,\lambda}^+(z) \cdot w_1^\lambda + r_\nu \right) \cdot w_1^\nu.$$

By setting

$$h_m^+(z) := \sum_{\nu=1}^{m-1} t_+^{-\nu} \cdot Q_{\nu,m-\nu}^+(z),$$

we have that the coefficient of w_1^m in the expansion of $F_0|_{V_+^*}$ is $h_m^+(z) + t_+^{-m} (p_m(x(z)) + r_m)$. The function h_m^+ can be regarded as a function obtained by pulling back a function on U_+^* by the local projection $(z, w_1) \mapsto z$, which coincides with $(x, y) \mapsto x$ in this locus. Note that $\{h_m^+\}_{m \leq n}$ are regarded as known functions, since h_m^+ depends only on the data $\{q_\nu\}_{\nu=1}^{m-1}$. By a simple observation, it turns out that one should construct $a_{j,n+1}$'s so that

$$b_{+,n+1}(z) = h_{n+1}^+(z) + t_+^{-n-1} (p_{n+1}(x(z)) + r_{n+1}) - a_{1,n+1}(z)$$

holds on U_+^* in order for the inductive assumption to hold for $n+1$.

Similarly, we have that

$$F_0|_{V_-^*} = \sum_{\nu=1}^{\infty} a_{0,\nu}(x, y) \cdot w_0^\nu = \sum_{\nu=1}^{\infty} t_-^{-\nu} \cdot \left(\sum_{\lambda=1}^{\infty} P_{\nu,\lambda}^-(z) \cdot w_1^\lambda + q_\nu(y(z)) + r_\nu \right) \cdot w_1^\nu$$

on V_-^* . By setting

$$h_m^-(z) := \sum_{\nu=1}^{m-1} t_-^{-\nu} \cdot P_{\nu,m-\nu}^-(z),$$

we have that the coefficient of the expansion of $h_m^-(z)$ in w_1^m is $h_m^-(z) + t_-^{-m} (q_m(y(z)) + r_m)$. The function h_m^- can be regarded as a function obtained by pulling back a function on U_-^* by the local projection $(z, w_1) \mapsto z$, which coincides with $(x, y) \mapsto y$ in this locus. Note that $\{h_m^-\}_{m \leq n}$ are regarded as known functions, since h_m^- depends only on the data $\{p_\nu\}_{\nu=1}^{m-1}$. By a simple observation, it turns out that one should construct $a_{j,n+1}$'s so that

$$b_{-,n+1}(z) = h_{n+1}^-(z) + t_-^{-n-1} \cdot (q_{n+1}(y(z)) + r_{n+1}) - a_{1,n+1}(z)$$

holds on U_-^* in order for the inductive assumption to hold for $n+1$.

By the observations above, we have that $b_{\pm,n+1}(z) - h_{n+1}^\pm(z)$ is known function after we finish defining $\{a_{j,\nu}\}_{j=0,1,\nu \leq n}$. Therefore, we can define $\{(U_0^*, a_{0,n+1}(x, y) = p_{n+1}(x) + q_{n+1}(y) + r_n), (U_1^*, a_{1,n+1}(z))\}$ by considering the equations

$$\begin{cases} t_+^{-n-1} a_{0,n+1}(z) - a_{1,n+1}(z) = b_{+,n+1}(z) - h_{n+1}^+(z) & (\text{on } U_+^*) \\ t_-^{-n-1} a_{0,n+1}(z) - a_{1,n+1}(z) = b_{-,n+1}(z) - h_{n+1}^-(z) & (\text{on } U_-^*). \end{cases}$$

As $H^0(C, N_{C/S}^-) = H^1(C, N_{C/S}^-) = 0$ (see §2.1.2), we actually have the unique solution.

3.1.4. *Proof of Proposition 3.3: (Step 2: Estimate of the coefficient functions).* As $V_{\pm}^* \Subset V^{\pm}$, there exists a constant M such that

$$\max \left\{ \sup_{V_+^*} |F_+|, \sup_{V_-^*} |F_-| \right\} < M.$$

In what follows, we assume that $M > 1$. Fix a positive constant R sufficiently larger than $1/\delta, 1/\varepsilon, \sup_{V_+^*} |w_0/y|, \sup_{V_+^*} |w_1/y|, \sup_{V_+^*} |w_1/y|, \sup_{V_-^*} |w_0/x|, \sup_{V_-^*} |w_1/x|$, and the inverses of these. Then we may assume that

$$\{(z, w_1) \mid \varepsilon < |z| < 2\varepsilon, |w_1| = 1/R\} \subset V_+^*$$

and

$$\{(z, w_1) \mid 1/(2\varepsilon) < |z| < 1/\varepsilon, |w_1| = 1/R\} \subset V_-^*$$

hold (see also Remark 3.4).

Let $B(X) = X + \sum_{\nu=2}^{\infty} B_{\nu} X^{\nu}$ be the formal power series defined by

$$(1) \quad \sum_{\nu=2}^{\infty} |1 - t^{\nu-1}| \cdot B_{\nu} X^{\nu} = KRM \frac{B(X)^2}{1 - RB(X)},$$

where the constant K is a positive real number as in Lemma 3.5. Note that it follows from the argument in [Sie] that $B(X)$ has a positive radius of convergence (see also [U83, Lemma 5]). Define a convergent power series $A(X) = \sum_{\nu=1}^{\infty} A_{\nu} X^{\nu}$ by $A_n := B_{n+1}$ ($n \geq 1$): i.e. $B(X) = X + XA(X)$. In this step, we show that

$$(2) \quad \max_{j=0,1} \sup_{p \in U_j^*} |a_{j,\nu}(p)| \leq A_{\nu}$$

holds for each ν by induction.

