# Character Integral Representation of Zeta function in $\mathrm{AdS}_{d+1}$:

## I. Derivation of the general formula

**Thomas BASILE**[a] **Euihun JOUNG**[a] **Shailesh LAL**[b] **Wenliang LI**[a]

[a] *Department of Physics and Research Institute of Basic Science, Kyung Hee University, Seoul 02447, Korea*

[b] *Centro de Fisica do Porto e Departamento de Fisica e Astronomia Faculdade de Ciencias da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal*

*E-mail:* thomas.basile@khu.ac.kr, euihun.joung@khu.ac.kr, slal@fc.up.pt, lii.wenliang@gmail.com

ABSTRACT: The zeta function of an arbitrary field in $(d+1)$-dimensional anti-de Sitter (AdS) spacetime is expressed as an integral transform of the corresponding $so(2,d)$ representation character, thereby extending the results of [1603.05387] for $\mathrm{AdS}_4$ and $\mathrm{AdS}_5$ to arbitrary dimensions. The integration in the variables associated with the $so(d)$ part of the character can be recast into a more explicit form using derivatives. The explicit derivative expressions are presented for $\mathrm{AdS}_{d+1}$ with $d = 2, 3, 4, 5, 6$.

## Contents

## 1 Introduction

The one-loop free energy is one of the simplest physical quantities capturing non-trivial quantum effects. It is divergent due to the contribution from the modes having infinite energy but can be neatly regularized by making use of the spectral zeta function, namely in the scheme of the zeta function regularization. The zeta function for any field (massive or massless, and of arbitrary spin) in four-dimensional Anti-de-Sitter (AdS) spacetime has been first calculated by Camporesi and Higuchi in [1] and generalized to higher dimensions by the same authors in [2, 3].

Zeta functions in AdS are useful in the test of certain AdS/CFT dualities: the one-loop free energy or vacuum energy in AdS spacetime corresponds to the non-planar contribution of the CFT free energy on the boundary. Since the typical AdS theories under consideration contain infinitely many fields, computing their one-loop free energy is a non-trivial task.[1] When the AdS theory is a higher-spin gravity, the one-loop free energy is calculable in an analytic manner, even though the field content still contains infinitely many elements. An interesting observation made in [9, 10] is that the summation of the zeta functions over the field content is convergent while that of the regularized one-loop free energy is divergent. This is interesting as it signifies that the zeta function regularization renders finite both the high energy divergence and the spectrum sum divergence.

The viability of the one-loop free energy computation in higher-spin gravities heavily relies on the simple structure of the spectrum: e.g. in the case of the non-minimal type-A theory, first constructed in four dimensions [11, 12] and later extended to arbitrary dimensions in [13], the spectrum consists of massless fields of all integer spins. The summation over a field content becomes quite cumbersome if the content itself does not have a simple expression. This kind of difficulty was encountered in the computation of the one-loop free energy of the AdS fields dual to the operators tri- and quadri-linear in free conformal scalar fields [14]. In the absence of the single-trace condition — that is, the cyclic projection on the operators — the field content could be expressed in a few lines, which could be used to calculate the one-loop free energy although it required quite burdensome works. What is worse is that the field content with cyclic projection does not have any manageable expression, hence it seems impossible to proceed in this way.

A key observation to bypass this problem is that the field content of an AdS theory dual to a free CFT can be derived group theoretically. One of the most efficient and general methods for such a derivation is the use of Lie algebra character. In fact, the spectrum of the four-dimensional type-A higher-spin gravity was obtained using the $so(2,3)$ character — namely, the Flato-Fronsdal theorem [15] (later generalized in arbitrary dimensions in [16, 17]). Since both the zeta function and the character are determined uniquely by the labels of $so(2,d)$ representations, we can devise a linear map which send the character of an $so(2,d)$ representation to the zeta function of the AdS field carrying the same representation. If the linear map itself does not depend on the labels of the representation, we can use it for

[1] If instead one considers the change in free energy by taking an alternative boundary condition for one of the AdS fields, this technical difficulty does not arise. This quantity is matched to the double-trace deformation of the corresponding CFT [4–8].

the character of a reducible representation without decomposing it into irreducible pieces. Such a map was explicitly constructed for bosonic fields in $AdS_4$ and any fields in $AdS_5$ and was named as "character integral representation of zeta function (CIRZ)" in [14]. The CIRZ turned out to be very efficient in evaluating the one-loop free energies. Notably, the computation of the non-minimal type-A higher-spin gravity becomes almost trivial. Moreover, the CIRZ allowed to tackle the one-loop free energy computation of the stringy AdS theory dual to free matrix model CFTs in the $N \to \infty$ limit [14, 18–20]. The CIRZ also proved useful in other vector model dualities: the "colored" higher-spin gravity (where the four-dimensional CIRZ was generalized to fermionic fields) [21] and the type-J higher-spin theories (whose conjectured dual are free vector model based on a massless spin-$j$ field) [22]. Some key elements of the CIRZ were also used in [23].

In this paper, we aim to derive the CIRZ in dimensions different from $AdS_4$ and $AdS_5$. More precisely, we seek for the formula extending the CIRZ in $AdS_{d+1}$ with arbitrary integer $d \geq 2$. Although the dimensional dependence in higher-spin gravity is rather minimal, most of the results in the literature [9, 10, 24–27] concern only specific dimensions for technical reasons, except in [23] where the results of the type-A higher-spin gravity are extended to arbitrary (non-integer) dimensions. From the viewpoint of physical applications, one might not need to care about higher dimensions yet, but it is at the same time tempting to obtain results with parametric dependence on $d$. As usual, generalities may provide new and valuable lessons on what is considered to be well-understood. It is actually the case here for the CIRZ: in the course of its derivation for general dimensions, we find many new insights on the zeta function, $so(2,d)$ character and the relations between them.

The general CIRZ formula we obtain is an integral transformation of $so(2,d)$ character and has a quite simple structure: it is in a sense even simpler than the original CIRZ expressions obtained in $AdS_4$ and $AdS_5$ in [14]. In order to deliver a flavor of our results, let us write down the expressions of the general CIRZ derived in Section 3. Firstly, for odd AdS dimensions $d+1 = 2r+1$ (or equivalently, even boundary dimensions $d = 2r$), we obtain

$$\zeta_{\mathcal{H}}(z) = \ln R \int_0^\infty \frac{\mathrm{d}\beta}{\Gamma(z)^2} \sum_{k=0}^{r} \oint \mu(\boldsymbol{\alpha}) \left(\left(\tfrac{\beta}{2}\right)^2 + \left(\tfrac{\alpha_k}{2}\right)^2\right)^{z-1} \times \\ \times \left[\prod_{\substack{0\leqslant j\leqslant r \\ j\neq k}} \frac{\cosh\beta - \cos\alpha_j}{\cos\alpha_k - \cos\alpha_j}\right] \chi_{\mathcal{H}}^{so(2,2r)}(\beta;\vec{\alpha}_k)\,. \tag{1.1}$$

where

$$\mu(\boldsymbol{\alpha}) := \prod_{n=0}^{r} \frac{\mathrm{d}\alpha_n}{2\pi\, i\, \alpha_n} \qquad \text{and} \qquad \vec{\alpha}_k := (\alpha_0, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_r)\,. \tag{1.2}$$

Secondly, for even AdS dimensions $d+1 = 2r+2$ (or equivalently, odd boundary dimensions

$d = 2r + 1$), we obtain

$$\zeta_{1,\mathcal{H}}(z) = \int_0^\infty \frac{\mathrm{d}\beta\,\beta^{2z-1}}{\Gamma(2z)} \sum_{k=0}^{r} \oint \mu(\boldsymbol{\alpha})\, \frac{\sinh\frac{\beta}{2}\,(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}\,(\cos\frac{\alpha_k}{2})^{\frac{1-\epsilon}{2}}}{2\,(\cosh\beta-\cos\alpha_k)} \times$$
$$\times \left[\prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}} \frac{\cosh\beta-\cos\alpha_j}{\cos\alpha_k-\cos\alpha_j}\right] \chi_{\mathcal{H}}^{so(2,2r+1)}(\beta;\vec{\alpha}_k)\,, \tag{1.3}$$

where $\epsilon = +1/-1$ for bosonic/fermionic spectrum. The subscript 1 in the zeta function means that it is the *primary* contribution to the actual zeta function. The remaining contributions will be introduced later in this paper but they are irrelevant in all the cases that we are interested in. Both of the formulae (1.1) and (1.3) involve complex contour integrals in $\alpha_n$ with $n = 0, 1, \ldots, r$. The contour of each $\alpha_n$ is a circle enclosing the origin counter-clockwise with $|\alpha_0| < \cdots < |\alpha_r|$. We would like to stress that the expressions (1.1) and (1.3) allow for a large room for various complex integral tricks: in the companion paper [28], we will compute the zeta function of partially-massless higher-spin gravities in arbitrary dimensions using complex integrals.

The contour integrals in $\alpha_i$ appearing in the above formulae (1.1) and (1.3) may be evaluated by the residue theorem, and hence reduce to a linear combination of derivatives in $\alpha_i$. The expression in terms of the $\alpha_i$ contour integral is compact and useful in many applications but it might be not explicit enough in other cases. For instance, if one wants to implement the formula in a computer program, the other expression in terms of the $\alpha_i$ derivatives would be more convenient. We derive the latter expression by adapting the standard tools used to derive the dimension formula from the Weyl character formula. Applying the expressions to $d = 2, 3, 4, 5, 6$, we provide the explicit form of the CIRZ in the dimensions which are the most relevant for physical applications.

The organization of the paper is as follows. In Section 2, we start by reviewing the one-loop free energy and zeta function in $\mathrm{AdS}_{d+1}$, then rewrite the latter as an integral of the dimension formula of an $so(d+2)$ representation. In Section 3, we derive the CIRZ formula in arbitrary dimensions, i.e. we show that the zeta function of any field in $\mathrm{AdS}_{d+1}$ can be written as an integral transform of its $so(2,d)$ character. In Section 4, we spell out an alternative form of the CIRZ where the previously mentioned contour integrals are replaced by a linear combination of derivatives of the character. Section 5 contains a brief summary and concluding remarks of the paper. Finally, some definitions and technical details are presented in Appendix A and B.

## 2 Zeta functions in AdS

In this section, we shall review the basics of the one-loop free energy and zeta function, and the integral expression of the zeta function in $\mathrm{AdS}_{d+1}$ obtained in [3]. After re-expressing the numerator of the integrand in terms of the dimension formula of an $so(d+2)$ representation, we shall discuss how the first derivative of the zeta function can be expressed in a "spectral integral" form.

### 2.1 One-loop free energy and zeta function

The one-loop free energy of a quantum field is given by the logarithm of the one-loop path integral:

$$\Gamma^{(1)}_{[m^2;\mathcal{V}]} = \frac{\epsilon}{2}\,\log\det_{\mathcal{V}}(\Box + m^2)\,, \tag{2.1}$$

where $\mathcal{V}$ is the space of the off-shell fields which are traceless and transverse, and the sign $\epsilon$ is $+1$ for a boson and $-1$ for a fermion. The operator $\Box + m^2$ is what appears in the quadratic Lagrangian, so one can regard the field practically as a free one. When the field has a gauge symmetry, we have to subtract the corresponding ghost contribution. The one-loop free energy (2.1) can be related to the zeta function

$$\zeta_{[m^2;\mathcal{V}]}(z) = \operatorname*{Tr}_{\mathcal{V}}\left[\frac{1}{(\Box + m^2)^z}\right], \tag{2.2}$$

where the trace is convergent for a sufficiently large value of $\mathrm{Re}(z)$. Once we obtain $\zeta_{[m^2;\mathcal{V}]}(z)$, the log det formula can be related to the zeta function by analytically continuing the value of $z$ to zero as

$$\log\det_{\mathcal{V}}(\Box + m^2) = \operatorname*{Tr}_{\mathcal{V}}\log(\Box + m^2) \;\;\to\;\; -\frac{\epsilon}{2}\,\zeta'_{[m^2;\mathcal{V}]}(0), \tag{2.3}$$

where we have used the zeta function regularization and the last expression is the finite part of the free energy. The UV divergence of the free energy corresponds to $\zeta_{[m^2;\mathcal{V}]}(0)$. The zeta function can be also related to the integrated propagator in the coincidence point limit:

$$G_{[m^2;\mathcal{V}]} = \operatorname*{Tr}_{\mathcal{V}}\left[\frac{1}{\Box + m^2}\right] \;\;\to\;\; \lim_{z\to 1}\zeta_{[m^2;\mathcal{V}]}(z)\,. \tag{2.4}$$

As we have just seen, both the free energy $\Gamma^{(1)}_{[m^2;\mathcal{V}]}$ and the propagator $G_{[m^2;\mathcal{V}]}$ can be obtained from the zeta function (2.2), hence this allows us to focus on the zeta function for a given field space $\mathcal{V}$ with the mass squared $m^2$.

In $\mathrm{AdS}_{d+1}$, free fields can be classified by the irreducible representations (irreps) they carry for the isometry algebra $so(2,d)$. The massive and massless [29–31] irreps[2] are the lowest-weight modules labeled by $[\Delta;\mathbb{Y}]$, where $\Delta$ is the lowest eigenvalue of the energy operator generating $so(2)$, and $\mathbb{Y} := (s_1,\ldots,s_r)$ is the highest weight of the rotational symmetry $so(d)$ classifying the traceless and transverse tensors and can be interpreted as the spin of the field. The zeta function for the module $[\Delta;\mathbb{Y}]$ has been calculated in [2, 3] (see also e.g. [25, 26] for a review) and its expression is

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \frac{\mathrm{Vol}(AdS_{d+1})}{\mathrm{Vol}(S_d)}\,\frac{\dim^{so(d)}_{\mathbb{Y}}}{2^{d-1}\,\Gamma(\frac{d+1}{2})^2}\int_0^\infty \mathrm{d}u\,\frac{\mu_{\mathbb{Y}}(u)}{\left[u^2 + (\Delta - \frac{d}{2})^2\right]^z}\,. \tag{2.5}$$

The volume of the $d$-sphere and the (regularized) volume of the $(d+1)$-dimensional AdS spacetime are given respectively by

$$\mathrm{Vol}(S_d) = \frac{2\,\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}\,, \qquad \mathrm{Vol}(AdS_{d+1}) = \begin{cases} \frac{2\,(-1)^r\,\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\,\ln R & [d=2r] \\ \pi^{d/2}\,\Gamma(-\frac{d}{2}) & [d=2r+1] \end{cases}\,, \tag{2.6}$$

[2] Notice that there exists an "exotic" class of field in $\mathrm{AdS}_{d+1}$, namely *continuous spin* fields [32–34]. It was shown that the partition function of such fields is equal to one.

where $R$ is the radius of the $\text{AdS}_{d+1}$. The function $\mu_{\mathbb{Y}}(u)$ appearing in the numerator of the integrand is given by

$$\mu_{\mathbb{Y}}(u) = \prod_{k=1}^{r}(u^2+\ell_k^2)\times\begin{cases} 1 & [d=2r] \\ u\tanh^{\epsilon}(\pi\, u) & [d=2r+1] \end{cases}, \tag{2.7}$$

where $\ell_k = s_k + \frac{d}{2} - k$, and the sign $\epsilon$ is positive for bosonic fields ($s_k \in \mathbb{N}$) and negative for fermionic ones ($s_k \in \frac{1}{2}\mathbb{N}$). The combination $\dim_{\mathbb{Y}}^{so(d)} \mu_{\mathbb{Y}}(u)$ in (2.5) is related to the dimension formula (which is also referred to as Weyl dimension formula) of an $so(d+2)$ irrep:

- For even $d = 2r$, the dimension of the $so(d+2)$ irrep $(s_0, \mathbb{Y}) = (s_0, s_1, \ldots, s_r)$ is

$$\dim_{(s_0,\mathbb{Y})}^{so(d+2)} = \prod_{0\leqslant i<j\leqslant r} \frac{(s_i - s_j + j - i)(s_i + s_j + d - i - j)}{(j-i)(d-i-j)}. \tag{2.8}$$

  This can be expressed in terms of the dimension of the $so(d)$ irrep $\mathbb{Y}$ as

$$\dim_{(s_0,\mathbb{Y})}^{so(d+2)} = \frac{2\,\dim_{\mathbb{Y}}^{so(d)}}{d!}\prod_{k=1}^{r}(s_0 - s_k + k)(s_0 + s_k + d - k). \tag{2.9}$$

