On minimal Besicovitch arrangements

Blondeau Da Silva Stéphane

October 26, 2021

Abstract

In this paper we focus on minimal Besicovitch arrangements to highlight some of their properties. An appropriate probability space enables us to find again in an elegant way some straightforward equalities associated with these arrangements. Resulting inequalities are also brought out. A connection with arrangements of lines in \mathbb{R}^2 is eventually made, where possible.

Introduction

Let \mathbb{F}_q be the q elements finite field where q is a prime power. A Besicovitch arrangement B is a set of lines that contains at least one line in each direction. A minimal Besicovitch arrangement is a Besicovitch arrangement that is the union of exactly $q + 1$ lines in \mathbb{F}_q^2 (see [\[2\]](#page-14-0)).

Blondeau Da Silva proved (in [\[2\]](#page-14-0))that the expected value of the number of points of multiplicity m in \mathbb{F}_q^2 , for $0 \le m \le q+1$, with respect to a randomly chosen arrangement of $q + 1$ lines with different slopes: $(1/(m!e))q^2 + O(q)$, as $q \to \infty$. He also demonstrated that the distance between the number of points in such a randomly chosen arrangement and $(1/(m!e))q^2$ is lower than q ln q with probability close to 1 for large q .

In the first section of the paper, we take advantage of the probability space defined by Faber (in [\[4\]](#page-14-1)) and also of certain specific random variables to identify some of the constraints that such an arrangement is subject to. Equalities and inequalities between the multiplicities of all points in \mathbb{F}_q^2 determined by this arrangement will be brought to light. Other properties of these arrangements will be emphasised in the second section, allowing us to make a connection with quantities specific to arrangements of $q + 1$ lines in \mathbb{R}^2 .

1 Some constraints minimal Besicovitch arrangements are subject to

1.1 The chosen probability space

Let q be a prime power and \mathbb{F}_q be the q elements finite field.

A line, in \mathbb{F}_q^2 , is a one-dimensional affine subspace. For $s \in \mathbb{F}_q \cup \{\infty\},$ $b \in \mathbb{F}_q$, let $l(s, b)$ denote the line :

$$
\begin{cases}\n y = sx + b & \text{if } s \in \mathbb{F}_q, \\
 x = b & \text{if } s = \infty.\n\end{cases}
$$

Recall from [\[4\]](#page-14-1) that a Besicovitch set in \mathbb{F}_q^2 is a set of points $E \subset \mathbb{F}_q^2$ such that:

$$
\forall i \in \mathbb{F}_q \cup \{\infty\}, \quad \exists b_i \in \mathbb{F}_q \quad \text{so that } l(i, b_i) \subset E.
$$

Let B be a Besicovitch arrangement. Henceforth we denote by \tilde{B} the Besicovitch set formed by all the points belonging to the lines of the Besicovitch arrangement B.

We define our probability space Ω , as was done by Faber (in [\[4\]](#page-14-1)), *i.e.*: $\Omega = \bigoplus_{i \in \mathbb{F}_q \cup \{\infty\}} \mathbb{F}_q$, assigning probability $q^{-(q+1)}$ to each element in Ω . The associated probability law is denoted by P.

From now on, we identify each element $\sum_{i\in\mathbb{F}_q\cup\{\infty\}} b_i$ in Ω with the minimal Besicovitch arrangement $B = \bigcup_{i \in \mathbb{F}_q \cup \{\infty\}} l(i, b_i)$.

For a Besicovitch arrangement B, for $0 \leq m \leq q+1$, we define x_m^B as the number of points in \mathbb{F}_q^2 through which exactly m lines pass, *i.e.* the number of points of multiplicity m in \mathbb{F}_q^2 .

For $0 \leq m \leq q+1$, let us denote by X_m the random variable from Ω to $\mathbb N$ that maps each element $\sum_{i\in\mathbb{F}_q\cup\{\infty\}} b_i$ to x_m^B , where $B=\bigcup_{i\in\mathbb{F}_q\cup\{\infty\}} l(i,b_i)$.

1.2 The three equalities

We can take advantage of the probability space defined above to bring out some equalities associated with minimal Besicovitch arrangements. To this end, we will reuse the notations of the proof of $[2,$ Theorem 1...

For P in \mathbb{F}_q^2 , let M_P be the random variable that maps $B \in \Omega$ to the multiplicity of P in B. For $0 \le m \le q+1$, let $f_{m,P}$: $\Omega \to \mathbb{R}$ be the random variable defined by:

$$
f_{m,P}(B) = \begin{cases} 1 & \text{if } M_P(B) = m, \\ 0 & \text{otherwise.} \end{cases}
$$

It follows that for $0 \le m \le q+1$ and for $B \in \Omega$:

$$
X_m(B) = \sum_{P \in \mathbb{F}_q^2} f_{m,P}(B). \tag{1}
$$

1.2.1 The first equality

We first consider the random variable S that maps $B \in \Omega$ to $\frac{1}{q^2} \sum_{m=0}^{q+1} X_m(B)$. We have the following results:

Proposition 1.1. $E(S) = 1$ and $Var(S) = 0$.

Proof. We first compute the expected value of S :

$$
E(S) = \frac{1}{q^2} \sum_{m=0}^{q+1} E(X_m) = \frac{1}{q^2} \sum_{m=0}^{q+1} \left(\binom{q+1}{m} \left(\frac{1}{q}\right)^m \left(1 - \frac{1}{q}\right)^{q+1-m} \right) q^2 = 1,
$$

according to the proof of [\[2,](#page-14-0) Theorem 1.].

