# THE LIND-LEHMER CONSTANT FOR $\mathbb{Z}_2^r \times \mathbb{Z}_4^s$

MICHAEL J. MOSSINGHOFF, VINCENT PIGNO, AND CHRISTOPHER PINNER

ABSTRACT. We show that the minimal positive logarithmic Lind-Mahler measure for a group of the form  $G = \mathbb{Z}_2^r \times \mathbb{Z}_4^s$  with  $|G| \ge 4$  is  $\frac{1}{|G|} \log(|G| - 1)$ . We also show that for  $G = \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$  with  $n \ge 3$  this value is  $\frac{1}{|G|} \log 9$ . Previously the minimal measure was only known for 2-groups of the form  $\mathbb{Z}_2^k$  or  $\mathbb{Z}_{2^k}$ .

#### 1. INTRODUCTION

Recall that for a polynomial  $F(x_1, \ldots, x_k)$  in  $\mathbb{Z}[x_1, \ldots, x_k]$ , one defines the traditional logarithmic Mahler measure by

$$m(F) = \int_0^1 \cdots \int_0^1 \log |F(e^{2\pi i x_1}, \dots, e^{2\pi i x_k})| \, dx_1 \cdots dx_k$$

In 2005, Lind [6] viewed  $[0,1]^k$  as the group  $(\mathbb{R}/\mathbb{Z})^k$  and generalized the Mahler measure to arbitrary compact abelian groups. In particular, for the finite abelian group

$$G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$$

and  $F \in \mathbb{Z}[x_1, \ldots, x_k]$ , we define the logarithmic Lind-Mahler measure by

$$m_G(F) = \frac{1}{|G|} \sum_{x_1=1}^{n_1} \cdots \sum_{x_k=1}^{n_k} \log |F(e^{2\pi i x_1/n_1}, \dots, e^{2\pi i x_k/n_k})|.$$

The close connection to the group determinant is explored by Vipismakul [8]. Writing

$$w_n := e^{2\pi i/n},$$

we plainly have

$$m_G(F) = \frac{1}{|G|} \log |M_G(F)|,$$

where

$$M_G(F) := \prod_{j_1=1}^{n_1} \cdots \prod_{j_k=1}^{n_k} F\left(w_{n_1}^{j_1}, \dots, w_{n_k}^{j_k}\right)$$

will be in  $\mathbb{Z}$ . Analogous to the classical Lehmer problem [4], we can ask for the minimal  $m_G(F) > 0$ , and to this end we define the *Lind-Lehmer constant* for G by

$$\lambda(G) := \min\{|M_G(F)| > 1 \mid F \in \mathbb{Z}[x_1, \dots, x_k]\}$$

Date: July 10, 2021.

<sup>2010</sup> Mathematics Subject Classification. Primary: 11R06; Secondary: 11B83, 11C08, 11G50, 11R09, 11T22, 43A40.

Key words and phrases. Lind-Lehmer constant, Mahler measure.

This work was supported in part by a grant from the Simons Foundation (#426694 to M. J. Mossinghoff).

We use  $|M_G(F)|$  rather than  $m_G(F)$  or  $|M_G(F)|^{1/|G|}$  so that we are dealing with integers; of course the minimal positive logarithmic measure will be  $\frac{1}{|G|} \log \lambda(G)$ . As Lind observed, for  $|G| \geq 3$  we always have the trivial bound

(1) 
$$\lambda(G) \le |G| - 1,$$

achieved, for example, by

$$F(x_1,...,x_k) = -1 + \prod_{i=1}^k \left(\frac{x_i^{n_i} - 1}{x_i - 1}\right).$$

Lind also showed that for prime powers  $p^\alpha$  with  $\alpha \geq 1$  we have

(2) 
$$\lambda(\mathbb{Z}_{p^{\alpha}}) = \begin{cases} 3, & \text{if } p = 2\\ 2, & \text{if } p \ge 3 \end{cases}$$

achieved with  $x^2 + x + 1$  if p = 2 and x + 1 if  $p \ge 3$ . Lind's results for cyclic groups were extended by Kaiblinger [3] and Pigno & Pinner [7] so that  $\lambda(\mathbb{Z}_m)$  is now known if 892 371 480  $\nmid m$ . The value for the *p*-group  $\mathbb{Z}_p^k$  was recently established by De Silva & Pinner [2], but little is known for direct products involving at least one term  $\mathbb{Z}_{p^{\alpha}}$  with  $\alpha \ge 2$ .

Here we are principally interested in the case of 2-groups

(3) 
$$G = \mathbb{Z}_{2^{\alpha_1}} \times \cdots \times \mathbb{Z}_{2^{\alpha_k}}.$$

It was shown in [2] that for all  $k \geq 2$ 

(4) 
$$\lambda(\mathbb{Z}_2^k) = 2^k - 1.$$

a case of equality in (1). We establish two main results regarding the Lind-Lehmer constant for groups of the form (3). First, we prove that equality occurs in (1) whenever G is a 2-group whose factors are all  $\mathbb{Z}_2$  or  $\mathbb{Z}_4$ .

**Theorem 1.1.** If  $G = \mathbb{Z}_2^r$  or  $\mathbb{Z}_4^s$  or  $\mathbb{Z}_2^r \times \mathbb{Z}_4^s$ , then

$$\lambda(G) = \max\{3, |G| - 1\}.$$

Second, we show that this is not true for all 2-groups: if we allow  $\alpha_i \geq 3$  in (3) then (1) need not be sharp.