  For $s_0 = iu - \frac{d}{2}$, the relation reduces to

$$\dim_{(iu-\frac{d}{2},\mathbb{Y})}^{so(d+2)} = \frac{2\,(-1)^r}{d!}\,\dim_{\mathbb{Y}}^{so(d)}\ \mu_{\mathbb{Y}}(u). \tag{2.10}$$

- For odd $d = 2r+1$, the dimension of the $so(d+2)$ irrep $(s_0, \mathbb{Y})$ is

$$\dim_{(s_0,\mathbb{Y})}^{so(d+2)} = \prod_{k=0}^{r}\frac{2s_k + d - 2k}{d-2k}\prod_{0\leqslant i<j\leqslant r}\frac{(s_i - s_j + j - i)(s_i + s_j + d - i - j)}{(j-i)(d-i-j)}, \tag{2.11}$$

  and can also be related to the dimension of the $so(d)$ irrep $\mathbb{Y}$ through

$$\dim_{(s_0,\mathbb{Y})}^{so(d+2)} = \frac{\dim_{\mathbb{Y}}^{so(d)}}{d!}\,(2s_0+d)\prod_{k=1}^{r}(s_0 - s_k + k)(s_0 + s_k + d - k). \tag{2.12}$$

  For $s_0 = iu - \frac{d}{2}$, the relation becomes

$$\frac{i}{2}\,\tanh^{\epsilon}(\pi u)\,\dim_{(iu-\frac{d}{2};\mathbb{Y})}^{so(d+2)} = \frac{(-1)^{r+1}}{d!}\,\dim_{\mathbb{Y}}^{so(d)}\ \mu_{\mathbb{Y}}(u). \tag{2.13}$$

Making use of the above information, the zeta function can be written as

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \int_0^{\infty}\frac{\mathrm{d}u\,\rho_{\epsilon}(u)}{\left[u^2 + (\Delta - \frac{d}{2})^2\right]^z}\,\dim_{(iu-\frac{d}{2};\mathbb{Y})}^{so(d+2)}, \tag{2.14}$$

with the function $\rho_{\epsilon}(u)$,

$$\rho_{\epsilon}(u) = \begin{cases} \dfrac{\ln R}{\pi} & [\text{even } d] \\ \dfrac{i}{2}\,\tanh^{\epsilon}(\pi\, u) & [\text{odd } d] \end{cases}. \tag{2.15}$$

The fact that the zeta function can be written in terms of the $so(d+2)$ irrep dimension $\dim^{so(d+2)}_{(iu-\frac{d}{2};\mathbb{Y})}$ helps us to make a link between the zeta function and the $so(2,d)$ character. Before establishing such a connection, let us first explore a few interesting properties of the zeta functions.

### 2.2 Spectral integral form of the zeta function

The zeta function $\zeta_{[m^2;\mathcal{V}]}(z)$, defined generally as (2.2), enjoys a simple identity,

$$\frac{\partial}{\partial m^2}\zeta_{[m^2;\mathcal{V}]}(z) = \frac{1}{2m}\frac{\partial}{\partial m}\zeta_{[m^2;\mathcal{V}]}(z) = -z\,\zeta_{[m^2;\mathcal{V}]}(z+1)\,, \tag{2.16}$$

which is nothing but the spectral version of the Hurwitz zeta function identity,

$$\frac{\partial}{\partial a}\,\zeta(z,a) = -z\,\zeta(z+1,a)\,, \tag{2.17}$$

with

$$\zeta(z,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^z}. \tag{2.18}$$

The identity (2.16) simply implies

$$\frac{\partial}{\partial m^2}\zeta_{[m^2;\mathcal{V}]}(0) = -\lim_{z\to 0} z\,\zeta_{[m^2;\mathcal{V}]}(z+1)\,, \tag{2.19}$$

and

$$\frac{\partial}{\partial m^2}\zeta'_{[m^2;\mathcal{V}]}(0) = -\,\mathrm{F.p.}\lim_{z\to 0}\zeta_{[m^2;\mathcal{V}]}(z+1)\,, \tag{2.20}$$

where F.p. refers to the finite part in the limit $z \to 0$, i.e. the constant term in the Laurent expansion in $z$. These two formulae provide the derivatives with respect to $m^2$ of the UV divergent and finite part of the free energy. The second equation (2.20) can be viewed as the regularized version of the formal expression,

$$\frac{\partial}{\partial m^2}\underset{\mathcal{V}}{\mathrm{Tr}}\log(\Box + m^2) = \underset{\mathcal{V}}{\mathrm{Tr}}\left[\frac{1}{\Box + m^2}\right]. \tag{2.21}$$

Now considering the AdS background, the identity (2.16) becomes

$$\frac{1}{2(\Delta-\frac{d}{2})}\frac{\partial}{\partial\Delta}\zeta_{[\Delta;\mathbb{Y}]}(z) = -z\,\zeta_{[\Delta;\mathbb{Y}]}(z+1)\,, \tag{2.22}$$

and we also have relations analogous to (2.20). These identities prove useful since it is easier to study the zeta function $\zeta_{[\Delta;\mathbb{Y}]}(z)$ near $z=1$ than $z=0$.

In the following, we shall make use of the identity (2.22) to show how a "spectral integral" form of the $\zeta'_{[\Delta;\mathbb{Y}]}(0)$ can be obtained. In the context of $\mathrm{AdS}_{2r+1}/\mathrm{CFT}_{2r}$ correspondence, this spectral integral formula was used to show the direct relation between $\zeta'_{[\Delta;\mathbb{Y}]}(0)$ and conformal anomaly coefficients. It first appeared in the case of totally symmetric representations in [35] and subsequently mixed symmetry representations in $\mathrm{AdS}_7$ [36], then generalized to arbitrary representations in [26]. Below, we provide a short derivation of the spectral integral formula in $\mathrm{AdS}_{d+1}$ for both even $d=2r$ and odd $d=2r+1$.

**Even $d$**

For even values of $d$, the zeta function has the form,

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \int_0^\infty \frac{\mathrm{d}u\, h_{\mathbb{Y}}(u)}{\left[u^2+\bar{\Delta}^2\right]^z}\,, \tag{2.23}$$

where $\bar{\Delta} = \Delta - \frac{d}{2}$. Since $h_{\mathbb{Y}}(u) = \frac{\ln R}{\pi}\,\dim^{so(d+2)}_{(iu-\frac{d}{2};\mathbb{Y})}$ is an even polynomial of order $2r$, the integral (2.23) is the same as one half of the integral from $u=-\infty$ to $u=\infty$ with the same integrand. The integral is convergent in the region $z > r+\frac{1}{2}$, so we can close the contour by adding the infinite upper half-circle, then shrink it down to enclose the branch cut singularity (i.e. the line defined by $\mathrm{Arg}(u - i|\bar{\Delta}|) = \pi$),

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \frac{1}{2}\oint_{i\,|\bar{\Delta}|} \frac{\mathrm{d}u\, h_{\mathbb{Y}}(u)}{\left[u^2+\bar{\Delta}^2\right]^z}\,. \tag{2.24}$$

This contour integral is convergent for any value of $z$, hence we can directly replace $z$ by the value we want. If we put $z=0$, the integrand becomes analytic and we get

$$\zeta_{[\Delta;\mathbb{Y}]}(0) = 0\,. \tag{2.25}$$

If we put $z=1$, the integrand has a simple pole at $u=+i\,|\bar{\Delta}|$ and gives

$$\zeta_{[\Delta;\mathbb{Y}]}(1) = \frac{1}{2}\,2\pi i\,\frac{h_{\mathbb{Y}}(i\,|\bar{\Delta}|)}{2i\,|\bar{\Delta}|} = \pi\,\frac{h_{\mathbb{Y}}(i\,\bar{\Delta})}{2\,|\bar{\Delta}|}\,. \tag{2.26}$$

Using (2.22), this implies

$$\frac{\partial}{\partial\Delta}\zeta'_{[\Delta;\mathbb{Y}]}(0) = -2\,\bar{\Delta}\,\zeta_{[\Delta;\mathbb{Y}]}(1) = -\pi\,\mathrm{sgn}(\bar{\Delta})\,h_{\mathbb{Y}}(i\,\bar{\Delta})\,. \tag{2.27}$$

From the fact that $\zeta'_{[\frac{d}{2};\mathbb{Y}]}(0)=0$ we can derive the expression

$$\zeta'_{[\Delta;\mathbb{Y}]}(0) = -\pi\int_0^{|\bar{\Delta}|}\mathrm{d}x\, h_{\mathbb{Y}}(i\,x) = -\ln R\int_0^{|\bar{\Delta}|}\mathrm{d}x\,\dim^{so(d+2)}_{(-x-\frac{d}{2};\mathbb{Y})}\,, \tag{2.28}$$

where the absolute value $|\bar{\Delta}|$ appears as a result of $\mathrm{sgn}(\bar{\Delta})$. The result is even in $\bar{\Delta}$ like the original form (2.23) hence insensitive to its sign. In physical term, the sign of $\bar{\Delta}$ determines whether the underlying field takes Dirichlet or Neumann boundary condition. Since the two boundary conditions should give different results, we need to modify the above definition of the zeta function.

By noticing that the expression (2.28) is not analytic on the imaginary axis of $\bar{\Delta}$, we can consider another expression where we analytically continue the value of $\bar{\Delta}$ from positive $\mathrm{Re}(\bar{\Delta})$ to negative one. This simply amounts to replacing $|\bar{\Delta}|$ by $\bar{\Delta}$ in (2.28). At the level of the contour integral representation (2.24), this "new" definition of the zeta function corresponds to the modification,

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \frac{1}{2}\oint_{i\,\bar{\Delta}} \frac{\mathrm{d}u\, h_{\mathbb{Y}}(u)}{\left[u^2+\bar{\Delta}^2\right]^z}\,, \tag{2.29}$$

where the contour encircles counter-clockwise the branch cut starting at $i\,\bar{\Delta}$ rather than $i\,|\bar{\Delta}|$. The zeta function (2.29) is what has been used in the literature.

**Odd $d$**

For odd values of $d$, the zeta function has the form,

$$\zeta_{[\Delta;\mathbb{Y}]}(z)=\int_0^\infty \frac{{\rm d}u\,\tanh^\epsilon(\pi u)\,h_{\mathbb{Y}}(u)}{\left[u^2+\bar\Delta^2\right]^z}\,, \tag{2.30}$$

where $h_{\mathbb{Y}}(u)=\frac{i}{2}\,{\rm dim}^{so(d+2)}_{(i\,u-\frac{d}{2};\mathbb{Y})}$ is now an odd function. Similarly to the even $d$ case, we can rewrite the above expression as a contour integral by adding to the real line the infinite radius upper-half circle. The function $h_{\mathbb{Y}}(u)$ is analytic again but $\tanh^\epsilon(\pi\,u)$ has infinitely many simple poles on the imaginary axis. We can separate those contributions as

$$\zeta_{[\Delta;\mathbb{Y}]}(z)=\frac{1}{2}\left(\oint_{i\,\bar\Delta}+\sum_{n=1}^\infty\oint_{i\left(n-\frac{1+\epsilon}{4}\right)}\right)\frac{{\rm d}u\,\tanh^\epsilon(\pi u)\,h_{\mathbb{Y}}(u)}{\left[u^2+\bar\Delta^2\right]^z}\,. \tag{2.31}$$

Here, we take the prescription that the zeta function is analytic in $\bar\Delta$. Due to the presence of infinitely many simple poles of $\tanh^\epsilon(\pi u)$, it is not easy to simplify further the above expression as opposed to the even $d$ case. However, if we take the difference between $\zeta_{[\Delta;\mathbb{Y}]}(z)$ and $\zeta_{[d-\Delta;\mathbb{Y}]}(z)$, we can cancel the cumbersome contribution and end up with

$$\zeta_{[\Delta;\mathbb{Y}]}(z)-\zeta_{[d-\Delta;\mathbb{Y}]}(z)=\oint_{i\,\bar\Delta}\frac{{\rm d}u\,\tanh^\epsilon(\pi u)\,h_{\mathbb{Y}}(u)}{\left[u^2+\bar\Delta^2\right]^z}\,. \tag{2.32}$$

Note that we do not have a perfect cancellation due to the prescription of analytic continuation in $\Delta$. If $\Delta$ is not an integer/half-integer for boson/fermion, taking $z=0$ and $z=1$ limit, we find

$$\zeta_{[\Delta;\mathbb{Y}]}(0)-\zeta_{[d-\Delta;\mathbb{Y}]}(0)=0\,, \tag{2.33}$$

and

$$\zeta_{[\Delta;\mathbb{Y}]}(1)-\zeta_{[d-\Delta;\mathbb{Y}]}(1)=\pi\,\frac{\tanh^\epsilon(\pi\,i\,\bar\Delta)\,h_{\mathbb{Y}}(i\,\bar\Delta)}{\bar\Delta}\,. \tag{2.34}$$

The first equation (2.33) means that the UV divergence does not depend on the sign of $\bar\Delta$, or in physical terms, the choice of the boundary conditions. Applying the second equation (2.34) to the zeta function identity (2.22), we reach the result,

$$\begin{aligned}\zeta'_{[\Delta;\mathbb{Y}]}(0)-\zeta'_{[d-\Delta;\mathbb{Y}]}(0)\ &=-2\pi\int_0^{\bar\Delta}{\rm d}x\,\tanh^\epsilon(\pi\,i\,x)\,h_{\mathbb{Y}}(i\,x)\\ &=\epsilon\,\pi\int_0^{\bar\Delta}{\rm d}x\,\tan^\epsilon(\pi\,x)\,{\rm dim}^{so(d+2)}_{(-x-\frac{d}{2};\mathbb{Y})}\,.\end{aligned} \tag{2.35}$$

Hence, the free energy difference between $\Delta$ and $d-\Delta$ is given by a "spectral integral" where the integrand involves the dimension of an $so(d+2)$ irrep.

If $\Delta$ is an integer/half-integer (or equivalently $\bar\Delta$ is an half-integer/integer) for boson/fermion, the difference of the zeta zero (2.33) does not vanish anymore but gives

$$\zeta_{[\Delta;\mathbb{Y}]}(0)-\zeta_{[d-\Delta;\mathbb{Y}]}(0)=2\,i\,h_{\mathbb{Y}}(i\,\bar\Delta)=-\,{\rm dim}^{so(d+2)}_{(-\Delta;\mathbb{Y})}\,. \tag{2.36}$$

Moreover, the equations (2.34) should be also modified because the integral (2.32) with $z = 1$ now involves a double pole, and consequently (2.35) should be modified as well. After all, the necessary modification in (2.34) and (2.35) for (half-)integer $\Delta$ amounts to removing the singularity arising in the limit where $\Delta$ approaches to an integer or half-integer. The exceptionality of the (half-)integer $\Delta$ was first observed in [35], where the focus was on the massless case with $\Delta = s + d - 2$.

Let us delve a little further in the consequences of (2.36). Let $w_I$ and $h_I$ denote the number of columns and rows contained in the $I$-th block[3] of $\mathbb{Y}$, and define $p_I = h_1 + h_2 + \cdots + h_I$ with $p_0 = 0$. Then, $w_I = s_{p_{I-1}+1} = \ldots = s_{p_I}$ and $\mathbb{Y}$ can be denoted by $\mathbb{Y} = (w_1^{h_1}, w_2^{h_2}, \ldots)$. Now, if we assume that $\Delta$ is an (half-)integer satisfying

$$w_{I+1} - p_I \leq \Delta - d \leq w_I - p_I - 1\,, \tag{2.37}$$

then we can use the identity

$$\dim^{so(d+2)}_{(-\Delta, w_1^{h_1}, w_2^{h_2}, \ldots)} = (-1)^{p_I+1} \dim^{so(d+2)}_{((w_1-1)^{h_1}, \ldots, (w_I-1)^{h_I}, \Delta-d+p_I, w_{I+1}^{h_{I+1}}, \ldots)}\,. \tag{2.38}$$

Note that the coefficient $\dim^{so(d+2)}_{(-\Delta;\mathbb{Y})}$ appearing in (2.36) does not vanish for a generic (half-)integer except for the points $\Delta = s_k + d - k$ with $k = 1, \ldots, r$. In particular, the massive fields with $\Delta \geq s_1 + d$ give the result $\dim^{so(d+2)}_{(\Delta-d;\mathbb{Y})}$. This AdS result could be reproduced in the CFT side from the zero modes of the effective kinetic operator of the Hubbard-Stratonovich field. In [35], the eigenvalues of such an operator in 3d has been calculated for $s = 0, 1, 2$ and conjectured for arbitrary integer spins as (the equation (3.24) of [35])

$$k_{n,0} = c_s(\Delta)\,\frac{\Gamma(n-1+\Delta)}{\Gamma(n+2-\Delta)}\,, \qquad k_{n,i} = \frac{\Gamma(2-\Delta)}{\Gamma(\Delta-1)}\,\frac{\Gamma(-1+i+\Delta)}{\Gamma(2+i-\Delta)}\,k_{n,0}\,, \tag{2.39}$$

where $n$ and $i$ range from $s+1$ to infinity and $-s$ to $s$, respectively. The eigenvalue $k_{n,i}$ vanishes if $n \leq \Delta - 2$ and the degeneracy for a fixed $n$ and $i$ is $n^2 - i^2$. Hence the total number of zero modes is

$$\sum_{n=s+1}^{\Delta-2} \sum_{i=-s}^{s} (n^2 - i^2) = \frac{(2s+1)\,(2\,\Delta-3)\,(\Delta-s-2)\,(\Delta+s-1)}{3!} = \dim^{so(5)}_{(\Delta-3,s)}\,. \tag{2.40}$$

Indeed, one can see that the number of the zero modes coincides with the AdS result (2.36).

