Noether's Problem for Some Subgroups of S_{14}: the Modular Case

Hang Fu · Ming-chang Kang · Baoshan Wang · Jian Zhou

Abstract. Let G be a subgroup of S_n, the symmetric group of degree n. For any field k, G acts naturally on the rational function field $k(x_1, \cdots, x_n)$ via k-automorphisms defined by $\sigma \cdot x_i := x_{\sigma \cdot i}$ for any $\sigma \in G$ and $1 \leq i \leq n$. In this article, we will show that if G is a solvable transitive subgroup of S_{14} and $\mathrm{char}(k) = 7$, then the fixed subfield $k(x_1, \cdots, x_{14})^G$ is rational (i.e., purely transcendental) over k. In proving the above theorem, we rely on the Kuniyoshi–Gaschütz Theorem or some ideas in its proof.

Keywords. Noether's problem · Lüroth's problem · Rationality problem · Kuniyoshi's Theorem · Gaschütz's Theorem

Mathematics Subject Classification. 13A50 · 14E08

1 Introduction

Let k be a field, L a finitely generated extension field of k. The field L is called *k-rational* (or rational over k) if L is purely transcendental over k, i.e., L is k-isomorphic to the quotient field of some polynomial ring over k. L is called *stably k-rational* if $L(x_1, \cdots, x_n)$ is k-rational for some $x_1, \cdots, x_n$, which are algebraically independent over L. L is called *k-unirational* if L is k-isomorphic to a subfield of some k-rational field. It is obvious that "k-rational" $\Rightarrow$ "stably k-rational" $\Rightarrow$ "k-unirational". Lüroth's problem asks, under what situations, the converse is true, i.e., "k-unirational" $\Rightarrow$ "k-rational". For a survey of this famous problem in algebraic geometry, see [MT].

Let G be a finite group, k a field, V a finite-dimensional vector space over k, and $\rho : G \to \mathrm{GL}(V)$ a faithful representation of G. Then G acts on the function field $k(V)$ by k-automorphisms. Noether's problem asks, under what situations, the fixed subfield $k(V)^G := \{f \in k(V) : \sigma \cdot f = f \text{ for any } \sigma \in G\}$ is k-rational. Noether's problem is a special form of Lüroth's problem. It is also related to the inverse Galois problem. See [Sw] for details.

When $\rho : G \to \mathrm{GL}(V_{\mathrm{reg}})$ is the regular representation of G over k, we will write $k(G)$ for the fixed subfield $k(V_{\mathrm{reg}})^G$. Explicitly, let G act on the rational function field $k(x_g : g \in G)$ by $h \cdot x_g := x_{hg}$ for any $g, h \in G$. Then $k(G) := k(x_g : g \in G)^G$, the fixed subfield of $k(x_g : g \in G)$.

When G is a subgroup of S_n, the symmetric group of degree n, the permutation representation associated to G induces a natural action of G on the rational function field $k(x_1, \cdots, x_n)$ via k-automorphisms defined by $\sigma \cdot x_i := x_{\sigma \cdot i}$ for any $\sigma \in G$ and $1 \leq i \leq n$. Noether's problem becomes the form: whether the fixed subfield $k(x_1, \cdots, x_n)^G := \{f \in k(x_1, \cdots, x_n) : \sigma \cdot f = f \text{ for any } \sigma \in G\}$ is k-rational. The main purpose of this paper is to study the k-rationality of $k(x_1, \cdots, x_n)^G$, where k is a field, G is a transitive subgroup of S_n, and n is a "small" positive integer.

Recall some previously known results.

Theorem 1.1. *Let k be a field, S_n the symmetric group of degree n, and $k(x_1, \cdots, x_n)$ the rational function field of n variables over k. Suppose that $G \leq S_n$ acts on $k(x_1, \cdots, x_n)$ by k-automorphisms defined by $\sigma \cdot x_i := x_{\sigma \cdot i}$ for any $\sigma \in G$ and $1 \leq i \leq n$.*

(i) [KW, Theorem 1.3] *If $1 \leq n \leq 5$, then $k(x_1, \cdots, x_n)^G$ is k-rational for any subgroup $G \leq S_n$.*

(ii) [KWZ, Theorem 1.2] *If* $n = 6$ *and* G *is a transitive subgroup of* S_6 *other than* $\mathrm{PSL}_2(\mathbb{F}_5)$, $\mathrm{PGL}_2(\mathbb{F}_5)$, *or* A_6*, then* $k(x_1, \cdots, x_6)^G$ *is k-rational.*

(iii) [KW, Theorem 1.4] *If* $n = 7$ *and* G *is a transitive subgroup of* S_7 *other than* $\mathrm{PSL}_2(\mathbb{F}_7)$ *or* A_7*, then* $k(x_1, \cdots, x_7)^G$ *is k-rational. If* $k \supseteq \mathbb{Q}(\sqrt{-7})$*, then* $k(x_1, \cdots, x_7)^{\mathrm{PSL}_2(\mathbb{F}_7)}$ *is k-rational.*

(iv) [WZ, Theorem 1.2] *If* $n = 8$ *and* G *is a solvable transitive subgroup of* S_8 *other than* C_8*, then* $k(x_1, \cdots, x_8)^G$ *is k-rational.*

(v) [KW, Theorem 1.5] *If* $n = 11$ *and* G *is a solvable transitive subgroup of* S_{11}*, then* $k(x_1, \cdots, x_{11})^G$ *is k-rational.*

We remark that, (i) for any subgroup $G \le S_6$ other than A_6, $\mathbb{C}(x_1, \cdots, x_6)^G$ is $\mathbb{C}$-rational, and $k(x_1, \cdots, x_6)^G$ is stably k-rational (for any field k); (ii) for any transitive subgroup $G \le S_7$ other than A_7, $\mathbb{C}(x_1, \cdots, x_7)^G$ is $\mathbb{C}$-rational; (iii) $k(x_1, \cdots, x_8)^{C_8}$ is k-rational if and only if either $\mathrm{char}(k) = 2$ or $[k(\zeta_8) : k] \le 2$ (if $\mathrm{char}(k) \ne 2$). For details, see [KW], [KWZ], [EM, Proposition 3.9], and [Sa, Theorem 5.11]. We also remark that the rationality problem for the transitive subgroups of S_{10} was solved and a manuscript is in preparation.

This paper arose when we attempted to solve the rationality problem for the subgroups of S_{14}. We note that this problem was investigated in [WW] for some subgroups. As noted in [KW, Theorem 3.2], extra efforts are required in the situation when $\mathrm{char}(k) = p > 0$ and the group order $|G|$ is divisible by p. If G is a p-group, such a "modular" case may be solved by the classical theorem of Kuniyoshi and Gaschütz, which is recalled in the following theorem.