Theorem 1.2. For  $n \ge 3$ 

$$\lambda(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) = 9,$$

achieved with  $F(x, y) = y^2 + y + 1$ .

Crucial to our proofs of these statements will be a congruence satisfied by  $M_G(F)$  when G is a p-group. This is a generalization of [2, Lemma 2.1] (see also [8, Theorem 2.1.2]).

Lemma 1.1. If p is a prime and

(5) 
$$G = \mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}},$$

then

$$M_G(F) \equiv F(1,\ldots,1)^{|G|} \mod p^k.$$

Notice that for the p-group (5) we have

$$M_G(F) = \prod_{t_1=0}^{\alpha_1} \cdots \prod_{t_k=0}^{\alpha_k} N_{t_1,\dots,t_k}(F),$$

where

$$N_{t_1,\dots,t_k}(F) = \prod_{\substack{j_1=1\\(j_1,p^{\alpha_1})=p^{t_1}}}^{p^{\alpha_1}} \cdots \prod_{\substack{j_k=1\\(j_k,p^{\alpha_k})=p^{t_k}}}^{p^{\alpha_k}} F(w_{p^{\alpha_1}}^{j_1},\dots,w_{p^{\alpha_k}}^{j_k}) \in \mathbb{Z}.$$

Since  $|1 - w_{p^{\alpha}}^{j}|_{p} < 1$  and the  $N_{t_{1},...,t_{k}}(F)$  are integers, we have

$$N_{t_1,\ldots,t_k}(F) \equiv F(1,\ldots,1)^{\varphi(p^{\alpha_1-t_1})\cdots\varphi(p^{\alpha_k-t_k})} \mod p.$$

In particular if p | F(1,...,1) we have  $p | N_{t_1,...,t_k}(F)$  for all  $t_i$  and  $|G|p^k | M_G(F)$ . So, in view of (1), we can assume for the *p*-group (5) that  $p \nmid F(1,...,1)$  for any *F* achieving  $\lambda(G)$ .

Thus, in the case of 2-groups we can assume an F with minimal measure has  $F(1, \ldots, 1)$  odd, and by Lemma 1.1 we see that

(6) 
$$M_G(F) \equiv 1 \mod 2^k.$$

Note this immediately produces (4).

Similarly for 3-groups we can assume that an F with minimal measure has  $3 \nmid F(1,\ldots,1)$  and  $M_G(F) \equiv \pm 1 \mod 3^k$ . This produces another case of equality in (1):

$$\lambda(\mathbb{Z}_3^k) = 3^k - 1,$$

as observed in [2]. For  $G = \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ , we have  $M_G(F) \equiv \pm 1 \mod 9$  and so we immediately obtain the minimal measure for an additional family of 3-groups.

Theorem 1.3. For  $n \ge 1$ 

$$\lambda(\mathbb{Z}_3 \times \mathbb{Z}_{3^n}) = 8,$$

achieved with F(x, y) = y + 1.

Section 2 of this article is devoted to the proof of Lemma 1.1, Section 3 establishes Theorem 1.1, and Section 4 proves Theorem 1.2.

#### 2. Proof of Lemma 1.1

We proceed by induction on  $\alpha_1 + \cdots + \alpha_k$ . For  $G = \mathbb{Z}_p$  we use that  $|w_p - 1|_p = p^{-1/(p-1)} < 1$ . Since  $M_G(F) \in \mathbb{Z}$  and  $M_G(F) \equiv F(1)^p \mod (1 - w_p)$  we see that  $M_G(F) \equiv F(1)^p \mod p$ .

Set

$$g(x_1, \dots, x_k) = \prod_{l_1=1}^{p^{\alpha_1}} \cdots \prod_{l_k=1}^{p^{\alpha_k}} F(x_1^{l_1}, \dots, x_k^{l_k})$$

and let *I* be the ideal in  $\mathbb{Z}[x_1, \ldots, x_n]$  generated by  $x_1^{p^{\alpha_1}} - 1, \ldots, x_k^{p^{\alpha_k}} - 1$ . Expanding, we have

$$g(x_1, \dots, x_k) = \sum_{0 \le \ell_1 < p^{\alpha_1}} \cdots \sum_{0 \le \ell_k < p^{\alpha_k}} a(\ell_1, \dots, \ell_k) x_1^{\ell_1} \cdots x_k^{\ell_k} \mod I.$$

We set

4

$$S := \sum_{j_1=1}^{p^{\alpha_1}} \cdots \sum_{j_k=1}^{p^{\alpha_k}} g(w_{p^{\alpha_1}}^{j_1}, \dots, w_{p^{\alpha_k}}^{j_k})$$
$$= \sum_{0 \le \ell_1 < p^{\alpha_1}} \cdots \sum_{0 \le \ell_k < p^{\alpha_k}} a(\ell_1, \dots, \ell_k) \sum_{j_1=1}^{p^{\alpha_1}} \cdots \sum_{j_k=1}^{p^{\alpha_k}} w_{p^{\alpha_1}}^{j_1 \ell_1} \cdots w_{p^{\alpha_k}}^{j_k \ell_k}$$
$$= a(0, \dots, 0) p^{\alpha_1 + \dots + \alpha_k}.$$