Considering now the (mixed-)symmetric (partially-)massless fields with $\Delta_{\text{PM}} = w_I + d - p_I - t$ with $1 \leqslant t \leqslant w_I - w_{I+1}$, i.e. $\Delta_{\text{PM}}$ satisfies (2.37), we obtain

$$\zeta_{[\Delta_{\text{PM}};\mathbb{Y}]}(0) - \zeta_{[d-\Delta_{\text{PM}};\mathbb{Y}]}(0) = (-1)^{p_I} \dim^{so(d+2)}_{\mathbb{Y}_{\text{KT}}}\,, \tag{2.41}$$

where $\mathbb{Y}_{\text{KT}}$ is the $so(d+2)$ irrep carried by the associated Killing tensors:

$$\mathbb{Y}_{\text{KT}} = ((w_1-1)^{h_1}, \ldots, (w_I-1)^{h_I}, w_I - t, w_{I+1}^{h_{I+1}}, \ldots)\,. \tag{2.42}$$

[3]In other words, the $I$-th block of $\mathbb{Y}$ is a succession of rows with the same length, and a diagram can be described as an aggregate of blocks ordered by decreasing length when examined from top to bottom (i.e. the first block is the one at the top of the diagram). In the notation introduced above, the $I$-th block is of length $w_I$ and height $h_I$, meaning it is composed of $h_I$ rows which are all of length $w_I$.

Considering the gauge parameter of the same field having $\Delta_{\rm GP} = w_I + d - p_I$ and $\mathbb{Y}_{\rm GP} = (w_1^{h_1}, \dots, w_{I-1}^{h_{I-1}}, {w_I}^{h_I - 1}, w_I - t, w_{I+1}^{h_{I+1}}, \dots)$, we find again the dimension of the Killing tensors:

$$\zeta_{[\Delta_{\rm GP};\mathbb{Y}_{\rm GP}]}(0) - \zeta_{[d-\Delta_{\rm GP};\mathbb{Y}_{\rm GP}]}(0) = (-1)^{p_I+1}\,\dim^{so(d+2)}_{\mathbb{Y}_{\rm KT}}\,, \tag{2.43}$$

but with opposite sign. Since the full zeta function is the difference between the physical mode and the gauge mode contributions, the net result becomes two times of (2.41). Like in the massive integral $\Delta$ case, the above AdS result for (partially-)massless field could be reproduced from the zero modes of the effective CFT kinetic operators. On top of these, the contributions of the ghost zero modes, giving rise again to the dimension of Killing tensors, should be appended to both sides of AdS and CFT, and it was shown in [35] that they match each other as a consequence of 'AdS Killing tensor = Conformal Killing tensor'.

## 3 Contour integral expression of the CIRZ

Now we turn to the main objective of the current paper — the derivation of the character integral representation of the zeta function in any dimensional AdS spacetime. Our goal is to express the zeta function (4.10) in terms of the $so(2,d)$ character so that the dependence on $\Delta$ and $\mathbb{Y}$ in $\zeta_{[\Delta;\mathbb{Y}]}$ enters only through the corresponding character $\chi^{so(2,d)}_{(\Delta,\mathbb{Y})}$ (see e.g. [37–40] for more details on $so(2,d)$ characters). It turns out that it is sufficient to consider the character solely over (possibly reducible) generalized Verma modules of the conformal algebra. These representations, when irreducible, correspond to massive fields in AdS for which the one-loop partition function takes the form (2.1). The formalism we develop extends trivially to massless fields [18, 41], which are described by irreducible representation defined as quotients of generalized Verma modules.

### 3.1 General dimensions

The character of the $so(2,d)$ generalized Verma module $\mathcal{V}(\Delta;\mathbb{Y})$ takes the form,

$$\chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}) = e^{-\beta\Delta}\,\chi^{so(d)}_{\mathbb{Y}}(\vec{\alpha})\,\mathcal{P}_d(i\beta;\vec{\alpha})\,, \tag{3.1}$$

where $\mathcal{P}_d$ is defined as

$$\mathcal{P}_d(\alpha_0;\vec{\alpha}) = \frac{e^{-i\,\frac{d}{2}\,\alpha_0}}{2^{d-r}}\prod_{k=1}^{r}\frac{1}{\cos\alpha_0 - \cos\alpha_k} \times \begin{cases} 1 & [\text{even } d] \\ \dfrac{i}{\sin\frac{\alpha_0}{2}} & [\text{odd } d] \end{cases}, \tag{3.2}$$

with $r = [d/2]$ and $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$. We first note that the dimension of the $so(d+2)$ irrep is given by the corresponding character evaluated at $\boldsymbol{\alpha} = \mathbf{0}$:

$$\dim^{so(d+2)}_{(iu-\frac{d}{2};\mathbb{Y})} = \left[\chi^{so(d+2)}_{(iu-\frac{d}{2};\mathbb{Y})}(\boldsymbol{\alpha})\right]_{\boldsymbol{\alpha}=\mathbf{0}}\,, \tag{3.3}$$

where $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_r)$ and $r = [d/2]$. Then, the $so(d+2)$ character can be related to that of $so(d)$ using the following identity (see Appendix A for additional details on the identity):

$$\chi^{so(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) = \sum_{k=0}^{r} \Big( e^{-i\alpha_k\, s_0}\, \chi^{so(d)}_{\mathbb{Y}_-}(\vec{\alpha}_k) + (-1)^d\, e^{i\alpha_k\,(s_0+d)}\, \chi^{so(d)}_{\mathbb{Y}_+}(\vec{\alpha}_k) \Big)\, \mathcal{P}_d(\alpha_k;\vec{\alpha}_k)\,, \tag{3.4}$$

where $\vec{\alpha}_k = (\alpha_0,\ldots,\alpha_{k-1},\alpha_{k+1},\ldots,\alpha_r)$ and $\mathbb{Y}_\pm = (s_1,\ldots,s_{r-1},(\pm)^{d+1}s_r)$. Another key trick is based on the identity,

$$\frac{1}{\left[u^2+\bar{\Delta}^2\right]^z} = \frac{\sqrt{\pi}}{\Gamma(z)} \int_0^\infty \mathrm{d}\beta \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta\, u)\, e^{-\beta\,\bar{\Delta}}\,, \tag{3.5}$$

holding for $\mathrm{Re}(\bar{\Delta})>0$ and $\mathrm{Re}(z)>0$. Note here that for the convergence of the $\beta$ integral, we ought to use $e^{-\beta\,|\bar{\Delta}|}$ in the $\beta$ integral, but in such a case the zeta function will be insensitive to the sign of $\bar{\Delta}$ and becomes incapable of distinguishing different boundary conditions. Hence, like the discussion below (2.28), we first derive the formula assuming $\mathrm{Re}(\bar{\Delta})>0$ then analytically continue $\bar{\Delta}$ to the negative $\mathrm{Re}(\bar{\Delta})$ region. Combining all these elements, we get

$$\begin{aligned}\zeta_{[\Delta;\mathbb{Y}]}(z) &= \frac{\sqrt{\pi}}{\Gamma(z)} \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}\beta\, \rho_\epsilon(u) \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta\, u) \\ &\times \left[ \sum_{k=0}^{r} e^{\frac{d}{2}(\beta+i\alpha_k)}\, \frac{\mathcal{P}_d(\alpha_k;\vec{\alpha}_k)}{\mathcal{P}_d(i\beta;\vec{\alpha}_k)} \Big( e^{u\alpha_k}\, \chi^{so(2,d)}_{(\Delta;\mathbb{Y}_+)}(\beta;\vec{\alpha}_k) + (-1)^d\, e^{-u\alpha_k}\, \chi^{so(2,d)}_{(\Delta;\mathbb{Y}_-)}(\beta;\vec{\alpha}_k) \Big) \right]_{\boldsymbol{\alpha}=\mathbf{0}}\,,\end{aligned} \tag{3.6}$$

where we used the factor $e^{-\beta(\Delta-\frac{d}{2})}$ to reconstruct the $so(2,d)$ characters of the irreps $[\Delta;\mathbb{Y}_\pm]$ according to (3.1). We can simplify the above formula using the identity of the $so(d)$ characters,

$$\chi^{so(d)}_{\mathbb{Y}_-}(\alpha_1,\ldots,\alpha_{k-1},-\alpha_k,\alpha_{k+1},\ldots,\alpha_{r-1},\alpha_r) = \chi^{so(d)}_{\mathbb{Y}_+}(\alpha_1,\ldots,\alpha_{k-1},\alpha_k,\alpha_{k+1},\ldots,\alpha_r)\,, \tag{3.7}$$

for $k=1,\ldots,r$. We obtain

$$\begin{aligned}\zeta_{[\Delta;\mathbb{Y}]}(z) &= \frac{\sqrt{\pi}}{\Gamma(z)} \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}\beta \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta\, u) \\ &\times \left[ \sum_{k=0}^{r} \tilde{\nu}_\epsilon(u,\beta,\alpha_k) \left(\frac{\sinh\frac{\beta}{2}}{\sin\frac{\alpha_k}{2}}\right)^{d-2r} \prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}} \frac{\cosh\beta-\cos\alpha_j}{\cos\alpha_k-\cos\alpha_j}\, \chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}_k) \right]_{\boldsymbol{\alpha}=\mathbf{0}}\,,\end{aligned} \tag{3.8}$$

where the function $\tilde{\nu}_\epsilon(u,\beta,\alpha)$ is defined to be

$$\tilde{\nu}_\epsilon(u,\beta,\alpha) = \begin{cases} \dfrac{2\,\ln R}{\pi}\cosh(\alpha\, u) & [\text{even } d] \\ -\tanh^\epsilon(\pi\, u)\,\sinh(\alpha\, u) & [\text{odd } d] \end{cases}. \tag{3.9}$$

Note here that each summand in the second line of (3.8) diverges in the limit $\boldsymbol{\alpha}\to\mathbf{0}$. Only the sum of the $r+1$ terms is regular in the limit $\boldsymbol{\alpha}\to\mathbf{0}$. For this reason, we cannot exchange the order of the summation and the evaluation $\boldsymbol{\alpha}=\mathbf{0}$.

To further simplify the formula, it is convenient to replace the evaluation of the character at $\boldsymbol{\alpha}=\mathbf{0}$ by the contour integrals,

$$\dim^{so(d+2)}_{(s_0,\mathbb{Y})} = \prod_{k=0}^{r} \oint_{\mathscr{C}_k} \frac{\mathrm{d}\alpha_k}{2\,\pi\,i\,\alpha_k}\,\chi^{so(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha})\,, \tag{3.10}$$

where $\mathscr{C}_k$ are contours encircling the origin counter-clockwise such that the contour $\mathscr{C}_k$ lies inside of $\mathscr{C}_{k+1}$: e.g. the circular contours with $|\alpha_k|<|\alpha_{k+1}|$ (see Figure 3.2).

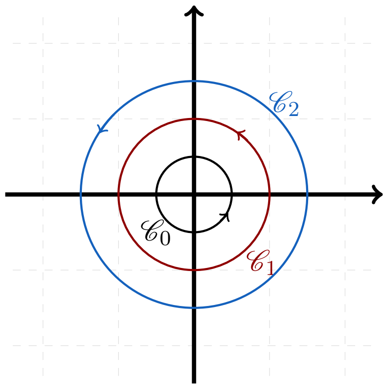


**Figure 1**. Example of "ordered" contours with $r=2$.

The advantage of the contour integral representation is that now we can perform the contour integration before the summation — but keeping the order of contours fixed. In this way we get

$$\begin{aligned}\zeta_{[\Delta;\mathbb{Y}]}(z) &= \frac{\sqrt{\pi}}{\Gamma(z)} \int_0^\infty \mathrm{d}\beta \int_0^\infty \mathrm{d}u \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta\,u) \\ &\times \sum_{k=0}^{r} \oint_C \mu(\boldsymbol{\alpha})\,\tilde{\nu}_\epsilon(u,\beta,\alpha_k) \left(\frac{\sinh\frac{\beta}{2}}{\sin\frac{\alpha_k}{2}}\right)^{d-2r} \prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}} \frac{\cosh\beta-\cos\alpha_j}{\cos\alpha_k-\cos\alpha_j}\,\chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}_k)\,,\end{aligned} \tag{3.11}$$

where we have interchanged the order of $u$ and $\beta$ integrations and $\mu(\boldsymbol{\alpha})$ denotes the measure for $(r+1)$-variable complex integral,

$$\oint_C \mu(\boldsymbol{\alpha}) = \prod_{k=0}^{r} \oint_{|\alpha_k|<|\alpha_{k+1}|} \frac{\mathrm{d}\alpha_k}{2\,\pi\,i\,\alpha_k}\,. \tag{3.12}$$

Finally, exchanging the order of the $u$ and $\boldsymbol{\alpha}$ integrations, we obtain

$$\begin{aligned}\zeta_{[\Delta;\mathbb{Y}]}(z) &= \frac{\sqrt{\pi}}{\Gamma(z)} \int_0^\infty \mathrm{d}\beta \sum_{k=0}^{r} \oint_C \mu(\boldsymbol{\alpha})\,\nu_\epsilon(z,\beta,\alpha_k)\,\times \\ &\times \left(\frac{\sinh\frac{\beta}{2}}{\sin\frac{\alpha_k}{2}}\right)^{d-2r} \prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}} \frac{\cosh\beta-\cos\alpha_j}{\cos\alpha_k-\cos\alpha_j}\,\chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}_k)\,,\end{aligned} \tag{3.13}$$

where the function $\nu_\epsilon(z,\beta,\alpha)$ is defined to be

$$\begin{aligned}\nu_\epsilon(z,\beta,\alpha) &= \int_0^\infty \mathrm{d}u\, \tilde{\nu}_\epsilon(u,\beta,\alpha)\, \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta\, u) \\ &= \int_0^\infty \mathrm{d}u\, \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta\, u) \times \begin{cases} \dfrac{2\,\ln R}{\pi}\, \cosh(\alpha\, u) & [\text{even } d] \\ -\tanh^{\epsilon}(\pi\, u)\, \sinh(\alpha\, u) & [\text{odd } d] \end{cases}.\end{aligned} \tag{3.14}$$

Evaluation of the above function requires separate considerations for even and odd $d$.