First, by Cauchy's inequality, we have that

$$\sup_{z \in U_{\pm}^*} |b_{\pm,1}(z)| \leq M \cdot R.$$

Therefore, the inequality (2) for $\nu = 1$ follows from Lemma 3.5 below.

Next we show the inequality (2) for $\nu = n + 1$ by assuming that it holds for $\nu = 1, 2, \dots, n$. As it holds that $|h_{n+1}^+(z)| \leq \sum_{\nu=1}^n |Q_{\nu, n+1-\nu}^+(z)|$, we have that

$$\sup_{U_+^*} |Q_{\nu,\lambda}^+| \leq A_{\nu} \cdot R^{\lambda}$$

holds by Cauchy's inequality. Therefore it follows that

$$\sup_{U_+^*} |h_{n+1}^+(z)| \leq \sum_{\nu=1}^n A_{\nu} \cdot R^{n+1-\nu} = \text{the coefficient of } X^{n+1} \text{ in the expansion of } \frac{RXA(X)}{1 - RX}.$$

Note that the same estimate holds also for h_{n+1}^- . As it holds that

$$\sup_{z \in U^+ \cap U_{\pm}^*} |b_{\pm, n+1}(z)| \leq MR^{n+1} = \text{the coefficient of } X^{n+1} \text{ in the expansion of } \frac{MRX}{1 - RX},$$

it follows from Lemma 3.5 that

$$\max_{j=0,1} \sup_{U_j^*} |a_{j, n+1}| \leq \text{the coefficient of } X^{n+1} \text{ in the expansion of } \frac{1}{|1 - t^{n+1}|} \cdot \frac{KRX(A(X) + M)}{1 - RX}.$$

As $M \geq 1$, we have that

$$\begin{aligned}
& \text{the coefficient of } X^{n+1} \text{ in the expansion of } \frac{1}{|1-t^{n+1}|} \cdot \frac{KRX(A(X)+M)}{1-RX} \\
\leq & \text{the coefficient of } X^{n+1} \text{ in the expansion of } \frac{1}{|1-t^{n+1}|} \cdot \frac{KRMB(X)}{1-RX} \\
= & \text{the coefficient of } X^{n+2} \text{ in the expansion of } \frac{1}{|1-t^{n+1}|} \cdot \frac{KRMB(X)}{1-RX} \\
\leq & \text{the coefficient of } X^{n+2} \text{ in the expansion of } \frac{KRM}{|1-t^{n+1}|} \cdot \frac{B(X)^2}{1-RB(X)}.
\end{aligned}$$

Thus we have the inequality (2) for $\nu = n + 1$ by the equation (1).

3.1.5. *Proof of Proposition 3.3 (Step 3: Convergence of F_j 's).* Let us shrink δ so that it is smaller than the radius of convergence of the poser series $A(X)$. Then it clearly holds that $\sup_{V_1^*} |a_{1,\nu}| \leq A_\nu$ when we regard $a_{1,\nu}$ as a function V_1^* by the rule we mentioned above. For $(x, y) \in V_0^*$,

$$\begin{aligned}
|a_{0,\nu}(x, y)| &= |p_\nu(x) + q_\nu(y) + r_\nu| \leq |p_\nu(x) + r_\nu| + |q_\nu(y) + r_\nu| + |r_\nu| \\
&\leq \sup_{x \in U^+ \cap U_0^*} |a_{0,\nu}(x)| + \sup_{y \in U^- \cap U_0^*} |a_{0,\nu}(y)| + |a_{0,\nu}(0, 0)| \leq 3A_\nu.
\end{aligned}$$

Thus we can regard F_j as a holomorphic function defined on V_j^* . By construction, we have that $F_0|_{V_\pm^*} - F_1|_{V_\pm^*} = F_\pm|_{V_\pm^*}$ holds on V_\pm^* . \square

REMARK 3.4. In Step 2 of the proof above, we applied Cauchy's inequality in several times, in which we used the fact that the circle $\{(z, w_1) \in V_1^* \mid z = z_0, |w_1| = 1/R\}$ is included in V_0^* for each $z_0 \in U_\pm^*$. For this, we need to choose V_j^* 's and its coordinates appropriately as we did in §2.2 and at the beginning of the proof. One of the most important property of our coordinates is that the projection $(z, w_1) \mapsto z$ coincides with $(x, y) \mapsto x$ on V_+^* and with $(x, y) \mapsto y$ on V_-^* . On the other hand, we used an open covering of a neighborhood of C taken by using a general theory (Siu's theorem [Siu]) in [K2]. Here we had to refine and shrink the open sets in order to take R as a constant, see also [K2, Remark 4.3]. We here remark that one can slightly simplify the proof of [K2, Theorem 1.4] by replacing the open covering with $\{V_j^*\}$ we used in the present paper.

LEMMA 3.5 ([K2, §4.2.3, 4.2.4]). *Let n be a positive integer, b_\pm a holomorphic function on U_\pm^* , and a_j be a function on U_j^* for $j = 0, 1$ such that*

$$\begin{cases} t_+^{-n} \cdot a_0 - a_1 = b_+ & (\text{on } U_+^*) \\ t_-^{-n} \cdot a_0 - a_1 = b_- & (\text{on } U_-^*). \end{cases}$$

Then there exists a constant $K = K(C, \{U_j^\})$ which does not depend on neither n , a_j nor b_\pm such that*

$$\max_{j=0,1} \sup_{U_j^*} |a_j| \leq \frac{K}{|1-t^n|} \cdot \max \left\{ \sup_{x \in U_+^*} |b_+(x)|, \sup_{y \in U_-^*} |b_-(y)| \right\}$$

holds.

