CONNECTIONS AND RESTRICTIONS TO CURVES

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Abstract. We construct a vector bundle E on a smooth complex projective surface X with the property that the restriction of E to any smooth closed curve in X admits an algebraic connection while E does not admit any algebraic connection.

1. Introduction

Let X be an irreducible smooth complex projective variety with cotangent bundle Ω^1_X and E a vector bundle on X. The coherent sheaf of local sections of E will also be denoted by E. A connection on E is a k-linear homomorphism of sheaves $D : E \longrightarrow E \otimes \Omega^1_X$ satisfying the Leibniz identity which says that $D(fs) = fD(s) + s \otimes df$, where s is a local section of E and f is a locally defined regular function.

Consider the sheaf of differential operators $\text{Diff}^i_X(E, E)$, of order i on E, and the associated symbol homomorphism $\sigma : \text{Diff}^1_X(E, E) \longrightarrow \text{End}(E) \otimes TX$. The inverse image

$$\text{At}(E) := \sigma^{-1}(\text{Id}_E \otimes TX)$$

is the Atiyah bundle for E. The resulting short exact sequence

$$0 \longrightarrow \text{Diff}^0_X(E, E) = \text{End}(E) \longrightarrow \text{At}(E) \xrightarrow{\sigma} TX \longrightarrow 0 \qquad (1.1)$$

is called the Atiyah exact sequence for E. A connection on E is a splitting of (1.1). We refer the reader to [At] for the details, in particular, see [At, p. 187, Theorem 1] and [At, p. 194, Proposition 9].

When X is a complex curve, Weil and Atiyah proved the following [We], [At]:

A vector bundle V on an irreducible smooth projective curve defined over $\mathbb{C}$ admits a connection if and only if the degree of each indecomposable component of V is zero.

This was first proved in [We]; see also [Gr, p. 69, THÉORÈME DE WEIL] for an exposition of it. The above criterion also follows from [At, p. 188, Theorem 2], [At, p. 201, Theorem 8] and [At, Theorem 10].

A semistable vector bundle V on a smooth complex projective variety X admits a connection if all the rational Chern classes of E vanish [Si, p. 40, Corollary 3.10]. On the other hand, a vector bundle W on X is semistable if and only if the restriction of W to a general complete intersection curve, which is an intersection of hyperplanes of sufficiently large degrees, is semistable [Fl, p. 637, Theorem 1.2], [MR, p. 221, Theorem 6.1]. On the other hand, any vector bundle E whose restriction to every curve is semistable actually satisfies very strong conditions [BB]; for example, if X is simply connected, then E must be of the form $L^{\oplus r}$ for some line bundle L.

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The following is a natural question to ask:

Question 1.1. *Let E be a vector bundle on X such that for every smooth closed curve $C \subset X$, the restriction $E|_C$ admits a connection. Does E admit a connection?*

Our aim is to show that in general the above vector bundle E does not admit a connection.

To produce an example of such a vector bundle, we construct a smooth complex projective surface X with $\text{Pic}(X) = \mathbb{Z}$ such that X admits an ample line bundle L_0 with $H^1(X, L_0) \neq 0$. Since $\text{Pic}(X) = \mathbb{Z}$, the ample line bundles on X are naturally parametrized by positive integers. Let L be the smallest ample line bundle (with respect to this parametrization) with the property that $H^1(X, L) \neq 0$. Let E be a nontrivial extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

We prove that the vector bundle $\text{End}(E)$ has the property that the restriction of it to every smooth closed curve in X admits a connection, while $\text{End}(E)$ does not admit a connection; see Theorem 3.1.

A surface X of the above type is constructed by taking a hyper-Kähler 4–fold X' with $\text{Pic}(X') = \mathbb{Z}$. Let $Y \subset X'$ be a smooth ample hypersurface such that $H^j(X', \mathcal{O}_{X'}(Y)) = 0$ for $j = 1, 2$, and let Z be a very general ample hypersurface of X' such that $H^j(X', \mathcal{O}_{X'}(Z)) = 0$ for $j = 1, 2$ and $H^2(X', \mathcal{O}_{X'}(Z-Y)) = 0$. Now take the surface X to be the intersection $Y \cap Z$.

2. Construction of a surface

We will construct a smooth complex projective surface S with Picard group $\mathbb{Z}$ that has an ample line bundle L with $H^1(S, L) \neq 0$.

