On the integrality of Seshadri constants of abelian surfaces

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Abstract. In this paper we consider the question of when Seshadri constants on abelian surfaces are integers. Our first result concerns self-products $E \times E$ of elliptic curves: If E has complex multiplication in $\mathbb{Z}[i]$ or in $\mathbb{Z}[\frac{1}{2}(1+i\sqrt{3})]$ or if E has no complex multiplication at all, then it is known that for every ample line bundle L on $E \times E$, the Seshadri constant $\varepsilon(L)$ is an integer. We show that, contrary to what one might expect, these are in fact the only elliptic curves for which this integrality statement holds. Our second result answers the question how – on any abelian surface – integrality of Seshadri constants is related to elliptic curves.

1. Introduction

For an ample line bundle L on a smooth projective variety X, the *Seshadri constant* of L at a point $x \in X$ is by definition the real number

$$\varepsilon(L,x) = \inf\left\{ \frac{L \cdot C}{\operatorname{mult}_x(C)} \;\middle|\; C \text{ irreducible curve through } x \right\}.$$

On abelian varieties, where they are independent of the chosen point x, these invariants have been the focus of a great deal of attention [8, 11, 1]. In the two-dimensional case, they are completely understood in the case when the Picard number of the abelian surface is one [3]. At the other extreme, self-products $E \times E$ of elliptic curves were studied in [4], where E is either an elliptic curve without complex multiplication or with $\operatorname{End}(E) = \mathbb{Z}[i]$ or $\operatorname{End}(E) = \mathbb{Z}[\frac{1}{2}(1+i\sqrt{3})]$. In those cases, the Seshadri constants $\varepsilon(L)$ of all ample line bundles L on $E \times E$ were found to be integers – they are in fact computed by elliptic curves. It is natural to expect that this should in effect hold on all surfaces $E \times E$, where E is an elliptic curve. Surprisingly, however, it turns out that the exact opposite is the case: Fractional Seshadri constants do occur on all self-products $E \times E$ except for the ones considered so far. The following theorem provides the complete picture:

Theorem 1 *Let E be an elliptic curve with complex multiplication. Then the following conditions are equivalent:*

(i) *For every ample line bundle L on $E \times E$, the Seshadri constant $\varepsilon(L)$ is an integer.*

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(ii) *Either* $\mathrm{End}(E) = \mathbb{Z}[i]$ *or* $\mathrm{End}(E) = \mathbb{Z}[\frac{1}{2}(1 + i\sqrt{3})]$.

For the proof of Theorem 1 we will first establish how integrality is related to elliptic curves. One direction is obvious: If all Seshadri constants on a given abelian surface are computed by elliptic curves, then certainly those numbers are all integers. It is however less clear to what extent the converse statement holds true. The following theorem answers this question; it holds on any abelian surface, regardless of whether it splits as a product or not.

Theorem 2 *Let X be an abelian surface. The following conditions are equivalent:*

(i) *For every ample line bundle L on X, the Seshadri constant $\varepsilon(L)$ is an integer.*
(ii) *For every ample line bundle L on X, either $\varepsilon(L) = \sqrt{L^2}$ and $\sqrt{L^2}$ is an integer, or $\varepsilon(L)$ is computed by an elliptic curve, i.e., there exists an elliptic curve $E \subset X$ such that*

$$\varepsilon(L) = L \cdot E .$$

If one is interested in constructing explicit examples of line bundles with fractional Seshadri constants on products $E \times E$, then a natural approach is to look for irreducible principal polarizations on these surfaces. In other words, one asks under which circumstances $E \times E$ is the Jacobian of a smooth genus 2 curve. This question was first studied by Hayashida and Nishi [7] in the case where the Endomorphism ring is the maximal order in the Endomorphism algebra. We extend their result in Prop. 4.5 to cases which include non-maximal orders.

Looking at Theorem 2, one might be tempted to hope that an analogous equivalence might hold for *each individual* line bundle. However, as we will show in Prop. 4.8, $\varepsilon(L)$ can be an integer without this being accounted for by the conditions in (ii).

We work throughout over the field of complex numbers.

