THE LIND-LEHMER CONSTANT FOR $\mathbb{Z}^r_2\times\mathbb{Z}^s_4$

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Abstract. We show that the minimal positive logarithmic Lind-Mahler measure for a group of the form $G = \mathbb{Z}_2^r \times \mathbb{Z}_4^s$ with $|G| \geq 4$ is $\frac{1}{|G|} \log(|G| - 1)$. We also show that for $G = \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ with $n \geq 3$ this value is $\frac{1}{|G|} \log 9$. Previously the minimal measure was only known for 2-groups of the form \mathbb{Z}_2^k or \mathbb{Z}_{2^k} .

1. INTRODUCTION

Recall that for a polynomial $F(x_1, \ldots, x_k)$ in $\mathbb{Z}[x_1, \ldots, x_k]$, one defines the traditional logarithmic Mahler measure by

$$
m(F) = \int_0^1 \cdots \int_0^1 \log |F(e^{2\pi ix_1}, \ldots, e^{2\pi ix_k})| dx_1 \cdots dx_k.
$$

In 2005, Lind [\[6\]](#page-9-0) viewed $[0,1]^k$ as the group $(\mathbb{R}/\mathbb{Z})^k$ and generalized the Mahler measure to arbitrary compact abelian groups. In particular, for the finite abelian group

$$
G=\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_k}
$$

and $F \in \mathbb{Z}[x_1,\ldots,x_k]$, we define the *logarithmic Lind-Mahler measure* by

$$
m_G(F) = \frac{1}{|G|} \sum_{x_1=1}^{n_1} \cdots \sum_{x_k=1}^{n_k} \log |F(e^{2\pi i x_1/n_1}, \ldots, e^{2\pi i x_k/n_k})|.
$$

The close connection to the group determinant is explored by Vipismakul [\[8\]](#page-9-1). Writing

$$
w_n := e^{2\pi i/n},
$$

we plainly have

$$
m_G(F) = \frac{1}{|G|} \log|M_G(F)|,
$$

where

$$
M_G(F) := \prod_{j_1=1}^{n_1} \cdots \prod_{j_k=1}^{n_k} F(w_{n_1}^{j_1}, \ldots, w_{n_k}^{j_k})
$$

will be in \mathbb{Z} . Analogous to the classical Lehmer problem [\[4\]](#page-9-2), we can ask for the minimal $m_G(F) > 0$, and to this end we define the Lind-Lehmer constant for G by

$$
\lambda(G) := \min\{ |M_G(F)| > 1 \mid F \in \mathbb{Z}[x_1, \dots, x_k] \}.
$$

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We use $|M_G(F)|$ rather than $m_G(F)$ or $|M_G(F)|^{1/|G|}$ so that we are dealing with integers; of course the minimal positive logarithmic measure will be $\frac{1}{|G|} \log \lambda(G)$. As Lind observed, for $|G| \geq 3$ we always have the trivial bound

$$
\lambda(G) \le |G| - 1,
$$

achieved, for example, by

$$
F(x_1,...,x_k) = -1 + \prod_{i=1}^k \left(\frac{x_i^{n_i} - 1}{x_i - 1} \right).
$$

Lind also showed that for prime powers p^{α} with $\alpha \geq 1$ we have

(2)
$$
\lambda(\mathbb{Z}_{p^{\alpha}}) = \begin{cases} 3, & \text{if } p = 2, \\ 2, & \text{if } p \ge 3, \end{cases}
$$

achieved with $x^2 + x + 1$ if $p = 2$ and $x + 1$ if $p \ge 3$. Lind's results for cyclic groups were extended by Kaiblinger [\[3\]](#page-9-3) and Pigno & Pinner [\[7\]](#page-9-4) so that $\lambda(\mathbb{Z}_m)$ is now known if 892 371 480 $\nmid m$. The value for the *p*-group \mathbb{Z}_p^k was recently established by De Silva & Pinner [\[2\]](#page-9-5), but little is known for direct products involving at least one term $\mathbb{Z}_{p^{\alpha}}$ with $\alpha \geq 2$.

Here we are principally interested in the case of 2-groups

(3)
$$
G = \mathbb{Z}_{2^{\alpha_1}} \times \cdots \times \mathbb{Z}_{2^{\alpha_k}}.
$$

It was shown in [\[2\]](#page-9-5) that for all $k \geq 2$

$$
\lambda(\mathbb{Z}_2^k) = 2^k - 1,
$$

a case of equality in [\(1\)](#page-1-0). We establish two main results regarding the Lind-Lehmer constant for groups of the form (3) . First, we prove that equality occurs in (1) whenever G is a 2-group whose factors are all \mathbb{Z}_2 or \mathbb{Z}_4 .

Theorem 1.1. If $G = \mathbb{Z}_2^r$ or \mathbb{Z}_4^s or $\mathbb{Z}_2^r \times \mathbb{Z}_4^s$, then

$$
\lambda(G) = \max\{3, |G| - 1\}.
$$

Second, we show that this is not true for all 2-groups: if we allow $\alpha_i \geq 3$ in [\(3\)](#page-1-1) then [\(1\)](#page-1-0) need not be sharp.