If  $(j_1, p^{\alpha_1}) = \cdots = (j_k, p^{\alpha_k}) = 1$ , then for these  $\varphi(p^{\alpha_1}) \cdots \varphi(p^{\alpha_k})$  values we have  $g(w_{p^{\alpha_1}}^{j_1}, \dots, w_{p^{\alpha_k}}^{j_k}) = M_G(F).$ 

Suppose that  $(j_1, p^{\alpha_1}) = p^{t_1}, \ldots, (j_k, p^{\alpha_k}) = p^{t_k}$  with at least one  $t_j \neq 0$ , and with  $L \geq 0$  of the  $t_i = \alpha_i$ . Suppose without loss of generality that  $t_i = \alpha_i$  for any  $i = 1, \ldots, L$  and  $t_i < \alpha_i$  for any  $i = L + 1, \ldots, k$ . For these  $\varphi(p^{\alpha_{L+1}-t_{L+1}}) \cdots \varphi(p^{\alpha_k-t_k})$  values, applying the induction hypothesis to  $G' = \mathbb{Z}_{p^{\alpha_{L+1}-t_{L+1}}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k-t_k}}$ , we have

$$g(w_{p^{\alpha_1}}^{j_1}, \dots, w_{p^{\alpha_k}}^{j_k}) = M_{G'} \left( F(1, \dots, 1, x_{L+1}, \dots, x_k) \right)^{p^{t_1 + \dots + t_k}}$$
$$= \left( F(1, \dots, 1)^{p^{(\alpha_{L+1} - t_{L+1}) + \dots + (\alpha_k - t_k)}} + hp^{k-L} \right)^{p^{t_1 + \dots + t_k}}$$
$$\equiv F(1, \dots, 1)^{|G|} \mod p^{k-L+\alpha_1 + \dots + \alpha_L + t_{L+1} + \dots + t_k}.$$

Hence these  $(p-1)^{k-L} p^{(\alpha_{L+1}-t_{L+1}-1)+\cdots+(\alpha_k-t_k-1)}$  terms contribute

$$\varphi(p^{\alpha_{L+1}-t_{L+1}})\cdots\varphi(p^{\alpha_k-t_k})F(1,\ldots,1)^{|G|} \mod p^{\alpha_1+\cdots+\alpha_k}$$

to S. Thus

$$0 \equiv \varphi(p^{\alpha_1}) \cdots \varphi(p^{\alpha_k}) M_G(F) + \left(p^{\alpha_1 + \dots + \alpha_k} - \varphi(p^{\alpha_1}) \cdots \varphi(p^{\alpha_k})\right) F(1, \dots, 1)^{|G|}$$
$$\equiv (p-1)^k p^{\alpha_1 + \dots + \alpha_k - k} \left( M_G(F) - F(1, \dots, 1)^{|G|} \right) \mod p^{\alpha_1 + \dots + \alpha_k}$$

and the statement follows.

### 3. Proof of Theorem 1.1

To prove Theorem 1.1, we require the following lemma.

**Lemma 3.1.** Suppose that  $F \in \mathbb{Z}[x_1, \ldots, x_n]$ , and let I denote the ideal of  $\mathbb{Z}[x_1, \ldots, x_n]$  generated by  $x_1^{n_1} - 1, \ldots, x_k^{n_k} - 1$ . Then  $F(w_{n_1}^{j_1}, \ldots, w_{n_k}^{j_k}) = 0$  for all  $1 \le j_i \le n_i$  if and only if  $F \in I$ .

*Proof.* Plainly any F in I will have  $F\left(w_{n_1}^{j_1}, \ldots, w_{n_k}^{j_k}\right) = 0$  for all  $0 \leq j_i < n_i$ . Conversely, suppose that  $F\left(w_{n_1}^{j_1}, \ldots, w_{n_k}^{j_k}\right) = 0$  for all  $0 \leq j_i < n_i$ . Clearly any F can be reduced mod I to a polynomial of degree less than  $n_i$  in each  $x_i$ :

$$F(x_1, \dots, x_k) = \sum_{t_1=0}^{n_1-1} \cdots \sum_{t_k=0}^{n_k-1} a(t_1, \dots, t_k) x_1^{t_1} \cdots x_k^{t_k} \mod I.$$

Since  $\sum_{j_i=0}^{n_i-1} w_{n_i}^{(t_i-T_i)j_i} = 0$  if  $t_i \neq T_i \mod n_i$  (and  $n_i$  otherwise) we have

$$a(T_1,\ldots,T_k) = \frac{1}{|G|} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_k=0}^{n_k-1} F\left(w_{n_1}^{j_1},\ldots,w_{n_k}^{j_k}\right) w_{n_1}^{-T_1j_1} \cdots w_{n_k}^{-T_kj_k}.$$

So  $a(T_1, \ldots, T_k) = 0$  for all  $0 \le T_i < n_i$  and  $F = 0 \mod I$ .

We now proceed to the proof of our first principal result.