2. Background

We will use a number of previous results concerning Seshadri constants on abelian surfaces in the proof of Thm. 1 and Thm. 2. In this preliminary section we briefly review some of these results. For more general background on Seshadri constants also see [9, Chapt. 5] and [5]

Submaximal divisors. For any ample line bundle L an a smooth projective surface X and any point $x \in X$, one has the basic bound $\varepsilon(L, x) \leqslant \sqrt{L^2}$. A divisor D on X is called *L-submaximal* at x if its Seshadri quotient $L \cdot D/\mathrm{mult}_0 D$ is strictly smaller than $\sqrt{L^2}$. In other words, a divisor D is L-submaximal if it forces $\varepsilon(L, x)$ to be smaller than the theoretical upper bound $\sqrt{L^2}$. It is a crucial observation that if D is an L-submaximal divisor that belongs to the linear series $|kL|$ for some integer $k \geqslant 1$, then every irreducible curve C that is L-submaximal at x must occur as a component of D. (see [3, Lemma 5.2 and Lemma 6.2]). It is for this fact that submaximal divisors are in many cases instrumental to explicitly determining Seshadri constants. One such situation is as follows: Suppose that C is an irreducible curve that is L-submaximal at x for some ample line bundle L. If the line bundle

$\mathcal{O}_X(C)$ is ample, then by [4, Prop. 1.2] the curve C is also $\mathcal{O}_X(C)$-submaximal at x, and in fact C *computes* $\varepsilon(\mathcal{O}_X(C),x)$ in the sense that

$$\varepsilon(\mathcal{O}_X(C),x) = \frac{\mathcal{O}_X(C)\cdot C}{\operatorname{mult}_x C}\,.$$

Symmetric divisors on abelian surfaces. Consider now an abelian surface X. A divisor D on X is *symmetric*, if D is invariant under the involution $x \mapsto -x$. Similarly, a line bundle L is *symmetric*, if $(-1)^*L = L$ in $\operatorname{Pic}(X)$. Symmetric line bundles enjoy important properties: The sixteen halfperiods on X are divided into *even* and *odd* halfperiods (with respect to L), and if D is a symmetric divisor with $\mathcal{O}_X(D) = L$, then the multiplicities of D at either of these sets of halfperiods are all even or all odd. (See [6, Sect. 4.7] for these facts and for further properties of symmetric line bundles.)

Let L be an ample line bundle on an abelian surface. As far as Seshadri constants are concerned we may assume that L is symmetric, since $\varepsilon(L)$ depends only on the algebraic equivalence class and every algebraic equivalence class contains contains symmetric line bundles. It was proven in [2] that if L is primitive and $\sqrt{L^2}$ is irrational, then $\varepsilon(L) < \sqrt{L^2}$. This is shown by constructing a symmetric submaximal divisor, the *Pell divisor*, $\operatorname{Pell}(L)$ in $|2kL|$ with multiplicity at least 2ℓ, where (k,ℓ) is the minimal solution of the Pell equation $\ell^2 - L^2k^2 = 1$. One has

$$\frac{L\cdot \operatorname{Pell}(L)}{\operatorname{mult}_0 \operatorname{Pell}(L)} \leqslant L^2\cdot\frac{k}{\ell} < \sqrt{L^2}\,.$$

3. Integral Seshadri constants

In this section we prove Theorem 2. As mentioned in the previous section, one has $\varepsilon(L) \leqslant \sqrt{L^2}$ for any ample line bundle. We start by giving an example showing that in condition (ii) of the theorem it can in fact happen that $\varepsilon(L) = \sqrt{L^2}$, even though $\varepsilon(L)$ is not computed by an elliptic curve.

Example 3.1 For any positive integer n consider a polarized abelian surface (X,L) of type $(1,2n^2)$ with $\rho(X)=1$. As $L^2 = 4n^2$ is a perfect square, one has $\varepsilon(L) = \sqrt{L^2} = 2n$ by [12, Prop. 1], but of course $\varepsilon(L)$ is not computed by an elliptic curve, since there are no such curves on X.