Then we determine the variance of S:

$$
Var(S) = Var(\frac{1}{q^2} \sum_{m=0}^{q+1} X_m) = E((\frac{1}{q^2} \sum_{m=0}^{q+1} X_m)^2) - E(S)^2
$$

= $\frac{1}{q^4} \sum_{(i,j) \in \{0, ..., q+1\}^2} \sum_{B \in \Omega} (\sum_{P \in \mathbb{F}_q^2} f_{i,P}(B) \sum_{Q \in \mathbb{F}_q^2} f_{j,Q}(B)) \mathbf{P}{B} - 1^2$
= $\frac{1}{q^4} \sum_{(i,j) \in \{0, ..., q+1\}^2} \sum_{B \in \Omega} \sum_{P,Q \in \mathbb{F}_q^2} (f_{i,P}(B) f_{j,Q}(B) \mathbf{P}{B}) - 1$
= $\frac{1}{q^4} \sum_{(i,j) \in \{0, ..., q+1\}^2} \sum_{P,Q \in \mathbb{F}_q^2} \mathbf{P}{f_{i,P} = f_{j,Q} = 1} - 1.$

Recall henceforth multinomial coefficients definition. Let n and p be positive integers. For $i \in \{1, 2, ..., p\}$, let k_i be positive integers such as $\sum_{i=1}^p k_i = n$. We have:

$$
\binom{n}{k_1, k_2, ..., k_p} = \frac{n!}{k_1! k_2! ... k_p!}.
$$

We note that the multinomial coefficient $\begin{pmatrix} a \\ b,c,d \end{pmatrix}$ (where $(a, b, c, d) \in \mathbb{N}^* \times \mathbb{Z}^3$) will be considered to be zero, if b , c or d is strictly negative.

We could demonstrate, in the same way as in the proof of [\[2,](#page-14-0) Theorem 1.], that for two distinct points P and Q in \mathbb{F}_q^2 and for $(i, j) \in \{1, ..., q\}^2$ such that $i + j \leq q$:

$$
\text{P}\{f_{i,P}=f_{j,Q}=1\}= \\ \binom{q}{i-1,j-1,q-i-j+2}\frac{(q-2)^{q-(i-1)}-(j-1)}{q^{q+1}}+\binom{q}{i,j,q-i-j}\frac{(q-1)(q-2)^{q-i-j}}{q^{q+1}}.
$$

The first term of the above sum corresponds to the case where $(PQ) \in B$, the second term to the case where $(PQ) \notin B$.

Let us consider the cases not examined yet:

- If $i = 0$ or $j = 0$ (and $i + j \le q$), the above equality is still valid; indeed, for all $B \in \Omega$, $(PQ) \notin B$ (by analogy with the proof of [\[2,](#page-14-0) Theorem 1.]), the first term of the equality is 0.
- ∘ If $i+j = q+1$ (respectively $q+2$), for all $B \in Ω$, the case where $(PQ) \notin B$ (by analogy with [\[2,](#page-14-0) Theorem 1.]) cannot occur. Otherwise (PQ) would intersect with at least $i + j = q + 1$ lines (respectively $q + 2$), which is impossible in a minimal Besicovitch arrangement composed of $q + 1$ lines. The above equality is still valid, its second term being 0. Moreover the particular case where $i + j = q + 1$ and i or j is 0 follows the same rule as in the above item.
- If $i + j > q + 2$, (PQ) would intersect with at least $q + 1$ lines $((i 1) +$ $(j - 1) \geq q + 1$, which is impossible as we just saw.

Therefore the equality is always valid. Then, since $P{f_{i,P} = f_{j,Q} = 1}$ does not depend on P and Q and $P{f_{i,P}=1}$ does not depend on P, we have:

$$
Var(S) = \frac{q^2 - 1}{q^2} \sum_{(i,j) \in \{0, \dots, q+1\}^2} P\{f_{i,P} = f_{j,Q} = 1\} + \frac{1}{q^2} \sum_{i \in \{0, \dots, q+1\}} P\{f_{i,P} = 1\} - 1.
$$
\n(2)

The first term of [\(2\)](#page-2-0) cuts into two parts. One checks that the first part is:

$$
(1 - \frac{1}{q^2}) \frac{1}{q^{q+1}} \sum_{(i,j) \in \{0,\dots,q+1\}^2} {q \choose i-1, j-1, q-i-j+2} (q-2)^{q-(i-1)-(j-1)}
$$

=
$$
(1 - \frac{1}{q^2}) \frac{q^q}{q^{q+1}}.
$$

The second part is:

$$
(1 - \frac{1}{q^2})\frac{q-1}{q^{q+1}} \sum_{(i,j) \in \{0,\dots,q+1\}^2} \binom{q}{i,j,q-i-j} (q-2)^{q-i-j} = (1 - \frac{1}{q^2})\frac{(q-1)q^q}{q^{q+1}}.
$$

The second term of [\(2\)](#page-2-0) is, thanks to the value of $P{f_{i,P} = 1}$ in the proof of [\[2,](#page-14-0) Theorem 1.]:

$$
\frac{1}{q^2}\sum_{i\in\{0,...,q+1\}}(\binom{q+1}{i}(\frac{1}{q})^i(1-\frac{1}{q})^{q+1-i})=\frac{1}{q^2}
$$

We finally obtain:

$$
Var(S) = (1 - \frac{1}{q^2})\left(\frac{q^q}{q^{q+1}} + \frac{(q-1)q^q}{q^{q+1}}\right) + \frac{1}{q^2} - 1 = 0.
$$

Hence, we can deduce from this proposition that, for all $B \in \Omega$, $S(B) = 1$ which can also be written as:

$$
\forall B \in \Omega, \qquad \sum_{m=0}^{q+1} x_m^B = q^2. \tag{3}
$$

This first trivial equality is based on the fact that $|\mathbb{F}_q^2| = q^2$, regardless of the arrangement chosen.

1.2.2 The second equality

Let us also consider the random variable M that maps $B \in \Omega$ to the mean multiplicity of the points in \mathbb{F}_q^2 , associated with the minimal Besicovitch arrangement B:

$$
\overline{M} = \frac{1}{q^2} \sum_{m=0}^{q+1} m X_m.
$$

We have the following results:

Proposition 1.2. $E(\overline{M}) = 1 + \frac{1}{q}$ and $Var(\overline{M}) = 0$.