Theorem 1.2. *Suppose that* k *is a field with* $\mathrm{char}(k) = p > 0$ *and* G *is a p-group.*

(i) [Ku2, Ku1, Ku3] *The fixed subfield* $k(G)$ *is k-rational.*

(ii) [Ga] *For any faithful representation* $\rho : G \to \mathrm{GL}(V)$*, where* V *is a finite-dimensional vector space over* k*, the fixed subfield* $k(V)^G$ *is k-rational.*

We note that, when G is cyclic and $\rho : G \to \mathrm{GL}(V_{\mathrm{reg}})$ is the regular representation, a method to find explicitly a transcendental basis of $k(G)$ was proposed in [Ha2].

Here is a result generalizing Kuniyoshi–Gaschütz Theorem for the case of the regular representation (note that it is unnecessary to assume that G is a p-group).

Theorem 1.3. [KP, Theorem 1.1] *Let* k *be a field with* $\mathrm{char}(k) = p > 0$*,* G *a finite group, and* $\tilde{G}$ *a group extension given by* $1 \to \mathbb{Z}/p\mathbb{Z} \to \tilde{G} \to G \to 1$*. Then* $k(\tilde{G})$ *is rational over* $k(G)$*.*

The referee told us that Section 4 of Saltman's paper [Sa] contains results similar to the above Theorem 1.3.

However, when we consider rationality problems of subgroups in S_{14} with the field k satisfying that $\mathrm{char}(k) = 7$, new situations may arise for which Theorem 1.2 and 1.3 will not work any more. For example, here is a typical question.

Question 1.4. *Let* k *be a field with* $\mathrm{char}(k) = 7$*, and* $k(t, x_1, \cdots, x_6)$ *the rational function field of* 7 *variables over* k *(i.e.,* $\mathrm{trdeg}_k k(t, x_1, \cdots, x_6) = 7$*). Suppose that* $G := \langle \sigma, \tau \rangle$ *is a group acting on* $k(t, x_1, \cdots, x_6)$ *by the monomial k-automorphisms defined by*

$$\sigma : t \mapsto t, x_1 \mapsto x_2 \mapsto \cdots \mapsto x_6 \mapsto t^2/\textstyle\prod_{i=1}^{6} x_i,$$
$$\tau : t \mapsto -t, x_i \mapsto x_{3i},$$

where the indices of x_i *are taken modulo* 7*. Is the fixed subfield* $k(t, x_1, \cdots, x_6)^G$ *k-rational? Is it stably k-rational?*

The reader may find the definition of monomial k-automorphisms in [HK1, page 805].

This example prompts us to reexamine Kuniyoshi–Gaschütz Theorem and to improve the old techniques for this new situation. As an illustration, the following theorem, which is a generalization of [WZ, Lemma 2.7], will solve Question 1.4 affirmatively for the stable rationality.

Theorem 1.5. *Let k be a field, n a positive integer, d an integer relatively prime to n, $k(t,s,x_0,x_1,\cdots,x_{n-1})$ a field satisfying that $\mathrm{trdeg}_k k(t,s,x_0,x_1,\cdots,x_{n-1}) = n+1$ and $t^d = \prod_{i=0}^{n-1} x_i$. Let G be a subgroup of S_n and $\chi : G \to k^\times$ a linear character with $\chi^d = 1$ (the possibility that χ is the trivial character is allowed). Suppose that G acts on $k(t,s,x_0,x_1,\cdots,x_{n-1})$ by k-automorphisms defined by $\sigma \cdot t := \chi(\sigma)t$, $\sigma \cdot s := s$, $\sigma \cdot x_i := x_{\sigma \cdot i}$ for any $\sigma \in G$ and $0 \le i \le n-1$. Then there exists some element $u \in k(t,s,x_0,x_1,\cdots,x_{n-1})$ such that $\sigma \cdot u = u$ for any $\sigma \in G$, and $k(t,s,x_0,x_1,\cdots,x_{n-1})^G = k(x_0/s, x_1/s, \cdots, x_{n-1}/s)^G(u)$.*

Note that the field $k(t,s,x_0,x_1,\cdots,x_{n-1})$ with $t^d = \prod_{i=0}^{n-1} x_i$ is nothing but the rational function field $k(t,s,x_1,\cdots,x_{n-1})$ in the variables $t,s,x_1,\cdots,x_{n-1}$. With Theorem 1.5 in hand, we may solve Question 1.4 by introducing a new element x_0 such that $t^2 = \prod_{i=0}^{6} x_i$. Applying Theorem 1.5, we find that the rationality problem of $k(t,x_1,\cdots,x_6)^G(s)$ is reduced to that of $k(x_0/s,x_1/s,\cdots,x_6/s)^G(u)$. Then we may apply Theorem 1.1.

In Theorem 1.5, we may as well consider the case $k(t,s,x_0,x_1,\cdots,x_{p-1})$ is a field satisfying that, p is a prime number, $\mathrm{trdeg}_k k(t,s,x_0,x_1,\cdots,x_{p-1}) = p+1$, and $t^{d'} = \prod_{i=0}^{p-1} x_i$ with d' divisible by p. See Question 3.1.

Before stating the next result, we digress to define some k-automorphisms.

Definition 1.6. Let k be a field, p an odd prime number, a an integer satisfying that $(\mathbb{Z}/p\mathbb{Z})^\times = \langle \bar{a} \rangle$, S_p the symmetric group of degree p, and $k(x_i, y_i : 0 \le i \le p-1)$ the rational function field of $2p$ variables over k. We define six elements $\sigma_1,\sigma_2,\lambda_1,\lambda_2,\rho_1,\rho_2 \in S_p$ such that they act on $k(x_i,y_i : 0 \le i \le p-1)$ by

$$\begin{aligned}
&\sigma_1 : x_i \mapsto x_{i+1}, y_i \mapsto y_i, &&\sigma_2 : x_i \mapsto x_i, y_i \mapsto y_{i+1}, &&\lambda_1 : x_i \mapsto y_{-i}, y_i \mapsto x_i,\\
&\lambda_2 : x_i \leftrightarrow y_i, &&\rho_1 : x_i \mapsto x_{ai}, y_i \mapsto y_i, &&\rho_2 : x_i \mapsto x_i, y_i \mapsto y_{ai},
\end{aligned}$$

where $0 \le i \le p-1$, and the indices of x_i and y_i are taken modulo p.