Proof of Theorem 2. The implication (ii) $\Rightarrow$ (i) being obvious, let us suppose (i). Assume by way of contradiction that there are ample line bundles L on X whose Seshadri constant is less than $\sqrt{L^2}$ and not computed by elliptic curves. Consider a primitive such line bundle L. Replacing L by an algebraically equivalent line bundle, we may assume that L is symmetric. We will now make use of the Pell divisor of L, i.e., the divisor $D = \operatorname{Pell}(L) \in |2kL|$ with $\operatorname{mult}_0(D) \geqslant 2\ell$, where (k,ℓ) is the minimal solution of Pell's equation

$$\ell^2 - 2dk^2 = 1\,,$$

having the property that

$$\frac{L\cdot D}{\operatorname{mult}_0(D)} < \sqrt{L^2}$$

(see [2]). It follows from [3, Lemma 5.2] that every irreducible curve computing $\varepsilon(L)$ is a component of D. Let C be one of these curves (which by assumption is not elliptic). As C is sub-maximal for L, it follows from [4, Prop. 1.2] that C computes its own Seshadri constant $\varepsilon(\mathcal{O}_X(C))$. The curves C and $(-1)^*C$ have the same multiplicity at the point 0 and they are algebraically equivalent. Therefore, by applying [3, Lemma 5.2] to the bundle $\mathcal{O}_X(C)$, we see that these two curves must coincide, i.e., that C is symmetric. So C descends to a curve $\overline{C}$ on the smooth Kummer surface of X. With an argument as in the proof of [3, Thm. 6.1], this curve $\overline{C}$ must be a (-2)-curve. (Otherwise C would move in a pencil of L-submaximal curves, but there can only be finitely many of those.) The upshot of these considerations is that the multiplicities $m_i = \operatorname{mult}_{e_i}(C)$ of C at the sixteen halfperiods e_i of X satisfy the equation

$$C^2 - \sum_{i=1}^{16} m_i^2 = -4\,. \tag{1}$$

On the other hand, one has

$$C^2 - m_1^2 < 0 \tag{2}$$

(where $m_1 = \operatorname{mult}_0(C)$), since otherwise

$$\frac{L\cdot C}{m_1} \geqslant \frac{\sqrt{L^2}\sqrt{C^2}}{m_1} \geqslant \frac{\sqrt{L^2}\cdot m_1}{m_1} = \sqrt{L^2}$$

contradicting the fact that C is submaximal for L. We claim now that

$$C^2 - m_1^2 = -1 \quad \text{or} \quad C^2 - m_1^2 = -4\,. \tag{3}$$

For the proof of (3), note first that, by (1) and (2), the only other possibilities for $C^2 - m_1^2$ are -2 and -3. In the case where $C^2 - m_1^2 = -2$, we see that the number m_1 must be even and we have $\sum_{i=2}^{16} m_i^2 = 2$ by (1). So there are exactly two half-periods at which C has odd multiplicity. But this cannot happen since a symmetric divisor can only have $4, 6, 10$ or 12 half-periods with odd multiplicity (see [10, Sect. 2, Cor. 3] or [6, Prop. 4.7.5]). In the other case, $C^2 - m_1^2 = -3$, the number m_1 is odd and we have $\sum_{i=2}^{16} m_i^2 = 1$ by (1). This leads to the same kind of contradiction as before.

We know that C computes its own Seshadri constant, i.e.,

$$\varepsilon(\mathcal{O}_X(C)) = \frac{C\cdot C}{\operatorname{mult}_0 C}\,.$$

But by (3), this number equals

$$\frac{m_1^2 - s}{m_1} = m_1 - \frac{s}{m_1}$$

where $s = 1$ or $s = 4$. As by assumption $\varepsilon(\mathcal{O}_X(C))$ is a positive integer, this means that necessarily

$$m_1 = 4 \quad \text{and} \quad C^2 = 12\,.$$

In this case the multiplicities m_i at the non-zero half-periods are all zero. So in particular, all multiplicities m_i are even. Therefore the line bundle $\mathcal{O}_X(C)$ is totally symmetric, and hence it is a square of another line bundle (see [10, Sect. 2, Cor. 4]). But because of $C^2 = 12$ this is impossible. So we have arrived at a contradiction, and this completes the proof of the theorem. □