In the rest of this subsection, we give a proof of Lemma 3.5 for the convenience of the reader, although its statement is nothing but a summary of some arguments in [K2, §4.2.3, 4.2.4] intrinsically. Note that $t^n \neq 1$ and $H^0(C, N_{C/S}^{-n}) = H^1(C, N_{C/S}^{-n}) = 0$ hold (as we mentioned in §2.1.2), since $N_{C/S}$ is non-torsion. Therefore, a_j 's are uniquely determined by b_\pm .

Proof of Lemma 3.5. Set $M := \max \left\{ \sup_{x \in U_+^*} |b_+(x)|, \sup_{y \in U_-^*} |b_-(y)| \right\}$. Let r be the value of a_0 at the nodal point. Then there uniquely exists a function p on $U^+ \cap U_0^*$ and q on $U^- \cap U_0^*$ such that

$$a_0 = \begin{cases} p + r & (\text{on } U^+ \cap U_0^*) \\ q + r & (\text{on } U^- \cap U_0^*). \end{cases}$$

Define 1-forms ω_0 and ω_1 by

$$\omega_0 := da_0 = \begin{cases} p'(x)dx & (\text{on } U^+ \cap U_0^*) \\ q'(y)dy & (\text{on } U^- \cap U_0^*) \end{cases}$$

and $\omega_1 := da_1 = a'_1(z)dz$. By the assumption, we have that $t_\pm^{-n} \cdot \omega_0 - \omega_1 = db_\pm (= b'_\pm(z)dz)$ on U_\pm^* . Define a new open covering $\{U_j^{**}\}$ by

$$U_0^{**} := \left\{ (x, y) \in U_0^* \mid \max\{|x|, |y|\} < \frac{5\varepsilon}{3} \right\}, \quad U_1^{**} := \left\{ (z, 0) \in U_1^* \mid \frac{4\varepsilon}{3} < |z| < \frac{3}{4\varepsilon} \right\}.$$

As $U_j^{**} \Subset U_j^*$, we have that

$$\sup_{z \in U^\pm \cap U_{01}^{**}} |b'_\pm(z)| \leq K_1 \cdot M$$

holds on a constant $K_1 > 0$, where $U_{01}^{**} := U_0^{**} \cap U_1^{**}$. By Lemma 3.6, we have that

$$\max \left\{ \sup_{x \in U_0^{**} \cap U^+} |p'(x)|, \sup_{y \in U_0^{**} \cap U^-} |q'(y)|, \sup_{z \in U_1^{**}} |a'_1(z)| \right\} \leq K_0 K_1 M$$

holds for a constant K_0 . By considering the path integral from the nodal point, we have that

$$\max \left\{ \sup_{x \in U_0^{**} \cap U^+} |p(x)|, \sup_{y \in U_0^{**} \cap U^-} |q(y)| \right\} \leq \frac{5\varepsilon}{3} K_0 K_1 M.$$

By fixing point z_\pm from $U_{01}^{**} \cap U^\pm$ and letting $C_\pm := b_\pm(z_\pm)$ and $C_1 := a_1(z_+)$, respectively, we have that

$$b_\pm(z) = C_\pm + \int_{z_\pm}^z b'_\pm(\zeta) d\zeta, \quad a_1(z) = C_1 + \int_{z_+}^z a'_1(\zeta) d\zeta.$$

Note that

$$\sup_{z \in U_1^{**}} \left| \int_{z_+}^z a'_1(\zeta) d\zeta \right| \leq K_2 \cdot \sup_{z \in U_1^{**}} |a'_1(z)| \leq K_0 K_1 K_2 M$$

holds for a constant K_2 which depends only on the diameter of U_1^{**} (or equivalently, only on ε). As it follows

$$\begin{cases} t_+^{-n} \cdot (-p(z_+) + r) - C_1 = C_+ \\ t_-^{-n} \cdot (-q(z_-) + r) - \left(\int_{z_+}^{z_-} a'_1(z) dz + C_1 \right) = C_-, \end{cases}$$

we have that $r = \frac{1}{t_+^{-n} - t_-^{-n}} \cdot (D_+ - D_-)$ and $C_1 = \frac{1}{t_+^{-n} - t_-^{-n}} \cdot (t_-^{-n} D_+ - t_+^{-n} D_-)$, where $D_+ := t_+^{-n} p(z_+) + C_+$ and $D_- := t_-^{-n} q(z_-) + C_- + \int_{z_+}^{z_-} a'_1(\zeta) d\zeta$. Note that

$$|D_+| \leq |p(z_+)| + |C_+| \leq \left(1 + \frac{5\varepsilon}{3} K_0 K_1\right) M$$

and

$$|D_-| \leq |q(z_-)| + |C_-| + \sup_{z \in U_1^{**}} \left| \int_{z_+}^z a'_1(\zeta) d\zeta \right| \leq \left(1 + \frac{5\varepsilon}{3} K_0 K_1 + K_0 K_1 K_2\right) M.$$

Let us denote by K_3 the constant $2 + \frac{10\varepsilon}{3} K_0 K_1 + K_0 K_1 K_2$. Then it follows from the arguments above that

$$\sup_{z \in U_1^{**}} |a_1(z)| \leq |C_1| + \sup_{z \in U_1^{**}} \left| \int_{z_+}^z a'_1(\zeta) d\zeta \right| \leq K_3 \cdot \left(1 + \frac{1}{|1 - t^n|}\right) \cdot M$$

and

$$\sup_{z \in U_0^{**}} |a_0(z)| \leq K_3 \cdot \left(1 + \frac{1}{|1 - t^n|}\right) \cdot M.$$