Proof of Theorem 1.1. Suppose that  $G = \mathbb{Z}_{2^{\alpha_1}} \times \cdots \times \mathbb{Z}_{2^{\alpha_k}}$  with  $2^{\alpha_i} = 4$  for  $1 \leq i \leq s$  and  $2^{\alpha_i} = 2$  for  $s + 1 \leq i \leq k$ . We write r = k - s. In view of (2) and (4) we may assume that  $k \geq 2$  and  $s \geq 1$ . Suppose that  $F(x_1, \ldots, x_k)$  has

$$1 < |M_G(F)| < |G| - 1 = 2^{k+s} - 1.$$

Suppose that  $F(x_1, \ldots, x_k)$  is a non-unit with at least one of the  $x_j$  complex, say  $x_1 = \pm i$ , and set  $G' = \mathbb{Z}_{2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{2^{\alpha_k}}$ . Plainly we may write

$$M_G(F) = AB,$$

with

$$A := M_{\mathbb{Z}_2 \times G'}(F), \quad B := M_{G'}(F(i, x_2, \dots, x_k)F(-i, x_2, \dots, x_k)).$$

From (6) we know that  $M_G(F)$  and A, and hence B, are all congruent to 1 mod  $2^k$ . Also B will be of the form  $|a + ib|^2$  and hence cannot be negative. Since it contains a non-unit we have B > 1, hence  $B \ge 2^k + 1$ . If  $A \ne 1$  then  $|A| \ge 2^k - 1$  and  $|M_G(F)| \ge (2^k - 1)(2^k + 1) = 4^k - 1 \ge |G| - 1$ , so we must have A = 1. Thus if  $F(x_1, x_2, \ldots, x_k)$  is a non-unit with  $x_j = \pm i$ , then we may assume  $F(y_1, \ldots, y_k)$  is a unit if  $y_j = \pm 1$ . We have two possibilities:

Case (a). There is at least one non-unit  $F(x_1, \ldots, x_k)$  with some  $x_j = \pm i$ . Case (b).  $F(x_1, \ldots, x_k)$  is a unit whenever any of the  $x_j = \pm i$ .

With *I* denoting the ideal generated by the  $x_j^{2^{\alpha_j}} - 1$ , and splitting the  $x_1$  dependence into even and odd exponents  $p(x_1) = \alpha(x_1^2) + x_1\beta(x_1^2)$ , we can write

$$F(x_1, \dots, x_k) = \sum_{\substack{0 \le \varepsilon_2, \dots, \varepsilon_s \le 3, \\ 0 \le \varepsilon_1, \varepsilon_{s+1}, \dots, \varepsilon_k \le 1}} a(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)(x_1^2) \ x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k} \mod I.$$

Since  $F(1, ..., 1) = \sum a(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)(1)$  is odd, we know that at least one of the  $a(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)(1)$  is odd. Replacing F by  $x_1^{\delta_1} \cdots x_n^{\delta_n} F$  with  $0 \le \delta_1, \delta_{s+1}, ..., \delta_k \le 1$  and  $0 \le \delta_2, ..., \delta_s \le 3$ , and reducing mod I, we can reshuffle the  $a(\varepsilon_1, ..., \varepsilon_k)(x_1^2)$  and assume that a(0, ..., 0)(1) is odd. Replacing F by -F we can assume that  $F(1, ..., 1) \equiv 1 \mod 4$ .

**Case (a).** Suppose we have non-units with complex  $x_j$ . Reordering and taking  $x_j \mapsto \pm x_1 x_j$  for  $2 \leq j \leq s$  and  $x_j \mapsto \pm x_j$  for  $s < j \leq k$  as necessary, we assume that the first of these is  $\gamma_1 = F(i, 1, ..., 1)$ . If (after the transformations) there are other non-units with complex entries in positions other than the first, by reordering and substituting  $x_j \mapsto x_j x_2$  as necessary for  $j \geq 3$ , we may assume that  $\gamma_2 = F(\pm i, i, \pm 1, ..., \pm 1)$ . We repeat this  $1 \leq h \leq s$  times until we have h non-units  $\gamma_j = F(a_{j1}, ..., a_{jk})$  with  $a_{jj} = i$ ,  $a_{j\ell} = \pm i$  for  $1 \leq \ell < j$  and  $a_{j\ell} = \pm 1$  for  $h < \ell \leq k$ , and  $F(x_1, ..., x_k)$  is a unit whenever  $x_\ell = \pm i$  with  $h < \ell \leq s$  if h < s.

Since the  $F(\pm 1, x_2, \ldots, x_k)$  are all units, with  $F(1, \ldots, 1) = 1$ , and

$$a(0,\ldots,0)(1) = \frac{2}{|G|} \sum_{\substack{x_2,\ldots,x_s=\pm i,\pm 1\\x_1,x_{s+1},\ldots,x_k=\pm 1}} F(x_1,\ldots,x_k)$$

is odd, plainly the  $F(\pm 1, x_2, \ldots, x_k)$  must all be 1. Applying Lemma 3.1, we may therefore assume that

$$F(x_1, \dots, x_k) = 1 + (x_1^2 - 1) \sum_{\substack{0 \le \varepsilon_2, \dots, \varepsilon_s \le 3, \\ 0 \le \varepsilon_1, \varepsilon_{s+1}, \dots, \varepsilon_k \le 1}} a(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.$$