4. Products of elliptic curves with complex multiplication

In this section we prove Theorem 1. Let E be an elliptic curve that has complex multiplication, i.e., $\mathrm{End}(E)\otimes\mathbb{Q}=\mathbb{Q}(\sqrt{d})$ for some square-free integer $d<0$. The endomorphism ring $\mathrm{End}(E)$ is then an order in $\mathbb{Q}(\sqrt{d})$, and hence it is of the form

$$\mathrm{End}(E)\simeq\mathbb{Z}[f\omega]$$

where f is a positive integer and

$$\omega=\begin{cases}\sqrt{d} & \text{if } d\equiv 2,3 \pmod 4\\ \frac{1}{2}(1+\sqrt{d}) & \text{if } d\equiv 1 \pmod 4\,.\end{cases}$$

It turns out that for our purposes it is more practical to use an equivalent but slightly different description: We write $\mathrm{End}(E)=\mathbb{Z}[\sigma]$, where

$$\begin{aligned} &\sigma=\sqrt{-e} && \text{with } e\in\mathbb{N}\\ \text{or}\quad &\sigma=\tfrac{1}{2}(1+\sqrt{-e}) && \text{with } e\in\mathbb{N} \text{ and } e\equiv 3 \pmod 4\,.\end{aligned}$$

On the product surface $E\times E$, denote by $F_1=\{0\}\times E$ and $F_2=E\times\{0\}$ the fibers of the projections, by Δ the diagonal, and by Γ the graph of the endomorphism corresponding to σ. The classes of these four curves form a basis of $\mathrm{NS}(E\times E)$ (see [13, Thm. 22] or [6, Thm. 11.5.1]).

Proposition 4.1 *Let E be an elliptic curve with complex multiplication. Write $\mathrm{End}(E)=\mathbb{Z}[\sigma]$ with σ as above. Then the intersection matrix of (F_1,F_2,Δ,Γ) is*

$$\begin{pmatrix} 0 & 1 & 1 & 1\\ 1 & 0 & 1 & |\sigma|^2\\ 1 & 1 & 0 & |1-\sigma|^2\\ 1 & |\sigma|^2 & |1-\sigma|^2 & 0\end{pmatrix}.$$

Proof. All four curves are elliptic, so we have

$$F_1^2=F_2^2=\Delta^2=\Gamma^2=0\,.$$

As each curve intersects the other ones transversely, it is enough to count the number of intersection points. So we have

$$F_1\cdot F_2=F_1\cdot\Delta=F_1\cdot\Gamma=F_2\cdot\Delta=1\,,$$

since these curves intersect only in the origin. For F_2 and Γ one has

$$F_2\cdot\Gamma=\#\{(x,0)\,|\,x\in E\}\cap\{(x,\sigma x)\,|\,x\in E\}\,,$$

and this shows that we need to count the number of solutions $x\in E$ of the equation $\sigma x=0$. But this number equals the degree of the isogeny $\sigma:E\to E$, and so we get

$$F_2\cdot\Gamma=\deg\sigma=|\sigma|^2\,.$$

Finally, for Δ and Γ we have

$$\Delta\cdot\Gamma=\#\{(x,x)\,|\,x\in E\}\cap\{(x,\sigma x)\,|\,x\in E\}\,,$$

and this is the number of fixed points of the isogeny σ. Hence by the Holomorphic Lefschetz Fixed-Point Formula [6, Thm. 13.1.2], we have

$$\Delta \cdot \Gamma = \#\,\mathrm{Fix}(\sigma) = |1-\sigma|^2 ,$$

and this concludes the proof of the proposition. □

We will need an explicit description of the ample cone of $E \times E$:

Proposition 4.2 *Let E be an elliptic curve with complex multiplication. Write* $\mathrm{End}(E) = \mathbb{Z}[\sigma]$ *as above. A line bundle*

$$L = \mathcal{O}_{E\times E}(a_1F_1 + a_2F_2 + a_3\Delta + a_4\Gamma)$$

is ample if and only if the two inequalities

$$\begin{aligned} a_1 + a_2 + 2a_3 + (|\sigma|^2+1)a_4 &> 0 \\ a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + |\sigma|^2a_2a_4 + |1-\sigma|^2a_3a_4 &> 0 \end{aligned}$$

are satisfied.