Theorem 1.2. For $n \geq 3$

$$
\lambda(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) = 9,
$$

achieved with $F(x, y) = y^2 + y + 1$.

Crucial to our proofs of these statements will be a congruence satisfied by $M_G(F)$ when G is a p-group. This is a generalization of $[2, \text{Lemma } 2.1]$ (see also $[8, \text{Theorem } 2.1]$) $2.1.2$].

Lemma 1.1. If p is a prime and

(5)
$$
G = \mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}},
$$

then

$$
M_G(F) \equiv F(1,\ldots,1)^{|G|} \mod p^k.
$$

Notice that for the p -group (5) we have

$$
M_G(F) = \prod_{t_1=0}^{\alpha_1} \cdots \prod_{t_k=0}^{\alpha_k} N_{t_1,\ldots,t_k}(F),
$$

where

$$
N_{t_1,\ldots,t_k}(F) = \prod_{\substack{j_1=1 \ (j_1,p^{\alpha_1})=p^{t_1}}}^{p^{\alpha_1}} \cdots \prod_{\substack{j_k=1 \ (j_k,p^{\alpha_k})=p^{t_k}}}^{p^{\alpha_k}} F(w_{p^{\alpha_1}}^{j_1},\ldots,w_{p^{\alpha_k}}^{j_k}) \in \mathbb{Z}.
$$

Since $|1-w^j_{p^{\alpha}}|_p < 1$ and the $N_{t_1,\dots,t_k}(F)$ are integers, we have

$$
N_{t_1,\ldots,t_k}(F) \equiv F(1,\ldots,1)^{\varphi(p^{\alpha_1-t_1})\cdots\varphi(p^{\alpha_k-t_k})} \bmod p.
$$

In particular if $p \mid F(1, \ldots, 1)$ we have $p \mid N_{t_1, \ldots, t_k}(F)$ for all t_i and $|G|p^k \mid M_G(F)$. So, in view of [\(1\)](#page-1-0), we can assume for the p-group [\(5\)](#page-1-2) that $p \nmid F(1, \ldots, 1)$ for any F achieving $\lambda(G)$.

Thus, in the case of 2-groups we can assume an F with minimal measure has $F(1, \ldots, 1)$ odd, and by Lemma [1.1](#page-1-3) we see that

(6)
$$
M_G(F) \equiv 1 \mod 2^k.
$$

Note this immediately produces [\(4\)](#page-1-4).

Similarly for 3-groups we can assume that an F with minimal measure has $3 \nmid$ $F(1,\ldots,1)$ and $M_G(F) \equiv \pm 1 \mod 3^k$. This produces another case of equality in (1) :

$$
\lambda(\mathbb{Z}_3^k) = 3^k - 1,
$$

as observed in [\[2\]](#page-9-5). For $G = \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$, we have $M_G(F) \equiv \pm 1 \mod 9$ and so we immediately obtain the minimal measure for an additional family of 3-groups.

Theorem 1.3. For $n \geq 1$

$$
\lambda(\mathbb{Z}_3\times\mathbb{Z}_{3^n})=8,
$$

achieved with $F(x, y) = y + 1$.

Section [2](#page-2-0) of this article is devoted to the proof of Lemma [1.1,](#page-1-3) Section [3](#page-3-0) establishes Theorem [1.1,](#page-1-5) and Section [4](#page-7-0) proves Theorem [1.2.](#page-1-6)

2. Proof of Lemma [1.1](#page-1-3)

We proceed by induction on $\alpha_1 + \cdots + \alpha_k$. For $G = \mathbb{Z}_p$ we use that $|w_p - 1|_p =$ $p^{-1/(p-1)} < 1$. Since $M_G(F) \in \mathbb{Z}$ and $M_G(F) \equiv F(1)^p \mod{1 - w_p}$ we see that $M_G(F) \equiv F(1)^p \mod p.$

Set

$$
g(x_1, \ldots, x_k) = \prod_{l_1=1}^{p^{\alpha_1}} \cdots \prod_{l_k=1}^{p^{\alpha_k}} F(x_1^{l_1}, \ldots, x_k^{l_k})
$$

and let I be the ideal in $\mathbb{Z}[x_1,\ldots,x_n]$ generated by $x_1^{p^{\alpha_1}}-1,\ldots,x_k^{p^{\alpha_k}}-1$. Expanding, we have

$$
g(x_1,\ldots,x_k)=\sum_{0\leq \ell_1
$$

We set

$$
S := \sum_{j_1=1}^{p^{\alpha_1}} \cdots \sum_{j_k=1}^{p^{\alpha_k}} g(w_{p^{\alpha_1}}^{j_1}, \ldots, w_{p^{\alpha_k}}^{j_k})
$$

=
$$
\sum_{0 \leq \ell_1 < p^{\alpha_1}} \cdots \sum_{0 \leq \ell_k < p^{\alpha_k}} a(\ell_1, \ldots, \ell_k) \sum_{j_1=1}^{p^{\alpha_1}} \cdots \sum_{j_k=1}^{p^{\alpha_k}} w_{p^{\alpha_1}}^{j_1 \ell_1} \cdots w_{p^{\alpha_k}}^{j_k \ell_k}
$$