Note that ρ_1 and ρ_2 depend on a. In the proof of Section 5.3, we will use the same ρ_1 or ρ_2, but the values of a will be different.

Let $\sigma_1,\sigma_2,\lambda_1,\lambda_2,\rho_1,\rho_2$ be the k-automorphisms introduced in Definition 1.6. Here is another theorem to be used in Section 5.

Theorem 1.7. *Let p be an odd prime number, k a field with $\mathrm{char}(k) = p > 0$, and $k(x_i,y_i : 0 \le i \le p-1)$ the rational function field of $2p$ variables over k. Then $k(x_i,y_i : 0 \le i \le p-1)^G$ is k-rational when G is $\langle \sigma_1,\sigma_2,\lambda_1,\rho_1\rho_2 \rangle$, $\langle \sigma_1,\sigma_2,\lambda_1,\rho_1^{-1}\rho_2 \rangle$, $\langle \sigma_1,\sigma_2,\lambda_2,\rho_1\rho_2 \rangle$, or $\langle \sigma_1,\sigma_2,\lambda_2,\rho_1^{-1}\rho_2 \rangle$.*

As an application of Theorem 1.5 and 1.7, we will prove the following theorem.

Theorem 1.8. *Let k be a field with $\mathrm{char}(k) = 7$ and G a solvable transitive subgroup of S_{14} acting naturally on $k(x_1,\cdots,x_{14})$, the rational function field of 14 variables over k. Then $k(x_1,\cdots,x_{14})^G$ is k-rational.*

Another version of Theorem 1.8 may be found in Theorem 4.1, which employs a description of the transitive subgroups of S_{14}. In Theorem 1.8, if G is a non-solvable transitive subgroup of S_{14}, the rationality of $k(x_1,\cdots,x_{14})^G$ is known only for a few cases (see Theorem 4.1 for details).

We remark that Theorem 1.5 and 1.7 may be applied also to prove the rationality problem of $k(x_1,\cdots,x_{10})^G$ when $\mathrm{char}(k) = 5$ and G is a solvable transitive subgroup of S_{10}. It may be true as well for many cases of $k(x_1,\cdots,x_{2p})^G$ when $\mathrm{char}(k) = p$ is an odd prime number.

The "non-modular" situation of Theorem 1.8 (i.e., for the case $\mathrm{char}(k) \ne 7$) is an ongoing research program. Hopefully it will be finished very soon.

This article is organized as follows. Section 2 contains some known results which will be applied in Section 3 and 5. The proofs of Theorem 1.5 and 1.7 will be given in Section 3. A list of transitive subgroups of S_{14} (up to conjugation within S_{14}) is provided in Section 4. Theorem 4.1 is a sharp form of Theorem 1.8 and the proof of Theorem 4.1 may be found in Section 5.

Standing terminology. Throughout this paper, G is a finite group and k is a field. Recall that $k(G)$ is defined at the beginning of this section. We will denote by S_n, A_n, C_n, and D_n the symmetric group of degree n, the alternating group of degree n, the cyclic group of order n, and the dihedral group of order $2n$ respectively. When we say that $k(x_1, \cdots, x_n)$ is a rational function field over a field k, we mean that $k(x_1, \cdots, x_n)$ is purely transcendental over k, and $\{x_1, \cdots, x_n\}$ is a transcendental basis, equivalently, $\mathrm{trdeg}_k k(x_1, \cdots, x_n) = n$. If $\sigma : k(x_1, \cdots, x_n) \to k(x_1, \cdots, x_n)$ is an automorphism and $u \in k(x_1, \cdots, x_n)$, then $\sigma \cdot u$ denotes $\sigma(u)$, the image of u under σ. By a linear character $\chi : G \to k^\times$ we mean a group homomorphism from G to $k^\times$. Whenever we write $\chi : G \to k^\times$, it is assumed that, for any $\sigma \in G$, $\chi(\sigma) \in k$ automatically.

Although we work on the case $\mathrm{char}(k) = p > 0$ sometimes, we also work on general cases. We will always state the assumptions of the field k explicitly. The readers should be aware not to confuse the groups $G(21)$ and G_{21}: $G(21)$ is the 21st transitive subgroup of S_{14} in Section 4, while G_{21} is the group G_{pd} with $(p, d) = (7, 3)$ and is defined in Definition 2.3.

Acknowledgments. We thank the referee whose sharp and insightful comments are very helpful to the presentation of this article.

2 Preliminaries

In this section, we recall several known results. These results will be used in the proofs of Section 3 and 5.

Theorem 2.1. [HK2, Theorem 1] *Let G be a finite group acting on $L(x_1, \cdots, x_n)$, the rational function field of n variables over a field L. Suppose that*

- *for any $\sigma \in G$, $\sigma(L) \subseteq L$;*
- *the restriction of the action of G to L is faithful;*
- *for any $\sigma \in G$,*
$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma),$$
where $A(\sigma) \in \mathrm{GL}_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over L.

Then there exist $z_1, \cdots, z_n \in L(x_1, \cdots, x_n)$ such that $L(x_1, \cdots, x_n) = L(z_1, \cdots, z_n)$ with $\sigma(z_i) = z_i$ for any $\sigma \in G$ and $1 \le i \le n$.

Theorem 2.2. [AHK, Theorem 3.1] *Let G be a finite group acting on $L(x)$, the rational function field of one variable over a field L. Suppose that for any $\sigma \in G$, $\sigma(L) \subseteq L$ and $\sigma(x) = a_\sigma x + b_\sigma$, where $a_\sigma \in L^\times$ and $b_\sigma \in L$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]^G$.*

Definition 2.3. [KW, Definition 3.1] Let k be a field, p an odd prime number, a an integer satisfying that $(\mathbb{Z}/p\mathbb{Z})^\times = \langle \bar{a} \rangle$, and $k(x_0, x_1, \cdots, x_{p-1})$ the rational function field of p variables over k. Define the k-automorphisms σ, τ on $k(x_0, x_1, \cdots, x_{p-1})$ by $\sigma : x_i \mapsto x_{i+1}$ and $\tau : x_i \mapsto x_{ai}$, where the indices of x_i are taken modulo p. Let d be a positive divisor of $p - 1$ and write $p - 1 = de$. Define a group $G_{pd} := \langle \sigma, \tau^e \rangle$. As abstract groups, $G_p \simeq C_p$ the cyclic group (when $d = 1$), $G_{2p} \simeq D_p$ the dihedral group (when $d = 2$), and $G_{p(p-1)}$ is isomorphic to the maximal solvable transitive subgroup of the symmetric group S_p. In general, G_{pd} is a semidirect product.