Proof. This follows from the fact that a line bundle L is ample if and only if both L^2 and the intersection of L with the ample line bundle $\mathcal{O}_{E\times E}(F_1+F_2)$ are positive (by the improvement of the Nakai-Moishezon criterion valid on abelian varieties [6, Cor. 4.3.3]). □

Next, we apply a change of basis to the Néron–Severi group to make calculations easier by choosing two basis elements which are orthogonal to F_1 and F_2. We define $\nabla := \Delta - F_1 - F_2$ and $\Sigma := \Gamma - |\sigma|^2F_1 - F_2$. The intersection matrix of $(F_1, F_2, \nabla, \Sigma)$ is then

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2\,\mathrm{Re}(\sigma) \\ 0 & 0 & -2\,\mathrm{Re}(\sigma) & -2|\sigma|^2 \end{pmatrix} .$$

In terms of this new basis, the ampleness condition for a line bundle

$$L = \mathcal{O}_{E\times E}(a_1F_1 + a_2F_2 + a_3\nabla + a_4\Sigma)$$

is expressed by the two inequalities

$$\begin{aligned} a_1 + a_2 &> 0 \\ a_1a_2 - a_3^2 - |\sigma|^2a_4^2 - 2\,\mathrm{Re}(\sigma)a_3a_4 &> 0\,. \end{aligned}$$

It was shown in [4] that if $\mathrm{End}(E) = \mathbb{Z}[i]$ or $\mathrm{End}(E) = \mathbb{Z}[\frac{1}{2}(1+i\sqrt{3})]$, then the Seshadri constants on $E \times E$ are computed by elliptic curves, and hence they are integers. We now show that in all other cases there exist ample line bundles on $E\times E$, whose Seshadri constants cannot be computed by elliptic curves. With Theorem 2 this will imply that there are line bundles with fractional Seshadri constants on the surface.

Proposition 4.3 *Let E be an elliptic curve with complex multiplication. Write* $\mathrm{End}(E) = \mathbb{Z}[\sigma]$ *as above. If $\sigma \notin \left\{i, \frac{1}{2}(1 + i\sqrt{3})\right\}$, then there exist ample line bundles L on $E \times E$ such that $\sqrt{L^2}$ is not an integer and such that $\varepsilon(L)$ is not computed by an elliptic curve.*

Proof. Our strategy is to exhibit ample line bundles L whose intersection number with any nef line bundle – and therefore in particular with every elliptic curve – is bigger than $\sqrt{L^2}$. For such L, the Seshadri constant cannot be computed by an elliptic curve, since $\varepsilon(L) \leqslant \sqrt{L^2}$ (see [9, Prop. 5.1.9]).

We first treat the case $\sigma = \sqrt{-e}$ with $e \neq 1$. For $k \in \mathbb{Z}$ consider the line bundle

$$L_k := \mathcal{O}_{E\times E}(2e\, F_1 + 2e\, F_2 + e\, \nabla + k\, \Sigma)\,.$$

As $L_k \cdot (F_1 + F_2) = 4e$, the line bundle L_k is ample if and only if $L_k^2 = 6e^2 - 2ek^2 > 0$. (This is a consequence of [6, Cor. 4.3.3].) Let M be an arbitrary line bundle numerically written as $M \equiv a_1F_1 + a_2F_2 + a_3\nabla + a_4\Sigma$. Then the intersection number of L_k and M is given by

$$L_k \cdot M = 2ea_2 + 2ea_1 - 2ea_3 - 2eka_4\,.$$

The crucial point in this construction is that $L_k \cdot M$ is a multiple of $2e$. So in particular the intersection number of L_k with any elliptic curve on $E \times E$ is at least $2e$, if L_k is ample. We will show that we can choose k in such a way that

(i) L_k is ample and $\sqrt{L_k^2} < 2e$,
(ii) L_k^2 is not a perfect square.