Proof. Using the proof of [\[2,](#page-14-0) Theorem 1.], we first compute the expected value of \overline{M} :

$$
E(\overline{M}) = \frac{1}{q^2} \sum_{m=0}^{q+1} mE(X_m) = \frac{1}{q^2} \sum_{m=0}^{q+1} m\left(\binom{q+1}{m} \left(\frac{1}{q}\right)^m (1 - \frac{1}{q})^{q+1-m}\right) q^2
$$

=
$$
\frac{q+1}{q} \sum_{m=1}^{q+1} \binom{q}{m-1} \left(\frac{1}{q}\right)^{m-1} (1 - \frac{1}{q})^{q-(m-1)} = 1 + \frac{1}{q}.
$$

Then we determine the variance of \overline{M} :

$$
Var(\overline{M}) = Var(\frac{1}{q^2} \sum_{m=0}^{q+1} mX_m) = E((\frac{1}{q^2} \sum_{m=0}^{q+1} mX_m)^2) - E(\overline{M})^2
$$

= $\frac{1}{q^4} \sum_{(i,j) \in \{0,\dots,q+1\}^2} \sum_{B \in \Omega} (\sum_{P \in \mathbb{F}_q^2} i f_{i,P}(B) \sum_{Q \in \mathbb{F}_q^2} j f_{j,Q}(B)) \mathbb{P}{B} - (1 + \frac{1}{q})^2$
= $\frac{1}{q^4} \sum_{(i,j) \in \{0,\dots,q+1\}^2} \sum_{P,Q \in \mathbb{F}_q^2} i j \mathbb{P}{f_{i,P} = f_{j,Q} = 1} - (1 + \frac{1}{q})^2.$

Using the same reasoning as in Proposition [1.1,](#page-1-0) we have:

$$
Var(\overline{M}) = \frac{q^2 - 1}{q^2} \sum_{(i,j) \in \{0,\dots,q+1\}^2} ij \, P\{f_{i,P} = f_{j,Q} = 1\} + \frac{1}{q^2} \sum_{i \in \{0,\dots,q+1\}} i^2 \, P\{f_{i,P} = 1\} - (1 + \frac{1}{q})^2.
$$
\n⁽⁴⁾

The first term of [\(4\)](#page-4-0) cuts into two parts. Knowing that $ij = (i - 1)(j - 1) +$ $(i-1) + (j-1) + 1$, one checks that the first part is:

$$
(1 - \frac{1}{q^2}) \sum_{(i,j) \in \{0,\dots,q+1\}^2} ij \binom{q}{i-1, j-1, q-i-j+2} \frac{(q-2)^{q-(i-1)-(j-1)}}{q^{q+1}}
$$

=
$$
(1 - \frac{1}{q^2}) \frac{1}{q^{q+1}} \left(q(q-1)q^{q-2} + qq^{q-1} + qq^{q-1} + q^q \right)
$$

=
$$
(1 - \frac{1}{q^2}) \left(\frac{4q-1}{q^2} \right)
$$

The second part is:

$$
(1 - \frac{1}{q^2}) \sum_{(i,j) \in \{0, \dots, q+1\}^2} ij \binom{q}{i, j, q-i-j} \frac{(q-1)(q-2)^{q-i-j}}{q^{q+1}}
$$

=
$$
(1 - \frac{1}{q^2}) \frac{1}{q^{q+1}} ((q-1)q(q-1)q^{q-2})
$$

The second term of [\(4\)](#page-4-0) is, thanks to the value of $P{f_{i,P} = 1}$ in the proof of [\[2,](#page-14-0) Theorem 1.] (knowing that $i^2 = i(i-1) + i$):

$$
\frac{1}{q^2} \sum_{i \in \{0, \dots, q+1\}} \left(i(i-1) + i \right) \binom{q+1}{i} \left(\frac{1}{q} \right)^i \left(1 - \frac{1}{q} \right)^{q+1-i} = \frac{1}{q^2} \left(\frac{(q+1)q}{q^2} + \frac{q+1}{q} \right)
$$
\n
$$
= \frac{2(q+1)}{q^3}.
$$

Adding the different terms of [\(4\)](#page-4-0), we get:

$$
Var(\overline{M}) = (1 - \frac{1}{q^2})(\frac{4q - 1}{q^2} + \frac{(q - 1)^2}{q^2}) + (\frac{2(q + 1)}{q^3}) - 1 - \frac{1}{q^2} - \frac{2}{q} = 0.
$$

Hence, we can deduce from this proposition that, for all $B \in \Omega$, $\overline{M}(B) =$ $1 + \frac{1}{q}$, which can also be written as:

$$
\forall B \in \Omega, \qquad \sum_{m=0}^{q+1} mx_m^B = q(q+1). \tag{5}
$$

This result can simply be interpreted as follows: the sum of the multiplicity of all points in \mathbb{F}_q^2 is $q(q+1)$, regardless of the arrangement of $q+1$ lines chosen. Indeed, there are q points on each of the $q + 1$ lines, and therefore, there are $q(q + 1)$ points counted with multiplicity.

Remark 1.3. Let us denote by \mathscr{P} the subset of N constituted of all prime powers. Let us consider, for $P \in \mathbb{F}_q^2$, the random variable M_P defined in the beginning of the section. Recall that M_P follows a binomial distribution with parameters $q+1$ and $1/q$ (see the proof of [\[2,](#page-14-0) Theorem 1.]), its expected value being $1+\frac{1}{q}$ and its standard deviation being $\sqrt{\frac{(q+1)(q-1)}{q^2}}$. $\{M_P: P \in \mathbb{F}_q^2\}$ is a set of identically distributed random variables (indeed, the value of $P{f_{i,P}=1}$ doesnot depend on P). Let us denote by $(S_q)_{q \in \mathscr{P}}$ the family of random variables where, for $q \in \mathscr{P}$:

$$
S_q = \frac{(\sum_{P \in \mathbb{F}_q^2} M_P) - q^2 (1 + \frac{1}{q})}{q \sqrt{\frac{(q+1)(q-1)}{q^2}}} = \frac{(\sum_{P \in \mathbb{F}_q^2} M_P) - q(q+1)}{\sqrt{q^2-1}}.
$$

If $\{M_P: P \in \mathbb{F}_q^2\}$ is a set of independent random variables, the central limit theorem will state that the indexed family $(S_q)_{q\in\mathscr{P}}$ converges in law towards the standard normal distribution.