=
$$
a(0, \ldots, 0) p^{\alpha_1 + \cdots + \alpha_k}.
$$

If $(j_1, p^{\alpha_1}) = \cdots = (j_k, p^{\alpha_k}) = 1$, then for these $\varphi(p^{\alpha_1}) \cdots \varphi(p^{\alpha_k})$ values we have $\scriptstyle j_1$ j_k

$$
g(w_{p^{\alpha_1}}^{j_1},\ldots,w_{p^{\alpha_k}}^{j_k})=M_G(F).
$$

Suppose that $(j_1, p^{\alpha_1}) = p^{t_1}, \ldots, (j_k, p^{\alpha_k}) = p^{t_k}$ with at least one $t_j \neq 0$, and with $L \geq 0$ of the $t_i = \alpha_i$. Suppose without loss of generality that $t_i = \alpha_i$ for any $i =$ $1, \ldots, L$ and $t_i < \alpha_i$ for any $i = L+1, \ldots, k$. For these $\varphi(p^{\alpha_{L+1}-t_{L+1}}) \cdots \varphi(p^{\alpha_k-t_k})$ values, applying the induction hypothesis to $G' = \mathbb{Z}_{p^{\alpha_{L+1}-t}L+1} \times \cdots \times \mathbb{Z}_{p^{\alpha_k-t_k}}$, we have

$$
g(w_{p^{\alpha_1}}^{j_1}, \dots, w_{p^{\alpha_k}}^{j_k}) = M_{G'}(F(1, \dots, 1, x_{L+1}, \dots, x_k))^{p^{t_1 + \dots + t_k}}
$$

=
$$
(F(1, \dots, 1)^{p^{(\alpha_{L+1} - t_{L+1}) + \dots + (\alpha_k - t_k)}} + hp^{k-L})^{p^{t_1 + \dots + t_k}}
$$

\equiv
$$
F(1, \dots, 1)^{|G|} \mod p^{k-L+\alpha_1 + \dots + \alpha_L + t_{L+1} + \dots + t_k}.
$$

Hence these $(p-1)^{k-L} p^{(\alpha_{L+1}-t_{L+1}-1)+\cdots+(\alpha_k-t_k-1)}$ terms contribute

$$
\varphi(p^{\alpha_{L+1}-t_{L+1}})\cdots \varphi(p^{\alpha_k-t_k})F(1,\ldots,1)^{|G|} \bmod p^{\alpha_1+\cdots+\alpha_k}
$$

to S. Thus

$$
0 \equiv \varphi(p^{\alpha_1}) \cdots \varphi(p^{\alpha_k}) M_G(F) + (p^{\alpha_1 + \cdots + \alpha_k} - \varphi(p^{\alpha_1}) \cdots \varphi(p^{\alpha_k})) F(1, \ldots, 1)^{|G|}
$$

$$
\equiv (p-1)^k p^{\alpha_1 + \cdots + \alpha_k - k} \left(M_G(F) - F(1, \ldots, 1)^{|G|} \right) \bmod p^{\alpha_1 + \cdots + \alpha_k}
$$

and the statement follows. $\hfill \square$

3. Proof of Theorem [1.1](#page-1-5)

To prove Theorem [1.1,](#page-1-5) we require the following lemma.

Lemma 3.1. Suppose that $F \in \mathbb{Z}[x_1,\ldots,x_n]$, and let I denote the ideal of $\mathbb{Z}[x_1,\ldots,x_n]$ generated by $x_1^{n_1} - 1, ..., x_k^{n_k} - 1$. Then $F(w_{n_1}^{j_1}, ..., w_{n_k}^{j_k}) = 0$ for all $1 \leq j_i \leq n_i$ if and only if $F \in I$.

Proof. Plainly any F in I will have $F(w_{n_1}^{j_1},...,w_{n_k}^{j_k}) = 0$ for all $0 \leq j_i < n_i$. Conversely, suppose that $F(w_{n_1}^{j_1},...,w_{n_k}^{j_k})=0$ for all $0 \leq j_i < n_i$. Clearly any F can be reduced mod I to a polynomial of degree less than n_i in each x_i :

$$
F(x_1,\ldots,x_k)=\sum_{t_1=0}^{n_1-1}\cdots\sum_{t_k=0}^{n_k-1}a(t_1,\ldots,t_k)x_1^{t_1}\cdots x_k^{t_k} \text{ mod } I.
$$

Since $\sum_{j_i=0}^{n_i-1} w_{n_i}^{(t_i-T_i)j_i} = 0$ if $t_i \not\equiv T_i \mod n_i$ (and n_i otherwise) we have

$$
a(T_1,\ldots,T_k)=\frac{1}{|G|}\sum_{j_1=0}^{n_1-1}\cdots\sum_{j_k=0}^{n_k-1}F(v_{n_1}^{j_1},\ldots,v_{n_k}^{j_k})w_{n_1}^{-T_1j_1}\cdots w_{n_k}^{-T_kj_k}.
$$

So $a(T_1, \ldots, T_k) = 0$ for all $0 \le T_i < n_i$ and $F = 0$ mod I.