### 3.2 $\mathbf{AdS}_{2r+1}$

In even $d = 2r$, the function $\nu_\epsilon(z,\beta,\alpha)$ can be evaluated as

$$\nu_\epsilon(z,\beta,\alpha) = \frac{2\,\ln R}{\pi} \int_0^\infty \mathrm{d}u\, \cosh(\alpha\, u)\, \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta\, u) \;=\; \frac{\ln R}{\sqrt{\pi}}\, \frac{\left(\left(\frac{\beta}{2}\right)^2 + \left(\frac{\alpha}{2}\right)^2\right)^{z-1}}{\Gamma(z)}\,. \tag{3.15}$$

This integral was evaluated in the region $\mathrm{Re}(\alpha) = \mathrm{Im}(\beta) = 0$, $\beta > 0$ and $\mathrm{Re}(z) > 0$, then analytically continued to other region. Since an additional factor of $1/\Gamma(z)$ is generated, we can express the result as

$$\Gamma(z)\, \zeta_{[\Delta;\mathbb{Y}]}(z) = \ln R \int_0^\infty \frac{\mathrm{d}\beta}{\Gamma(z)} \left(\frac{\beta}{2}\right)^{2(z-1)} f_{[\Delta;\mathbb{Y}]}(z,\beta), \tag{3.16}$$

where the function $f_{[\Delta;\mathbb{Y}]}(z,\beta)$ is defined as

$$f_{[\Delta;\mathbb{Y}]}(z,\beta) = \sum_{k=0}^{r} \oint_C \mu(\boldsymbol{\alpha})\, \left(1 + \left(\tfrac{\alpha_k}{\beta}\right)^2\right)^{z-1} \prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}} \frac{\cosh\beta - \cos\alpha_j}{\cos\alpha_k - \cos\alpha_j}\, \chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}_k)\,. \tag{3.17}$$

Note that the contour of $\alpha_k$ does not contain $\pm i\,\beta$. The function $\beta^{2(z-1)}$ has a branch cut on the negative real axis of $\beta$ with the phase factor $e^{4\pi z i}$. Using this information, we can rewrite (3.16) as

$$\Gamma(z)\, \zeta_{[\Delta;\mathbb{Y}]}(z) = \ln R \oint \frac{\mathrm{d}\beta}{\Gamma(z)\, 2\, i\, \sin(2\pi z)} \left(-\frac{\beta}{2}\right)^{2(z-1)} f_{[\Delta;\mathbb{Y}]}(z,\beta)\,, \tag{3.18}$$

where the contour integral is defined by the Hankel contour (see Figure 2).
Therefore, the first derivative of the zeta function reduces to

$$\zeta'_{[\Delta;\mathbb{Y}]}(0) = \frac{\ln R}{2} \oint \frac{\mathrm{d}\beta}{2\pi\, i} \left(\frac{\beta}{2}\right)^{-2} f_{[\Delta;\mathbb{Y}]}(0,\beta)\,, \tag{3.19}$$

whereas $\zeta_{[\Delta;\mathbb{Y}]}(0) = 0$. Notice that, because the integrand of the above integral is devoid of branch cut, the Hankel contour can be deformed to a closed contour encircling the origin of the complex plane. In the end, we arrived at the relation between the zeta function $\zeta_{[\Delta;\mathbb{Y}]}(z)$ and the character $\chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha})$ for an arbitrary lowest weight representation $[\Delta;\mathbb{Y}]$. Since the relation (3.19) is a linear map independent of the representation, we can apply the same formula to a generic *reducible* representation of the $so(2,d)$ algebra, e.g. the representation corresponding to the field content of any theory.

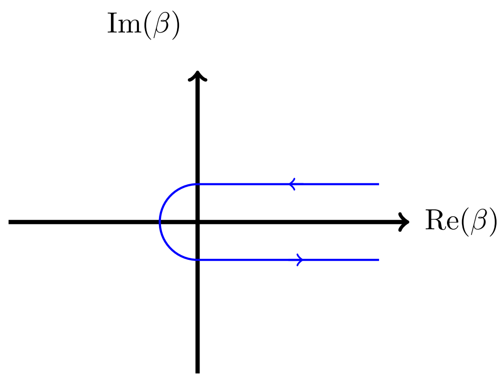


**Figure 2**. Hankel contour.

### 3.3 $\mathbf{AdS}_{2r+2}$

In odd $d=2r+1$, the function $\nu_\epsilon(z,\beta,\alpha)$ becomes

$$\nu_\epsilon(z,\beta,\alpha) = -\frac{\left(\frac{\beta}{2}\right)^{2z-1}}{\Gamma(z+\frac{1}{2})}\,\lambda_\epsilon(z,\beta,\alpha), \tag{3.20}$$

with

$$\lambda_\epsilon(z,\beta,\alpha) = \int_0^\infty \mathrm{d}u\,\tanh^\epsilon(\pi\,u)\,\sinh(\alpha\,u)\,{}_0F_1\big(z+\tfrac{1}{2},-\tfrac{u^2\,\beta^2}{4}\big)\,. \tag{3.21}$$

Here, we have used the identity,

$$\left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(u\,\beta) = \frac{\left(\frac{\beta}{2}\right)^{2z-1}}{\Gamma(z+\frac{1}{2})}\,{}_0F_1\big(z+\tfrac{1}{2},-\tfrac{u^2\,\beta^2}{4}\big)\,. \tag{3.22}$$

In terms of the function $\lambda_\epsilon(z,\beta,\alpha)$, the zeta function is given by

$$\begin{aligned}\zeta_{[\Delta;\mathbb{Y}]}(z) = &-\int_0^\infty \mathrm{d}\beta\,\frac{\beta^{2z-1}}{\Gamma(2z)}\sum_{k=0}^{r}\oint_C \mu(\boldsymbol{\alpha})\,\lambda_\epsilon(z,\beta,\alpha_k)\,\frac{\sinh\frac{\beta}{2}}{\sin\frac{\alpha_k}{2}}\times \\ &\times\prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}}\frac{\cosh\beta-\cos\alpha_j}{\cos\alpha_k-\cos\alpha_j}\,\chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}_k)\,.\end{aligned} \tag{3.23}$$

If we expand $\lambda_\epsilon(z,\beta,\alpha)$ in $z$ as

$$\lambda_\epsilon(z,\beta,\alpha) = \lambda_\epsilon(0,\beta,\alpha) + z\,\lambda'_\epsilon(0,\beta,\alpha) + \mathcal{O}(z^2)\,, \tag{3.24}$$

where $\lambda'_\epsilon(z,\beta,\alpha) = \frac{\partial}{\partial z}\lambda_\epsilon(z,\beta,\alpha)$, the higher order terms $\mathcal{O}(z^2)$ contribute neither to $\zeta_{[\Delta;\mathbb{Y}]}(0)$ nor to $\zeta'_{[\Delta;\mathbb{Y}]}(0)$ — the parts of the zeta function that are relevant to physics. The second term $z\,\lambda'_\epsilon(0,\beta,\alpha)$ does not contribute to $\zeta_{[\Delta;\mathbb{Y}]}(0)$, but it does to $\zeta'_{[\Delta;\mathbb{Y}]}(0)$. However, as we shall explain below, its contribution vanishes if the character is even in $\beta$. Moreover, even when they contribute, their contribution is a rational number. For this reason, their role is rather marginal compared to the term $\lambda_\epsilon(0,\beta,\alpha)$. The zeta function obtained only with $\lambda_\epsilon(0,\beta,\alpha)$ neglecting the $\mathcal{O}(z)$ term will be referred to as the *primary* contribution $\zeta_{1,[\Delta;\mathbb{Y}]}(z)$,[4] while the contribution related to $z\,\lambda'_\epsilon(0,\beta,\alpha)$ as the *secondary* one $\zeta_{2,[\Delta;\mathbb{Y}]}(z)$. To summarize, the physically relevant parts of the zeta function are given by

$$\zeta_{[\Delta;\mathbb{Y}]}(0) = \zeta_{1,[\Delta;\mathbb{Y}]}(0)\,, \qquad \zeta'_{[\Delta;\mathbb{Y}]}(0) = \zeta'_{1,[\Delta;\mathbb{Y}]}(0) + \zeta'_{2,[\Delta;\mathbb{Y}]}(0)\,, \tag{3.25}$$

where $\zeta_{1,[\Delta;\mathbb{Y}]}(z)$ and $\zeta_{2,[\Delta;\mathbb{Y}]}(z)$ are calculated hereafter.

### Primary contribution

Let us consider first the primary contribution to the zeta function. It is given through $\lambda_\epsilon(0,\beta,\alpha)$ in the form of an integral,

$$\lambda_\epsilon(0,\beta,\alpha) = \int_0^\infty \mathrm{d}u\, \tanh^\epsilon(\pi\, u)\, \sinh(\alpha\, u)\, \cos(\beta\, u)\,, \tag{3.26}$$

where we have used the fact that ${}_0F_1(\frac{1}{2}, -\frac{x^2}{4}) = \cos x$. The above integral is not convergent but can be considered as a distribution (see e.g. Chapter 6 of [42]). By multiplying the integrand with $e^{-\sigma\, u}$, we can evaluate the integral in the region $\mathrm{Re}(\sigma) > |\mathrm{Re}(\alpha)| + |\mathrm{Im}(\beta)|$. By taking the limit $\sigma \to 0^+$, we obtain a finite result (hence the original integral is Abel summable [42]),

$$\lambda_\epsilon(0,\beta,\alpha) = -\frac{\sin\frac{\alpha}{2}\,(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}(\cos\frac{\alpha}{2})^{\frac{1-\epsilon}{2}}}{\cosh\beta - \cos\alpha}\,. \tag{3.27}$$

We can analytically continue the above to other region of $\alpha, \beta$. Plugging this into (3.23), we obtain the primary contribution of the zeta function as

$$\zeta_{1,[\Delta;\mathbb{Y}]}(z) = \int_0^\infty \frac{\mathrm{d}\beta\,\beta^{2z-1}}{\Gamma(2z)}\, f_{1,[\Delta;\mathbb{Y}]}(\beta), \tag{3.28}$$

where the function $f_{1,[\Delta;\mathbb{Y}]}(\beta)$ is given by

$$f_{1,[\Delta;\mathbb{Y}]}(\beta) = \sum_{k=0}^{r} \oint_C \mu(\boldsymbol{\alpha})\, \frac{\sinh\frac{\beta}{2}\,(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}(\cos\frac{\alpha_k}{2})^{\frac{1-\epsilon}{2}}}{(\cosh\beta - \cos\alpha_k)} \prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}} \frac{\cosh\beta - \cos\alpha_j}{\cos\alpha_k - \cos\alpha_j}\, \chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}_k)\,. \tag{3.29}$$

We finally reached a formula where the relation itself does not involve any dependency on $\Delta$ and $\mathbb{Y}$.

### Secondary contribution

The second term $z\,\lambda'_\epsilon(0,\beta,\alpha)$ in (3.24) is linear in $z$, hence contributes to the derivative of zeta function as a residue:

$$\zeta'_{2,[\Delta;\mathbb{Y}]}(0) = \oint \frac{\mathrm{d}\beta}{2\pi i\,\beta}\, f_{2,[\Delta;\mathbb{Y}]}(\beta)\,, \tag{3.30}$$

[4] The primary contribution $\zeta_{1,[\Delta;\mathbb{Y}]}(z)$ was referred to as the modified zeta function $\tilde{\zeta}_{[\Delta;\mathbb{Y}]}(z)$ in [14].

with

$$f_{2,[\Delta;\mathbb{Y}]}(\beta) = -\sum_{k=0}^{r} \oint_C \mu(\boldsymbol{\alpha})\, \lambda'_\epsilon(0,\beta,\alpha_k) \times \\ \times \frac{\sinh\frac{\beta}{2}}{\sin\frac{\alpha_k}{2}} \prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}} \frac{\cosh\beta-\cos\alpha_j}{\cos\alpha_k-\cos\alpha_j}\, \chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}_k)\,. \tag{3.31}$$

Remark that, since the second line of (3.31) is regular in $\beta$, only terms with negative powers in the Laurent series of $\lambda'_\epsilon(0,\beta,\alpha)$ can contribute to the residue in (3.30). To isolate these negative power terms, let us split $\tanh^\epsilon(\pi u)$ in the integrand of $\lambda_\epsilon(z,\beta,\alpha)$ given in (3.21) into two terms as

$$\tanh^\epsilon(\pi u) = 1-\frac{2}{1+\epsilon\, e^{2\pi u}}\,. \tag{3.32}$$

Focusing on the second term of (3.32), the $u$ integral (3.21) gives a regular function of $\beta$, so all the negative power terms come from the first term "1". Now moving to the first term of (3.32), the corresponding integral can be evaluated for $\mathrm{Re}(z)>0$ as

$$\int_0^\infty \mathrm{d}u\, \sinh(\alpha\, u)\, {}_0F_1\big(z+\tfrac{1}{2}, -\tfrac{u^2\beta^2}{4}\big) = (2z-1)\,\frac{\alpha}{\beta^2}\, {}_2F_1\Big(1,\tfrac{3}{2}-z;\tfrac{3}{2};-\tfrac{\alpha^2}{\beta^2}\Big)\,, \tag{3.33}$$

from which we obtain

$$\lambda'_\epsilon(0,\beta,\alpha) = \frac{2\,\beta\, \arctan\big(\frac{\alpha}{\beta}\big)}{\beta^2+\alpha^2} + \mathcal{O}(1)\,. \tag{3.34}$$

Here, $\mathcal{O}(1)$ means a regular function of $\beta$ and it is related to the second term of (3.32). Again, both of the $u$ integrals arising from the first and second terms in (3.32) are evaluated in the region where $\mathrm{Re}(\alpha)=\mathrm{Im}(\beta)=0$, then analytically continued to other values. In the end, we obtain a compact expression of the secondary contribution:

$$\zeta'_{2,[\Delta;\mathbb{Y}]}(0) = -\oint \frac{\mathrm{d}\beta}{2\pi i} \sum_{k=0}^{r} \oint_C \mu(\boldsymbol{\alpha})\, \frac{2\arctan\big(\frac{\alpha_k}{\beta}\big)}{\beta^2+\alpha_k^2}\, \frac{\sinh\frac{\beta}{2}}{\sin\frac{\alpha_k}{2}} \prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}} \frac{\cosh\beta-\cos\alpha_j}{\cos\alpha_k-\cos\alpha_j}\, \chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta;\vec{\alpha}_k)\,, \tag{3.35}$$

where the $\beta$ contour now encloses anti-clockwise the branch cut from $-i\,\alpha_k$ to $i\,\alpha_k$. Remark that if the character is even in $\beta$, then the whole integrand is even in $\beta$ and the residue vanishes trivially. Therefore, in such a case, the secondary contribution of the zeta function vanishes. Furthermore, the secondary contribution is independent of $\epsilon$, i.e. whether the spectrum is bosonic or fermionic.

### 3.4 Cross-check

In the above, we have derived the CIRZ in any dimensions. The derivation required many technical steps, hence it would be good if we can cross-check the formula in the end. In principle, if the zeta function of a field with an arbitrary mass and spin is reproduced from CIRZ by inserting the character of the corresponding field, it will be a sufficient test for our formula. In this section, we show how the spectral integral expressions for $\zeta'_{[\Delta,\mathbb{Y}]}(0)$ given in (2.28) and (2.35) for odd and even dimensional AdS can be reproduced starting from the CIRZ formula obtained in the previous section. This, besides being a check, displays interesting analogies between the spectral and Hurwitz zeta functions.

**AdS$_{2r+1}$**

Let us consider first the CIRZ formula (3.19) in odd dimensional AdS. By inserting the $so(2,d)$ character, we obtain

$$\begin{aligned}\zeta'_{[\Delta;\mathbb{Y}]}(0) &= \frac{\ln R}{2^{r-1}}\oint\frac{d\beta}{2\pi\, i}\sum_{k=0}^{r}\oint_C \mu(\boldsymbol{\alpha})\,\frac{e^{-\bar{\Delta}\,\beta}\,\chi^{so(d)}_{\mathbb{Y}}(\vec{\alpha}_k)}{(\beta^2+\alpha_k^2)}\prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}}\frac{1}{(\cos\alpha_k-\cos\alpha_j)}\\ &= -\frac{\ln R}{2^{r-1}}\sum_{k=0}^{r}\oint_C \mu(\boldsymbol{\alpha})\,\frac{\sin(\alpha_k\,\bar{\Delta})}{\alpha_k}\,\chi^{so(d)}_{\mathbb{Y}}(\vec{\alpha}_k)\prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}}\frac{1}{(\cos\alpha_k-\cos\alpha_j)}\,,\end{aligned} \tag{3.36}$$

where $\bar{\Delta}=\Delta-\frac{d}{2}$ and we have performed the $\beta$ integration. From the sine function in the integrand, it is manifest that

$$\zeta'_{[\frac{d}{2};\mathbb{Y}]}(0)=0\,. \tag{3.37}$$

By taking a derivative with respect to $\Delta$, the formula becomes

$$\frac{\partial}{\partial\Delta}\zeta'_{[\Delta;\mathbb{Y}]}(0) = -\frac{\ln R}{2^{r-1}}\sum_{k=0}^{r}\oint_C \mu(\boldsymbol{\alpha})\,\cos(\bar{\Delta}\,\alpha_k)\,\chi^{so(d)}_{\mathbb{Y}}(\vec{\alpha}_k)\prod_{\substack{0\leqslant j\leqslant r\\ j\neq k}}\frac{1}{(\cos\alpha_k-\cos\alpha_j)}\,, \tag{3.38}$$

Using (3.2),(3.4) and (3.10), one can show that

$$\frac{\partial}{\partial\Delta}\zeta'_{[\Delta;\mathbb{Y}]}(0) = -\ln R\,\dim^{so(d+2)}_{(-\Delta;\mathbb{Y})}\,. \tag{3.39}$$

Therefore, the odd dimensional CIRZ formula (3.19) correctly reproduces the known result (2.27).