However, Proposition [1.2](#page-3-0) shows us that, for all $q \in \mathscr{P}$, S_q is the constant random variable with value O. Thus, the random variables of $\{M_P: P \in \mathbb{F}_q^2\}$ are dependent.

1.2.3 The third equality

Let us eventually consider the random variable V that maps $B \in \Omega$ to the second moment of the multiplicity of the points in \mathbb{F}_q^2 , associated with the minimal Besicovitch arrangement B:

$$
V = \frac{1}{q^2} \sum_{m=0}^{q+1} m^2 X_m.
$$

We have the following results:

Proposition 1.4. $E(V) = 2 + \frac{2}{q}$ and $Var(V) = 0$.

Proof. We first compute the expected value of V , using the proof of [\[2,](#page-14-0) Theorem 1.]:

$$
E(V) = \frac{1}{q^2} \sum_{m=0}^{q+1} m^2 E(X_m) = \frac{1}{q^2} \sum_{m=1}^{q+1} m^2 \left(\binom{q+1}{m} \left(\frac{1}{q}\right)^m (1 - \frac{1}{q})^{q+1-m} \right) q^2 = 2 + \frac{2}{q},
$$

the last equality being given by the calculation of the second term of [\(4\)](#page-4-0) in the proof of Proposition [1.2.](#page-3-0)

Then we determine the variance of V :

$$
Var(V) = Var(\frac{1}{q^2} \sum_{m=0}^{q+1} m^2 X_m) = E((\frac{1}{q^2} \sum_{m=0}^{q+1} m^2 X_m)^2) - E(V)^2
$$

= $\frac{1}{q^4} \sum_{(i,j) \in \{0,...,q+1\}^2} \sum_{B \in \Omega} (\sum_{P \in \mathbb{F}_q^2} i^2 f_{i,P}(B) \sum_{Q \in \mathbb{F}_q^2} j^2 f_{j,Q}(B)) P\{B\}$
 $- (2 + \frac{2}{q})^2$
= $\frac{1}{q^4} \sum_{(i,j) \in \{0,...,q+1\}^2} \sum_{P,Q \in \mathbb{F}_q^2} i^2 j^2 P\{f_{i,P} = f_{j,Q} = 1\} - (2 + \frac{2}{q})^2.$

Using the same reasoning as in Proposition [1.1,](#page-1-0) we have:

$$
Var(V) = \frac{q^2 - 1}{q^2} \sum_{(i,j) \in \{0,\dots,q+1\}^2} i^2 j^2 P\{f_{i,P} = f_{j,Q} = 1\}
$$

+
$$
\frac{1}{q^2} \sum_{i \in \{0,\dots,q+1\}} i^4 P\{f_{i,P} = 1\} - (2 + \frac{2}{q})^2.
$$
 (6)

The first term of [\(6\)](#page-6-0) cuts into two parts. Knowing that:

$$
i2j2 = i1i2j1j2 + 3i1i2j1 + 3i1j1j2 + 9i1j1 + i1i2 + j1j2 + 3i1 + 3j1 + 1,
$$

where $i_1 = i - 1$, $i_2 = i - 2$, $j_1 = j - 1$ and $j_2 = j - 2$, one checks that the first part is:

$$
(1 - \frac{1}{q^2}) \sum_{(i,j) \in \{0, \dots, q+1\}^2} i^2 j^2 {q \choose i-1, j-1, q-i-j+2} \frac{(q-2)^{q-(i-1)-(j-1)}}{q^{q+1}}
$$

=
$$
(1 - \frac{1}{q^2}) (\frac{q(q-1)(q-2)(q-3)}{q^5} + \frac{6q(q-1)(q-2)}{q^4} + \frac{11q(q-1)}{q^3} + \frac{6q}{q^2} + \frac{1}{q})
$$

=
$$
\frac{25q^5 - 35q^4 - 2q^3 + 29q^2 - 23q + 6}{q^6}.
$$

Knowing that $i^2 j^2 = i(i-1)j(j-1) + i(i-1)j + ij(j-1) + ij$, the second part is:

$$
(1 - \frac{1}{q^2}) \sum_{(i,j) \in \{0, \ldots, q+1\}^2} i^2 j^2 {q \choose i, j, q-i-j} \frac{(q-1)(q-2)^{q-i-j}}{q^{q+1}}
$$

=
$$
(1 - \frac{1}{q^2}) (\frac{q(q-1)^2(q-2)(q-3)}{q^5} + \frac{2q(q-1)^2(q-2)}{q^4} + \frac{q(q-1)^2}{q^3})
$$

=
$$
\frac{4q^6 - 17q^5 + 24q^4 - 4q^3 - 22q^2 + 21q - 6}{q^6}.
$$

The second term of [\(6\)](#page-6-0) is, thanks to the value of $P{f_{i,P} = 1}$ in the proof of [\[2,](#page-14-0) Theorem 1.] (and with the help of the following equality $i^4 = i(i-1)(i-2)(i-$

$$
3) + 6i(i - 1)(i - 2) + 7i(i - 1) + i):
$$

\n
$$
\frac{1}{q^2} \sum_{i \in \{1, \dots, q+1\}} i^4 {q+1 \choose m} (\frac{1}{q})^m (1 - \frac{1}{q})^{q+1-m}
$$

\n
$$
= \frac{1}{q^2} (\frac{(q+1)q(q-1)(q-2)}{q^4} + \frac{6(q+1)q(q-1)}{q^3} + \frac{7(q+1)q}{q^2} + \frac{q+1}{q})
$$

\n
$$
= \frac{15q^3 + 6q^2 - 7q + 2}{q^5}.
$$

Adding the different terms of [\(6\)](#page-6-0), we obtain the expected result.