We now proceed to the proof of our first principal result.

Proof of Theorem [1.1.](#page-1-5) Suppose that $G = \mathbb{Z}_{2^{\alpha_1}} \times \cdots \times \mathbb{Z}_{2^{\alpha_k}}$ with $2^{\alpha_i} = 4$ for $1 \leq$ $i \leq s$ and $2^{\alpha_i} = 2$ for $s + 1 \leq i \leq k$. We write $r = k - s$. In view of [\(2\)](#page-1-7) and [\(4\)](#page-1-4) we may assume that $k \geq 2$ and $s \geq 1$. Suppose that $F(x_1, \ldots, x_k)$ has

$$
1 < |M_G(F)| < |G| - 1 = 2^{k+s} - 1.
$$

Suppose that $F(x_1, \ldots, x_k)$ is a non-unit with at least one of the x_j complex, say $x_1 = \pm i$, and set $G' = \mathbb{Z}_{2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{2^{\alpha_k}}$. Plainly we may write

$$
M_G(F) = AB,
$$

with

$$
A := M_{\mathbb{Z}_2 \times G'}(F), \quad B := M_{G'}(F(i, x_2, \ldots, x_k)F(-i, x_2, \ldots, x_k)).
$$

From [\(6\)](#page-2-1) we know that $M_G(F)$ and A, and hence B, are all congruent to 1 mod 2^k . Also B will be of the form $|a + ib|^2$ and hence cannot be negative. Since it contains a non-unit we have $B > 1$, hence $B \ge 2^k + 1$. If $A \ne 1$ then $|A| \ge 2^k - 1$ and $|M_G(F)| \ge (2^k - 1)(2^k + 1) = 4^k - 1 \ge |G| - 1$, so we must have $A = 1$. Thus if $F(x_1, x_2, \ldots, x_k)$ is a non-unit with $x_j = \pm i$, then we may assume $F(y_1, \ldots, y_k)$ is a unit if $y_j = \pm 1$. We have two possibilities:

Case (a). There is at least one non-unit $F(x_1, \ldots, x_k)$ with some $x_j = \pm i$. Case (b). $F(x_1, \ldots, x_k)$ is a unit whenever any of the $x_i = \pm i$.

With I denoting the ideal generated by the $x_j^{2^{\alpha_j}} - 1$, and splitting the x_1 dependence into even and odd exponents $p(x_1) = \alpha(x_1^2) + x_1\beta(x_1^2)$, we can write

$$
F(x_1,\ldots,x_k) = \sum_{\substack{0 \le \varepsilon_2,\ldots,\varepsilon_s \le 3, \\ 0 \le \varepsilon_1,\varepsilon_{s+1},\ldots,\varepsilon_k \le 1}} a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k)(x_1^2) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k} \mod I.
$$

Since $F(1,\ldots,1)=\sum a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k)(1)$ is odd, we know that at least one of the $a(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ (1) is odd. Replacing F by $x_1^{\delta_1} \cdots x_n^{\delta_n} F$ with $0 \leq \delta_1, \delta_{s+1}, \ldots, \delta_k \leq 1$ and $0 \le \delta_2, \ldots, \delta_s \le 3$, and reducing mod I, we can reshuffle the $a(\epsilon_1, \ldots, \epsilon_k)(x_1^2)$ and assume that $a(0, \ldots, 0)(1)$ is odd. Replacing F by $-F$ we can assume that $F(1,\ldots,1)\equiv 1 \bmod 4.$

Case (a). Suppose we have non-units with complex x_j . Reordering and taking $x_j \mapsto \pm x_1x_j$ for $2 \leq j \leq s$ and $x_j \mapsto \pm x_j$ for $s < j \leq k$ as necessary, we assume that the first of these is $\gamma_1 = F(i, 1, \ldots, 1)$. If (after the transformations) there are other non-units with complex entries in positions other than the first, by reordering and substituting $x_j \mapsto x_j x_2$ as necessary for $j \geq 3$, we may assume that $\gamma_2 = F(\pm i, i, \pm 1, \dots, \pm 1)$. We repeat this $1 \leq h \leq s$ times until we have h non-units $\gamma_j = F(a_{j1}, \ldots, a_{jk})$ with $a_{jj} = i$, $a_{j\ell} = \pm i$ for $1 \leq \ell < j$ and $a_{j\ell} = \pm 1$ for $h < \ell \leq k$, and $F(x_1, \ldots, x_k)$ is a unit whenever $x_{\ell} = \pm i$ with $h < \ell \leq s$ if $h < s$.