Theorem 2.4. [KW, Theorem 3.2] *Let $k(x_0, x_1, \cdots, x_{p-1})$ and G_{pd} be the same as in Definition 2.3. If* $\mathrm{char}(k) = p > 0$*, then the fixed subfield $k(x_0, x_1, \cdots, x_{p-1})^{G_{pd}}$ is k-rational.*

Definition 2.5. [KWZ, Definition 3.1 and 3.2] Let G and H be finite groups such that G acts on a finite set X from the left. Let A be the set of all functions from X to H, then G acts naturally on A by $(g \cdot a)(x) := a(g^{-1} \cdot x)$, where $g \in G$, $a \in A$, and $x \in X$. Note that A may be identified with the direct product of $|X|$ copies of H, thus any $a \in A$ may be written as $a = (a_x : x \in X)$. Under this identification, the group G acts on A by $(g \cdot a)_x := a_{g^{-1}\cdot x}$ where $g \in G$, $a \in A$, and $(g \cdot a)_x$ is the x-component of $g \cdot a$.

The *wreath product* $H \wr_X G$ is defined as the semidirect product $A \rtimes G$, where A is the normal subgroup with an action of G from the left.

Furthermore, when G and H are groups acting on the sets X and Y from the left, the wreath product $H \wr_X G := A \rtimes G$ acts on $Y \times X$ by defining $(a, g) \cdot (y, x) := (a_{g\cdot x} \cdot y, g \cdot x)$ for any $g \in G$, $a \in A$, $x \in X$, and $y \in Y$.

Adopting the notation of Definition 2.5, we write $X_m := \{1, \cdots, m\}$ (the set consisting of m elements), and $Y_n := \{1, \cdots, n\}$ (the set consisting of n elements). If $G \le S_m$ and $H \le S_n$, then we may regard $H \wr_{X_m} G$ as a subgroup of S_{mn} because $Y_n \times X_m$ is a set consisting of mn elements. With this understanding, we have the following theorem.

Theorem 2.6. [KWZ, Theorem 3.5] *Let k be a field, $G \le S_m$ and $H \le S_n$ act on the rational function fields $k(x_1, \cdots, x_m)$ and $k(y_1, \cdots, y_n)$ respectively. Assume that both $k(x_1, \cdots, x_m)^G$ and $k(y_1, \cdots, y_n)^H$ are k-rational. Then the wreath product $H \wr_{X_m} G$ acts on the rational function field $k(z_{i,j} : 1 \le i \le m, 1 \le j \le n)$, and the fixed subfield $k(z_{i,j} : 1 \le i \le m, 1 \le j \le n)^{H \wr_{X_m} G}$ is k-rational.*

In a similar way, if G and H act on the sets X and Y respectively from the left, then the direct product $G \times H$ acts on $X \times Y$ by $(g, h) \cdot (x, y) := (g \cdot x, h \cdot y)$.

Theorem 2.7. [KWZ, Theorem 3.6] *Let k, G, and H be the same as in Theorem 2.6. Then the direct product $G \times H$ acts on the rational function field $k(z_{i,j} : 1 \le i \le m, 1 \le j \le n)$ such that the fixed subfield $k(z_{i,j} : 1 \le i \le m, 1 \le j \le n)^{G\times H}$ is k-rational.*

Lemma 2.8. [Ha1, page 245] *Let k be a field, $k(x_1, \cdots, x_{n-1})$ the rational function field of $n-1$ variables over k. Suppose that σ is a k-automorphism on $k(x_1, \cdots, x_{n-1})$ defined by $\sigma : x_1 \mapsto x_2 \mapsto \cdots \mapsto x_{n-1} \mapsto 1/\prod_{i=1}^{n-1} x_i$. Then there exist $y_1, \cdots, y_{n-1} \in k(x_1, \cdots, x_{n-1})$ such that $k(x_1, \cdots, x_{n-1}) = k(y_1, \cdots, y_{n-1})$ and $\sigma : y_1 \mapsto y_2 \mapsto \cdots \mapsto y_{n-1} \mapsto 1 - \sum_{i=1}^{n-1} y_i$.*

Proof. For the convenience of the reader, we include the proof of [Ha1]. Define $w := 1 + x_1 + x_1x_2 + \cdots + x_1x_2\cdots x_{n-1}$, $y_1 := 1/w$, and $y_i := x_1x_2\cdots x_{i-1}/w$ for $2 \le i \le n$. It follows that $\sum_{i=1}^{n} y_i = 1$ and $\sigma : y_1 \mapsto y_2 \mapsto \cdots \mapsto y_n \mapsto y_1$. ■

Lemma 2.9. *Let k be a field, p an odd prime number, $k(y_1, \cdots, y_{p-1})$ the rational function field of $p-1$ variables over k. Suppose that σ is a k-automorphism on $k(y_1, \cdots, y_{p-1})$ defined by $\sigma : y_1 \mapsto y_2 \mapsto \cdots \mapsto y_{p-1} \mapsto 1 - \sum_{i=1}^{p-1} y_i$.*

(i) *If $\mathrm{char}(k) = p$, then the fixed subfield $k(y_1, \cdots, y_{p-1})^{\langle\sigma\rangle}$ is k-rational.*

(ii) *If $\mathrm{char}(k) \ne p$, then the field $k(y_1, \cdots, y_{p-1})^{\langle\sigma\rangle}(z)$ is k-isomorphic to $k(C_p)$, where z is an element transcendental over $k(y_1, \cdots, y_{p-1})^{\langle\sigma\rangle}$.*

Proof. (i) Define $u := \sum_{i=1}^{p-1}(p-i)y_i$. Then $\sigma : u \mapsto u + 1$. Apply Theorem 2.1 to $k(y_1, \cdots, y_{p-1}) = L(y_2, \cdots, y_{p-1})$ with $L := k(u)$. Since $L^{\langle\sigma\rangle}$ is k-rational by Lüroth's theorem, done.

(ii) Since $1/p$ exists and belongs to k, we may define $z_i := y_i - 1/p$ for $1 \le i \le p-1$. It follows that $\sigma : z_1 \mapsto z_2 \mapsto \cdots \mapsto z_{p-1} \mapsto -\sum_{i=1}^{p-1} z_i$. On the other hand, write $k(C_p) := k(x_0, x_1, \cdots, x_{p-1})^{\langle\sigma\rangle}$, where $\sigma : x_i \mapsto x_{i+1}$ (the indices of x_i are taken modulo p). Define $v := \sum_{i=0}^{p-1} x_i$ and $v_i := x_i - v/p$ for $0 \le i \le p-1$. Note that $\sum_{i=0}^{p-1} v_i = 0$. Define $K := k(v_0, v_1, \cdots, v_{p-1})$. Then $k(x_0, x_1, \cdots, x_{p-1}) = K(v)$ and apply Theorem 2.2. Hence the result. ■

3 Proofs of Theorem 1.5 and 1.7

We will prove Theorem 1.5 and 1.7 in this section.