Let X be a hyper-Kähler 4–fold with Picard group $\mathbb{Z}$. For example a sufficiently general deformation of $\text{Hilb}^2(M)$, where M is a polarized $K3$ surface, will have this property. Let $Y \subset X$ be a smooth ample hypersurface. Note that the vanishing theorem of Kodaira says that

$$H^j(X, \mathcal{O}_X(Y)) = 0 \tag{2.1}$$

for all $j > 0$, because K_X is trivial [Ko]. Let Z be a very general ample hypersurface of X such that both the line bundles $\mathcal{O}_X(Z)$ and $\mathcal{O}_X(Z-Y)$ are ample. In view of the vanishing theorem of Kodaira, the ampleness of $\mathcal{O}_X(Z)$ implies that

$$H^j(X, \mathcal{O}_X(Z)) = 0 \tag{2.2}$$

for all $j > 0$, while that of $\mathcal{O}_X(Z-Y)$ implies that

$$H^j(X, \mathcal{O}_X(Z-Y)) = 0 \tag{2.3}$$

for all $j > 0$. Let

$$\iota : S := Y \cap Z \hookrightarrow X$$

be the intersection and

$$L := \mathcal{O}_X(Y)|_S$$

the restriction of it. Note that L is ample.

Let $\mathcal{I} := \mathcal{O}_X(-S) \subset \mathcal{O}_X$ be the ideal sheaf for S. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_S \longrightarrow 0$$

by $\mathcal{O}_X(Y)$ we get an exact sequence

$$0 \longrightarrow \mathcal{I}(Y) \longrightarrow \mathcal{O}_X(Y) \longrightarrow \iota_* L \longrightarrow 0\,. \tag{2.4}$$

The natural inclusion of $\mathcal{O}_X(-Z)$ in $\mathcal{O}_X$ and $\mathcal{O}_X(Y-Z)$ together produce an inclusion of $\mathcal{O}_X(-Z)$ in $\mathcal{O}_X \oplus \mathcal{O}_X(Y-Z)$. Consequently, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-Z) \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_X(Y-Z) \longrightarrow \mathcal{I}(Y) \longrightarrow 0\,. \tag{2.5}$$

In view of (2.1), the connecting homomorphism

$$H^1(S,\, L) \longrightarrow H^2(X,\, \mathcal{I}(Y)) \tag{2.6}$$

in the long exact sequence of cohomologies associated to (2.4) is an isomorphism.

Since the canonical line bundle of X is trivial, Serre duality gives

$$H^{2+j}(X,\, \mathcal{O}_X(-Z))^* \;=\; H^{2-j}(X,\, \mathcal{O}_X(Z))\,.$$

So using (2.2) we conclude that the left-hand side vanishes for $j \;=\; 0,\, 1$. Again by Serre duality

$$H^2(X,\, \mathcal{O}_X(Y-Z))^* \;=\; H^2(X,\, \mathcal{O}_X(Z-Y)) \;=\; 0$$

(see (2.3)).

Thus in the long exact sequence of cohomologies associated to (2.5), we have

$$H^2(X,\, \mathcal{O}_X(-Z)) \;=\; 0 \;=\; H^{2+j}(X,\, \mathcal{O}_X(-Z))\,, \quad \text{and} \quad H^2(X,\, \mathcal{O}_X(Y-Z)) \;=\; 0\,.$$

Hence this long exact sequence of cohomologies associated to (2.5) gives an isomorphism

$$H^2(X,\, \mathcal{O}_X) \stackrel{\sim}{\longrightarrow} H^2(X,\, \mathcal{I}(Y))\,;$$

so combining this with the isomorphism in (2.6) it now follows that $H^1(S,\, L)$ is isomorphic to $H^2(X,\, \mathcal{O}_X)$. We have $\dim H^2(X,\, \mathcal{O}_X) \;=\; 1$, so

$$\dim H^1(S,\, L) \;=\; 1\,. \tag{2.7}$$

By Grothendieck–Lefschetz hyperplane theorem for Picard group, the restriction map $\text{Pic}(X) \longrightarrow \text{Pic}(Y)$ is an isomorphism [SGA2, Exposeé XII]; in fact, a weaker version given in [Ha, Chapter IV, p. 179, Corollary 3.2] suffices for our purpose. By the generalized Noether–Lefschetz theorem (see [Jo, p. 121, Theorem 5.1]), the restriction map $\text{Pic}(Y) \longrightarrow \text{Pic}(S)$ is also an isomorphism. Thus $\text{Pic}(S)$ is isomorphic to $\mathbb{Z}$. Combining this with (2.7) it follows that the surface S has the desired properties.

3. Question 1.1 in special cases

In this section will will first use the construction in Section 2 to show that Question 1.1 in the introduction has a negative answer in general. Then we will show that in some particular cases the answer is affirmative.

3.1. **Example with a negative answer.** We will construct a smooth projective surface X and a vector bundle E on it that does not admit any connection while the restriction of E to every smooth curve in X admits a connection.