Since the $F(\pm 1, x_2, \ldots, x_k)$ are all units, with $F(1, \ldots, 1) = 1$, and

$$
a(0, ..., 0)(1) = \frac{2}{|G|} \sum_{\substack{x_2, ..., x_s = \pm i, \pm 1 \\ x_1, x_{s+1}, ..., x_k = \pm 1}} F(x_1, ..., x_k)
$$

is odd, plainly the $F(\pm 1, x_2, \ldots, x_k)$ must all be 1. Applying Lemma [3.1,](#page-3-1) we may therefore assume that

$$
F(x_1,\ldots,x_k) = 1 + (x_1^2 - 1) \sum_{\substack{0 \le \varepsilon_2,\ldots,\varepsilon_s \le 3, \\ 0 \le \varepsilon_1,\varepsilon_{s+1},\ldots,\varepsilon_k \le 1}} a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.
$$

Notice that the $F(\pm i, x_2, \ldots, x_k) \in \mathbb{Z}[i]$ will all have odd real part and even imaginary part. Moreover, writing $u = (1 - i)$ where $u^2 \mid 2$ and $x_j \equiv 1 \mod u$ for any $x_j = \pm 1$ or $\pm i$, the $F(\pm i, x_2, \ldots, x_k)$ must all be congruent mod u^3 in $\mathbb{Z}[i]$. Since $|u|_2 = 2^{-1/2}$ plainly two units $\pm 1, \pm i$, in $\mathbb{Z}[i]$ cannot be congruent mod u^3 unless they are equal. If $h \geq 2$ then we know that the $F(\pm i, \pm 1, x_3, \ldots, x_k)$ will all be units and so must be all 1 or all -1 . Replacing F by x_1^2 F we can assume that they are all 1. Applying Lemma [3.1](#page-3-1) we get

$$
F(x_1,\ldots,x_k) = 1 + (x_1^2 - 1)(x_2^2 - 1) \sum_{\substack{0 \le \varepsilon_3,\ldots,\varepsilon_s \le 3, \\ 0 \le \varepsilon_1,\varepsilon_2,\varepsilon_{s+1},\ldots,\varepsilon_k \le 1}} a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.
$$

Likewise, if $h \geq 3$ we have that $F(\pm i, \pm i, \pm 1, x_4, \ldots, x_k)$ are all units and 1 mod 4, so these must all equal 1. Applying the lemma and repeating up to $F(\pm i,\ldots,\pm i,\pm 1,x_{h+1},\ldots,x_k)$, we deduce that

$$
F(x_1,\ldots,x_k) = 1 + \prod_{j=1}^h (x_j^2 - 1) \sum_{\substack{0 \le \varepsilon_{h+1}, \ldots, \varepsilon_s \le 3, \\ 0 \le \varepsilon_1, \ldots, \varepsilon_h, \varepsilon_{s+1}, \ldots, \varepsilon_k \le 1}} a(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.
$$

If $s > h$, we further have that the $F(\pm i, \ldots, \pm i, x_{h+2}, \ldots, x_k)$ are all units. If $h \geq 2$ they will all be 1 mod 4 and so must all equal 1. If $h = 1$ then they are all 1 or all -1 and, by replacing F by x_1^2 F if necessary, we may assume they are all 1. Separating into real and imaginary parts, applying Lemma [3.1,](#page-3-1) then repeating for each variable, we find

$$
F(x_1,\ldots,x_k) = 1 + \prod_{j=1}^h (x_j^2 - 1) \prod_{j=h+1}^s (x_j^2 + 1) \sum_{0 \le \varepsilon_1,\ldots,\varepsilon_k \le 1} a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.
$$

Suppose that there are $t \geq 1$ conjugate pairs of non-units $F(a_{j1}, \ldots, a_{jk}) = \gamma_j$. Then plainly

(7)
$$
\gamma_j = a_j + ib_j, \quad a_j \equiv 1 \mod 2^s, \quad b_j \equiv 0 \mod 2^s.
$$

Trivially we have $|\gamma_j|^2 \geq 5$, and if $t \geq r + s$ then

$$
|M_G(F)| \ge 5^t \ge 5^r \cdot 5^s > 2^r \cdot 4^s - 1,
$$

so we can assume that

$$
(8) \t t \leq r + s - 1.
$$

If $t \leq r$ then, by using the transformation $x_{\ell} \mapsto x_{\ell}x_j$ if $x_j = -1$ to remove $x_{\ell} = -1$ with $\ell > j$, we can assume that the r-tuples $(x_{s+1},...,x_k)$ achieving the γ_j take the form

$$
(1, \ldots, 1), (\pm 1, 1, \ldots, 1), (\pm 1, \pm 1, 1, \ldots, 1), \ldots, (\underbrace{\pm 1, \ldots, \pm 1}_{t-1}, 1, \ldots, 1).
$$

In particular, $F(x_1, \ldots, x_k)$ will be a unit if $x_j = -1$ for any $s+t \leq j \leq k$. (If $s \geq 2$) the units will all be 1; if $s = 1$ we may need to take $x_1^2 F$ to make the value when

 $x_{s+t} = -1$ and hence the rest equal 1.) Successively applying the lemma again, we find

$$
F(x_1, \dots, x_k) = 1 + \prod_{j=1}^h (x_j^2 - 1) \prod_{j=h+1}^s (x_j^2 + 1) \prod_{j=s+t}^k (x_j + 1) R
$$

with

$$
R = \sum_{0 \leq \varepsilon_1, \dots, \varepsilon_{s+t-1} \leq 1} a(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{s+t-1}) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{s+t-1}^{\varepsilon_{s+t-1}}.
$$

Hence we obtain that

$$
\gamma_j = a_j + ib_j
$$
, $a_j \equiv 1 \mod 2^{s+r+1-t}$, $b_j \equiv 0 \mod 2^{s+r+1-t}$.