Thus we have

$$\max_{j=0,1} \sup_{U_j^{**}} |a_j| < \frac{3K_3}{|1 - t^n|} \cdot M.$$

When $z \in U_1^* \setminus U_1^{**}$, it holds that $z \in U^+ \cap U_0^{**}$ or $z \in U^- \cap U_0^{**}$. In the former case, we have that

$$|a_1(z)| = |t_+^{-n} a_0(z) - b_+(z)| \leq |a_0(z)| + |b_+(z)| \leq \left(1 + \frac{3K_3}{|1 - t^n|}\right) \cdot M.$$

By the same arguments for the other cases, the lemma follows by letting $K := 2 + 3K_3$. \square

LEMMA 3.6. *Let n be a positive integer and $i: \widetilde{C} \rightarrow C$ be the normalization such that the preimage of the nodal point is $\{0, \infty\} \subset \mathbb{P}^1 = \widetilde{C}$. Denote by \widetilde{U}_j^{**} the preimage $i^{-1}(U_j^{**})$ and \widetilde{U}^\pm the preimage $i^{-1}(U^\pm)$. Let η_\pm be 1-forms on $\widetilde{U}^\pm \cap \widetilde{U}_{01}^{**}$ such that the Čech cohomology class $\{(\widetilde{U}^\pm \cap \widetilde{U}_{01}^{**}, \eta_\pm)\} \in \check{H}^1(\{\widetilde{U}_j^{**}\}, K_{\widetilde{C}} \otimes i^* N_{C/S}^{-n})$ is trivial. Denote by ω_j the 1-form on \widetilde{U}_j^{**} for $j = 0, 1$ uniquely determined by*

$$\begin{cases} t_+^{-n} \cdot \omega_0 - \omega_1 = \eta_+ & (\text{on } \widetilde{U}^+ \cap \widetilde{U}_{01}^{**}) \\ t_-^{-n} \cdot \omega_0 - \omega_1 = \eta_- & (\text{on } \widetilde{U}^- \cap \widetilde{U}_{01}^{**}). \end{cases}$$

*Then there exists a constant $K_0 = K_0(C, \{U_j^{**}\})$ which does not depend on neither n nor η_\pm such that*

$$\begin{aligned} & \max \left\{ \sup_{x \in \widetilde{U}_0^{**} \cap \widetilde{U}^+} |g_0^+(x)|, \sup_{y \in \widetilde{U}_0^{**} \cap \widetilde{U}^-} |g_0^-(y)|, \sup_{z \in \widetilde{U}_1^{**}} |g_1(z)| \right\} \\ & \leq K_0 \cdot \max \left\{ \sup_{z \in \widetilde{U}^+ \cap \widetilde{U}_{01}^{**}} |h_+(z)|, \sup_{z \in \widetilde{U}^- \cap \widetilde{U}_{01}^{**}} |h_-(z)| \right\}, \end{aligned}$$

where $\omega_1 = g_1(z)dz$,

$$\omega_0 = \begin{cases} g_0^+(x)dx & (\text{on } \widetilde{U}^+ \cap \widetilde{U}_{01}^{**}) \\ g_0^-(y)dy & (\text{on } \widetilde{U}^- \cap \widetilde{U}_{01}^{**}), \end{cases}$$

and $\eta_{\pm} = h_{\pm}(z)dz$.

PROOF. By replacing ω_0 with

$$\begin{cases} t_+^{-n} \cdot \omega_0 & (\text{on } \widetilde{U}^+ \cap \widetilde{U}_{01}^*) \\ t_-^{-n} \cdot \omega_0 & (\text{on } \widetilde{U}^- \cap \widetilde{U}_{01}^*), \end{cases}$$

the proof of the lemma is reduced to the case of $n = 0$, which follows from [KS, Lemma 2]. \square

REMARK 3.7. Lemma 3.5 also holds in the case where C is a cycle of multiple rational curves (see [K2, §4.2.3, 4.2.4] for details). Note that [K2, Lemma 4.2] is used for the estimate of the constants appears in the proof of the general statement which corresponds to the constant C_1 and r in the proof above.

3.2. Proof of Proposition 3.3 when C is a cycle of multiple rational curves.

Let C be a cycle of rational curves consists of n irreducible components ($n \geq 2$). As Proposition 3.3 for this C is shown by intrinsically the same arguments as in the previous section, here we only explain the outline.

Denote by $C_{(1)}, C_{(2)}, \dots, C_{(n-1)}, C_{(n)} = C_{(0)}$ the irreducible components of C . For $\nu = 0, 1, 2, \dots, n-1$, Fix a small neighborhood V_{ν} of $C_{(\nu)} \cap C_{\text{reg}}$ and $V_{\nu, \nu+1}$ of $C_{(\nu)} \cap C_{(\nu+1)}$. We may assume that $V_{\nu} \subset i(\widetilde{V}_{(\nu)})$, and that $V_{\nu, \nu+1}$ is included in the image of $\widetilde{V}_{0(\nu+1)}^+ = F_{\nu+1, \nu}^{-1}(\widetilde{V}_{0(\nu)}^-)$ by i , where we are using the notation in Remark 2.5. Define coordinates (z_{ν}, w_{ν}) of V_{ν} by $i^*z_{\nu} = S_{(\nu)}$ and $i^*w_{\nu} = S_{(\nu)} \cdot \xi_{0(\nu)} = T_{(\nu)} \cdot \xi_{\infty(\nu)}$, and $(x_{\nu}, x_{\nu+1})$ of $V_{\nu, \nu+1}$ by $i^*x_{\nu+1} = S_{(\nu+1)}$ and $i^*x_{\nu} = F_{\nu+1, \nu}^*T_{(\nu)}$. Let