**AdS$_{2r+2}$**

In even dimensional AdS, we show how the difference of the zeta function can be obtained from the CIRZ (3.28). In fact, the derivation shows an interesting analogy with one well-known property of Hurwitz zeta function:

$$i^{-z}\,\zeta(z,a)+i^{z}\,\zeta(z,1-a)=\frac{(2\pi)^z}{\Gamma(z)}\,\mathrm{Li}_{1-z}(e^{-2\pi i\,a})\,. \tag{3.40}$$

It can be derived from the integral representation,

$$\zeta(z,a)=\int_0^\infty\frac{d\beta\,\beta^{z-1}}{\Gamma(z)}\,\frac{e^{-a\,\beta}}{1-e^{-\beta}} = -\Gamma(1-z)\oint\frac{d\beta}{2\pi i}\,\frac{(-\beta)^{z-1}\,e^{-(a-\frac{1}{2})\,\beta}}{2\,\sinh\frac{\beta}{2}}\,, \tag{3.41}$$

where the integral is along the Hankel contour. For $\mathrm{Re}(z)<0$ and $a\in(0,1)$, we can add to the contour an infinite clockwise circle. Then, the contour can be deformed to encircle the infinitely many simple poles arising from $1/\sinh\frac{\beta}{2}$: they are at $\beta = 2\pi\, i\, n$ for all integer $n \neq 0$. Hence, we obtain

$$\begin{aligned}\zeta(z,a) &= \Gamma(1-z)\sum_{n=1}^{\infty}\Big[(-1)^n\,(2\pi\, i\, n)^{z-1}\,e^{(a-\frac{1}{2})\,2\pi\, i\, n} + (-1)^n\,(-2\pi\, i\, n)^{z-1}\,e^{-(a-\frac{1}{2})\,2\pi\, i\, n}\Big] \\ &= \Gamma(1-z)\,(2\pi)^{z-1}\left[i^{z-1}\,\mathrm{Li}_{1-z}(e^{2\pi\, i\, a}) + i^{1-z}\,\mathrm{Li}_{1-z}(e^{-2\pi\, i\, a})\right], \end{aligned} \tag{3.42}$$

from which we can easily derive (3.40). In particular, it allows to cancel the divergence of the zeta function at $z=1$ and evaluate the finite part:

$$\lim_{z\to 1}\left[i^{-z}\,\zeta(z,a) + i^{z}\,\zeta(z,1-a)\right] = 2\pi\,\mathrm{Li}_0(e^{-2\pi\, i\, a}) = -\pi\, i\,\frac{e^{-\pi\, i\, a}}{\sin(\pi\, a)}\,. \tag{3.43}$$

Now let us move to the difference of the spectral zeta function. Since the difference of the character $\chi^{so(2,d)}_{(\Delta;\mathbb{Y})}(\beta,\vec{\alpha}) - \chi^{so(2,d)}_{(d-\Delta;\mathbb{Y})}(\beta,\vec{\alpha})$ is even in $\beta$, the corresponding secondary contribution simply vanishes: $\zeta'_{2,[\Delta;\mathbb{Y}]}(0) - \zeta'_{2,[d-\Delta;\mathbb{Y}]}(0) = 0$. Focussing on the primary contribution (3.28), we first find that the integrand function $f_{1,[\Delta;\mathbb{Y}]}$ reduces to

$$f_{1,[\Delta;\mathbb{Y}]}(\beta) = \sum_{k=0}^{r}\oint_C \mu(\boldsymbol{\alpha})\,\frac{\chi^{so(d)}_{\mathbb{Y}}(\vec{\alpha}_k)}{2^{r+1}\,\Pi_{\substack{0\leqslant j\leqslant r\\ j\neq k}}(\cos\alpha_k - \cos\alpha_j)}\,\frac{(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}\,(\cos\frac{\alpha_k}{2})^{\frac{1-\epsilon}{2}}\,e^{-\bar{\Delta}\,\beta}}{\cosh\beta - \cos\alpha_k}\,, \tag{3.44}$$

by inserting the expression of $\chi^{so(2,d)}_{(\Delta;\mathbb{Y})}$. By changing the order of the $\beta$ and $\boldsymbol{\alpha}$ integrals:

$$\zeta_{1,[\Delta,\mathbb{Y}]}(z) = \sum_{k=0}^{r}\oint_C \mu(\boldsymbol{\alpha})\,\frac{\chi^{so(d)}_{\mathbb{Y}}(\vec{\alpha}_k)\,\xi(2z,\bar{\Delta},\alpha_k)}{2^{r+1}\,\Pi_{\substack{0\leqslant j\leqslant r\\ j\neq k}}(\cos\alpha_k - \cos\alpha_j)}\,, \tag{3.45}$$

with

$$\xi(z,\bar{\Delta},\alpha) = \int_0^{\infty}\frac{\mathrm{d}\beta\,\beta^{z-1}}{\Gamma(z)}\,\frac{(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}\,(\cos\frac{\alpha}{2})^{\frac{1-\epsilon}{2}}\,e^{-\bar{\Delta}\,\beta}}{\cosh\beta - \cos\alpha}\,, \tag{3.46}$$

we can focus first on the function $\xi(z,\bar{\Delta})$ which plays an analogous role as the Hurwitz zeta function in the relation (3.40). In particular, $\frac{\partial}{\partial\bar{\Delta}}\xi(z,\bar{\Delta},\alpha) = -z\,\xi(z+1,\bar{\Delta},\alpha)$. We recast the integral (3.46) as the integral over the Hankel contour,

$$\xi(z,\bar{\Delta},\alpha) = -\Gamma(1-z)\oint\frac{\mathrm{d}\beta\,(-\beta)^{z-1}}{2\pi i}\,\frac{(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}\,(\cos\frac{\alpha}{2})^{\frac{1-\epsilon}{2}}\,e^{-\bar{\Delta}\,\beta}}{2\,\sinh\frac{\beta+i\,\alpha}{2}\,\sinh\frac{\beta-i\,\alpha}{2}}\,. \tag{3.47}$$

Like in the case of Hurwitz zeta function, we add to the contour an infinite clockwise circle and shrink it to enclose the infinite many simple poles arising, at this time, from $\sinh\frac{\beta\pm i\,\alpha}{2}$: they are at $\beta = (2\pi\, n \mp \alpha)\, i$. By collecting the residues, we get

$$\xi(z,\bar{\Delta},\alpha) = \frac{\Gamma(1-z)\,e^{\bar{\Delta}\,\alpha\, i}}{2\, i\,\sin\frac{\alpha}{2}}\sum_{n=-\infty}^{\infty}\left[-(2\pi\, n - \alpha)\, i\right]^{z-1}(-\epsilon)^n\,e^{-2\pi\, n\,\bar{\Delta}\, i} + (\alpha\leftrightarrow -\alpha)\,, \tag{3.48}$$

which is divergent in the $z \to 1$ limit. The divergence can be canceled by taking the difference,

$$\lim_{z\to 1}\left[\xi(z,\bar{\Delta},\alpha)-\xi(z,-\bar{\Delta},\alpha)\right] = \frac{\pi}{2}\,\frac{e^{\bar{\Delta}\,\alpha\, i}}{\sin\frac{\alpha}{2}}\,\tanh^{\epsilon}(\pi\, i\,\bar{\Delta})+(\alpha\leftrightarrow-\alpha)$$
$$= \pi\, i\,\frac{\sin(\bar{\Delta}\,\alpha)}{\sin\frac{\alpha}{2}}\,\tanh^{\epsilon}(\pi\, i\,\bar{\Delta})\,. \tag{3.49}$$

Using the above result in (3.45) together with (3.2), (3.4) and (3.10), we can reproduce (2.35).

## 4 Derivative expression of the CIRZ

In this section, we present a different expression of CIRZ in terms of derivatives in $\alpha_i$. As explained in Section 2, the combination $\dim^{so(d)}_{\mathbb{Y}}\,\mu_{\mathbb{Y}}(u)$ in the zeta function (2.5) is related to the Weyl dimension formula which can be obtained as a limit of the $so(d+2)$ characters. In Section 3, this limit was taken as a contour integral. The expression with contour integrals in $\alpha_i$ variables is useful — see the companion paper [28] — but sometimes not explicit enough. For instance, if one wants to implement the CIRZ formula in a computer program, it will be more convenient to have an expression, where all the $\alpha_i$ contour integrals are already evaluated using the residue theorem, involving $\alpha_i$ derivatives of the $so(d+2)$ characters. In fact, for an expression in terms of $\alpha_i$ derivatives, it is simpler to re-derive the CIRZ by taking the limit of $so(d+2)$ characters using a generalized L'Hôpital's rule. Below, we demonstrate how to obtain such an expression. The CIRZ for $\mathrm{AdS}_4$ and $\mathrm{AdS}_5$ originally presented in [14] are recovered as special cases.

### 4.1 General dimensions

In order to recover the Weyl character formula (3.3), we need to evaluate the $so(d+2)$ characters in the limit $\boldsymbol{\alpha}\to\mathbf{0}$. It is actually subtle to perform the evaluation since the $so(d+2)$ characters take the form,

$$\chi^{so(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) = \frac{\mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha})}{\mathsf{D}^{(d+2)}(\boldsymbol{\alpha})}\,, \tag{4.1}$$

and both the numerator and denominator vanish as $\boldsymbol{\alpha}\to\mathbf{0}$:

$$\lim_{\boldsymbol{\alpha}\to\mathbf{0}}\mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) = \lim_{\boldsymbol{\alpha}\to\mathbf{0}}\mathsf{D}^{(d+2)}(\boldsymbol{\alpha}) = 0\,. \tag{4.2}$$

The explicit expressions for the numerator $\mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}$ and the denominator $\mathsf{D}^{(d+2)}$ are given in (A.3) and (A.2). Despite the apparent singularity, the limit of the $so(d+2)$ character does exist, and can be obtained by using a generalisation of L'Hôpital's rule (see Appendix B for more details):

$$\dim^{so(d+2)}_{(s_0,\mathbb{Y})} = \lim_{\boldsymbol{\alpha}\to\mathbf{0}}\chi^{so(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) = \frac{\mathscr{D}_{\Phi^{d+2}}\,\mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}}}{\mathscr{D}_{\Phi^{d+2}}\,\mathsf{D}^{(d+2)}(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}}}\,, \tag{4.3}$$

where the differential operator $\mathscr{D}_{\Phi^{d+2}}$ is given, in the notation $\partial_i = \partial_{\alpha_i}$, by

$$\mathscr{D}_{\Phi^{d+2}} = \prod_{0\leqslant i<j\leqslant r} (\partial_i^2 - \partial_j^2) \times \begin{cases} 1 & [d=2r] \\ \prod_{k=0}^{r} \partial_k & [d=2r+1] \end{cases} . \tag{4.4}$$

Firstly, the denominator of (4.3) depends only on $d$:

$$\mathscr{D}_{\Phi^{d+2}}\, \mathsf{D}^{(d+2)}(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\mathbf{0}} = c_d \tag{4.5}$$

and can be explicitly evaluated as explained in Appendix B.1. The result reads

$$c_d = (-1)^{\frac{r(r+1)}{2}}\, 2^r (r+1)! \prod_{0\leqslant i<j\leqslant r} (d-i-j)(j-i) \times \begin{cases} 1 & [d=2r] \\ 2\, i^{r+1} \prod_{k=0}^{r} (d/2-k) & [d=2r+1] \end{cases} . \tag{4.6}$$

Secondly, the numerator of (4.3) can be recast, as explained in Appendix B.2, into

$$\mathscr{D}_{\Phi^{d+2}}\, \mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}} = 2(r+1)\,(-1)^d\, u^{d-2r} \sum_{n=0}^{r} u^{2n}\, \mathfrak{D}_{(n)} \mathsf{N}^{(d)}_{\mathbb{Y}}(\vec{\alpha})\Big|_{\boldsymbol{\alpha}=\mathbf{0}}, \tag{4.7}$$

where the differential operator $\mathfrak{D}_{(n)}$ is defined as

$$\mathfrak{D}_{(n)} := \bar{\partial}_{(n)} \mathscr{D}_{\Phi^d}\,, \tag{4.8}$$

with

$$\bar{\partial}_{(n)} = (-1)^{r-n} \sum_{1\leq i_1<i_2<\cdots<i_{r-n}\leq r} \partial_{i_1}^2 \dots \partial_{i_{r-n}}^2 \,. \tag{4.9}$$

Using (4.5) and (4.7), we can write the zeta function as

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \frac{2(r+1)}{c_d}\,(-1)^d \sum_{n=0}^{r} \int_0^\infty \frac{\mathrm{d}u\, \rho_\epsilon(u)}{[u^2+(\Delta-\frac{d}{2})^2]^z}\, u^{2n+d-2r}\, \mathfrak{D}_{(n)}\, \mathsf{N}^{(d)}_{\mathbb{Y}}(\vec{\alpha})\Big|_{\vec{\alpha}=\vec{0}}, \tag{4.10}$$

where $\rho_\epsilon(u)$ is given in (2.15). At this point, introducing the $\beta$-integral (3.5) will give rise to a factor $e^{-\beta\,\Delta}$. By reconstructing the $so(d+2)$ character from $e^{-\beta\,\Delta}$ and $\mathsf{N}^{(d)}_{\mathbb{Y}}$ and using the identity (A.8), we obtain

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \frac{2(r+1)}{\Gamma(z)\, c_d} \int_0^\infty \mathrm{d}\beta \sum_{n=0}^{r} \varphi_{\epsilon,n}(z,\beta)\, \mathfrak{D}_{(n)} \Big[\mathsf{D}^{(d+2)}(i\beta,\vec{\alpha})\, \chi^{so(2,d)}_{(\Delta,\mathbb{Y})}(\beta,\vec{\alpha})\Big]\Big|_{\vec{\alpha}=\vec{0}}, \tag{4.11}$$

with

$$\varphi_{\epsilon,n}(z,\beta) = \sqrt{\pi} \int_0^\infty \mathrm{d}u \left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-\frac{1}{2}}(\beta\, u) \times \begin{cases} \frac{\ln R}{\pi}\, u^{2n} & [d=2r] \\ \frac{i}{2}\, \tanh^\epsilon(\pi\, u)\, u^{2n+1} & [d=2r+1] \end{cases} . \tag{4.12}$$

Remark that the function $\varphi_{\epsilon,n}(z,\beta)$ is related to $\nu_\epsilon(z,\beta,\alpha)$ defined in (3.14) as

$$\varphi_{\epsilon,n}(z,\beta) = \frac{\sqrt{\pi}}{2} \times \begin{cases} \partial_\alpha^{2n}\, \nu_\epsilon(z,\beta,\alpha)\,\big|_{\alpha=0} & [d=2r] \\ -i\,\partial_\alpha^{2n+1}\, \nu_\epsilon(z,\beta,\alpha)\,\big|_{\alpha=0} & [d=2r+1] \end{cases}. \tag{4.13}$$

Hence, the expression (4.11) can be considered as the result of the $\alpha_i$ contour integrals of (3.13).