If this is achieved, then we have an ample line bundle as claimed in the proposition.

Turning to the proof of that claim, note that (i) is equivalent to the condition that k lies in the interval $(\sqrt{e}, \sqrt{3e})$. So we have to show that if $e \neq 1$ then this interval contains an integer k such that also condition (ii) is fulfilled. The subsequent Lemma 4.4 shows that if L_k^2 is a perfect square, then L_{k+1}^2 cannot be a perfect square. As the interval $(\sqrt{e}, \sqrt{3e})$ contains at least two integers when $e \geqslant 6$, we are thus reduced to treating the range $2 \leq e \leq 5$. For these values of e we can do explicit calculations, which show that integers k as required exist:

e	2	3	4	5
k	2	2	3	3
L_k^2	8	30	24	60

Now we treat the second case, i.e., $\sigma = \frac{1}{2}(1 + \sqrt{-e})$ with $e \equiv 3 \pmod 4$ and $e \neq 3$. In this case, we consider for odd $k \in \mathbb{Z}$ the line bundles

$$L_k := \mathcal{O}_{E\times E}(2e\, F_1 + 2e\, F_2 + (e - k)\, \nabla + 2k\, \Sigma)\,.$$

Analogously to the case before, L_k is ample if and only if $L_k^2 = 6e^2 - 2ek^2 > 0$. Since k is odd, it follows that the intersection number of L_k and M, which is given by

$$L_k \cdot M = 2ea_2 + 2ea_1 - 2ea_3 - e(k+1)a_4\,,$$

is a multiple of $2e$. If $e \neq 3$, then it is possible to choose an integer $k \in (\sqrt{e}, \sqrt{3e})$. This ensures that L_k is ample and that $\sqrt{L_k^2} < 2e$. (Note that the interval does not contain an odd integer for $e = 3$.)

By the subsequent Lemma 4.4 we know that if L_k^2 is a perfect square then L_{k+2}^2 is not. Since the interval $(\sqrt{e}, \sqrt{3e})$ contains at least four integers when $e \geqslant 30$, we are reduced to the cases $7 \leq e \leq 27$. We finish the proof by providing explicit values of k for the remaining six cases:

e	7	11	15	19	23	27
k	3	5	5	5	5	7
L_k^2	168	176	600	1216	2024	1728

This completes the proof of the proposition. □

Lemma 4.4 *Let $e \geqslant 2$ be an integer and let $e = m^2 n$ be its unique representation as a product of a square and a square-free integer. For positive integers k we define $A_k := 6e^2 - 2ek^2$. Then either A_k or A_{k+1} is not a perfect square, and if furthermore $n \geqslant 3$, then either A_k or A_{k+2} is not a perfect square.*

Proof. First, we treat the case $n \geqslant 3$. Suppose that $A_k = m^2 n\,(6m^2 n - 2k^2)$ is a perfect square. Since n is square-free, it follows that $6m^2 n - 2k^2 = nr^2$ for an integer r. We deduce that either n or $n/2$ is a divisor of k^2, and hence it is a divisor of k. Consequently, neither A_{k+1} nor A_{k+2} can be a perfect square, because n and $n/2$, respectively, cannot be a divisor of $k + 1$ and $k + 2$.

Next, we consider the case $n = 2$. Suppose that $A_k = 4m^2\,(6m^2 - k^2)$ is a perfect square. Then the factor $6m^2 - k^2$ must itself be a perfect square, and this implies that the equation $k^2 + r^2 = 6m^2$ has an integral solution (k, r, m). By cancelling common factors we then find also a solution with $\gcd(k, r) = 1$. We will now obtain a contradiction by considering the equation modulo 8: On the one hand, $r^2 + k^2$ is either 1, 2 or 5 modulo 8, and on the other hand $6m^2$ is either 0 or 6 modulo 8.