Notice that the  $F(\pm i, x_2, \ldots, x_k) \in \mathbb{Z}[i]$  will all have odd real part and even imaginary part. Moreover, writing u = (1 - i) where  $u^2 \mid 2$  and  $x_j \equiv 1 \mod u$  for any  $x_j = \pm 1$  or  $\pm i$ , the  $F(\pm i, x_2, \ldots, x_k)$  must all be congruent mod  $u^3$  in  $\mathbb{Z}[i]$ . Since  $|u|_2 = 2^{-1/2}$  plainly two units  $\pm 1, \pm i$ , in  $\mathbb{Z}[i]$  cannot be congruent mod  $u^3$  unless they are equal. If  $h \geq 2$  then we know that the  $F(\pm i, \pm 1, x_3, \ldots, x_k)$  will all be units and so must be all 1 or all -1. Replacing F by  $x_1^2 F$  we can assume that they are all 1. Applying Lemma 3.1 we get

$$F(x_1,\ldots,x_k) = 1 + (x_1^2 - 1)(x_2^2 - 1) \sum_{\substack{0 \le \varepsilon_3, \ldots, \varepsilon_s \le 3, \\ 0 \le \varepsilon_1, \varepsilon_2, \varepsilon_{s+1}, \ldots, \varepsilon_k \le 1}} a(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.$$

Likewise, if  $h \ge 3$  we have that  $F(\pm i, \pm 1, x_4, \ldots, x_k)$  are all units and 1 mod 4, so these must all equal 1. Applying the lemma and repeating up to  $F(\pm i, \ldots, \pm i, \pm 1, x_{h+1}, \ldots, x_k)$ , we deduce that

$$F(x_1,\ldots,x_k) = 1 + \prod_{j=1}^h (x_j^2 - 1) \sum_{\substack{0 \le \varepsilon_{h+1},\ldots,\varepsilon_s \le 3, \\ 0 \le \varepsilon_1,\ldots,\varepsilon_h,\varepsilon_{s+1},\ldots,\varepsilon_k \le 1}} a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.$$

If s > h, we further have that the  $F(\pm i, \ldots, \pm i, x_{h+2}, \ldots, x_k)$  are all units. If  $h \ge 2$  they will all be 1 mod 4 and so must all equal 1. If h = 1 then they are all 1 or all -1 and, by replacing F by  $x_1^2 F$  if necessary, we may assume they are all 1. Separating into real and imaginary parts, applying Lemma 3.1, then repeating for each variable, we find

$$F(x_1, \dots, x_k) = 1 + \prod_{j=1}^h (x_j^2 - 1) \prod_{j=h+1}^s (x_j^2 + 1) \sum_{0 \le \varepsilon_1, \dots, \varepsilon_k \le 1} a(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.$$

Suppose that there are  $t \ge 1$  conjugate pairs of non-units  $F(a_{j1}, \ldots, a_{jk}) = \gamma_j$ . Then plainly

(7) 
$$\gamma_j = a_j + ib_j, \quad a_j \equiv 1 \mod 2^s, \quad b_j \equiv 0 \mod 2^s.$$

Trivially we have  $|\gamma_i|^2 \ge 5$ , and if  $t \ge r+s$  then

$$|M_G(F)| \ge 5^t \ge 5^r \cdot 5^s > 2^r \cdot 4^s - 1,$$

so we can assume that

$$(8) t \le r+s-1.$$

If  $t \leq r$  then, by using the transformation  $x_{\ell} \mapsto x_{\ell} x_j$  if  $x_j = -1$  to remove  $x_{\ell} = -1$  with  $\ell > j$ , we can assume that the *r*-tuples  $(x_{s+1}, \ldots, x_k)$  achieving the  $\gamma_j$  take the form

$$(1,\ldots,1), (\pm 1,1,\ldots,1), (\pm 1,\pm 1,1,\ldots,1), \ldots, (\underbrace{\pm 1,\ldots,\pm 1}_{t-1},1,\ldots,1).$$

In particular,  $F(x_1, \ldots, x_k)$  will be a unit if  $x_j = -1$  for any  $s + t \le j \le k$ . (If  $s \ge 2$  the units will all be 1; if s = 1 we may need to take  $x_1^2 F$  to make the value when

 $x_{s+t} = -1$  and hence the rest equal 1.) Successively applying the lemma again, we find

$$F(x_1, \dots, x_k) = 1 + \prod_{j=1}^h (x_j^2 - 1) \prod_{j=h+1}^s (x_j^2 + 1) \prod_{j=s+t}^k (x_j + 1)R$$

with

$$R = \sum_{0 \le \varepsilon_1, \dots, \varepsilon_{s+t-1} \le 1} a(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{s+t-1}) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{s+t-1}^{\varepsilon_{s+t-1}}.$$

Hence we obtain that

$$\gamma_j = a_j + ib_j, \quad a_j \equiv 1 \mod 2^{s+r+1-t}, \ b_j \equiv 0 \mod 2^{s+r+1-t}.$$