 \Box

We can deduce again from this proposition that, for all $B \in \Omega$, $V(B) = 2 + \frac{2}{q}$, which can also be written as:

$$
\forall B \in \Omega, \qquad \sum_{m=0}^{q+1} m^2 x_m^B = 2q(q+1). \tag{7}
$$

Remark 1.5. We can recover the Incidence Formula in [\[4\]](#page-14-1), $|\tilde{B}| = \frac{q(q+1)}{2} +$ $\sum_{m=1}^{q+1} \frac{(m-1)(m-2)}{2}$ $\frac{2(m-2)}{2}x_m^B$, from equalities [\(5\)](#page-5-0) and [\(7\)](#page-7-0). Indeed, for B a minimal Besicovitch arrangement we have:

$$
|\tilde{B}| = \sum_{m=1}^{q+1} x_m^B = x_1^B + x_2^B + \sum_{m=3}^{q+1} x_m^B.
$$

We deduce from equalities [\(5\)](#page-5-0) and [\(7\)](#page-7-0):

$$
x_1^B = \sum_{m=3}^{q+1} (m^2 - 2m)x_m^B \text{ and } x_2^B = \frac{q(q+1)}{2} + \sum_{m=3}^{q+1} \frac{-m^2 + m}{2} x_m^B. \tag{8}
$$

Then $|\tilde{B}|$ is equal to $\frac{q(q+1)}{2} + \sum_{m=3}^{q+1} \frac{m^2 - 3m + 2}{2} x_m^B$.

We note that the second equality in [\(8\)](#page-7-1) can also be found as follows. If all the intersection points between two lines of our $q + 1$ lines arrangement are distinct, there will be $\binom{q+1}{2}$ points of multiplicity 2. Let *i* be an integer such that $3 \leq i \leq q+1$. A point of multiplicity i is a point through which i lines pass. It implies that the number of points of multiplicity 2 is reduced by $\binom{i}{2}$ for each of these points. Hence:

$$
x_2^B = {q+1 \choose 2} - \sum_{i=3}^{q+1} {i \choose 2} x_i^B.
$$

1.2.4 Topological invariants in minimal Besicovitch arrangements in $\mathbb{F}_q^{\;\;2}$ and examples

The three equalities $((3), (5)$ $((3), (5)$ $((3), (5)$ $((3), (5)$ and $(7))$ associated with minimal Besicovitch arrangements show some of the constraints they are subject to:

Corollary 1.6. For $B \in \Omega$:

$$
\left\{\begin{array}{ll} \sum_{m=0}^{q+1} x_m^B &= q^2\\ \sum_{m=1}^{q+1} m x_m^B &= q(q+1)\\ \sum_{m=0}^{q+1} m^2 x_m^B &= 2q(q+1). \end{array}\right.
$$

Remark 1.7. It may be noted that the values of $\sum_{m=0}^{q+1} m^3 x_m^B$ are far from being always the same for B in \mathbb{F}_q^2 .

Let us point out the existence of two particular examples which will be useful in the remainder of this section. Here is the first one:

Example 1.8. It is the extreme case where all the lines of the minimal Besicovitch arrangement B_0 are concurrent: $B_0 = \bigcup_{i \in \mathbb{F}_q \cup \{\infty\}} l(i,0)$ for example. We have $x_1^{B_0} = q^2 - 1, x_{q+1}^{B_0} = 1$ and $x_i^{B_0} = 0$ if $i \neq 1, q+1$.

Faber produces the second one in [\[4,](#page-14-1) Example.]:

Example 1.9. It is the minimal Besicovitch arrangement defined as below: $B_1 = \left(\bigcup_{i \in \mathbb{F}_q} l(i, -i^2)\right) \cup l(\infty, 0).$

If q is odd, $x_0^{B_1} = \frac{(q-1)^2}{2}$ $\left(x_1^{2-1}\right)^2$, $x_1^{B_1} = \frac{3q-3}{2}$, $x_2^{B_1} = \frac{q^2-2q+3}{2}$, $x_3^{B_1} = \frac{q-1}{2}$ and $x_i^{B_1} = 0$ if $i \neq 0, 1, 2, 3$. If q is even, $x_0^{B_1} = \frac{q(q-1)}{2}$ $\frac{a^{(n-1)}}{2}, x_1^{B_1} = 0, x_2^{B_1} = \frac{q(q+1)}{2}$ $\frac{x+1}{2}$ and $x_i^{B_1} = 0$ if $i \neq 0, 1, 2$.

1.3 Resulting inequalities

The following proposition can be deduced from the foregoing:

Proposition 1.10. Let m and q be positive integers such that $m \leq q+1$. We have:

$$
\begin{aligned}\n\max(\{x_m^B : B \in \Omega\}) &\leq \frac{2q(q+1)}{m^2} \quad \text{for } m \geq 2, \\
\max(\{x_0^B : B \in \Omega\}) &\leq \frac{q(q-1)}{2} \quad \text{and} \quad \max(\{x_1^B : B \in \Omega\}) = q^2 - 1.\n\end{aligned}
$$

Furthermore, as $q \to \infty$:

$$
\max(\{x_0^B : B \in \Omega\}) \backsim 1/2q^2.
$$

Proof. Let $B \in \Omega$ then we obtain thanks to equalities [\(3\)](#page-3-1) and [\(5\)](#page-5-0):

$$
x_0^B + q = \sum_{m=2}^{q+1} (m-1)x_m^B.
$$

Thanks to equalities [\(5\)](#page-5-0) and [\(7\)](#page-7-0), we get as well:

$$
q(q+1) = \sum_{m=2}^{q+1} m(m-1)x_m^B \ge 2\sum_{m=2}^{q+1} (m-1)x_m^B = 2(x_0^B + q).
$$

Thus:

$$
x_0^B \le \frac{q(q-1)}{2}.
$$

Example [1.9](#page-8-0) allows us to conclude that $max({x_0^B : B \in \Omega}) \sim 1/2q^2$, as q goes to infinity.