Proof of Theorem 1.5. The action of G on $k(t, s, x_0, x_1, \cdots, x_{n-1})$ is well-defined because $\sigma \cdot t^d = \chi^d(\sigma) t^d = t^d$ for any $\sigma \in G$. Let $H := \ker(\chi)$. Then

$$\begin{aligned}
& k(t, s, x_0, x_1, \cdots, x_{n-1})^G \\
= \ & k(t, s, x_0/s, x_1/s, \cdots, x_{n-1}/s)^G \\
= \ & k(t^d/s^n, t^a s^b, x_0/s, x_1/s, \cdots, x_{n-1}/s)^G \text{ for some } a, b \in \mathbb{Z} \text{ such that } an + bd = 1 \\
= \ & k(t^a s^b, x_0/s, x_1/s, \cdots, x_{n-1}/s)^G \text{ since } t^d/s^n = \textstyle\prod_{i=0}^{n-1}(x_i/s) \\
= \ & \left(k(t^a s^b, x_0/s, x_1/s, \cdots, x_{n-1}/s)^H\right)^{G/H} \\
= \ & \left(k(x_0/s, x_1/s, \cdots, x_{n-1}/s)^H (t^a s^b)\right)^{G/H} \text{ since } H := \ker(\chi) \text{ fixes } t^a s^b \\
= \ & \left(k(x_0/s, x_1/s, \cdots, x_{n-1}/s)^H\right)^{G/H} (u) \text{ by applying Theorem 2.2} \\
= \ & k(x_0/s, x_1/s, \cdots, x_{n-1}/s)^G (u).
\end{aligned}$$

■

Proof of Theorem 1.7. Note that $\lambda_1^4 = \lambda_2^2 = 1$. Moreover, it is straightforward to check that

$$\begin{aligned}
&\lambda_1 \sigma_1 \lambda_1^{-1} = \sigma_2^{-1}, \quad && \lambda_1 \sigma_2 \lambda_1^{-1} = \sigma_1, \quad && \lambda_2 \sigma_1 \lambda_2^{-1} = \sigma_2, \quad && \lambda_2 \sigma_2 \lambda_2^{-1} = \sigma_1, \\
&\rho_1 \sigma_1 \rho_1^{-1} = \sigma_1^a, && \rho_1 \sigma_2 \rho_1^{-1} = \sigma_2, && \rho_2 \sigma_1 \rho_2^{-1} = \sigma_1, && \rho_2 \sigma_2 \rho_2^{-1} = \sigma_2^a.
\end{aligned}$$

Define $N := \langle \sigma_1, \sigma_2 \rangle \le G$, where G is one of the four groups we consider. Clearly N is a normal subgroup of G. Define

$$u_i := \sum_{j=0}^{p-1} j^i x_j \qquad \text{and} \qquad v_i := \sum_{j=0}^{p-1} j^i y_j,$$

where $0 \le i \le p-1$ (by convention we write $0^0 = 1$). We find that

$$k(x_i, y_i : 0 \le i \le p-1) = k(u_i, v_i : 0 \le i \le p-1),$$

and

$$\begin{aligned}
&\sigma_1 \cdot u_i = \sum_{j=0}^{p-1} j^i x_{j+1} = \sum_{j=0}^{p-1} (j-1)^i x_j, \qquad && \sigma_1 \cdot v_i = v_i, \\
&\sigma_2 \cdot u_i = u_i, && \sigma_2 \cdot v_i = \sum_{j=0}^{p-1} j^i y_{j+1} = \sum_{j=0}^{p-1} (j-1)^i y_j.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sigma_1 &: u_0 \mapsto u_0, u_1 \mapsto u_1 - u_0, u_2 \mapsto u_2 - 2u_1 + u_0, \cdots, \\
\sigma_2 &: v_0 \mapsto v_0, v_1 \mapsto v_1 - v_0, v_2 \mapsto v_2 - 2v_1 + v_0, \cdots.
\end{aligned}$$

Also note that

$$\lambda_1 : u_i \mapsto (-1)^i v_i, v_i \mapsto u_i, \quad \lambda_2 : u_i \leftrightarrow v_i, \quad \rho_1 : u_i \mapsto a^{-i} u_i, \quad \rho_2 : v_i \mapsto a^{-i} v_i.$$

We will show that $k(x_i, y_i : 0 \le i \le p-1)^G$ is k-rational. By Theorem 2.1, it suffices to show that $k(u_0, v_0, u_1, v_1)^G$ is k-rational. Recall that $N := \langle \sigma_1, \sigma_2 \rangle \le G$, and

$$\sigma_1 : \frac{u_1}{u_0} \mapsto \frac{u_1}{u_0} - 1, \frac{v_1}{v_0} \mapsto \frac{v_1}{v_0}, \quad \sigma_2 : \frac{u_1}{u_0} \mapsto \frac{u_1}{u_0}, \frac{v_1}{v_0} \mapsto \frac{v_1}{v_0} - 1.$$

We have

$$k(u_0,v_0,u_1,v_1)^N = k\left(u_0,v_0,\frac{u_1}{u_0},\frac{v_1}{v_0}\right)^N = k(u_0,v_0,u_1',v_1'),$$

where u_1' and v_1' are the Artin–Schreier elements defined by

$$u_1' := \left(\frac{u_1}{u_0}\right)^p - \frac{u_1}{u_0} \qquad \text{and} \qquad v_1' := \left(\frac{v_1}{v_0}\right)^p - \frac{v_1}{v_0}.$$

Note that

$$\left[k\left(u_0,v_0,\frac{u_1}{u_0},\frac{v_1}{v_0}\right) : k(u_0,v_0,u_1',v_1')\right] = p^2 = |N|,$$

because

$$u_1' = \prod_{i=0}^{p-1}\left(\frac{u_1}{u_0}+i\right) \qquad \text{and} \qquad v_1' = \prod_{i=0}^{p-1}\left(\frac{v_1}{v_0}+i\right).$$

Now we have

$$\lambda_1 : u_0 \leftrightarrow v_0, u_1' \mapsto -v_1', v_1' \mapsto u_1', \quad \lambda_2 : u_0 \leftrightarrow v_0, u_1' \leftrightarrow v_1', \quad \rho_1 : u_1' \mapsto a^{-1}u_1', \quad \rho_2 : v_1' \mapsto a^{-1}v_1'.$$

To show that $k(u_0,v_0,u_1',v_1')^{G/N}$ is k-rational, we apply Theorem 2.1 again. It remains to show that $k(u_1',v_1')^{G/N}$ is k-rational. By applying Theorem 2.2 to $k(u_1',v_1') = k(u_1'/v_1')(v_1')$, the problem is reduced to the rationality of $k(u_1'/v_1')^{G/N}$, which is the case because of Lüroth's theorem.