From [\(8\)](#page-5-0) and [\(7\)](#page-5-1) this is plainly also valid if $t > r$. Thus, we have

$$
|M_G(F)| = |\gamma_1| \cdots |\gamma_t| \ge (2^{r+s+1-t}-1)^{2t} > 2^{2t(r+s+1.5-t)} \ge 2^{2(r+s-1.5)} \ge 2^{r+2s}
$$

for $r \geq 1$. If $r = 0$ and $t \geq 2$ then we have $s \geq 2$, and from [\(7\)](#page-5-1) we obtain

$$
|M_G(F)| \ge (2^s - 1)^{2t} > 2^{2t(s - 0.5)} \ge 2^{4s - 2} \ge 4^s.
$$

Finally if $t = 1$ and $r = 0$ then, since $F(i, 1, \ldots, 1)$ and its conjugate are the only non-units, we know that $F(\pm i, -1, x_3, \ldots, x_k)$ are all units and so equal 1. Hence we can add an extra factor $(x_2 + 1)$ to get

$$
|M_G(F)| \ge (2^{s+1} - 1)^2 > 2^{2s}.
$$

Case (b). Since $a(0, \ldots, 0)(1)$ is odd, we know that $a(0, \ldots, 0)(-1)$ is odd. Since the $F(\pm i, x_2, \ldots, x_k)$ are all units and

$$
a(0, \ldots, 0)(-1) = \frac{1}{|G|/2} \sum_{\substack{x_1 = \pm i \\ x_2, \ldots, x_s = \pm i, \pm 1 \\ x_{s+1}, \ldots, x_k = \pm 1}} F(x_1, \ldots, x_k)
$$

is odd, plainly the $F(\pm i, x_2, \dots, x_k)$ must all be 1 or all be -1. Replacing F by x_1^2F we assume $F(\pm i, x_2, \ldots, x_k) = 1$. Applying Lemma [3.1](#page-3-1) to the real and imaginary parts we can assume that

$$
F(x_1,\ldots,x_k) = 1 + (x_1^2 + 1) \sum_{\substack{0 \le \varepsilon_2,\ldots,\varepsilon_s \le 3, \\ 0 \le \varepsilon_1,\varepsilon_{s+1},\ldots,\varepsilon_k \le 1}} a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.
$$

Notice that all the $F(\pm 1, x_2, \ldots, x_k) \equiv F(1, \ldots, 1) \equiv 1 \mod u^3$. Hence if $s > 1$ the units $F(\pm 1, \pm i, x_3, \ldots, x_k)$ are all 1. Applying the Lemma and repeating we obtain

$$
F(x_1,\ldots,x_k) = 1 + \prod_{j=1}^s (x_j^2 + 1) \sum_{0 \le \varepsilon_1,\ldots,\varepsilon_k \le 1} a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.
$$

Hence we have

$$
M_G(F) = M_{\mathbb{Z}_2^k}(f)
$$

where

$$
f(x_1,\ldots,x_k)=1+2^s\!\!\!\!\sum_{0\leq \varepsilon_1,\ldots,\varepsilon_k\leq 1}\!\!\!\!\!A(\varepsilon_1,\ldots,\varepsilon_k)x_1^{\varepsilon_1}\cdots x_k^{\varepsilon_k}.
$$

Suppose that there are t elements $f(\pm 1, \ldots, \pm 1)$ that are not ± 1 . If $t \geq k + s - 1$ then plainly $|M_G(F)| \geq 3^t \geq 3^{k+s-1} > 2^{k+s}$ since $k+s \geq 3$, so we assume that $t \leq k + s - 2$. Sending $x_j \mapsto -x_j$ we assume that one of them is $f(1, \ldots, 1) = \gamma_1$. If $t > 1$ then, reordering and mapping $x_{\ell} \mapsto x_{\ell}x_j$ if we have $\ell > j$ with $x_{\ell} =$ $x_j = -1$, we can assume that the remaining values are $\gamma_2 = f(-1, 1, \ldots, 1), \gamma_3 =$ $f(a_{31}, a_{32}, 1, \ldots, 1), \ldots, \gamma_t = f(a_{t1}, \ldots, a_{t(t-1)}, 1, \ldots, 1).$ If $t \leq k$ then we will have $f(x_1, \ldots, x_k) = 1$ whenever $x_j = -1$ for some $t \leq j \leq k$, and applying the lemma we find

$$
f(x_1,\ldots,x_k) = 1 + 2^s \prod_{j=t}^k (x_j+1) \sum_{0 \leq \varepsilon_1,\ldots,\varepsilon_{t-1} \leq 1} A(\varepsilon_1,\ldots,\varepsilon_{t-1}) x_1^{\varepsilon_1} \cdots x_{t-1}^{\varepsilon_{t-1}}.
$$