**Odd dimensional AdS**

For $d=2r$, the function $\nu_\epsilon(z,\beta,\alpha)$ has been computed exactly as (3.15). The corresponding $\varphi_{\epsilon,n}(\beta;z)$ is

$$\varphi_{\epsilon,n}(z,\beta) = \frac{\ln R}{2\sqrt{\pi}}\, \frac{\Gamma(n+\frac{1}{2})}{\Gamma(z-n)} \left(\frac{\beta}{2}\right)^{2(z-n-1)}, \tag{4.14}$$

hence, the zeta function can be expressed as

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \frac{\ln R}{\Gamma(z)}\,\frac{d+2}{2\,c_d} \int_0^\infty \mathrm{d}\beta \sum_{n=0}^{r} \frac{(2n)!}{4^n n!}\, \frac{(\frac{\beta}{2})^{2(z-n-1)}}{\Gamma(z-n)} \times \\ \times \mathfrak{D}_{(n)}\Big[\mathsf{D}^{(d+2)}(i\beta,\vec{\alpha})\,\chi^{so(2,d)}_{(\Delta,\mathbb{Y})}(\beta,\vec{\alpha})\Big]\Big|_{\vec{\alpha}=\vec{0}}. \tag{4.15}$$

The $\beta$ integral is convergent for $\mathrm{Re}(z)>d$, but one can analytically continue $z$ to other values. Since the only singularity of the integrand is at $\beta=0$, we can deform the $\beta$ integral to a complex integral with the contour encircling the origin counter-clockwise:

$$\int_0^\infty \mathrm{d}\beta\, \frac{(\frac{\beta}{2})^{2(z-1-n)}}{\Gamma(z-n)}\, f(\beta) = (-1)^n\, 2^{2n+1}\, n! \oint \frac{\mathrm{d}\beta}{2\,\pi\,i}\, \frac{f(\beta)}{\beta^{2(n+1)}} + \mathcal{O}(z)\,. \tag{4.16}$$

In the end, the first derivative of the zeta function in $\mathrm{AdS}_{2r+1}$ reads

$$\frac{\zeta'_{[\Delta,\mathbb{Y}]}(0)}{\ln\, R} = \frac{d+2}{c_d} \oint \frac{\mathrm{d}\beta}{2\,\pi\,i} \sum_{n=0}^{r} \frac{(-1)^n\,(2n)!}{\beta^{2(n+1)}}\, \mathfrak{D}_{(n)}\Big[\mathsf{D}^{(d+2)}(i\beta,\vec{\alpha})\,\chi^{so(2,d)}_{(\Delta,\mathbb{Y})}(\beta,\vec{\alpha})\Big]\Big|_{\vec{\alpha}=\vec{0}}, \tag{4.17}$$

which contains the $\alpha_i$-derivatives instead of the contour integrals in (3.17).

**Even dimensional AdS**

For $d=2r+1$, the functions $\varphi_{\epsilon,n}(z,\beta)$ is given by

$$\varphi_{\epsilon,n}(z,\beta) = \frac{i\sqrt{\pi}\,(\frac{\beta}{2})^{2z-1}}{2\,\Gamma(z+\frac{1}{2})}\, \mu_{\epsilon,n}(z,\beta)\,, \tag{4.18}$$

with

$$\mu_{\epsilon,n}(z,\beta) = \partial_\alpha^{2n+1}\lambda_\epsilon(z,\beta,\alpha)\,\big|_{\alpha=0}\,. \tag{4.19}$$

Following Section 3.3, we focus on the first two Taylor coefficients of $\lambda_\epsilon(z,\beta,\alpha)$ in $z$, which have been computed in (3.27) and (3.34). They immediately give $\mu_{\epsilon,n}(0,\beta)$ and $\mu'_{\epsilon,n}(0,\beta)=$

$\frac{\partial}{\partial z}\,\mu_{\epsilon,n}(z,\beta)\,|_{z=0}$ corresponding to the primary and secondary contributions to the zeta function. The primary contribution reads

$$\zeta_{[\Delta,\mathbb{Y}],1}(z) = \frac{i\,(d+1)}{2\,c_d}\int_0^\infty \mathrm{d}\beta\,\frac{\beta^{2z-1}}{\Gamma(2z)}\sum_{n=0}^{r}\mu_{\epsilon,n}(0,\beta)\,\mathfrak{D}_{(n)}\Big[\mathsf{D}^{(d+2)}(i\beta,\vec{\alpha})\,\chi^{so(2,d)}_{(\Delta,\mathbb{Y})}(\beta,\vec{\alpha})\Big]\Big|_{\vec{\alpha}=\vec{0}}\,. \tag{4.20}$$

and the secondary contribution is

$$\zeta'_{[\Delta,\mathbb{Y}],2}(0) = \frac{i\,(d+1)}{2\,c_d}\oint\frac{\mathrm{d}\beta}{2\pi\,i\,\beta}\sum_{n=0}^{r}\mu'_{\epsilon,n}(0,\beta)\,\mathfrak{D}_{(n)}\Big[\mathsf{D}^{(d+2)}(i\beta,\vec{\alpha})\,\chi^{so(2,d)}_{(\Delta,\mathbb{Y})}(\beta,\vec{\alpha})\Big]\Big|_{\vec{\alpha}=\vec{0}}\,. \tag{4.21}$$

### 4.2 Explicit expressions in low dimensions

In this section we spell out the explicit formulae for the zeta function in $AdS_3$, $AdS_5$ and $AdS_7$, and its primary and secondary contributions in $AdS_4$ and $AdS_6$. In order to display the various formulae in a compact way, let us introduce

$$\mathsf{f}^{d,(n)}_{(\Delta,\mathbb{Y})}(\beta) := \mathfrak{D}_{(n)}\Big[\mathsf{D}^{(d+2)}(i\beta,\vec{\alpha})\,\chi^{so(2,d)}_{(\Delta,\mathbb{Y})}(\beta,\alpha)\Big]\Big|_{\alpha=0}\times\begin{cases}\dfrac{(d+2)\,(2n)!}{2^{2n+1}\,n!\,c_d} & [d=2r]\\[2mm] \dfrac{i\,(d+1)}{2\,c_d} & [d=2r+1]\end{cases}\,. \tag{4.22}$$

Then, the zeta function in $AdS_{2r+1}$ reads

$$\zeta_{[\Delta;\mathbb{Y}]}(z) = \frac{\ln R}{\Gamma(z)}\int_0^\infty \mathrm{d}\beta\sum_{n=0}^{r}\frac{\left(\frac{\beta}{2}\right)^{2(z-n-1)}}{\Gamma(z-n)}\,\mathsf{f}^{2r,(n)}_{(\Delta,\mathbb{Y})}(\beta)\,, \tag{4.23}$$

whose first derivative is given by

$$\zeta'_{[\Delta,\mathbb{Y}]}(0) = \ln\,R\oint\frac{\mathrm{d}\beta}{2\,\pi\,i}\sum_{n=0}^{r}(-1)^n\,\frac{2^{2n+1}\,n!}{\beta^{2(n+1)}}\,\mathsf{f}^{2r,(n)}_{(\Delta,\mathbb{Y})}(\beta)\,. \tag{4.24}$$

On the other hand, the primary contribution to the zeta function in $AdS_{2r+2}$ reads

$$\zeta_{[\Delta;\mathbb{Y}],1}(z) = \int_0^\infty \mathrm{d}\beta\,\frac{\beta^{2z-1}}{\Gamma(2z)}\sum_{n=0}^{r}\mu_{\epsilon,n}(0,\beta)\,\mathsf{f}^{2r+1,(n)}_{(\Delta,\mathbb{Y})}(\beta)\,, \tag{4.25}$$

whereas the secondary contribution is given by

$$\zeta'_{[\Delta;\mathbb{Y}],2}(0) = \oint\frac{\mathrm{d}\beta}{2\pi\,i\,\beta}\sum_{n=0}^{r}\mu'_{\epsilon,n}(0,\beta)\,\mathsf{f}^{2r+1,(n)}_{(\Delta,\mathbb{Y})}(\beta)\,. \tag{4.26}$$

To derive the explicit expressions below, we will use (4.6), (4.8) and (A.2).

### 4.2.1 AdS$_3$

In order to write down explicitly the functions $\mathsf{f}^{2,(0)}_{\mathcal{H}}(\beta)$ and $\mathsf{f}^{2,(1)}_{\mathcal{H}}(\beta)$ relevant to the computation of the zeta function in AdS$_3$, we need to know the expression of the differential operator

$$\mathfrak{D}_{(0)} = \bar{\partial}_{(0)}\, \mathscr{D}_{\Phi^2}\,, \qquad \mathfrak{D}_{(1)} = \bar{\partial}_{(1)}\, \mathscr{D}_{\Phi^2}\ . \tag{4.27}$$

Since $\mathscr{D}_{\Phi^2}$ is the identity [5], we have

$$\mathfrak{D}_{(0)} = -\partial_\alpha^2\,, \qquad \mathfrak{D}_{(1)} = 1\,. \tag{4.28}$$

Together with

$$c_2 = -4\,, \qquad \mathsf{D}^{(4)}(i\beta,\alpha) = 2\,(\cosh\beta - \cos\alpha)\,, \tag{4.29}$$

we find

$$\mathsf{f}^{2,(0)}_{\mathcal{H}}(\beta) \ = \ \Big(1 + 2\sinh^2\tfrac{\beta}{2}\,\partial_\alpha^2\Big)\,\chi^{so(2,2)}_{\mathcal{H}}(\beta,\alpha)\big|_{\alpha=0}\,, \tag{4.30}$$

and

$$\mathsf{f}^{2,(1)}_{\mathcal{H}}(\beta) = -\sinh^2\tfrac{\beta}{2}\ \chi^{so(2,2)}_{\mathcal{H}}(\beta,0)\,. \tag{4.31}$$

Inserting these ingredients into (4.23), we obtain

$$\zeta_{\mathcal{H}}(z) = \frac{\ln R}{\Gamma(z)^2}\int_0^\infty \mathrm{d}\beta\,\Big(\frac{\beta}{2}\Big)^{2(z-1)}\Big[1 + \tfrac{4(1-z)}{\beta^2}\,\sinh^2\tfrac{\beta}{2} + 2\sinh^2\tfrac{\beta}{2}\,\partial_\alpha^2\Big]\chi^{so(2,2)}_{\mathcal{H}}(\beta,\alpha)\big|_{\alpha=0}\,, \tag{4.32}$$

and

$$\zeta'_{\mathcal{H}}(0) = \ln\,R\,\oint\frac{\mathrm{d}\beta}{2\,\pi\,i}\,\frac{2}{\beta^2}\,\Big(1 + \tfrac{4}{\beta^2}\,\sinh^2\tfrac{\beta}{2} + 2\sinh^2\tfrac{\beta}{2}\,\partial_\alpha^2\Big)\,\chi^{so(2,2)}_{\mathcal{H}}(\beta,\alpha)\big|_{\alpha=0}\,. \tag{4.33}$$

### 4.2.2 AdS$_4$

The relevant differential operators read

$$\mathscr{D}_{\Phi^3} = \partial_\alpha\,, \qquad \bar{\partial}_{(0)} = -\partial_\alpha^2\,, \qquad \bar{\partial}_{(1)} = 1\,, \tag{4.34}$$

Using

$$c_3 = 12\,, \qquad \mathsf{D}^{(5)}(i\beta,\alpha) = -8\,i\,\sinh\tfrac{\beta}{2}\,\sin\tfrac{\alpha}{2}\,\big(\cosh\beta - \cos\alpha\big)\,. \tag{4.35}$$

we obtain

$$\mathsf{f}^{3,(0)}_{\mathcal{H}}(\beta) = \tfrac{1}{3}\,\sinh\tfrac{\beta}{2}\Big(\sinh^2\tfrac{\beta}{2} - 6 - 12\sinh^2\tfrac{\beta}{2}\,\partial_\alpha^2\Big)\,\chi^{so(2,3)}_{\mathcal{H}}(\beta,\alpha)\big|_{\alpha=0}\,, \tag{4.36}$$

and

$$\mathsf{f}^{3,(1)}_{\mathcal{H}}(\beta) = \frac{4}{3}\sinh^3\tfrac{\beta}{2}\,\chi^{so(2,3)}_{\mathcal{H}}(\beta,0)\,. \tag{4.37}$$

The functions $\mu_{\epsilon,0}$ and $\mu_{\epsilon,1}$ read

$$\mu_{\epsilon,0}(0,\beta) = -\frac{(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}}{4\,\sinh^2\frac{\beta}{2}}\,, \qquad \mu_{\epsilon,1}(0,\beta) = \frac{(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}\big((5-3\,\epsilon)\sinh^2\frac{\beta}{2} + 12\big)}{32\sinh^4\frac{\beta}{2}}\,. \tag{4.38}$$

[5] Indeed, $so(2)$ being unidimensional, it does not have a root space decomposition like the higher dimensional orthogonal algebras.

According to (4.25), the primary contribution is

$$\zeta_{\mathcal{H},1}(z) = \int_0^\infty d\beta \, \frac{\beta^{2z-1}}{\Gamma(2z)} \, \frac{(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}}{\sinh\frac{\beta}{2}} \, \left(1 + \tfrac{1-\epsilon}{8}\,\sinh^2\tfrac{\beta}{2} + \sinh^2\tfrac{\beta}{2}\partial_\alpha^2\right) \chi_{\mathcal{H}}^{so(2,3)}(\beta,\alpha)\big|_{\alpha=0}\,, \tag{4.39}$$

By setting $\epsilon = +1$ or $-1$, we obtain the formulae derived in [14] and [21] for bosonic and fermionic spectrum, respectively. To compute the secondary contribution, we need the first derivative of $\mu_{\epsilon,n}(z,\beta)$:

$$\mu'_{\epsilon,0}(0,\beta) = \frac{2}{\beta^2} + \mathcal{O}(1)\,, \qquad \mu'_{\epsilon,1}(0,\beta) = -\frac{16}{\beta^4} + \mathcal{O}(1)\,. \tag{4.40}$$

Inserting the above ingredients into (4.26), we arrive at

$$\zeta'_{\mathcal{H},2}(0) = \oint \frac{d\beta}{2\pi\, i}\, \frac{2\sinh^3\frac{\beta}{2}}{\beta^3} \left(-\frac{32}{3\beta^2} - \frac{2}{\sinh^2\frac{\beta}{2}} + \frac{1}{3} - 4\,\partial_\alpha^2\right) \chi_{\mathcal{H}}^{so(2,3)}(\beta,\alpha)\Big|_{\alpha=0}\,. \tag{4.41}$$

Notice that, as we already pointed out, since the function of $\beta$ multiplying the character in the integrand is even, this secondary contribution vanishes if $\chi_{\mathcal{H}}^{so(2,3)}(\beta,\alpha)$ is also an even function of $\beta$.

### 4.2.3 AdS$_5$

The differential operators $\mathfrak{D}_{(n)}$ are composed of

$$\mathscr{D}_{\Phi^4} = \partial_1^2 - \partial_2^2\,, \qquad \bar{\partial}_{(0)} = \partial_1^2\,\partial_2^2\,, \qquad \bar{\partial}_{(1)} = -\big(\partial_1^2 + \partial_2^2\big), \qquad \bar{\partial}_{(2)} = 1\,. \tag{4.42}$$

After some computations, one obtains

$$\begin{aligned} \mathsf{f}_{\mathcal{H}}^{4,(0)}(\beta) \;=\; \Big[1 - \sinh^2\tfrac{\beta}{2}\,(\tfrac{1}{3}\sinh^2\tfrac{\beta}{2} - 1)\,(\partial_1^2 + \partial_2^2) \\ -\tfrac{1}{3}\,\sinh^4\tfrac{\beta}{2}\,\big(\partial_1^4 + \partial_2^4 - 12\,\partial_1^2\,\partial_2^2\big)\Big]\,\chi_{\mathcal{H}}^{so(2,4)}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,, \end{aligned} \tag{4.43}$$

$$\mathsf{f}_{\mathcal{H}}^{4,(1)}(\beta) = \sinh^2\tfrac{\beta}{2}\,\Big[\tfrac{1}{3}\sinh^2\tfrac{\beta}{2} - 1 - \sinh^2\tfrac{\beta}{2}\,(\partial_1^2 + \partial_2^2)\Big]\,\chi_{\mathcal{H}}^{so(2,4)}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,, \tag{4.44}$$

and

$$\mathsf{f}_{\mathcal{H}}^{4,(2)}(\beta) = \tfrac{1}{2}\,\sinh^4\tfrac{\beta}{2}\,\chi_{\mathcal{H}}^{so(2,4)}(\beta,\vec{0})\,. \tag{4.45}$$

Plugging these expressions into (4.23), we reproduce the CIRZ formula for AdS$_5$ derived in [14].