Finally, we treat the case $n = 1$. Suppose that $A_k = m^2\,(6m^2 - 2k^2)$ is a perfect square. As before, it follows that $6m^2 - 2k^2 = r^2$ for some integer r. Assume by way of contradiction that $A_{k+1} = m^2\,(r^2 - 4k - 2)$ is a perfect square as well. Then the factor $r^2 - 4k - 2$ must be a perfect square as well. This, however, cannot happen because it is either 2 or 3 modulo 4. □

Proof of Theorem 1. The implication (ii) ⇒ (i) follows from [4], where it is shown that all Seshadri constants are computed by elliptic curves in that case. Assume now that condition (i) holds. By Prop. 4.3 there exist ample line bundles L whose Seshadri constant is not computed by elliptic curves and such that $\sqrt{L^2}$ is not an integer. Theorem 2 then shows that there are ample line bundles on $E \times E$ with fractional Seshadri constants. □

The method of proof of Theorem 1 shows the existence of line bundles with fractional Seshadri constants, but does not construct them explicitly. One idea to find such line bundles very concretely is to look for principal polarizations on $E \times E$. Those are either irreducible, i.e., they arise from a smooth curve of genus 2, or they correspond to a sum of two elliptic curves (see [14, Thm. 2]). In the irreducible case one has a fractional Seshadri constant $\varepsilon(L) = \frac{4}{3}$ by [12, Prop. 2].

The problem of finding smooth genus two curves on $E \times E$ was first studied by Hayashida and Nishi in [7], where they show that if $\mathrm{End}(E)$ is isomorphic to the maximal order of $\mathbb{Q}(\sqrt{-m})$, then there exists such principal polarizations L if and only if $m \notin \{1, 3, 7, 15\}$. Note that this shows in particular that there are cases in which no such principal polarizations exist. We extend their result by exhibiting irreducible principal polarizations when $\mathrm{End}(E) = \mathbb{Z}[\sqrt{-e}]$ with $e \equiv 2, 3 \pmod 4$. (Note that these include non-maximal orders.)

Proposition 4.5 *Let E be an elliptic curve with complex multiplication such that $\mathrm{End}(E) = \mathbb{Z}[\sqrt{-e}]$ with $e \equiv 2, 3$ (mod 4). Then there exist irreducible principal polarizations L on $E \times E$. In particular, we have $\varepsilon(L) = \frac{4}{3}$ for these line bundles.*

Proof. Note to begin with that for an irreducible principal polarization L one has $\varepsilon(L) = \frac{4}{3}$: This was first shown by Steffens [12] when the Picard number is one, where the Seshadri constant is computed by a curve in $|4L|$. Thanks to the fact that this curve is irreducible, it also computes $\varepsilon(L)$ in the general case by [3, Lemma 6.2].

Turning to the proof of the proposition, we first treat the case $e \equiv 2 \pmod 4$. Writing $e = 4n + 2$, consider the line bundle

$$\begin{aligned} L_n &:= \mathcal{O}_{E\times E}(2(n+1)F_1 + 2F_2 + \nabla + \Sigma) \\ &= \mathcal{O}_{E\times E}(-(2n+1)F_1 + \Delta + \Gamma)\,. \end{aligned}$$

It is a consequence of $L_n \cdot (F_1 + F_2) = 2n + 4 > 0$ and $L_n^2 = 2$ that L_n is a principal polarization. Arguing as in the proof of Proposition 4.3, it follows that the intersection number of L_n with any elliptic curve $N \subset E \times E$ is a multiple of 2. So, $L \cdot N \neq 1$ and therefore L must be irreducible.

The case $e = 4n + 3$ can be dealt with analogously: In this case one can show that the line bundle

$$\begin{aligned} L_n &:= \mathcal{O}_{E\times E}(2(n+1)F_1 + 2F_2 + \Sigma) \\ &= \mathcal{O}_{E\times E}(-(2n+1)F_1 + F_2 + \Gamma) \end{aligned}$$

is an irreducible principal polarization. □

Theorem 1 provides the exact picture, on which surfaces $E\times E$ fractional Seshadri constants occur. It is important to point out that on the other hand there are always line bundles whose Seshadri constant are integral – in fact this happens for all bundles in the cone in $\mathrm{NS}(E \times E)$ generated by the classes of F_1, F_2, Δ, Γ:

Proposition 4.6 *For line bundles*

$$L = \mathcal{O}_{E\times E}(a_1F_1 + a_2F_2 + a_3\Delta + a_4\Gamma)$$

with non-negative coefficients a_i, one has

$$\begin{aligned} \varepsilon(L) &= \min\{L\cdot F_1, L\cdot F_2, L\cdot\Delta, L\cdot\Gamma\} \\ &= \min\{a_2 + a_3 + a_4, a_1 + a_3 + |\sigma|^2 a_4, \\ &\qquad a_1 + a_2 + |1-\sigma|^2 a_4, a_1 + |\sigma|^2 a_2 + |1-\sigma|^2 a_3\}\,. \end{aligned}$$

Proof. Let D be the divisor $a_1F_1 + a_2F_2 + a_3\Delta + a_4\Gamma$, and let C be any irreducible curve C passing through 0, which is not a component of D. As D is effective, we have

$$\frac{L \cdot C}{\mathrm{mult}_0 C} = \frac{D \cdot C}{\mathrm{mult}_0 C} \geqslant \frac{\mathrm{mult}_0 D \cdot \mathrm{mult}_0 C}{\mathrm{mult}_0 C} \geqslant a_1 + a_2 + a_3 + a_4$$
$$\geqslant a_2 + a_3 + a_4 = L \cdot F_1 \,.$$

This implies that $\varepsilon(L)$ is computed by one of the components of D. Their intersection numbers with L are computed using Prop. 4.1, and this yields the assertion of the proposition. □

As the following example shows, the line bundles in the cone generated by the classes of F_1, F_2, Δ, Γ are not the only ones with integral Seshadri constants.

Example 4.7 Consider the line bundle $L = \mathcal{O}_{E\times E}(4F_1 + 2F_2 - \Delta)$. It is ample by Prop. 4.2. The fact that $L \cdot F_1 = 1$ implies that its Seshadri constant is 1.

Finally, we will discuss whether or not the statement in Theorem 2 can be generalized such that the conditions hold for each individual line bundle. Clearly, if there exists a line bundle L with a fractional Seshadri constant, then a suitable multiple of L will lead to a line bundle, whose Seshadri constant is an integer but is not calculated by an elliptic curve. One might hope that it still holds for primitive line bundles. This, however, is not the case:

Proposition 4.8 *There exists an abelian surface X and a primitive line bundle L on X such that the Seshadri constant $\varepsilon(L)$ is an integer less than $\sqrt{L^2}$ and is calculated by a non-elliptic curve.*

Proof. Let E be an elliptic curve with complex multiplication such that $\mathrm{End}(E) = \mathbb{Z}[\sqrt{-2}]$. As noted in the proof of Prop. 4.5, the Seshadri constant of the principal polarization $L_0 = \mathcal{O}_{E\times E}(-F_1 + \Delta + \Gamma)$ on $E \times E$ is computed by an irreducible curve $C \in |4L_0|$ with $\mathrm{mult}_0(C) = 6$. We consider the primitive line bundle $L := \mathcal{O}_{E\times E}(D)$, where the divisor D is defined as

$$D := 3C + F_1 + F_2 + \Delta$$
$$= -11F_1 + F_2 + 13\,\Delta + 12\,\Gamma \,.$$

We claim that the Seshadri constant $\varepsilon(L)$ equals 20 and is calculated by C.

We have
$$\frac{L \cdot D}{\mathrm{mult}_0(D)} = \frac{D^2}{\mathrm{mult}_0 D} = \frac{438}{21} < \sqrt{438} = \sqrt{L^2}\,,$$

so D is submaximal for L. Therefore [3, Lemma 6.2] implies that the Seshadri constant of L is calculated by a component of D. So checking C, F_1, F_2 and Δ we see that
$$\varepsilon(L) = 20 = \frac{L \cdot C}{\mathrm{mult}_0 C} < 26 = L \cdot F_1 = L \cdot F_2 = L \cdot \Delta \,.$$

□

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