$$F^+(z_{\nu}, w_{\nu}) = \sum_{n=1}^{\infty} b_{\nu, \nu+1, n}^+(z_{\nu}) \cdot w_{\nu}^n$$

be a holomorphic function defined on $V_{\nu} \cap V_{\nu, \nu+1}$, and

$$F^-(z_{\nu+1}, w_{\nu+1}) = \sum_{n=1}^{\infty} b_{\nu, \nu+1, n}^-(z_{\nu+1}) \cdot w_{\nu+1}^n$$

be a holomorphic function defined on $V_{\nu+1} \cap V_{\nu, \nu+1}$. Then it is sufficient to find a holomorphic function F_{ν} on V_{ν} and $F_{\nu, \nu+1}$ on $V_{\nu, \nu+1}$ such that

$$\begin{cases} F_{\nu, \nu+1} - F_{\nu} = F^+ & (\text{on } V_{\nu} \cap V_{\nu, \nu+1}) \\ F_{\nu, \nu+1} - F_{\nu-1} = F^- & (\text{on } V_{\nu+1} \cap V_{\nu, \nu+1}) \end{cases}$$

by shrinking V . F_{ν} is constructed in the form of

$$F_{\nu}(z_{\nu}, w_{\nu}) = \sum_{n=1}^{\infty} a_{\nu, n}(z_{\nu}) \cdot w_{\nu}^n,$$

and $F_{\nu,\nu+1}$ is of

$$F_{\nu,\nu+1}(x_\nu, x_{\nu+1}) = \sum_{n=1}^{\infty} a_{\nu,\nu+1,n}(x_\nu, x_{\nu+1}) \cdot w_{\nu,\nu+1}(x_\nu, x_{x_\nu, x_{\nu+1}})^n,$$

where $w_{\nu,\nu+1}$ is the function defined by $i^*w_{x_\nu, x_{\nu+1}} = S_{(\nu)} \cdot \xi_{0(\nu)}$, and the functions $a_{\nu,n}(z_\nu)$ and $a_{\nu,\nu+1,n}$ are holomorphic functions defined on $C \cap V_\nu$ and $C \cap V_{\nu,\nu+1}$, respectively. Let $p_{\nu,n}^{\nu+1}(x_\nu)$ be a function on $C_{(\nu)} \cap V_{\nu,\nu+1}$ and $p_{\nu+1,n}^\nu(x_{\nu+1})$ be a function on $C_{(\nu+1)} \cap V_{\nu,\nu+1}$ such that

$$a_{\nu,\nu+1,n}(x_\nu, x_{\nu+1}) = \begin{cases} p_{\nu,n}^{\nu+1}(x_\nu) + r_{\nu,\nu+1,n} & (\text{on } C_{(\nu)} \cap V_{\nu,\nu+1}) \\ p_{\nu+1,n}^\nu(x_{\nu+1}) + r_{\nu,\nu+1,n} & (\text{on } C_{(\nu+1)} \cap V_{\nu,\nu+1}) \end{cases}$$

holds, where $r_{\nu,\nu+1,n} := a_{\nu,\nu+1,n}(0,0)$. The function $a_{\nu,\nu+1,n}$ is also regarded as a function defined on $V_{\nu,\nu+1}$ by $a_{\nu,\nu+1,n}(x_\nu, x_{\nu+1}) := p_{\nu,n}^{\nu+1}(x_\nu) + p_{\nu+1,n}^\nu(x_{\nu+1}) + r_{\nu,\nu+1,n}$. By setting $t_{\nu,\nu+1}^+ := 1$ and $t_{\nu,\nu+1}^- := t_{\nu+1,\nu}$, we have that

$$\begin{cases} w_\nu = t_{\nu,\nu+1}^+ \cdot w_{\nu,\nu+1} & (\text{on } C_{(\nu)} \cap V_{\nu,\nu+1}) \\ w_{\nu+1} = t_{\nu,\nu+1}^- \cdot w_{\nu,\nu+1} & (\text{on } C_{(\nu+1)} \cap V_{\nu,\nu+1}) \end{cases}$$

holds.

By the same argument as in Step 1 of the proof of Proposition 3.3, it follows that one should define $a_{\nu,n}$'s and $a_{\nu,\nu+1,n}$'s by

$$\begin{cases} (t_{\nu,\nu+1}^+)^{-n} a_{\nu,\nu+1,n+1} - a_{\nu,n} = b_{\nu,\nu+1,n}^+ - h_{\nu,\nu+1,n}^+ & (\text{on } C_{(\nu)} \cap V_{\nu,\nu+1}) \\ (t_{\nu,\nu+1}^-)^{-n} a_{\nu,\nu+1,n+1} - a_{\nu,n+1} = b_{\nu,\nu+1,n}^- - h_{\nu,\nu+1,n}^- & (\text{on } C_{(\nu+1)} \cap V_{\nu,\nu+1}). \end{cases}$$

Here the functions $h_{\nu,\nu+1,n}^\pm(z_\nu)$ are defined by

$$h_{\nu,\nu+1,n}^+(z_\nu) = \sum_{m=1}^{n-1} (t_{\nu,\nu+1}^+)^{-m} \cdot P_{\nu+1,n,n-m}^\nu(z_\nu)$$

and

$$h_{\nu,\nu+1,n}^-(z_{\nu+1}) = \sum_{m=1}^{n-1} (t_{\nu,\nu+1}^-)^{-m} \cdot P_{\nu,n,n-m}^{\nu+1}(z_{\nu+1}),$$

where

$$p_{\nu+1,n}^\nu(x_{\nu+1}(z_\nu, w_\nu)) = \sum_{\lambda=1}^{\infty} P_{\nu+1,n,\lambda}^\nu(z_\nu) \cdot w_\nu^\lambda$$

and

$$p_{\nu,n}^{\nu+1}(x_\nu(z_{\nu+1}, w_{\nu+1})) = \sum_{\lambda=1}^{\infty} P_{\nu,n,\lambda}^{\nu+1}(z_{\nu+1}) \cdot w_{\nu+1}^\lambda.$$