From (8) and (7) this is plainly also valid if t > r. Thus, we have

$$|M_G(F)| = |\gamma_1| \cdots |\gamma_t| \ge (2^{r+s+1-t} - 1)^{2t} > 2^{2t(r+s+.5-t)} \ge 2^{2(r+s-.5)} \ge 2^{r+2s}$$

for  $r \ge 1$ . If r = 0 and  $t \ge 2$  then we have  $s \ge 2$ , and from (7) we obtain

$$|M_G(F)| \ge (2^s - 1)^{2t} > 2^{2t(s - 0.5)} \ge 2^{4s - 2} \ge 4^s.$$

Finally if t = 1 and r = 0 then, since F(i, 1, ..., 1) and its conjugate are the only non-units, we know that  $F(\pm i, -1, x_3, ..., x_k)$  are all units and so equal 1. Hence we can add an extra factor  $(x_2 + 1)$  to get

$$|M_G(F)| \ge (2^{s+1} - 1)^2 > 2^{2s}.$$

**Case (b).** Since  $a(0,\ldots,0)(1)$  is odd, we know that  $a(0,\ldots,0)(-1)$  is odd. Since the  $F(\pm i, x_2, \ldots, x_k)$  are all units and

$$a(0,\ldots,0)(-1) = \frac{1}{|G|/2} \sum_{\substack{x_1=\pm i\\x_2,\ldots,x_s=\pm i,\pm 1\\x_{s+1},\ldots,x_k=\pm 1}} F(x_1,\ldots,x_k)$$

is odd, plainly the  $F(\pm i, x_2, \ldots, x_k)$  must all be 1 or all be -1. Replacing F by  $x_1^2 F$  we assume  $F(\pm i, x_2, \ldots, x_k) = 1$ . Applying Lemma 3.1 to the real and imaginary parts we can assume that

$$F(x_1, \dots, x_k) = 1 + (x_1^2 + 1) \sum_{\substack{0 \le \varepsilon_2, \dots, \varepsilon_s \le 3, \\ 0 \le \varepsilon_1, \varepsilon_{s+1}, \dots, \varepsilon_k \le 1}} a(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.$$

Notice that all the  $F(\pm 1, x_2, \ldots, x_k) \equiv F(1, \ldots, 1) \equiv 1 \mod u^3$ . Hence if s > 1 the units  $F(\pm 1, \pm i, x_3, \ldots, x_k)$  are all 1. Applying the Lemma and repeating we obtain

$$F(x_1,\ldots,x_k) = 1 + \prod_{j=1}^s (x_j^2 + 1) \sum_{0 \le \varepsilon_1,\ldots,\varepsilon_k \le 1} a(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k) x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}.$$

Hence we have

$$M_G(F) = M_{\mathbb{Z}_2^k}(f)$$

where

$$f(x_1,\ldots,x_k) = 1 + 2^s \sum_{0 \le \varepsilon_1,\ldots,\varepsilon_k \le 1} A(\varepsilon_1,\ldots,\varepsilon_k) x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}.$$

Suppose that there are t elements  $f(\pm 1, \ldots, \pm 1)$  that are not  $\pm 1$ . If  $t \ge k + s - 1$ then plainly  $|M_G(F)| \ge 3^t \ge 3^{k+s-1} > 2^{k+s}$  since  $k + s \ge 3$ , so we assume that  $t \le k + s - 2$ . Sending  $x_j \mapsto -x_j$  we assume that one of them is  $f(1, \ldots, 1) = \gamma_1$ . If t > 1 then, reordering and mapping  $x_\ell \mapsto x_\ell x_j$  if we have  $\ell > j$  with  $x_\ell =$   $x_j = -1$ , we can assume that the remaining values are  $\gamma_2 = f(-1, 1, \ldots, 1), \gamma_3 = f(a_{31}, a_{32}, 1, \ldots, 1), \ldots, \gamma_t = f(a_{t1}, \ldots, a_{t(t-1)}, 1, \ldots, 1)$ . If  $t \leq k$  then we will have  $f(x_1, \ldots, x_k) = 1$  whenever  $x_j = -1$  for some  $t \leq j \leq k$ , and applying the lemma we find

$$f(x_1, \dots, x_k) = 1 + 2^s \prod_{j=t}^k (x_j + 1) \sum_{0 \le \varepsilon_1, \dots, \varepsilon_{t-1} \le 1} A(\varepsilon_1, \dots, \varepsilon_{t-1}) x_1^{\varepsilon_1} \cdots x_{t-1}^{\varepsilon_{t-1}}$$

Thus the

8

$$\gamma_j \equiv 1 \mod 2^{s+k-t+1}$$

(with this trivially holding if  $k \leq t - 1$ ), and

$$M_G(F) \ge (2^{s+k+1-t} - 1)^t.$$

For t = 1 this gives

$$|M_G(F)| \ge 2^{s+k} - 1 = |G| - 1,$$

and for  $t\geq 2$ 

$$M_G(F)| \ge 2^{t(s+k+0.5-t)} \ge 2^{2s+2k-3} \ge 2^{s+k}.$$

## 4. Proof of Theorem 1.2

Using  $\Phi_j(x)$  to denote the *j*th cyclotomic polynomial and recalling (see [1] or [5]) that for j > k the resultant  $|\text{Res}(\Phi_j, \Phi_k)| = q^{\varphi(k)}$  if  $j = kq^{\alpha}$  for some prime q and 1 otherwise, we see that

$$M_{\mathbb{Z}_{2} \times \mathbb{Z}_{2^{n}}}(1+y+y^{2}) = M_{\mathbb{Z}_{2^{n}}}(\Phi_{3}(y))^{2} = \left(\prod_{j=0}^{n} |\operatorname{Res}(\Phi_{3}, \Phi_{2^{j}})|\right)^{2} = 9.$$