### 4.2.4 AdS$_6$

To define the differential operators $\mathfrak{D}_{(n)}$, we need the following building blocks:

$$\mathscr{D}_{\Phi^5} = \big(\partial_1^2 - \partial_2^2\big)\,\partial_1\partial_2\,, \quad \bar{\partial}_{(0)} = \partial_1^2\,\partial_2^2\,, \quad \bar{\partial}_{(1)} = -\big(\partial_1^2 + \partial_2^2\big)\,, \quad \bar{\partial}_{(2)} = 1\,. \tag{4.46}$$

Then we find

$$\begin{aligned} \mathsf{f}_{\mathcal{H}}^{5,(0)}(\beta) \;=\; \sinh\tfrac{\beta}{2}\,\Big[-2 + \tfrac{1}{3}\sinh^2\tfrac{\beta}{2} - \tfrac{3}{20}\sinh^4\tfrac{\beta}{2} + \tfrac{1}{2}\sinh^2\tfrac{\beta}{2}\,(\cosh\beta - 5)\,(\partial_1^2 + \partial_2^2) \\ +\tfrac{2}{3}\sinh^4\tfrac{\beta}{2}\,(\partial_1^4 + \partial_2^4 - 12\,\partial_1^2\,\partial_2^2)\Big]\,\chi_{\mathcal{H}}^{so(2,5)}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,, \end{aligned} \tag{4.47}$$

$$\mathfrak{f}^{5,(1)}_{\mathcal{H}}(\beta) = \tfrac{1}{3}\sinh^3\tfrac{\beta}{2}\Big(5-\cosh\beta+4\sinh^2\tfrac{\beta}{2}(\partial_1^2+\partial_2^2)\Big)\,\chi^{so(2,5)}_{\mathcal{H}}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,, \tag{4.48}$$

$$\mathfrak{f}^{5,(2)}_{\mathcal{H}}(\beta) = -\tfrac{4}{15}\sinh^5\tfrac{\beta}{2}\,\chi^{so(2,5)}_{\mathcal{H}}(\beta,\vec{0})\,. \tag{4.49}$$

The functions $\mu_{\epsilon,0}(0,\beta)$ and $\mu_{\epsilon,1}(0,\beta)$ were already computed in (4.38), whereas

$$\mu_{\epsilon,2}(0,\beta) = \frac{(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}}{256\,\sinh^6\frac{\beta}{2}}\Big[480+120(3-\epsilon)\sinh^2\tfrac{\beta}{2}+(19-30\,\epsilon+15\,\epsilon^2)\sinh^4\tfrac{\beta}{2}\Big]\,. \tag{4.50}$$

The primary contribution is

$$\begin{aligned}\zeta_{\mathcal{H},1}(z) = \int_0^\infty \mathrm{d}\beta\,\frac{\beta^{2z-1}}{\Gamma(2z)}\,\frac{(\cosh\frac{\beta}{2})^{\frac{1+\epsilon}{2}}}{64\sinh\frac{\beta}{2}}\,&\Big[96+(1-\epsilon)\Big(16-(3+\epsilon)\sinh^2\tfrac{\beta}{2}\Big)\sinh^2\tfrac{\beta}{2}\\ &+\tfrac{8}{3}\Big(24-(1+3\,\epsilon)\sinh^2\tfrac{\beta}{2}\Big)\sinh^2\tfrac{\beta}{2}\,(\partial_1^2+\partial_2^2)\\ &-\tfrac{32}{3}\sinh^4\tfrac{\beta}{2}\,(\partial_1^4+\partial_2^4-12\,\partial_1^2\,\partial_2^2)\Big]\,\chi^{so(2,5)}_{\mathcal{H}}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,.\end{aligned} \tag{4.51}$$

Using (4.40) and

$$\mu'_{\epsilon,2}(0,\beta) = \frac{368}{\beta^6}+\mathcal{O}(1)\,, \tag{4.52}$$

we obtain the secondary contribution as

$$\begin{aligned}\zeta'_{\mathcal{H},2}(0) = \oint\frac{\mathrm{d}\beta}{2\pi\,i}\,\frac{\sinh\frac{\beta}{2}}{\beta^3}\,\Big(&-4+\tfrac{2}{3}\sinh^2\tfrac{\beta}{2}-\tfrac{3}{10}\sinh^4\tfrac{\beta}{2}+\sinh^2\tfrac{\beta}{2}\,(\cosh\beta-5)(\partial^2_{\alpha_1}+\partial^2_{\alpha_2})\\ &+\tfrac{4}{3}\sinh^4\tfrac{\beta}{2}\,(\partial^4_{\alpha_1}+\partial^4_{\alpha_2}-12\,\partial^2_{\alpha_1}\partial^2_{\alpha_2})\\ &+\frac{16\,\sinh^2\frac{\beta}{2}}{3\,\beta^2}\,\big(\cosh\beta-5-4\sinh^2\tfrac{\beta}{2}(\partial^2_{\alpha_1}+\partial^2_{\alpha_2})\big)\\ &-\frac{1472\,\sinh^4\frac{\beta}{2}}{15\,\beta^4}\Big)\,\chi^{so(2,5)}_{\mathcal{H}}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,.\end{aligned} \tag{4.53}$$

### 4.2.5 AdS$_7$

Finally, the relevant differential operators for AdS$_7$ zeta functions is obtained by combining the differential operator associated to the (positive) root system of $so(6)$,

$$\mathscr{D}_{\Phi^6} = (\partial_1^2-\partial_2^2)\,(\partial_1^2-\partial_3^2)\,(\partial_2^2-\partial_3^2)\,, \tag{4.54}$$

with

$$\bar{\partial}_{(0)} = -\partial_1^2\,\partial_2^2\,\partial_3^2\,,\quad \bar{\partial}_{(1)} = \partial^2_{\alpha_1}\partial_2^2+\partial_1^2\partial_3^2+\partial_2^2\partial_3^2\,,\quad \bar{\partial}_{(2)} = -(\partial_1^2+\partial_2^2+\partial_3^2)\,,\quad \bar{\partial}_{(3)} = 1\,. \tag{4.55}$$

Using the above expressions, one can compute the building blocks $\mathfrak{f}^{6,(k)}_{\mathcal{H}}$ as

$$\begin{aligned}\mathfrak{f}^{6,(0)}_{\mathcal{H}}(\beta) = \Big[&1+\tfrac{1}{135}\sinh^2\tfrac{\beta}{2}\,(111-23\cosh\beta+2\cosh2\beta)\,(\partial_1^2+\partial_3^2+\partial_3^2)\\ &+\tfrac{2}{27}\sinh^4\tfrac{\beta}{2}\,(\cosh\beta-4)\,(\partial_1^4+\partial_2^4+\partial_3^4-6\,\partial_1^2\,\partial_2^2-6\,\partial_1^2\,\partial_3^2-6\,\partial_2^2\,\partial_3^2)\\ &+\tfrac{4}{135}\sinh^6\tfrac{\beta}{2}\,(\partial_1^6+\partial_2^6+\partial_3^6-15\,\partial_1^2\,\partial_2^4-15\,\partial_1^2\,\partial_3^4-15\,\partial_2^2\,\partial_1^4-15\,\partial_2^2\,\partial_3^4\\ &-15\,\partial_3^2\,\partial_1^4-15\,\partial_3^2\,\partial_2^4+270\,\partial_1^2\,\partial_2^2\,\partial_3^2)\Big]\,\chi^{so(2,6)}_{\mathcal{H}}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,,\end{aligned} \tag{4.56}$$

$$\mathsf{f}_{\mathcal{H}}^{6,(1)}(\beta) = -\tfrac{1}{90}\sinh^2\tfrac{\beta}{2}\Big[111 - 23\cosh\beta + 2\cosh 2\beta - 20\sinh^2\tfrac{\beta}{2}\,(\cosh\beta - 4)\,(\partial_1^2+\partial_2^2+\partial_3^2) \\ -20\sinh^4\tfrac{\beta}{2}(\partial_1^4+\partial_2^4+\partial_3^4 - 6\,\partial_1^2\,\partial_2^2 - 6\,\partial_1^2\,\partial_3^2 - 6\,\partial_2^2\,\partial_3^2)\Big]\,\chi_{\mathcal{H}}^{so(2,6)}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,, \quad (4.57)$$

$$\mathsf{f}_{\mathcal{H}}^{6,(2)}(\beta) = -\tfrac{1}{6}\sinh^4\tfrac{\beta}{2}\Big[\cosh\beta - 4 - 2\sinh^2\tfrac{\beta}{2}\,(\partial_1^2+\partial_2^2+\partial_3^2)\Big]\,\chi_{\mathcal{H}}^{so(2,6)}(\beta,\vec{\alpha})\big|_{\vec{\alpha}=\vec{0}}\,, \quad (4.58)$$

$$\mathsf{f}_{\mathcal{H}}^{6,(3)}(\beta) = -\tfrac{1}{6}\sinh^6\tfrac{\beta}{2}\,\chi_{\mathcal{H}}^{so(2,6)}(\beta,\vec{0})\,. \quad (4.59)$$

The zeta function can be readily derived using (4.23).

## 5 Summary and Conclusion

In this work, we derived the character integral representation of zeta function (CIRZ) in arbitrary dimensions. We started with a brief review of the AdS zeta functions in Section 2, which include its definition and some interesting identities. In Section 3, we expressed the CIRZ formula in terms of contour integrals in arbitrary dimensions. In Section 4, we derived a different CIRZ formula in terms of derivatives, generalizing the previous derivative expressions for $AdS_4$ and $AdS_5$ [14] to $AdS_3$, $AdS_6$ and $AdS_7$. This procedure also clearly generalizes to arbitrary dimensions.

As outlined in the Introduction, the CIRZ is particularly useful to deal with theories with an infinite spectrum. When the spectrum can be captured by some CFT data, as is the case for partially massless higher-spin theories [43], one can compute the free energy of the theory without necessarily knowing the detailed decomposition of the spectrum. This will be done in the companion paper [28], where we will establish the matching of the one-loop corrections of partially massless higher-spin gravities with the $1/N$ corrections of the dual CFTs in any dimensions. The CIRZ method could also prove efficient in computing one-loop effects of a Kaluza-Klein tower. Indeed, the corresponding spectrum is obtained from the branching rule of the higher-dimensional field, and hence should be fully encompassed by the character of the latter.

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## A Character identities

In this Appendix we shall derive the identity (3.4) used in the main text. The character formula for the $so(d+2)$ irrep labelled by the highest weight $(s_0, \mathbb{Y})$, with $\mathbb{Y} = (s_1, \dots, s_r)$, reads

$$\chi_{(s_0,\mathbb{Y})}^{so(d+2)}(\boldsymbol{\alpha}) = \frac{\mathsf{N}_{(s_0,\mathbb{Y})}^{(d+2)}(\boldsymbol{\alpha})}{\mathsf{D}^{(d+2)}(\boldsymbol{\alpha})}\,, \quad (A.1)$$

with the denominator

$$\mathsf{D}^{(d+2)}(\boldsymbol{\alpha}) = \prod_{0\leqslant i<j\leqslant r} 2\,(\cos\alpha_i - \cos\alpha_j) \times \begin{cases} 1 & [d=2r] \\ \displaystyle\prod_{k=0}^{r} 2\,i\,\sin\frac{\alpha_k}{2} & [d=2r+1] \end{cases}, \tag{A.2}$$

and the numerator

$$\mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) = \begin{cases} \frac{1}{2}\Big(\det\big[2i\sin(\alpha_i\,\ell_j)\big] + \det\big[2\cos(\alpha_i\,\ell_j)\big]\Big) & [d=2r] \\ \det\big[2i\sin(\alpha_i\,\ell_j)\big] & [d=2r+1] \end{cases}. \tag{A.3}$$

Here, $\ell_j := s_j + \frac{d}{2} - j$ and $\det[a_{ij}]$ denotes the determinant of a matrix whose matrix element is $a_{ij}$. The indices $i,j$ range from 0 to $r$. The determinants appearing in (A.3) may be expanded as an alternating sum of minors:

$$\mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) = \sum_{k=0}^{r}(-1)^k \times \begin{cases} \Big(i\sin(\alpha_k\,\ell_0)\,\det_k\big[2i\sin(\alpha_i\,\ell_j)\big] \\ \quad + \cos(\alpha_k\,\ell_0)\,\det_k\big[2\cos(\alpha_i\,\ell_j)\big]\Big) & [d=2r] \\ 2\,i\,\sin(\alpha_k\,\ell_0)\,\det_k\big[2i\sin(\alpha_i\,\ell_j)\big] & [d=2r+1] \end{cases}. \tag{A.4}$$

The above expression can be recast into recursive formulae in $d$ as

$$\mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) = \sum_{k=0}^{r}(-1)^k \times \begin{cases} \Big(e^{i\alpha_k\,\ell_0}\,\mathsf{N}^{(d)}_{\mathbb{Y}_+}(\vec{\alpha}_k) + e^{-i\alpha_k\,\ell_0}\,\mathsf{N}^{(d)}_{\mathbb{Y}_-}(\vec{\alpha}_k)\Big) & [d=2r] \\ 2i\,\sin(\alpha_k\,\ell_0)\,\mathsf{N}^{(d)}_{\mathbb{Y}}(\vec{\alpha}_k) & [d=2r+1] \end{cases}. \tag{A.5}$$

For $d=2r+1$, (A.5) is straightforward to prove. For $d=2r$, the formula (A.4) can be first expanded as

$$\begin{aligned} \mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) \;=\; \frac{1}{2}\sum_{k=0}^{r}(-1)^k\Big(& e^{i\alpha_k\,\ell_0}\,\big(\det_k[2i\sin(\alpha_i\,\ell_j)] + \det_k[2\cos(\alpha_i\,\ell_j)]\big) \\ & + e^{-i\alpha_k\,\ell_0}\,\big(\det_k[2\cos(\alpha_i\,\ell_j)] - \det_k[2i\sin(\alpha_i\,\ell_j)]\big)\Big), \end{aligned} \tag{A.6}$$

then the relative minus sign in the second line can be absorbed into the determinant by changing the last column from $\sin(\alpha_i\,\ell_r)$ to $\sin(-\alpha_i\,\ell_r)$.

Similarly, one can decompose the denominator of an $so(d+2)$ character as a multiple of the denominator of an $so(d)$ character expressed in terms of $\vec{\alpha}_k$:

$$\mathsf{D}^{(d+2)}(\boldsymbol{\alpha}) = (-1)^k\,\mathsf{D}^{(d)}(\vec{\alpha}_k)\prod_{\substack{0\leqslant i\leqslant r\\ i\neq k}} 2\,(\cos\alpha_k - \cos\alpha_i) \times \begin{cases} 1 & [d=2r] \\ 2i\sin\frac{\alpha_k}{2} & [d=2r+1] \end{cases}, \tag{A.7}$$

where the overall factor $(-1)^k$ comes from re-ordering part of the product containing the $\alpha_k$ variables in $\mathsf{D}^{(d+2)}(\boldsymbol{\alpha})$. Note that (A.7) can be expressed as

$$\mathsf{D}^{(d+2)}(\boldsymbol{\alpha}) = (-1)^k\,\mathsf{D}^{(d)}(\vec{\alpha}_k)\,\frac{(-1)^d\,e^{-i\alpha_k\,d/2}}{\mathcal{P}_d(\alpha_k;\vec{\alpha}_k)}\,. \tag{A.8}$$

by using the definition of the function $\mathcal{P}_d$ defined in (3.2). Combining (A.5) and (A.8), we can finally express the $so(d+2)$ characters in terms of $so(d)$ characters as

$$\chi^{so(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha}) = \sum_{k=0}^{r} \mathcal{P}_d(\alpha_k;\vec{\alpha}_k) \begin{cases} \left(e^{-i\alpha_k\, s_0}\, \chi^{so(d)}_{\mathbb{Y}_-}(\vec{\alpha}_k) + e^{i\alpha_k(s_0+d)}\, \chi^{so(d)}_{\mathbb{Y}_+}(\vec{\alpha}_k)\right) & [d=2r] \\ \left(e^{-i\alpha_k\, s_0} - e^{i\alpha_k(s_0+d)}\right) \chi^{so(d)}_{\mathbb{Y}}(\vec{\alpha}_k) & [d=2r+1] \end{cases} . \tag{A.9}$$

## B Generalized L'Hôpital's rule

In this Appendix, we will discuss some technical details of the generalized L'Hôpital's rule (4.3) (see e.g. [44] for a pedagogical introduction) which are crucial in obtaining the derivative expression of CIRZ in Section 4. The differential operator $\mathscr{D}_{\Phi^{d+2}}$ is defined as

$$\mathscr{D}_{\Phi^{d+2}} = \prod_{\theta\in\Phi^{d+2}} \partial_\theta \,. \tag{B.1}$$

In the orthonormal basis, the set of positive roots of $so(d+2)$ reads

$$\Phi^{d+2} = \begin{cases} \{\mathsf{e}_i \pm \mathsf{e}_j \,, 0\leqslant i<j\leqslant r\} & [d=2r] \\ \{\mathsf{e}_i \pm \mathsf{e}_j \,, 0\leqslant i<j\leqslant r\}\cup\{\mathsf{e}_k \,, 0\leqslant k\leqslant r\} & [d=2r+1] \end{cases} , \tag{B.2}$$

where $\{\mathsf{e}_k\}_{k=0,\dots,r}$ is a basis of unit orthonormal vector of $\mathbb{R}^{r+1}$. As a consequence, when acting on an $so(d+2)$ character, we have

$$\partial_\theta = \tfrac{1}{2}(\partial_{\alpha_i} \pm \partial_{\alpha_j}) \quad \text{for} \quad \theta = \mathsf{e}_i \pm \mathsf{e}_j \,, \quad \text{and} \quad \partial_\theta = \partial_{\alpha_k} \quad \text{for} \quad \theta = \mathsf{e}_k \,. \tag{B.3}$$

From now on, we will use the notation $\partial_i = \partial_{\alpha_i}$. Since the differential operator (B.1) acts on both the numerator and denominator, we can rescale this operator to eliminate the factor $\frac{1}{2}$ from $\partial_\theta$ with $\theta = \mathsf{e}_i \pm \mathsf{e}_j$. The differential operator $\mathscr{D}_{\Phi^{d+2}}$ then takes the form (4.4).