In addition, Example [1.8](#page-8-1) shows us that $max({x_1^B : B \in \Omega}) \ge q^2 - 1$. But the case where the number of simple points is equal to q^2 doesn't occur, because there is at least one point of multiplicity greater or equal to 2 in a minimal Besicovitch arrangement.

Eventually, for $B \in \Omega$ and for $2 \le m \le q+1$, equality [\(7\)](#page-7-0) gives us:

$$
x_m^B \le \frac{2q(q+1)}{m^2}.
$$

More precisely, we obtain the following two results. If q is even, $max({x_0^B :$ $B \in \Omega$ }) = $q(q-1)/2$ (Example [1.9\)](#page-8-0) and if q is odd, $max({x_0^B : B \in \Omega})$ = $(q-1)^2/2$ (see [\[1,](#page-14-2) Proposition 7] and also Example [1.9\)](#page-8-0).

For $B \in \Omega$, by considering only x_0^B , x_1^B and x_2^B , we highlight three inequalities:

Proposition 1.11. For $B \in \Omega$:

$$
\begin{cases}\n x_0^B + x_1^B + x_2^B \leq q^2 \\
 3x_0^B - x_2^B \leq q^2 - 2q \\
 3x_0^B + 2x_1^B + x_2^B \geq 2q^2 - q.\n\end{cases}
$$

Proof. The first inequality derives directly from equality [\(3\)](#page-3-1). The remaining two rely on equalities [\(8\)](#page-7-1):

$$
x_1^B = \sum_{m=3}^{q+1} (m^2 - 2m)x_m^B
$$
 and
$$
x_2^B = \frac{q(q+1)}{2} + \sum_{m=3}^{q+1} \frac{-m^2 + m}{2}x_m^B.
$$

Indeed:

$$
3x_0^B - x_2^B = 3(q^2 - \sum_{m=1}^{q+1} x_m^B) - x_2^B = 3q^2 - 3x_1^B - 4x_2^B - 3\sum_{m=3}^{q+1} x_m^B
$$

= $q^2 - 2q + \sum_{m=3}^{q+1} (-m^2 + 4m - 3)x_m^B \le q^2 - 2q$ as $m \ge 3$,

$$
3x_0^B + 2x_1^B + x_2^B = 3(q^2 - \sum_{m=1}^{q+1} x_m^B) + 2x_1^B + x_2^B = 3q^2 - x_1^B - 2x_2^B - 3\sum_{m=3}^{q+1} x_m^B
$$

= $2q^2 - q + \sum_{m=3}^{q+1} (m-3)x_m^B \ge 2q^2 - q$ as $m \ge 3$.

 \Box

Regarding Example [1.9,](#page-8-0) the sharpness of the three inequalities of Proposition [1.11](#page-9-0) can be emphasised: if q is even, all of these three inequalities are sharp.

Let us recall that, for $B \in \Omega$, $0 \leq x_i^B / q^2 \leq 1$ if $i = 0, 1, 2$ as $|\mathbb{F}_q^2| = q^2$. Let us then place ourselves in the unit cube of \mathbb{R}^3 and consider the subset of points (x, y, z) of \mathbb{R}^3 , displayed in green in the figure below, defined by:

$$
\begin{cases}\n x+y+z \leq 1 \\
 3x-z \leq 1 \\
 3x+2y+z \geq 2.\n\end{cases}
$$

As, for a sufficiently large q , the terms q or $2q$ of the inequalities of Proposition [1.11](#page-9-0) can be neglected compared to q^2 , all $(x_0^B/q^2, x_1^B/q^2, x_2^B/q^2)$, for $B \in \Omega$, are in the green subset with a $O(1/q)$ accuracy, as $q \to \infty$.

Most minimal Besicovitch arrangements would be represented by points located near the blue point with coordinates $(1/e, 1/e, 1/(2e))$, according to [\[2,](#page-14-0) Theorem 1.]. The point associated with Example [1.9](#page-8-0) and that one associated with Example [1.8](#page-8-1) tend respectively to the mauve and the yellow point, as q goes to infinity.

 \Box

Figure 1: The subset of the unit cube of \mathbb{R}^3 containing the points associated with the minimal Besicovitch arrangements.

$2 \quad \text{ connections between} \ \mathbb{F}_q^{\ 2} \ \text{and} \ \mathbb{R}^2$

In this section we will consider the definition of a finite incidence structure given in [\[5\]](#page-14-3) in a affine plane:

Definition 2.1. A finite incidence structure is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where \mathcal{P} and $\mathcal L$ are two finite disjoint sets and $\mathcal I$ is a subset such that $\mathcal I \subseteq \mathcal L \times \mathcal P$. Elements in P are the points of the affine plane, elements in $\mathcal L$ are the lines, whereas $\mathcal I$ is the incidence relation.

For $P \in \mathcal{P}$ and $L \in \mathcal{L}$, $(L, P) \in \mathcal{I}$ if and only if $P \in L$. For A an arrangement in an affine plane, we will use the following notation:

- $p_1(A)$ is the cardinality of A;
- $p_{01}(A)$ is the number of pairs consisting of a line of A and one of its point of intersection;
- $x_i(A)$ is the number of points of multiplicity i associated with A, for $i \geq 2$;
- $p_0(A)$ is the number of points defined by A (without counting multiplicity).

Note that $p_0(A) = \sum_{i=2}^{|A|} x_i(A)$. Henceforth, let us define the following equivalence relation:

Definition 2.2. Let A and B be two arrangements in the plane consisting of at least two lines.