To find the explicit generator we illustrate the case $G = \langle\sigma_1,\sigma_2,\lambda_1,\rho_1\rho_2\rangle$. Since $N = \langle\sigma_1,\sigma_2\rangle$, it follows that $G/N = \langle\lambda_1,\rho_1\rho_2\rangle$. Note that the action of G/N on $k(u_1'/v_1')$ is not faithful. In fact, $\rho_1\rho_2$ acts trivially on it and $\lambda_1 : u_1'/v_1' \mapsto -v_1'/u_1'$. We conclude that the fixed subfield $k(u_1'/v_1')^{G/N} = k(u_1'/v_1' - v_1'/u_1')$. ■

Consider a variant of Theorem 1.5. Let k be a field, p an odd prime number, $k(t,s,x_0,x_1,\cdots,x_{p-1})$ a field satisfying that $\mathrm{trdeg}_k k(t,s,x_0,x_1,\cdots,x_{p-1}) = p+1$ and $t^{d'} = \prod_{i=0}^{p-1} x_i$, where d' is an integer divisible by p. Write $d' = pd_0$ and define $X_i := x_i/t^{d_0}$, then $\prod_{i=0}^{p-1} X_i = 1$. Consider the action of the group G_{pd} in Definition 2.3. The rationality problem of $k(t,s,X_0,X_1,\cdots,X_{p-1})^{G_{pd}}$ is reduced, up to stable rationality, to the following question.

Question 3.1. *Let k be a field, p an odd prime number, $k(x_1,\cdots,x_{p-1})$ the rational function field of $p-1$ variables over k. Let a be an integer satisfying that $(\mathbb{Z}/p\mathbb{Z})^\times = \langle\bar{a}\rangle$. Define k-automorphisms σ,τ on $k(x_1,\cdots,x_{p-1})$ by $\sigma : x_1 \mapsto x_2 \mapsto \cdots \mapsto x_{p-1} \mapsto 1/\prod_{i=1}^{p-1} x_i$ and $\tau : x_i \mapsto x_{ai}$, where the indices of x_i are taken modulo p. For any positive divisor d of $p-1$, write $p-1 = de$ and define the group $G_{pd} := \langle\sigma,\tau^e\rangle$. We ask under what situations the fixed subfield $k(x_1,\cdots,x_{p-1})^{G_{pd}}$ is k-rational (resp., stably k-rational).*

Alternatively, we may formulate the above question in terms of G-lattices. The reader may consult Section 2 of [HKY] for unexplained terminology.

Let p be an odd prime number and a an integer satisfying that $(\mathbb{Z}/p\mathbb{Z})^\times = \langle\bar{a}\rangle$. Define the group G by $G := \langle\sigma,\tau\rangle$, where $\sigma^p = \tau^{p-1} = 1$ and $\tau\sigma\tau^{-1} = \sigma^a$. For any positive divisor d of $p-1$, write $p-1 = de$ and define the group $G_{pd} := \langle\sigma,\tau^e\rangle$.

Now define the lattices $N := \mathbb{Z}[G/\langle\tau\rangle]$ and $M := N/(\mathbb{Z}\sum_{i=0}^{p-1}\sigma^i)$. In other words, $0 \to \mathbb{Z} \to N \to M \to 0$ is a short exact sequence of G-lattices. Note that N may be described explicitly. Write $N' := \oplus_{i=0}^{p-1}\mathbb{Z}e_i$ with the G-action defined by $\sigma \cdot e_i = e_{i+1}$ and $\tau \cdot e_i = e_{ai}$. Then N and N' are isomorphic G-lattices.

If k is a field, consider $k(N)^{G_{pd}}$ and $k(M)^{G_{pd}}$ where the group acts trivially on the field k. Questions: Is $k(N)^{G_{pd}}$ rational? Is $k(M)^{G_{pd}}$ rational? Is $[M]^{fl}$ a permutation lattice or an invertible lattice?

We remark that, if $d = 1$ (i.e., $G_p \cong C_p$, the cyclic group of order p), then $k(M)^{G_p}$ is stably k-rational if and only if so is $k(G_p)$, because we may apply Lemma 2.8 and 2.9.

If $d = 2$ (i.e., $G_{2p} \cong D_p$, the dihedral group of order $2p$), then $k(M)^{G_{2p}}$ is stably k-rational if and only if so is $k(G_{2p})$ by [HKY, Theorem 5.6].

On the other hand, we don't know the answer whether $k(M)^{G_{pd}}$ is k-rational (resp., stably k-rational) if d is a divisor of $p-1$ other than 1 or 2. The situation of $k(N)^{G_{pd}}$ is unclear even when $k=\mathbb{Q}$, which was investigated by Breuer during 1920s [Br].

4 Transitive Subgroups of S_{14}

The transitive subgroups of S_{14}, up to conjugation within S_{14}, was classified by [Mi]. It was reconfirmed in [CHM]. The message was finally integrated into the libraries of [GAP]. In this paper, we will name these groups as those given in the GAP library.

The symmetric group S_{14} contains 63 transitive subgroups. They are labeled as $G(i)$, where $1 \le i \le 63$. Among the 63 subgroups, exactly 36 of them are solvable.

These subgroups are classified into 13 classes. Groups belonging to Class 1–6 are solvable, while groups belonging to Class 7–13 are non-solvable. Note that the classification into 13 classes is devised by us, not by the GAP library.