Let X be a smooth complex projective surface with $\text{Pic}(X) = \mathbb{Z}$ that admits an ample line bundle L with $H^1(X, L) \neq 0$; we saw in Section 2 that such a surface exists. Let $\mathcal{O}_X(1)$ denote the ample generator of $\text{Pic}(X)$. Then $L = \mathcal{O}_X(r) = \mathcal{O}_X(1)^{\otimes r}$ with r positive. We choose L with smallest possible r. Since $\text{Pic}(X) = \mathbb{Z}$, we have $H^1(X, \mathcal{O}_X) = 0$ because $H^1(X, \mathcal{O}_X) = 0$ is the (abelian) Lie algebra of the Lie group $\text{Pic}(X)$. On the other hand, the Kodaira vanishing theorem says that $H^1(X, \mathcal{O}_X(-k)) = 0$ for all $k > 0$. Therefore, it follows that

$$H^1(X,\, L\otimes \mathcal{O}_X(-d)) \,=\, 0\,, \forall \;\, d \,>\, 0\,. \tag{3.1}$$

Let

$$0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{3.2}$$

be the non-split extension corresponding to a non-zero element in $H^1(X, L)$.

Theorem 3.1. *The vector bundle* $\text{End}(E) = E \otimes E^*$ *in* (3.2) *has the property that the restriction of it to every smooth closed curve in* X *admits a connection. The vector bundle* $\text{End}(E)$ *does not admit a connection.*

Proof. Take any smooth closed curve $C \subset X$. So $C \in |\mathcal{O}_X(d)|$ with d positive. Consider the restriction homomorphism $H^1(X, L) \longrightarrow H^1(C, L|_C)$. Using the long exact sequence of cohomologies associated to

$$0 \longrightarrow L\otimes \mathcal{O}_X(-d) \longrightarrow L \longrightarrow L|_C \longrightarrow 0$$

we conclude that its kernel is $H^1(X, L\otimes \mathcal{O}_X(-d))$, which is zero by (3.1). In particular, the extension class for (3.2) has a nonzero image in $H^1(C, L|_C)$. Therefore, the restriction of the exact sequence (3.2) to C does not split.

We will show that $E|_C$ is indecomposable.

Assume that $E|_C = L_1 \oplus L_2$ with $\text{degree}(L_1) \geq \text{degree}(L_2)$. Since $\text{degree}(E|_C) = \text{degree}(L|_C) > 0 = \text{degree}(\mathcal{O}_C)$, the composition

$$L_1 \hookrightarrow E|_C \longrightarrow \mathcal{O}_C$$

is the zero homomorphism. Hence L_1 coincides with the subbundle $L|_C \subset E|_C$. This contradicts the earlier observation that the restriction of the exact sequence (3.2) to C does not split. Hence we conclude that $E|_C$ is indecomposable.

Consider the projective bundle $\mathbb{P}(E|_C) \longrightarrow C$. Let $E_{\text{PGL}(2)} \longrightarrow C$ be the principal $\text{PGL}(2,\mathbb{C})$–bundle corresponding to it. Since E is indecomposable, it follows that $E_{\text{PGL}(2)}$ admits an algebraic connection [AB, p. 342, Theorem 4.1]. The vector bundle $\text{End}(E|_C) \longrightarrow C$ is associated to $E_{\text{PGL}(2)}$ for the adjoint action of $\text{PGL}(2,\mathbb{C})$ on $\text{End}_{\mathbb{C}}(\mathbb{C}^2) = \text{M}(2,\mathbb{C})$. Therefore, a connection on $E_{\text{PGL}(2)}$ induces a connection on the vector bundle $\text{End}(E|_C)$. Hence, we conclude that $\text{End}(E|_C) = \text{End}(E)|_C$ admits an algebraic connection.

On the other hand, $c_2(\text{End}(E)) = -c_1(L)^2 \neq 0$. This implies that the vector bundle E on X does not admit a connection [At, Theorem 4]. ☐

3.2. **Special cases with positive answer.** Let S be a smooth complex projective curve, X a smooth complex projective variety and $p : X \longrightarrow S$ a smooth surjective morphism such that every fiber of p is rationally connected. Assume that there is a smooth closed curve $\widetilde{S} \subset X$ such that the restriction

$$p|_{\widetilde{S}} : \widetilde{S} \longrightarrow S$$

is an étale morphism.

Lemma 3.2. *Let E be a vector bundle on X whose restriction to every smooth curve on X admits a connection. Then E admits a connection.*

Proof. Let Y be a smooth complex projective rationally connected variety and V a vector bundle on Y, such that for every smooth rational curve $\mathbb{CP}^1 \stackrel{\iota}{\hookrightarrow} Y$ the restriction ι^*V has a connection. Any connection on a curve is flat, and $\mathbb{CP}^1$ is simply connected, so the above vector bundle ι^*V is trivial. This implies that the vector bundle V is trivial [BdS, Proposition 1.2].

From the above observation it follows that $E = p^*p_*E$. Therefore, it suffices to show that p_*E admits a connection. Now, by the given condition, the vector bundle $(p|_{\widetilde{S}})^*p_*E = E|_{\widetilde{S}}$ admits a connection. Fix a connection D on $E|_{\widetilde{S}}$. Averaging D over the fibers of p we get a connection on p_*E. This completes the proof. ☐

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