 $(\mathcal{P}_A, \mathcal{L}_A, \mathcal{I}_A)$ is the incidence structure of A and $(\mathcal{P}_B, \mathcal{L}_B, \mathcal{I}_B)$ is that of B. The equivalence relation R is defined by:

$$
(\mathcal{P}_A, \mathcal{L}_A, \mathcal{I}_A) \mathcal{R}(\mathcal{P}_B, \mathcal{L}_B, \mathcal{I}_B) \iff |A| = |B| \text{ and } \forall i \in \{2, ..., |A|\}, \ x_i(A) = x_i(B).
$$

The equivalence class of $(\mathcal{P}_A, \mathcal{L}_A, \mathcal{I}_A)$ will be denoted by $[x_2(A), ..., x_{|A|}(A)].$

This being so, let us study the connections between \mathbb{F}_q^2 and \mathbb{R}^2 , where q is a prime power and \mathbb{F}_q the q elements finite field.

Let us denote by $\mathcal{S}_{\mathbb{F}_{q}}$ the set of incidence structures in \mathbb{F}_{q}^2 and $\mathcal{S}_{\mathbb{R}^2}$ the set of incidence structures in \mathbb{R}^2 . Let Φ be the identity function from $\mathcal{S}_{\mathbb{F}_q}$ / \mathcal{R} to $\mathcal{S}_{\mathbb{R}^2}/\mathcal{R}$ mapping an element of $\mathcal{S}_{\mathbb{F}_q}$ and \mathcal{R} to that one of $\mathcal{S}_{\mathbb{R}^2}/\mathcal{R}$ with the same notation.

Our aim is here to identify some connections between an arrangement F of lines in \mathbb{F}_q^2 and an arrangement A in \mathbb{R}^2 , such that $A \in \Phi([x_2(F), ..., x_{|F|}(F)])$.

To this end, we define quantities specific to \mathbb{F}_q^2 and \mathbb{R}^2 .

For F an arrangement of lines in \mathbb{F}_q^2 :

• $x_i(F)$ is the number of points of multiplicity i associated with F, for $i \in \{0, 1\}.$

For A an arrangement of lines in \mathbb{R}^2 (see [\[3\]](#page-14-4)):

- $f_i(A)$ is the number of *i*-dimensional cells determined by A, for $i \in$ $\{0, 1, 2\}$, where 0-dimensional cells are the vertices of A, 1-dimensional cells the edges of A and 2-dimensional cells the faces of A ;
- $f_i^b(A)$ is the number of bounded *i*-dimensional cells for $i \in \{1, 2\}$.

There exist four relationships between the different mathematical quantities in \mathbb{R}^2 (see [\[3\]](#page-14-4)), listed below:

Proposition 2.3. For all arrangements A of lines in \mathbb{R}^2 :

$$
f_1(A) = p_1(A) + p_{01}(A), \quad f_2(A) = 1 - p_0(A) + p_1(A) + p_{01}(A),
$$

\n
$$
f_1^b(A) = -p_1(A) + p_{01}(A), \quad f_2^b(A) = 1 - p_0(A) - p_1(A) + p_{01}(A).
$$

Let F be an arrangement of $q+1$ lines in \mathbb{F}_q^2 . Assuming henceforth the existence of an arrangement A in \mathbb{R}^2 , such that $A \in \Phi([x_2(F), ..., x_{|F|}(F)])$.

One has $p_1(A) = p_1(F) = q + 1$ (be $A \in \Phi([x_2(F), ..., x_{q+1}(F)])$). We also have $p_0(F) = \sum_{i=2}^{q+1} x_i(F) = p_0(A)$, since for all $i \in \{2, ..., q+1\}$ $x_i(A) = x_i(F)$. For a fixed point \overline{P} in \mathbb{R}^2 or in \mathbb{F}_q^2 , the number of pairs consisting of a line of A passing through P and the point \overline{P} itself is by definition the multiplicity of this point. Therefore $p_{01}(A)$ and $p_{01}(F)$ are equal to the sum of the multiplicity of all points of intersection: $p_{01}(F) = \sum_{i=2}^{q+1} ix_i(F) = p_{01}(A)$.

In this context, we can outline the following theorem:

Theorem 2.4. Let F be an arrangement of $q + 1$ lines in \mathbb{F}_q^2 . Let us assume that there exists an arrangement A in \mathbb{R}^2 , such that:

$$
A \in \Phi([x_2(F), ..., x_{|F|}(F)]).
$$

Then:

$$
f_1^b(A) = (q+1)(q-1) - x_1(F)
$$
 and $f_2^b(A) = x_0(F)$.

Proof. According to Proposition [2.3,](#page-11-0) we know: $f_1^b(A) = -p_1(A) + p_{01}(A) =$ $-(q+1)+\sum_{i=2}^{q+1} ix_i(F)$. Equality [\(5\)](#page-5-0) gives us: $f_1^b(A) = -(q+1)+q(q+1)-x_1(F)$ i.e. the first equality of the theorem.

For the second equality, according to Proposition [2.3](#page-11-0) we have: $f_2^b(A)$ = $1 - p_0(A) - p_1(A) + p_{01}(A) = 1 - \sum_{i=2}^{q+1} x_i(F) - (q+1) + \sum_{i=2}^{q+1} ix_i(F)$. From the equalities $\sum_{i=0}^{q+1} x_i(F) = q^2$ and $\sum_{i=0}^{q+1} ix_i(F) = q(q+1)$ (which are true for all sets of $q + 1$ lines), we deduce that $f_2^b(A) = 1 - (q^2 - x_0(F) - x_1(F)) - (q +$ 1) + $(q(q + 1) - x_1(F)) = x_0(F)$. \Box