To describe these subgroups, we first define the following elements in S_{14}:

$$\begin{aligned}
\sigma_1 &= (1,3,5,7,9,11,13),\\
\sigma_2 &= (2,4,6,8,10,12,14),\\
\tau_1 &= (1,2)(3,8,5,14,9,12)(4,7,6,13,10,11),\\
\tau_2 &= (3,7,5,13,9,11)(4,8,6,14,10,12),\\
\lambda_1 &= (1,2)(3,14,13,4)(5,12,11,6)(7,10,9,8),\\
\lambda_2 &= (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14),\\
\lambda_3 &= (3,13)(5,11)(7,9),\\
\lambda_4 &= (4,14)(6,12)(8,10),\\
\lambda_5 &= (3,5,9)(7,13,11),\\
\lambda_6 &= (4,6,10)(8,14,12),\\
\mu_i &= (2i+1,2i+2),\ \text{for } 0 \le i \le 6,\\
\nu_1 &= (1,4)(2,3)(5,6)(7,8)(9,10)(11,12)(13,14),\\
\nu_2 &= (1,3)(2,4),\\
\nu_3 &= (1,6)(2,5)(3,11)(4,12)(7,8)(9,10).
\end{aligned}$$

Here is a list of these 63 subgroups $G(i)$ of S_{14}.

The notation G refers to some group $G(i)$ when we don't intend to specify its numeral. Remember that the groups G_{21} and G_{42} are subgroups of S_7 in Definition 2.3, don't confuse them with $G(21)$ and $G(42)$. We also remark that in this list, some groups may be isomorphic as abstract groups (we use the notation $\simeq$), but they are not conjugate within S_{14}. For example, consider the group $G(17)$ in Class 12 and the group $G(19)$ in Class 11. We write $G(17) \simeq C_2 \times \mathrm{PSL}_2(\mathbb{F}_7)$ to mean that $G(17)$ is isomorphic to a direct product of C_2 and $\mathrm{PSL}_2(\mathbb{F}_7)$ as abstract groups. On the other hand, we write $G(19) = \mathrm{PSL}_2(\mathbb{F}_7) \times C_2$ to mean that $G(19)$ is defined as a subgroup of S_{14}, where the first factor is a subgroup of S_7, the second factor is S_2, and the identification of the direct product as a subgroup of S_{14} is understood as the way given in Theorem 2.7. We adopt similar convention for groups in Classes 1, 8, and 11 (see the explanation in Definition 2.5, Theorem 2.6, and 2.7).

Class 1. The direct product and the wreath product of smaller groups:

$$\begin{array}{llll}
G(1) = C_7 \times C_2, & G(3) = D_7 \times C_2, & G(5) = G_{21} \times C_2, & G(7) = G_{42} \times C_2,\\
G(8) = C_7 \wr_{X_2} C_2, & G(20) = D_7 \wr_{X_2} C_2, & G(26) = G_{21} \wr_{X_2} C_2, & G(45) = G_{42} \wr_{X_2} C_2,\\
G(29) = C_2 \wr_{X_7} C_7, & G(38) = C_2 \wr_{X_7} D_7, & G(44) = C_2 \wr_{X_7} G_{21}, & G(48) = C_2 \wr_{X_7} G_{42}.
\end{array}$$

Class 2. G contains a normal subgroup $N_7 := \langle \sigma_1\sigma_2 \rangle \simeq C_7$:

$$G(2) = N_7 \rtimes \left\langle \tau_1^3 \right\rangle \simeq D_7, \quad G(4) = N_7 \rtimes \langle \tau_1 \rangle \simeq G_{42}.$$

Class 3. G contains a normal subgroup $N_{49} := \langle \sigma_1, \sigma_2 \rangle \simeq C_7^2$:

$$\begin{aligned}
&G(12) = N_{49} \rtimes \langle \lambda_1 \rangle, && G(13) = N_{49} \rtimes \langle \lambda_2, \lambda_3\lambda_4 \rangle, \\
&G(14) = N_{49} \rtimes \langle \lambda_2, \lambda_5\lambda_6 \rangle, && G(15) = N_{49} \rtimes \left\langle \lambda_2, \lambda_5^{-1}\lambda_6 \right\rangle, \\
&G(22) = N_{49} \rtimes \left\langle \lambda_1, \lambda_5^{-1}\lambda_6 \right\rangle, && G(23) = N_{49} \rtimes \langle \lambda_1, \lambda_5\lambda_6 \rangle, \\
&G(24) = N_{49} \rtimes \langle \lambda_2, \lambda_3\lambda_4, \lambda_5\lambda_6 \rangle, && G(25) = N_{49} \rtimes \left\langle \lambda_2, \lambda_3\lambda_4, \lambda_5^{-1}\lambda_6 \right\rangle, \\
&G(31) = N_{49} \rtimes \left\langle \lambda_2, \lambda_3, \lambda_4, \lambda_5^{-1}\lambda_6 \right\rangle, && G(32) = N_{49} \rtimes \langle \lambda_2, \lambda_3, \lambda_4, \lambda_5\lambda_6 \rangle, \\
&G(36) = N_{49} \rtimes \langle \lambda_1, \lambda_5, \lambda_6 \rangle, && G(37) = N_{49} \rtimes \langle \lambda_2, \lambda_3\lambda_4, \lambda_5, \lambda_6 \rangle.
\end{aligned}$$

Class 4. G contains a normal subgroup $N_8 := \langle \mu_0\mu_1\mu_2\mu_5, \mu_0\mu_2\mu_3\mu_4, \mu_0\mu_1\mu_4\mu_6 \rangle \simeq C_2^3$:

$$G(6) = N_8 \rtimes \langle \sigma_1\sigma_2 \rangle \simeq C_2^3 \rtimes C_7, \quad G(11) = N_8 \rtimes \left\langle \sigma_1\sigma_2, \tau_2^2 \right\rangle \simeq C_2^3 \rtimes G_{21}.$$

Class 5. G contains a normal subgroup $N_{16} := \langle \mu_0\mu_1\mu_2\mu_5, \mu_0\mu_2\mu_3\mu_4, \mu_0\mu_1\mu_4\mu_6, \mu_1\mu_2\mu_4 \rangle \simeq C_2^4$:

$$G(9) = N_{16} \rtimes \langle \sigma_1\sigma_2 \rangle \simeq C_2^4 \rtimes C_7, \quad G(18) = N_{16} \rtimes \left\langle \sigma_1\sigma_2, \tau_2^2 \right\rangle \simeq C_2^4 \rtimes G_{21}.$$