As one can estimate $|a_{\nu,n}|$ and $|a_{\nu,\nu+1,n}|$ by the same argument as in Step 2 of the proof of Proposition 3.3, the proposition holds (see also Remark 3.7). \square

4. PROOF OF THEOREM 1.1

4.1. Proof of Theorem 1.1 when C is a rational curve with a node. Let C be a rational curve with a node embedded in S such that the normal bundle satisfies the Diophantine assumption in Theorem 1.1. We use the notation in §2.2. Then it is sufficient to show that we may assume $G \equiv 1$ by changing the coordinates such as S and T . Let $g(S, \xi_0) := \frac{1}{2\pi\sqrt{-1}} \log G(S, \xi_0)$ be the branch such that $g(0, 0) = 0$. By applying Proposition 3.2 to $F_+ := -(V_0 \rightarrow \tilde{V}_0^+)^*g$ and $F_- := 0$, we have that, by shrinking \tilde{V} if necessary, there exist holomorphic functions $h_+ : \tilde{V}_0^+ \rightarrow \mathbb{C}$, $h_1 : \tilde{V}_1 \rightarrow \mathbb{C}$ and $h_- : \tilde{V}_0^- \rightarrow \mathbb{C}$ such that

$$\begin{cases} h_+ - h_1 = -g & (\text{on } \tilde{V}_1 \cap \tilde{V}_0^+) \\ h_- - h_1 = 0 & (\text{on } \tilde{V}_1 \cap \tilde{V}_0^-) \end{cases}$$

holds (Set $h_+ := (\tilde{V}_0^+ \rightarrow V_0)^*F_0$, $h_- := (\tilde{V}_0^- \rightarrow V_0)^*F_0$ and $h_1 := (\tilde{V}_1 \rightarrow V_1)^*F_1$, for the solution $\{(V_j, F_j)\}$ in Proposition 3.2). Define a function h on \tilde{V} by

$$h := \begin{cases} h_+ + g & (\text{on } \tilde{V}_0^+) \\ h_1 & (\text{on } \tilde{V}_1) \\ h_- & (\text{on } \tilde{V}_0^-). \end{cases}$$

As clearly it holds that $F^*h_- = h_+$ by definition, we have that $F^*(h|_{\tilde{V}_0^-}) = h|_{\tilde{V}_0^+} + g$. Denote by H the function $e^{2\pi\sqrt{-1}h}$. Define a new coordinate function \hat{S} on $\tilde{V}_0^+ \cup \tilde{V}_1$ by $\hat{S} := S \cdot H^{-1}$, \hat{T} on $\tilde{V}_0^- \cup \tilde{V}_1$ by $\hat{T} := T \cdot H$, $\hat{\xi}_0$ on a neighborhood of D_0 by $\hat{\xi}_0 := \tilde{w} \cdot \hat{S}^{-1} = \xi_0 \cdot H$, and $\hat{\xi}_\infty$ on a neighborhood of D_∞ by $\hat{\xi}_\infty := \tilde{w} \cdot \hat{T}^{-1} = \xi_\infty \cdot H^{-1}$. Then, as it follows $F^*(H|_{\tilde{V}_0^-}) = H|_{\tilde{V}_0^+} \cdot G$ by the construction, we have that

$$F^*\hat{T} = (F^*T) \cdot (F^*H) = \frac{t \cdot \xi_0}{G} \cdot (H \cdot G) = t \cdot (\xi_0 H) = t \cdot \hat{\xi}_0$$

and

$$F^*\hat{\xi}_\infty = (F^*\xi_\infty) \cdot (F^*H)^{-1} = (G \cdot S) \cdot (H \cdot G)^{-1} = S \cdot H^{-1} = \hat{S}$$

on $F^{-1}(\tilde{V}_0^- \cap \tilde{V}_1)$. Therefore, by replacing (S, ξ_0) and (T, ξ_∞) with $(\hat{S}, \hat{\xi}_0)$ and $(\hat{T}, \hat{\xi}_\infty)$ respectively, we have that $F(S, \xi_0) = (t \cdot \xi_0, S)$ holds, which proves the theorem. \square

4.2. Proof of Theorem 1.1 when C is a cycle of multiple rational curves. Here we use the notation in Remark 2.5.

First, we show that we may assume that $G_{\nu+1, \nu} \equiv 1$ holds for $\nu = 1, 2, n-2$ by changing the coordinates appropriately. Let $\{(\tilde{V}'_{(\nu)}, \tilde{C}'_{(\nu)}, (\tilde{V}'_{0(\nu)})^\pm, S'_{(\nu)}, T'_{(\nu)}, \xi'_{0(\nu)}, \xi'_{\infty(\nu)})\}_{\nu=1}^n$ be the n -copies of $(\tilde{V}, \tilde{C}, \tilde{V}_0^\pm, S, T, \xi_0, \xi_\infty)$ in Example 2.1. Denote by $i' : \coprod_{\nu=1}^n \tilde{V}'_{(\nu)} \rightarrow \tilde{V}'$ the quotient by the relation generated by the maps $(\tilde{V}'_{0(\nu+1)})^+ \rightarrow (\tilde{V}'_{0(\nu)})^-$ naturally induced by $\tilde{F}_{\nu+1, \nu}$'s for $\nu = 1, 2, \dots, n-1$. In what follows, we regard $\tilde{V}'_{(\nu)}$ as a subset of \tilde{V}' . Note that $\tilde{V}'_{(1)} \cap \tilde{V}'_{(n)} = \emptyset$ holds as subset of \tilde{V}' . Then it follows from a simple observation that