Thus the

$$
\gamma_j \equiv 1 \mod 2^{s+k-t+1}
$$

(with this trivially holding if $k \leq t-1$), and

$$
|M_G(F)| \ge (2^{s+k+1-t}-1)^t.
$$

For $t = 1$ this gives

$$
|M_G(F)| \ge 2^{s+k} - 1 = |G| - 1,
$$

and for $t\geq 2$

$$
|M_G(F)| \ge 2^{t(s+k+0.5-t)} \ge 2^{2s+2k-3} \ge 2^{s+k}.
$$

4. Proof of Theorem [1.2](#page-1-6)

Using $\Phi_i(x)$ to denote the jth cyclotomic polynomial and recalling (see [\[1\]](#page-9-6) or [\[5\]](#page-9-7)) that for $j > k$ the resultant $|\text{Res}(\Phi_j, \Phi_k)| = q^{\varphi(k)}$ if $j = kq^{\alpha}$ for some prime q and 1 otherwise, we see that

$$
M_{\mathbb{Z}_2 \times \mathbb{Z}_{2^n}}(1+y+y^2) = M_{\mathbb{Z}_{2^n}}(\Phi_3(y))^2 = \left(\prod_{j=0}^n |\text{Res}(\Phi_3, \Phi_{2^j})|\right)^2 = 9.
$$

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$. Reducing mod $x^2 - 1$, we can write our $F(x, y)$ in $\mathbb{Z}[x, y]$ in the form

$$
F(x, y) = A_0(y^2) + xA_1(y^2) + yA_2(y^2) + xyA_3(y^2).
$$

Plainly,

$$
M_G(F(x,y)) = M_{\mathbb{Z}_{2^n}}(F(1,y))M_{\mathbb{Z}_{2^n}}(F(-1,y)),
$$

where each of these measures is a product of $n + 1$ integers,

$$
M_{\mathbb{Z}_{2^n}}(f(y)) = \prod_{j=0}^n N_j(f), \quad N_j(f) := \text{Res}(f, \Phi_{2^j}),
$$

that is,

$$
N_0(f) = f(1),
$$
 $N_1(f) = f(-1),$ $N_2(f) = f(i)f(-i) = |f(i)|^2,$

and, writing $w_j := e^{2\pi i/2^j}$, for any $j = 3, \ldots, n$, we have

$$
N_j(f) = \prod_{\substack{k=1 \ k \text{ odd}}}^{2^j} f(w_j^k) = \prod_{\substack{k=1 \ k \text{ odd}}}^{2^{j-1}} f(w_j^k) f(-w_j^k) = |R_j(f)|^2,
$$

where

$$
R_j(f) := \prod_{\substack{k=1 \ k \equiv 1 \bmod 4}}^{2^{j-1}} f(w_j^k) f(-w_j^k) \in \mathbb{Z}[i], \quad 3 \le j \le n.
$$

Note $N_j(f)$ and $R_j(f)$ represent the norms of $f(w_j^k)$ from $\mathbb{Q}(w_j)$ to \mathbb{Q} and $\mathbb{Q}(i)$ respectively, and since they are algebraic integers they will be in $\mathbb Z$ and $\mathbb Z[i]$, respectively.

Since $|1 - w_j|_2 = 2^{-1/\varphi(2^j)}$, each $N_j(F(\pm 1, y)) \equiv F(1, 1)^{2^{j-1}} \mod 2$, and if $M_G(F) < 2^{2n+2}$ we can assume $F(1,1)$ and all the $N_j(F(\pm 1,y))$ are odd. Note that for all the $j \ge 2$ we have $N_j(F(\pm 1, y)) = |a + ib|^2 = a^2 + b^2 \equiv 1 \text{ mod } 4$.

If $|M_G(F)| < 9$ then $|M_{\mathbb{Z}_{2^n}}(F(1,y))|$ or $|M_{\mathbb{Z}_{2^n}}(F(-1,y))|$ must be 1. Replacing $x \mapsto -x$ as necessary we assume that

$$
1 < |M_{\mathbb{Z}_{2^n}}(F(1,y))| < 9, \quad |M_{\mathbb{Z}_{2^n}}(F(-1,y))| = 1.
$$

Since

$$
F(1,1) = A_0(1) + A_1(1) + A_2(1) + A_3(1)
$$

is odd, we can assume that at least one of the $A_i(1)$ is odd. Replacing F by xF or yF or xyF and reducing by $x^2 - 1$ as necessary, we may assume that $A_0(1)$ is odd. Replacing y by $-y$ and F by $-F$ as necessary, we may further assume that $|F(1, 1)| \ge |F(1, -1)|$ and $F(1, 1) > 0$.