Let  $G = \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ . Reducing mod  $x^2 - 1$ , we can write our F(x, y) in  $\mathbb{Z}[x, y]$  in the form

$$F(x,y) = A_0(y^2) + xA_1(y^2) + yA_2(y^2) + xyA_3(y^2).$$

Plainly,

$$M_G(F(x,y)) = M_{\mathbb{Z}_{2^n}}(F(1,y))M_{\mathbb{Z}_{2^n}}(F(-1,y)),$$

where each of these measures is a product of n + 1 integers,

$$M_{\mathbb{Z}_{2^n}}(f(y)) = \prod_{j=0}^n N_j(f), \quad N_j(f) := \operatorname{Res}(f, \Phi_{2^j}),$$

that is,

$$N_0(f) = f(1), \quad N_1(f) = f(-1), \quad N_2(f) = f(i)f(-i) = |f(i)|^2,$$

and, writing  $w_j := e^{2\pi i/2^j}$ , for any  $j = 3, \ldots, n$ , we have

$$N_j(f) = \prod_{\substack{k=1\\k \text{ odd}}}^{2^j} f(w_j^k) = \prod_{\substack{k=1\\k \text{ odd}}}^{2^{j-1}} f(w_j^k) f(-w_j^k) = |R_j(f)|^2,$$

where

$$R_j(f) := \prod_{\substack{k=1\\k\equiv 1 \mod 4}}^{2^{j-1}} f(w_j^k) f(-w_j^k) \in \mathbb{Z}[i], \quad 3 \le j \le n.$$

Note  $N_j(f)$  and  $R_j(f)$  represent the norms of  $f(w_j^k)$  from  $\mathbb{Q}(w_j)$  to  $\mathbb{Q}$  and  $\mathbb{Q}(i)$  respectively, and since they are algebraic integers they will be in  $\mathbb{Z}$  and  $\mathbb{Z}[i]$ , respectively.

Since  $|1 - w_j|_2 = 2^{-1/\varphi(2^j)}$ , each  $N_j(F(\pm 1, y)) \equiv F(1, 1)^{2^{j-1}} \mod 2$ , and if  $M_G(F) < 2^{2n+2}$  we can assume F(1, 1) and all the  $N_j(F(\pm 1, y))$  are odd. Note that for all the  $j \ge 2$  we have  $N_j(F(\pm 1, y)) = |a + ib|^2 = a^2 + b^2 \equiv 1 \mod 4$ .

If  $|M_G(F)| < 9$  then  $|M_{\mathbb{Z}_{2^n}}(F(1,y))|$  or  $|M_{\mathbb{Z}_{2^n}}(F(-1,y))|$  must be 1. Replacing  $x \mapsto -x$  as necessary we assume that

$$1 < |M_{\mathbb{Z}_{2n}}(F(1,y))| < 9, |M_{\mathbb{Z}_{2n}}(F(-1,y))| = 1.$$

Since

$$F(1,1) = A_0(1) + A_1(1) + A_2(1) + A_3(1)$$

is odd, we can assume that at least one of the  $A_i(1)$  is odd. Replacing F by xF or yF or xyF and reducing by  $x^2 - 1$  as necessary, we may assume that  $A_0(1)$  is odd. Replacing y by -y and F by -F as necessary, we may further assume that  $|F(1,1)| \ge |F(1,-1)|$  and F(1,1) > 0.

Since

j

$$F(1,-1) = A_0(1) + A_1(1) - A_2(1) - A_3(1),$$
  

$$F(-1,1) = A_0(1) - A_1(1) + A_2(1) - A_3(1),$$
  

$$F(-1,-1) = A_0(1) - A_1(1) - A_2(1) + A_3(1),$$

we have

$$A_{0}(1) = \frac{1}{4}(F(1,1) + F(1,-1) + F(-1,1) + F(-1,-1)),$$
  

$$A_{1}(1) = \frac{1}{4}(F(1,1) + F(1,-1) - F(-1,1) - F(-1,-1)),$$
  

$$A_{2}(1) = \frac{1}{4}(F(1,1) - F(1,-1) + F(-1,1) - F(-1,-1)),$$
  

$$A_{3}(1) = \frac{1}{4}(F(1,1) - F(1,-1) - F(-1,1) + F(-1,-1)).$$

Observe that

$$F(1, w_j^k)F(1, -w_j^k) = \left(A_0(w_j^{2k}) + A_1(w_j^{2k})\right)^2 - w_j^{2k} \left(A_2(w_j^{2k}) + A_3(w_j^{2k})\right)^2$$

and

$$F(-1, w_j^k)F(-1, -w_j^k) = \left(A_0(w_j^{2k}) - A_1(w_j^{2k})\right)^2 - w_j^{2k} \left(A_2(w_j^{2k}) - A_3(w_j^{2k})\right)^2$$

differ by

 $4\left(A_0(w_j^{2k})A_1(w_j^{2k}) - w_j^{2k}A_2(w_j^{2k})A_3(w_j^{2k})\right) \in 4\mathbb{Z}[w_{j-1}].$ 

Hence  $R_j(F(1,y))$  and  $R_j(F(-1,y))$  differ by an element of  $4\mathbb{Z}[w_{j-1}]$  and, since both are in  $\mathbb{Z}[i]$ , we conclude that

$$R_j(F(1,y)) - R_j(F(-1,y)) \in 4\mathbb{Z}[i].$$

Since  $N_j(F(-1, y)) = 1$ , we have  $R_j(F(-1, y)) = \pm 1$  or  $\pm i$ , and either  $R_j(F(1, y)) = R_j(F(-1, y))$  and  $N_j(F(1, y)) = 1$ , or  $N_j(F(1, y)) \ge (4 - 1)^2 = 9$ .