the quotient \tilde{C}' of $\coprod_{\nu=1}^n \tilde{C}'_{(n)}$ is a tree of rational curves with intersection matrix

$$\begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

As this matrix is negative definite, it follows from [L, Theorem 4.9] and Grauert's theorem [G] that \tilde{C}' admits a strictly pseudoconvex neighborhood \tilde{V}' whose maximal compact analytic subset is \tilde{C}' . Note that, by the arguments as in §2.3, it holds that $H^1(\tilde{C}', N_{\tilde{C}'/\tilde{V}'}^{-m}) = 0$ holds for each $m \geq 0$. Thus, it follows the same argument as in the proof of [K1, Proposition 3.1] that the restriction $H^1(\tilde{V}', \mathcal{O}_{\tilde{V}'}) \rightarrow H^1(\tilde{C}', \mathcal{O}_{\tilde{C}'})$ is injective. As $H^1(\tilde{C}', \mathcal{O}_{\tilde{C}'}) = 0$, it follows from the same arguments as in the previous subsection that we may assume $G_{\nu+1, \nu} \equiv 1$ for $\nu = 1, 2, n-2$.

Therefore, the problem is reduced to showing that we may assume $G_{1, n} \equiv 1$ by changing the coordinates. By replacing $\xi_{0(0)}$ with $G_{1, n}(0, 0)^{-1/n} \cdot \xi_{0(0)}$, we may assume that $G_{1, n}(0, 0) = 1$. Then the theorem follows by the same argument as in the previous subsection. \square

5. TOWARD THE GLUING CONSTRUCTION OF K3 SURFACES CORRESPONDING TO DEGENERATIONS OF K3 SURFACES OF TYPE III

Let (C, S) be the example as in Example 2.2, 2.6 or 2.7. Assume that the normal bundles $N_{C/S}$ is a $U(1)$ -flat line bundles which satisfies Diophantine condition. Then it follows from Theorem 1.1 that one can take a neighborhood V of C in S as in Example 2.1 or Example 2.4 with $n = 2$ or 3 . Define a function $\Phi: V \rightarrow \mathbb{R}$ by $i^*\Phi = |\tilde{w}|$ when C is a rational curve with a node, and by $(i^*\Phi)|_{\tilde{V}(\nu)} = |S(\nu) \cdot \xi_{0(\nu)}| = |T(\nu) \cdot \xi_{\infty(\nu)}|$ when C consists of two or three irreducible components.

Fix positive constants δ and R such that $R > 1$ and $\delta \ll 1$. By the same arguments as in [K3] we may assume $W := \{p \in V \mid \Phi(p) < \delta R\}$ are relatively compact subsets of V by shrinking V and changing the scaling of the coordinates. Denote by W^* the subset $\{p \in V \mid \delta/R < \Phi(p) < \delta R\}$ of W and set $\tilde{W} := i^{-1}(W)$ and $\tilde{W}^* := i^{-1}(W^*)$, where $i: \tilde{V} \rightarrow V$ is as in Example 2.1 or 2.4. The set W^* is a subset of $M := S \setminus \{p \in V \mid \Phi(p) \leq \delta/R\}$.

Define a meromorphic 2-form $\eta_{\tilde{W}}$ on \tilde{W} by

$$\eta_{\tilde{W}} := \frac{dS \wedge d\xi_0}{S \cdot \xi_0} = -\frac{dT \wedge d\xi_{\infty}}{T \cdot \xi_{\infty}}$$

when $n = 1$, and by

$$\eta_{\tilde{W}}|_{\tilde{W} \cap \tilde{V}(\nu)} := \frac{dS(\nu) \wedge d\xi_{0(\nu)}}{S(\nu) \cdot \xi_{0(\nu)}} = -\frac{dT(\nu) \wedge d\xi_{\infty(\nu)}}{T(\nu) \cdot \xi_{\infty(\nu)}}$$

for each ν when $n \geq 2$. As it holds that

$$F^* \eta_{\tilde{W}} = -F^* \frac{dT}{T} \wedge F^* \frac{d\xi_\infty}{\xi_\infty} = -\frac{d(t\xi_0)}{t\xi_0} \wedge \frac{dS}{S} = \frac{dS \wedge d\xi_0}{S \cdot \xi_0} = \eta_{\tilde{W}}$$

when $n = 1$ and

$$(F_{\nu+1,\nu})^* \eta_{\tilde{W}} = -(F_{\nu+1,\nu})^* \frac{dT(\nu)}{T(\nu)} \wedge (F_{\nu+1,\nu})^* \frac{d\xi_{\infty(\nu)}}{\xi_{\infty(\nu)}} = -\frac{d(t\xi_{0(\nu)})}{t\xi_{0(\nu)}} \wedge \frac{dS(\nu)}{S(\nu)} = \frac{dS(\nu) \wedge d\xi_{0(\nu)}}{S(\nu) \cdot \xi_{0(\nu)}} = \eta_{\tilde{W}}$$

when $n \geq 2$, it follows that there exists a meromorphic 2-form η_W on W with $i^* \eta_W = \eta_{\tilde{W}}$ in both the cases. Now we have the following:

PROPOSITION 5.1. *S admits a meromorphic 2-form η which has no zero and has poles only along C such that $\eta|_W = \eta_W$ holds.*

Proposition 5.1 is shown by the same argument as in the proof of [K3, Proposition 3.1]. Here we use the fact that any leaf of a compact Levi-flat hypersurface of W^* defined by $\{\tilde{w} = \text{constant}\}$ is dense (Therefore, it follows that $H^0(W, \mathcal{O}_W) \cong \mathbb{C}$ by the same arguments as in the proof of [K3, Lemma 3.2]).