The first equality of Theorem [2.4](#page-11-1) can be explained as follows. When we add a line to an arrangement, two cases appear. The first one is the case where each new point (resulting from the intersection of the new line and one of the others) leads at the same time to the appearance of a bounded one-dimensional cells for each of the two lines and the disappearance of a single point on each of these lines. The second one is the case where the new line passes through an already existing point; here, only one new bounded 1-dimensional cell appears and one single point disappears (on the new line). This explains the coefficient −1 of $x_1(F)$ in the first equality of the theorem. In other words, $x_1(F) + f_1^b(A)$ is a constant for a given q . In the already mentioned extreme case of concurrence of all lines of a minimal Besicovitch arrangement B (Example [1.8\)](#page-8-1), we have: $x_1(F) = (q+1) \times (q-1)$ and $f_1^b(A) = 0$. Thus we recover the first equality of Theorem [2.4.](#page-11-1)

Example 2.5. We give an example where $q = 5$. $\mathbb{Z}/5\mathbb{Z}$ being a field, we can use the following notations. To obtain a minimal Besicovitch arrangement F , we choose 6 lines in \mathbb{F}_5^2 with distinct slopes. Their equations are: $y = 0$, $y = x + 1$, $y = 2x + 1$, $y = 3x + 2$, $4x + 2$ and $x = 0$. The figure below shows us the different lines of F in \mathbb{F}_5^2 (on the left) and one of its associated arrangement A in \mathbb{R}^2 , verifying $A \in \Phi([x_2(F), ..., x_{|F|}(F)])$ (on the right):

Figure 2: An example illustrating the connections between \mathbb{F}_q^2 and \mathbb{R}^2 ; here $q=5$.

We can deduce from Theorem [2.4](#page-11-1) the following result: Remark 2.6. Let F be an arrangement of $q + 1$ lines in \mathbb{F}_q^2 . Let us assume that there exists an arrangement A in \mathbb{R}^2 such that:

$$
A \in \Phi([x_2(F), ..., x_{|F|}(F)]).
$$

Then:

$$
f_1(A) = (q+1)^2 - x_1(F)
$$
 and $f_2(A) = 2(q+1) + x_0(F)$.

Indeed, it is sufficient to note that, according to Proposition [2.3,](#page-11-0) $f_1(A)$ = $f_1^b(A) + 2p_1(A)$ and $f_2(A) = f_2^b(A) + 2p_1(A)$. Theorem [2.4](#page-11-1) and the fact that $p_1(A) = q + 1$ enable us to conclude.

As a result, we can define in \mathbb{F}_q^2 , in view of Remark [2.6,](#page-12-0) some quantities, equivalent to those defined in \mathbb{R}^2 , associated with an arrangement F of $q+1$ lines:

- $\tilde{f}_0(F) = \tilde{p}_0(F);$
- $\tilde{f}_1(F) = (q+1)^2 \tilde{x}_1(F);$
- $\tilde{f}_2(F) = 2(q+1) + \tilde{x}_0(F)$.

Euler's formula applied to an arrangement A in \mathbb{R}^2 gives us: $(f_0(A) + 1)$ – $f_1(A) + f_2(A) = \chi(S^2) = 2$, where $\chi(S^2)$ is the Euler-Poincaré characteristic of the sphere S^2 (see [\[3\]](#page-14-4)). In \mathbb{F}_q^2 , we obtain similarly, for all arrangements F of $q + 1$ lines:

$$
(\tilde{f}_0(F) + 1) - \tilde{f}_1(F) + \tilde{f}_2(F) = 2.
$$
\n(9)

Indeed, we have:

$$
(\tilde{f}_0(F) + 1) - \tilde{f}_1(F) + \tilde{f}_2(F) =
$$

$$
(\sum_{i=2}^{q+1} \tilde{x}_i(F) + 1) - (q^2 + 2q + 1 - \tilde{x}_1(F)) + (2q + 2 + \tilde{x}_0(F)),
$$

equality [\(3\)](#page-3-1) allowing us to obtain the expected result.

Extending the notation above to all sets of lines, we obtain that this equality is still valid for all arrangements in \mathbb{F}_q^2 . Indeed the proof of equality [\(9\)](#page-13-0) relies only on equality [\(3\)](#page-3-1), which is always true.

Conclusion

To conclude, equalities and inequalities brought to light in the first section enlarge the knowledge on minimal Besicovitch arrangements. In addition, in the second section, the values of x_0^B and x_1^B , for an arrangement of $q + 1$ lines B in \mathbb{F}_q^2 , the q elements finite field, emerge from Theorem [2.4](#page-11-1) through two new equalities implying a well-chosen arrangement in \mathbb{R}^2 . This first value is useful in the two-dimensional version of the finite Kakeya problema; indeed, it is the number of points in \mathbb{F}_q^2 through which at least one line of a minimal Besicovitch arrangement B passes (easy to determine from the value of x_0^B) which is expected in this issue (see $[4, 1]$ $[4, 1]$). The second value, for its part, is required in the determination of the complexity of self-dual normal bases (see [\[6,](#page-14-5) [2\]](#page-14-0)). Therefore, all these equalities and inequalities open up new prospects for future research in this two specific fields.

References

- [1] A. Blokhuis and F. Mazzocca. The finite field Kakeya problem. Building Bridges Between Mathematics and Computer Science, 19:205–218, 2008.
- [2] S. Blondeau Da Silva. On randomly chosen arrangements of $q+1$ lines with different slopes in \mathbb{F}_q^2 . Journal of Number Theory, 180C:533-543, 2017.
- [3] P. Cartier. Les arrangements d'hyperplans : un chapitre de géométrie combinatoire. Séminaire Bourbaki, 901, 1980/81:1-22, 1981.
- [4] X. W. C. Faber. On the finite field Kakeya problem in two dimensions. Journal of Number Theory, 124:248–257, 2007.
- [5] A. Pott. Finite Geometry and Character Theory. Springer, 1995. Lect. Notes in Math. 1601.
- [6] S. Vinatier. Éléments explicites en théorie algébrique des nombres. 2013. Habilitation à Diriger des Recherches (HDR).