Since

$$
F(1,-1) = A_0(1) + A_1(1) - A_2(1) - A_3(1),
$$

\n
$$
F(-1,1) = A_0(1) - A_1(1) + A_2(1) - A_3(1),
$$

\n
$$
F(-1,-1) = A_0(1) - A_1(1) - A_2(1) + A_3(1),
$$

we have

$$
A_0(1) = \frac{1}{4}(F(1,1) + F(1,-1) + F(-1,1) + F(-1,-1)),
$$

\n
$$
A_1(1) = \frac{1}{4}(F(1,1) + F(1,-1) - F(-1,1) - F(-1,-1)),
$$

\n
$$
A_2(1) = \frac{1}{4}(F(1,1) - F(1,-1) + F(-1,1) - F(-1,-1)),
$$

\n
$$
A_3(1) = \frac{1}{4}(F(1,1) - F(1,-1) - F(-1,1) + F(-1,-1)).
$$

Observe that

$$
F(1, w_j^k)F(1, -w_j^k) = (A_0(w_j^{2k}) + A_1(w_j^{2k}))^2 - w_j^{2k} (A_2(w_j^{2k}) + A_3(w_j^{2k}))^2
$$

and

$$
F(-1, w_j^k)F(-1, -w_j^k) = (A_0(w_j^{2k}) - A_1(w_j^{2k}))^2 - w_j^{2k} (A_2(w_j^{2k}) - A_3(w_j^{2k}))^2
$$

differ by

$$
4\left(A_0(w_j^{2k})A_1(w_j^{2k})-w_j^{2k}A_2(w_j^{2k})A_3(w_j^{2k})\right)\in 4\mathbb{Z}[w_{j-1}].
$$

Hence $R_j(F(1, y))$ and $R_j(F(-1, y))$ differ by an element of $4\mathbb{Z}[w_{j-1}]$ and, since both are in $\mathbb{Z}[i]$, we conclude that

$$
R_j(F(1, y)) - R_j(F(-1, y)) \in 4\mathbb{Z}[i].
$$

Since $N_i(F(-1, y)) = 1$, we have $R_i(F(-1, y)) = \pm 1$ or $\pm i$, and either $R_i(F(1, y)) =$ $R_j(F(-1, y))$ and $N_j(F(1, y)) = 1$, or $N_j(F(1, y)) \geq (4 - 1)^2 = 9$.

Thus if $|M_G(F)| < 9$ then we must have $N_j(F(1,y)) = N_j(F(-1,y)) = 1$ for $j = 3, \ldots, n$ and $M_G(F) = M_{\mathbb{Z}_2 \times \mathbb{Z}_4}(F)$. By Theorem [1.1](#page-1-5) and Lemma [1.1,](#page-1-3) we have $|M_{\mathbb{Z}_2\times\mathbb{Z}_4}(F)|\geq 7$ and $M_{\mathbb{Z}_2\times\mathbb{Z}_4}(F)\equiv 1 \bmod 4$, and so

$$
M_G(F) = M_{\mathbb{Z}_2 \times \mathbb{Z}_4}(F) = -7.
$$

Since $N_j(f) \equiv 1 \mod 4$ for $j \geq 2$ we must have $|F(1,1)F(1,-1)| = 7$ and $N_2(F(1, y)) = 1$ and

$$
F(1, 1) = 7
$$
, $F(1, -1)$, $F(-1, \pm 1) = \pm 1$, $F(\pm 1, \pm i) = \pm 1$ or $\pm i$,

with $R_i(F(1, y)) = R_i(F(-1, y)) = \pm 1$ or $\pm i$ for $j = 3, ..., n$. We have

$$
A_0(1) = \frac{1}{4}(F(1,1) + F(1,-1) + F(-1,1) + F(-1,-1)) = \frac{1}{4}(7 \pm 1 \pm 1 \pm 1)
$$

and, since $A_0(1)$ is odd, we must have $F(1, -1) = F(-1, \pm 1) = -1$ and $A_0(1) = 1$ and $A_1(1) = A_2(1) = A_3(1) = 2$. Hence

$$
F(x, y) = 1 + 2x + 2y + 2xy + (y^{2} - 1)(B_{0}(y^{2}) + xB_{1}(y^{2}) + yB_{2}(y^{2}) + xyB_{3}(y^{2})).
$$

Thus

Thus

$$
F(1,i) = 3 + 4i - 2(B_0(-1) + B_1(-1) + iB_2(-1) + iB_3(-1)),
$$

\n
$$
F(-1,i) = -1 - 2(B_0(-1) - B_1(-1) + iB_2(-1) - iB_3(-1)),
$$

and since $F(\pm 1, i)$ are units with odd real part and difference in $4\mathbb{Z}[i]$ they must be both be 1 or -1. By replacing F by y^2F as necessary, we may assume $F(\pm 1, i)$ = −1. Solving, we obtain $B_0(-1) = B_1(-1) = B_2(-1) = B_3(-1) = 1$ and

$$
F(x,y) = -1 + (1+x)(1+y)(1+y^2) + (y^4 - 1)(C_0(y^2) + xC_1(y^2) + yC_2(y^2) + xyC_3(y^2)).
$$

Therefore

$$
F(1, w_3)F(1, -w_3) = (1 + 2i - 2C_0(i) - 2C_1(i))^2 - 4i(1 + i - C_2(i) - C_3(i))^2
$$

and

$$
F(-1, w_3)F(-1, -w_3) = (-1 - 2C_0(i) + 2C_1(i))^2 - 4i(C_2(i) - C_3(i))^2.
$$

Since both are units and are members of $1+4\mathbb{Z}[i]$, these must both equal 1. However, their difference

$$
4((i - 2C_0(i))(1 + i - 2C_1(i)) - i(1 + i - 2C_3(i))(1 + i - 2C_2(i))) \in 4(1 + i + 2\mathbb{Z}[i])
$$

is not zero.

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