Thus if  $|M_G(F)| < 9$  then we must have  $N_j(F(1,y)) = N_j(F(-1,y)) = 1$  for  $j = 3, \ldots, n$  and  $M_G(F) = M_{\mathbb{Z}_2 \times \mathbb{Z}_4}(F)$ . By Theorem 1.1 and Lemma 1.1, we have  $|M_{\mathbb{Z}_2 \times \mathbb{Z}_4}(F)| \ge 7$  and  $M_{\mathbb{Z}_2 \times \mathbb{Z}_4}(F) \equiv 1 \mod 4$ , and so

$$M_G(F) = M_{\mathbb{Z}_2 \times \mathbb{Z}_4}(F) = -7.$$

Since  $N_j(f) \equiv 1 \mod 4$  for  $j \geq 2$  we must have |F(1,1)F(1,-1)| = 7 and  $N_2(F(1,y)) = 1$  and

$$F(1,1) = 7$$
,  $F(1,-1)$ ,  $F(-1,\pm 1) = \pm 1$ ,  $F(\pm 1,\pm i) = \pm 1$  or  $\pm i$ ,

with  $R_j(F(1,y)) = R_j(F(-1,y)) = \pm 1$  or  $\pm i$  for  $j = 3, \dots, n$ . We have

$$A_0(1) = \frac{1}{4}(F(1,1) + F(1,-1) + F(-1,1) + F(-1,-1)) = \frac{1}{4}(7 \pm 1 \pm 1 \pm 1)$$

and, since  $A_0(1)$  is odd, we must have  $F(1, -1) = F(-1, \pm 1) = -1$  and  $A_0(1) = 1$ and  $A_1(1) = A_2(1) = A_3(1) = 2$ . Hence

$$F(x,y) = 1 + 2x + 2y + 2xy + (y^2 - 1)(B_0(y^2) + xB_1(y^2) + yB_2(y^2) + xyB_3(y^2)).$$
  
Thus

Thus

$$F(1,i) = 3 + 4i - 2(B_0(-1) + B_1(-1) + iB_2(-1) + iB_3(-1)),$$
  

$$F(-1,i) = -1 - 2(B_0(-1) - B_1(-1) + iB_2(-1) - iB_3(-1)),$$

and since  $F(\pm 1, i)$  are units with odd real part and difference in  $4\mathbb{Z}[i]$  they must be both be 1 or -1. By replacing F by  $y^2F$  as necessary, we may assume  $F(\pm 1, i) =$ -1. Solving, we obtain  $B_0(-1) = B_1(-1) = B_2(-1) = B_3(-1) = 1$  and

$$F(x,y) = -1 + (1+x)(1+y)(1+y^2) + (y^4 - 1) \left( C_0(y^2) + xC_1(y^2) + yC_2(y^2) + xyC_3(y^2) \right)$$

Therefore

$$F(1, w_3)F(1, -w_3) = (1 + 2i - 2C_0(i) - 2C_1(i))^2 - 4i(1 + i - C_2(i) - C_3(i))^2$$

and

$$F(-1, w_3)F(-1, -w_3) = (-1 - 2C_0(i) + 2C_1(i))^2 - 4i(C_2(i) - C_3(i))^2$$

Since both are units and are members of  $1+4\mathbb{Z}[i]$ , these must both equal 1. However, their difference

$$4\left((i-2C_0(i))(1+i-2C_1(i))-i(1+i-2C_3(i))(1+i-2C_2(i))\right) \in 4(1+i+2\mathbb{Z}[i])$$
  
is not zero.

#### References

- T. M. Apostol, Resultants of cyclotomic polynomials, Proc. Amer. Math. Soc. 24 (1970) 457-462.
- [2] D. De Silva & C. Pinner, The Lind-Lehmer constant for Z<sup>n</sup><sub>p</sub>, Proc. Amer. Math. Soc. 142(6) (2014) 1935-1941.
- [3] N. Kaiblinger, On the Lehmer constant of finite cyclic groups, Acta Arith. 142(1) (2010) 79-84.
- [4] D.H. Lehmer, Factorization of certain cyclotomic functions, Ann. Math. 34(2) (1933) 461-479.
- [5] E. T. Lehmer, A numerical function applied to cyclotomy, Bull. Amer. Math. Soc. 36 (1930) 291-298.
- [6] D. Lind, Lehmer's problem for compact abelian groups, Proc. Amer. Math. Soc. 133 (2005) 1411-1416.
- [7] V. Pigno & C.G. Pinner, The Lind-Lehmer constant for cyclic groups of order less than 892,371,480, Ramanujan J. 33 (2014) 295-300.
- [8] W. Vipismakul, The stabilizer of the group determinant and bounds for Lehmer's conjecture on finite abelian groups, Ph. D. Thesis, University of Texas at Austin, 2013.

Department of Mathematics & Computer Science, Davidson College, Davidson, NC 28035-6996, USA

 $E\text{-}mail\ address:\ \texttt{mimossinghoff} @\texttt{davidson.edu}$ 

Department of Mathematics & Statistics, California State University, Sacramento, CA 95819, USA

 $E\text{-}mail\ address:\ \texttt{vincent.pigno@csus.edu}$ 

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA  $E\text{-}mail\ address:\ \texttt{pinner@math.ksu.edu}$