PROPOSITION 5.2. *It holds that $H_1(M, \mathbb{C}) = 0$.*

PROOF. Take a real number r with $\delta/R < r < \delta R$. As it is clear that W^* is homotopic to $H_r := \Phi^{-1}(r)$, it follows from Lemma 5.3 below that $H_1(W^*, \mathbb{C}) \cong \mathbb{C}^2$. By Mayer-Vietoris sequence corresponds to the open covering $\{W, M\}$ of S , we obtain an exact sequence

$$H_2(S, \mathbb{C}) \rightarrow H_1(W^*, \mathbb{C}) \rightarrow H_1(W, \mathbb{C}) \oplus H_1(M, \mathbb{C}) \rightarrow H_1(S, \mathbb{C}).$$

As it is easily observed that the image of the map $H_2(S, \mathbb{C}) \rightarrow H_1(W^*, \mathbb{C})$ is isomorphic to \mathbb{C} , we have that $H_1(M, \mathbb{C}) = 0$ (Note that $H_1(W, \mathbb{C}) \cong \mathbb{C}$ and $H_1(S, \mathbb{C}) = 0$). \square

LEMMA 5.3. *The Levi-flat manifold $H_r := \Phi^{-1}(r)$ is C^ω -diffeomorphic to $(\mathbb{C}^* \times \text{U}(1)) / \sim_{r,n}$ for sufficiently small r , where $\sim_{r,n}$ is the relation generated by*

$$(\eta, \lambda) \sim_{r,n} (r^n \cdot \lambda^n \cdot \eta, t(N_{C/S}) \cdot \lambda)$$

for $(\eta, \lambda) \in \mathbb{C}^* \times \text{U}(1)$.

PROOF. Let $\{(\widehat{V}(\nu), \widehat{C}(\nu), \widehat{V}_{0(\nu)}^\pm, S(\nu), T(\nu), \xi_{0(\nu)}, \xi_{\infty(\nu)})\}_{\nu=-\infty}^\infty$ be copies of $(\tilde{V}, \tilde{C}, \tilde{V}_0^\pm, S, T, \xi_0, \xi_\infty)$ in Example 2.1. Define a biholomorphism $F_{\nu+1,\nu}: \widehat{V}_{0(\nu+1)}^+ \rightarrow \widehat{V}_{0(\nu)}^-$ by $(F_{\nu+1,\nu})^*(T(\nu), \xi_{\infty(\nu)}) = (\xi_{0(\nu)}, S(\nu))$ for each ν . Define \widehat{V} by gluing $\widehat{V}(\nu)$'s by $F_{\nu+1,\nu}$'s. Note that there is the natural covering map $\widehat{V} \rightarrow V$, which can be regarded as the universal covering.

Consider the map $\widehat{g}_\nu: \{(\eta, \lambda) \in \mathbb{C}^* \times \text{U}(1) \mid 2r^{-\nu+1} < |\eta| < 2r^{-\nu-1}\} \rightarrow \widehat{V}(\nu)$ defined by

$$(\widehat{g}_\nu)^*(S(\nu), \xi_{0(\nu)}) = \left(r^\nu \cdot \lambda^\nu \cdot \eta, \frac{1}{r^{\nu-1} \cdot \lambda^{\nu-1} \cdot \eta} \right).$$

Then, by a simple argument, it follows that \widehat{g}_ν 's glue together to define an embedding $\widehat{g}: \mathbb{C}^* \times \text{U}(1) \rightarrow \widehat{V}$. As it follows that \widehat{g} induces an embedding $g: (\mathbb{C}^* \times \text{U}(1)) / \sim_{r,n} \rightarrow V$ and the image clearly coincides with H_r , the lemma follows. \square

Note that, by Lemma 5.3, H_r is diffeomorphic to $T_{g_n}^2$, which is the fiber bundle over $\text{U}(1)$ whose fiber is $T^2 := \text{U}(1) \times \text{U}(1)$ and the monodromy is $g_n: T^2 \ni (p, q) \mapsto (pq^n, q) \in T^2$. From this, one can have that $H_1(H_r, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$. We emphasize that H_r is not homeomorphic to $T^3 := \text{U}(1) \times \text{U}(1) \times \text{U}(1)$, since $H_1(T^3, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 3}$.

By Proposition 5.1 and 5.2, it seems to be natural to expect that one can construct a K3 surface by holomorphically gluing such models $\{(M_\nu, W_\nu^*, \eta_\nu|_{M_\nu})\}_\nu$ as obtained by the same construction as $(M, W^*, \eta|_M)$ (cf. [K3]).

QUESTION 5.4. Does there exist a non-singular K3 surface X with holomorphic 2-form σ which admits an open covering $X = \bigcup_\nu M_\nu$ such that $\sigma|_{M_\nu} = \eta_\nu|_{M_\nu}$ and $M_\nu \cap M_\mu \subset W_\nu^*$ for each $\nu \neq \mu$, where $(M_\nu, W_\nu^*, \eta_\nu|_{M_\nu})$'s are as above? \square

By considering the limit as the tab for gluing $\bigcup_\nu W_\nu^*$ goes to the set of zero measure (i.e. as $R \rightarrow 1$ and $\delta \rightarrow 0$), the K3 surfaces X should degenerate to a singular K3 surface which is the union of rational surfaces and whose singular part is the union of a cycle of rational curves. We remark that the affirmative answer to Question 5.4 implies the existence of a K3 surface which includes a Levi-flat hypersurface which is diffeomorphic to $T_{g_n}^2$ (and thus is not homeomorphic to T^3).

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