### B.1 Computing the denominator

According to (A.2), the denominator of the $so(d+2)$ character depends on the dimension $d+2$, but not the highest weight $(s_0,\mathbb{Y})$ of the representation. As a result, the denominator of the generalized L'Hôpital's rule is simply a $d$-dependent constant, which is denoted $c_d$ in (4.5). We will review a part of the derivation of the Weyl dimension formula, which can be found in e.g. [44] and enables us to obtain an explicit formula for $c_d$. First, recall that the denominator of an $so(d+2)$ character can be expressed as the product,

$$\mathsf{D}^{(d+2)} = \prod_{\theta\in\Phi^{d+2}} (e^{\theta/2} - e^{-\theta/2}) \,. \tag{B.4}$$

One can then show that the action of the differential operator (B.1) on $\mathsf{D}^{(d+2)}$ reads

$$\mathscr{D}_{\Phi^{d+2}} \prod_{\theta\in\Phi^{d+2}} (e^{\theta/2} - e^{-\theta/2}) = |\mathcal{W}_{d+2}| \prod_{\theta\in\Phi^{d+2}} \langle\theta,\rho\rangle \,, \tag{B.5}$$

where $\rho$ is the $so(d+2)$ Weyl vector:

$$\rho := \tfrac{1}{2} \sum_{\theta \in \Phi^{d+2}} \theta, \tag{B.6}$$

and $|\mathcal{W}_{d+2}|$ is the cardinal of the Weyl group of $so(d+2)$

$$|\mathcal{W}_{d+2}| = \begin{cases} 2^r\,(r+1)! & [d=2r] \\ 2^{r+1}\,(r+1)! & [d=2r+1] \end{cases}, \tag{B.7}$$

and $\langle\,,\rangle$ denotes the inner product on the root space. Since in this paper we have used $e^{i\theta}$, instead of $e^{\theta}$, for the variables of the $so(d+2)$ character, the constant $c_d$ is given by

$$c_d = i^{|\Phi^{d+2}|} |\mathcal{W}_{d+2}| \prod_{\theta \in \Phi^{d+2}} \langle \theta, \rho \rangle, \tag{B.8}$$

with $|\Phi^{d+2}|$ being the number of positive roots of $so(d+2)$,

$$|\Phi^{d+2}| = \begin{cases} r(r+1) & [d=2r] \\ (r+1)^2 & [d=2r+1] \end{cases}. \tag{B.9}$$

Since the Weyl vector of $so(d+2)$ in the orthonormal basis is

$$\rho = \sum_{k=0}^{r} (\tfrac{d}{2} - k)\, \mathsf{e}_k, \tag{B.10}$$

we finally obtain $c_d$ as given in (4.6).

### B.2 Simplifying the numerator

Using (A.5), we can express $\mathsf{N}^{(d+2)}$ in terms of $\mathsf{N}^{(d)}$, then the numerator of the generalized L'Hôpital's rule becomes

$$\mathscr{D}_{\Phi^{d+2}}\, \mathsf{N}^{(d+2)}_{(s_0,\mathbb{Y})}(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\mathbf{0}} = 2 \sum_{k=0}^{r} (-1)^{k+d}\, \mathscr{D}_{\Phi^{d+2}} \Big[\sigma_d(\alpha_k u)\, \mathsf{N}^{(d)}_{\mathbb{Y}}(\vec{\alpha}_k)\Big]\Big|_{\boldsymbol{\alpha}=\mathbf{0}} \tag{B.11}$$

with

$$\sigma_d(\alpha_k\, u) = \begin{cases} \cosh(\alpha_k\, u) & [d=2r] \\ \sinh(\alpha_k\, u) & [d=2r+1] \end{cases}. \tag{B.12}$$

Here, we took advantage of the facts that

$$\mathsf{N}^{(d)}_{\mathbb{Y}_-}(\alpha_1, \ldots, \alpha_{k-1}, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_r) = \mathsf{N}^{(d)}_{\mathbb{Y}_+}(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_r)\,, \tag{B.13}$$

and that for $d = 2r$ the differential operator $\mathscr{D}_{\Phi^{d+2}}$ is invariant under the $\alpha_k \to -\alpha_k$ transformation for any $k = 0, \ldots, r$. The differential operator $\mathscr{D}_{\Phi^{d+2}}$ can then be rewritten as:

$$\mathscr{D}_{\Phi^{d+2}} = (-1)^k\, \mathscr{D}_{\Phi^{d}|k} \prod_{\substack{0 \leqslant i \leqslant r \\ i \neq k}} (\partial_k^2 - \partial_i^2) \times \begin{cases} 1 & [d=2r] \\ \partial_k & [d=2r+1] \end{cases}, \tag{B.14}$$

where $\mathscr{D}_{\Phi^d|k}$ is expressed in terms of the unit vectors $\{\mathsf{e}_0, \dots, \mathsf{e}_{k-1}, \mathsf{e}_{k+1}, \dots, \mathsf{e}_r\}$ of $\mathbb{R}^r$. Expanding the last product, we can then isolate the derivative with respect to $\alpha_k$:

$$\prod_{\substack{0\leqslant i\leqslant r\\ i\neq k}} (\partial_k^2 - \partial_i^2) = \sum_{n=0}^{r} \partial_k^{2n}\, \bar{\partial}_{(n|k)}\,, \tag{B.15}$$

with

$$\bar{\partial}_{(n|k)} = (-1)^{r-n} \sum_{\substack{0\leq i_1<i_2<\cdots<i_{r-n}\leq r\\ i_j\neq k}} \partial_{i_1}^2 \dots \partial_{i_{r-n}}^2\,. \tag{B.16}$$

In particular, we have

$$\bar{\partial}_{(r|k)} = 1\,, \quad \bar{\partial}_{(r-1|k)} = -\sum_{\substack{0\leqslant i\leqslant r\\ i\neq k}} \partial_i^2\,, \quad \text{and} \quad \bar{\partial}_{(0|k)} = (-1)^r \prod_{\substack{0\leqslant i\leqslant r\\ i\neq k}} \partial_i^2\,. \tag{B.17}$$

With this decomposition at hand, we can write

$$\mathscr{D}_{\Phi^{d+2}}\Big[\sigma_d(\alpha_k u)\, \mathsf{N}_{\mathbb{Y}}^{(d)}(\vec{\alpha}_k)\Big]\Big|_{\boldsymbol{\alpha}=\mathbf{0}} = (-1)^k\, u^{d-2r} \sum_{n=0}^{r} u^{2n}\, \bar{\partial}_{(n|k)}\, \mathscr{D}_{\Phi^d|k}\, \mathsf{N}_{\mathbb{Y}}^{(d)}(\vec{\alpha}_k)\Big|_{\boldsymbol{\alpha}=\mathbf{0}}\,,$$

where we have used the fact that the dependency on the angle $\alpha_k$ is confined to the function $\sigma_d(\alpha_k u)$. On top of that, this enable us to extract the $u$ dependent part according to $\partial_k$. Notice finally that the remaining summands involve all variables $\{\alpha_i\}_{i=0,\dots,r}$ except for $\alpha_k$. As a consequence, the terms

$$\bar{\partial}_{(n|k)} \mathscr{D}_{\Phi^d|k} \mathsf{N}_{\mathbb{Y}}^{(d)}(\vec{\alpha}_k)\Big|_{\boldsymbol{\alpha}=\mathbf{0}} \tag{B.18}$$

all produce the same contribution for different $k = 0, \dots, r$. We can therefore drop the subscript $k$ in the above expressions. In the end, we obtain (4.7) with (4.8), where the factor $u^{d-2r}$ is from the additional $\partial_k$ in (B.14) for $d = 2r + 1$.

## References


[1] R. Camporesi and A. Higuchi, *Arbitrary spin effective potentials in anti-de Sitter space-time*, *Phys. Rev.* **D47** (1993) 3339–3344.

[2] R. Camporesi and A. Higuchi, *Spectral functions and zeta functions in hyperbolic spaces*, *J. Math. Phys.* **35** (1994) 4217–4246.

[3] R. Camporesi and A. Higuchi, *The Plancherel measure for p-forms in real hyperbolic spaces*, *J. Geom. Phys.* **15** (1994) 57–94.

[4] E. Witten, *Multitrace operators, boundary conditions, and AdS / CFT correspondence*, hep-th/0112258.

[5] M. Berkooz, A. Sever and A. Shomer, *'Double trace' deformations, boundary conditions and space-time singularities*, *JHEP* **05** (2002) 034, [hep-th/0112264].

[6] S. S. Gubser and I. Mitra, *Double trace operators and one loop vacuum energy in AdS / CFT*, *Phys. Rev.* **D67** (2003) 064018, [hep-th/0210093].

[7] S. S. Gubser and I. R. Klebanov, *A Universal result on central charges in the presence of double trace deformations*, *Nucl. Phys.* **B656** (2003) 23–36, [hep-th/0212138].

[8] D. E. Diaz and H. Dorn, *Partition functions and double-trace deformations in AdS/CFT*, *JHEP* **05** (2007) 046, [hep-th/0702163].

[9] S. Giombi and I. R. Klebanov, *One Loop Tests of Higher Spin AdS/CFT*, *JHEP* **12** (2013) 068, [1308.2337].

[10] S. Giombi, I. R. Klebanov and B. R. Safdi, *Higher Spin $AdS_{d+1}/CFT_d$ at One Loop*, *Phys. Rev.* **D89** (2014) 084004, [1401.0825].

[11] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B243** (1990) 378–382.

[12] M. A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B285** (1992) 225–234.

[13] M. A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, *Phys. Lett.* **B567** (2003) 139–151, [hep-th/0304049].

[14] J.-B. Bae, E. Joung and S. Lal, *One-loop test of free SU(N) adjoint model holography*, *JHEP* **04** (2016) 061, [1603.05387].

[15] M. Flato and C. Fronsdal, *One Massless Particle Equals Two Dirac Singletons: Elementary Particles in a Curved Space. 6.*, *Lett. Math. Phys.* **2** (1978) 421–426.

[16] E. Angelopoulos and M. Laoues, *Singletons on AdS(n)*, in *Conference Moshe Flato Dijon, France, September 5-8, 1999*, pp. 3–23, 1999.

[17] M. A. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, *JHEP* **12** (2004) 046, [hep-th/0404124].

[18] J.-B. Bae, E. Joung and S. Lal, *On the Holography of Free Yang-Mills*, *JHEP* **10** (2016) 074, [1607.07651].

[19] J.-B. Bae, E. Joung and S. Lal, *One-loop free energy of tensionless type IIB string in $AdS_5\times S^5$*, *JHEP* **06** (2017) 155, [1701.01507].

[20] J.-B. Bae, E. Joung and S. Lal, *Exploring Free Matrix CFT Holographies at One-Loop*, *Universe* **3** (2017) 77, [1708.04644].

[21] Y. Pang, E. Sezgin and Y. Zhu, *One Loop Tests of Supersymmetric Higher Spin $AdS_4/CFT_3$*, *Phys. Rev.* **D95** (2017) 026008, [1608.07298].

[22] J.-B. Bae, E. Joung and S. Lal, *A note on vectorial $AdS_5/CFT_4$ duality for spin-$j$ boundary theory*, *JHEP* **12** (2016) 077, [1611.00112].

[23] E. D. Skvortsov and T. Tran, *AdS/CFT in Fractional Dimension and Higher Spin Gravity at One Loop*, *Universe* **3** (2017) 61, [1707.00758].

[24] S. Giombi, I. R. Klebanov and A. A. Tseytlin, *Partition Functions and Casimir Energies in Higher Spin $AdS_{d+1}/CFT_d$*, *Phys. Rev.* **D90** (2014) 024048, [1402.5396].

[25] S. Giombi, I. R. Klebanov and Z. M. Tan, *The ABC of Higher-Spin AdS/CFT*, *Universe* **4** (2018) 18, [1608.07611].

[26] M. Günaydin, E. D. Skvortsov and T. Tran, *Exceptional $F(4)$ higher-spin theory in $AdS_6$ at one-loop and other tests of duality*, *JHEP* **11** (2016) 168, [1608.07582].

[27] C. Brust and K. Hinterbichler, *Partially Massless Higher-Spin Theory II: One-Loop Effective

*Actions*, JHEP **01** (2017) 126, [1610.08522].

[28] T. Basile, E. Joung, S. Lal and W. Li, *Character Integral Representation of Zeta function in $AdS_{d+1}$: II. Application to partially-massless higher-spin gravities*, JHEP **07** (2018) 132, [1805.10092].

[29] R. R. Metsaev, *Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time*, Phys. Lett. **B354** (1995) 78–84.

[30] R. R. Metsaev, *Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space*, Lect. Notes Phys. **524** (1999) 331–340, [hep-th/9810231].

[31] R. R. Metsaev, *Fermionic fields in the d-dimensional anti-de Sitter space-time*, Phys. Lett. **B419** (1998) 49–56, [hep-th/9802097].

[32] R. R. Metsaev, *Continuous spin gauge field in (A)dS space*, Phys. Lett. **B767** (2017) 458–464, [1610.00657].

[33] R. R. Metsaev, *Fermionic continuous spin gauge field in (A)dS space*, Phys. Lett. **B773** (2017) 135–141, [1703.05780].

[34] R. R. Metsaev, *Continuous-spin mixed-symmetry fields in AdS(5)*, J. Phys. **A51** (2018) 215401, [1711.11007].

[35] S. Giombi, I. R. Klebanov, S. S. Pufu, B. R. Safdi and G. Tarnopolsky, *AdS Description of Induced Higher-Spin Gauge Theory*, JHEP **10** (2013) 016, [1306.5242].

[36] M. Beccaria and A. A. Tseytlin, *Higher spins in $AdS_5$ at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT*, JHEP **11** (2014) 114, [1410.3273].

[37] F. A. Dolan, *Character formulae and partition functions in higher dimensional conformal field theory*, J. Math. Phys. **47** (2006) 062303, [hep-th/0508031].

[38] M. Beccaria, X. Bekaert and A. A. Tseytlin, *Partition function of free conformal higher spin theory*, JHEP **08** (2014) 113, [1406.3542].

[39] T. Basile, X. Bekaert and N. Boulanger, *Mixed-symmetry fields in de Sitter space: a group theoretical glance*, JHEP **05** (2017) 081, [1612.08166].

[40] A. Bourget and J. Troost, *The Conformal Characters*, JHEP **04** (2018) 055, [1712.05415].

[41] R. K. Gupta and S. Lal, *Partition Functions for Higher-Spin theories in AdS*, JHEP **07** (2012) 071, [1205.1130].

[42] R. Estrada and R. Kanwal, *A Distributional Approach to Asymptotics: Theory and Applications*. Birkhäuser Advanced Texts Basler Lehrbücher. Birkhäuser Boston, 2012.

[43] X. Bekaert and M. Grigoriev, *Higher order singletons, partially massless fields and their boundary values in the ambient approach*, Nucl. Phys. **B876** (2013) 667–714, [1305.0162].

[44] B. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer, 2003.