Class 6. G contains a normal subgroup $N_{64} := \langle \mu_0\mu_1, \mu_0\mu_2, \mu_0\mu_3, \mu_0\mu_4, \mu_0\mu_5, \mu_0\mu_6 \rangle \simeq C_2^6$:

$$\begin{aligned}
&G(21) = N_{64} \rtimes \langle \sigma_1\sigma_2 \rangle \simeq C_2^6 \rtimes C_7, && G(35) = N_{64} \rtimes \left\langle \sigma_1\sigma_2, \tau_2^2 \right\rangle \simeq C_2^6 \rtimes G_{21}, \\
&G(27) = N_{64} \rtimes \left\langle \sigma_1\sigma_2, \tau_1^3 \right\rangle \simeq C_2^6 \rtimes D_7, && G(28) = N_{64} \rtimes \left\langle \sigma_1\sigma_2, \tau_2^3 \right\rangle \simeq C_2^6 \rtimes D_7, \\
&G(40) = N_{64} \rtimes \langle \sigma_1\sigma_2, \tau_1 \rangle \simeq C_2^6 \rtimes G_{42}, && G(41) = N_{64} \rtimes \langle \sigma_1\sigma_2, \tau_2 \rangle \simeq C_2^6 \rtimes G_{42}.
\end{aligned}$$

Class 7. G is the whole group S_{14}:

$$G(63) = S_{14}.$$

Class 8. The direct product and the wreath product of smaller groups:

$$G(49) = S_7 \times C_2, \quad G(61) = S_7 \wr_{X_2} C_2, \quad G(57) = C_2 \wr_{X_7} S_7.$$

Class 9. G contains the normal subgroup N_{64} defined in Class 6:

$$G(54) = N_{64} \rtimes \langle \sigma_1\sigma_2, \nu_1 \rangle \simeq C_2^6 \rtimes S_7, \quad G(55) = N_{64} \rtimes \langle \sigma_1\sigma_2, \nu_2 \rangle \simeq C_2^6 \rtimes S_7.$$

Class 10. G is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$:

$$G(10) = \langle \sigma_1\sigma_2, \nu_3 \rangle \simeq \mathrm{PSL}_2(\mathbb{F}_7).$$

Class 11. The direct product and the wreath product of smaller groups:

$$\begin{aligned}
&G(19) = \mathrm{PSL}_2(\mathbb{F}_7) \times C_2, && G(52) = \mathrm{PSL}_2(\mathbb{F}_7) \wr_{X_2} C_2, && G(51) = C_2 \wr_{X_7} \mathrm{PSL}_2(\mathbb{F}_7), \\
&G(47) = A_7 \times C_2, && G(58) = A_7 \wr_{X_2} C_2, && G(56) = C_2 \wr_{X_7} A_7.
\end{aligned}$$

Class 12. G contains a normal elementary abelian 2-subgroup:

$G(17) \simeq C_2 \times \mathrm{PSL}_2(\mathbb{F}_7)$,

$G(33)$ is an extension of $\mathrm{PSL}_2(\mathbb{F}_7)$ by C_2^3, $\quad G(34) \simeq C_2^3 \rtimes \mathrm{PSL}_2(\mathbb{F}_7)$,

$G(42)$ is an extension of $\mathrm{PSL}_2(\mathbb{F}_7)$ by C_2^4, $\quad G(43) \simeq C_2^4 \rtimes \mathrm{PSL}_2(\mathbb{F}_7)$,

$G(50) \simeq C_2^6 \rtimes \mathrm{PSL}_2(\mathbb{F}_7)$, $\quad G(53) \simeq C_2^6 \rtimes A_7$.

Class 13. G does not contain any solvable normal subgroups:

$$\begin{aligned} &G(16) \simeq \mathrm{PGL}_2(\mathbb{F}_7), \quad G(30) \simeq \mathrm{PSL}_2(\mathbb{F}_{13}), \quad G(39) \simeq \mathrm{PGL}_2(\mathbb{F}_{13}), \\ &G(46) \simeq S_7, \qquad\quad G(59) \simeq A_7^2 \rtimes C_4, \qquad G(60) \simeq A_7^2 \rtimes C_2^2, \qquad G(62) = A_{14}. \end{aligned}$$

The following theorem is a strengthened version of Theorem 1.8.

Theorem 4.1. *Let k be a field and G a transitive subgroup of S_{14}. Then $k(x_1, \cdots, x_{14})^G$ is k-rational provided that*

- (i) $G = G(i)$ *belongs to Classes* 1, 2, 7, *and* 8*;*
- (ii) $\mathrm{char}(k) = 7$ *and* $G = G(i)$ *belongs to Class* 3*;*
- (iii) $\mathrm{char}(k) \neq 2$ *and* $G = G(i)$ *belongs to Classes* 4, 5, 6, *and* 9*;*
- (iv) $k \supseteq \mathbb{Q}(\sqrt{-7})$ *and* $G = G(i)$ *belongs to Class* 10.

We remark that we don't know whether $k(x_1, \cdots, x_{14})^G$ is k-rational if $G = G(i)$ belongs to Classes 11–13.

5 Proof of Theorem 4.1

5.1 $G = G(i)$ belonging to Classes 1, 7, and 8

Proof. Apply Theorem 1.1, 2.6, and 2.7. For example, consider $G(48) = C_2 \wr_{X_7} G_{42}$. Note that G_{42} is a subgroup of S_7 by Definition 2.3. We may identify C_2 with the symmetric group S_2. By Definition 2.5, $G(48)$ may be regarded as a subgroup of S_{14}. Now apply Theorem 1.1 and 2.6. ■

5.2 $G = G(i)$ belonging to Classes 2 and 10

Proof. Write $\sigma := \sigma_1\sigma_2$. Define $y_i := x_{2i+1} + x_{2i+2}$ for $0 \leq i \leq 6$.

To show that $k(x_1, \cdots, x_{14})^G$ is k-rational, we apply Theorem 2.1. Since G is faithful on $k(y_0, y_1, \cdots, y_6)$, it suffices to show that $k(y_0, y_1, \cdots, y_6)^G$ is k-rational.

It is routine to check that

$$\sigma : y_i \mapsto y_{i+1}, \quad \tau_1 : y_i \mapsto y_{3i}, \quad \nu_3 : y_0 \leftrightarrow y_2, y_1 \leftrightarrow y_5, y_i \mapsto y_i, \text{ for } i = 3, 4, 6,$$

where the indices of y_i are taken modulo 7.

The rationality of $k(y_0, y_1, \cdots, y_6)^G$ follows from Theorem 1.1. ■

5.3 $G = G(i)$ belonging to Class 3

Proof. We rename the original variables $\{x_1, \cdots, x_{14}\}$ by the new names $\{x_0, y_0, x_1, y_1, \cdots, x_6, y_6\}$. Then

$$\begin{aligned} &\sigma_1 : x_i \mapsto x_{i+1}, \quad \sigma_2 : y_i \mapsto y_{i+1}, \quad \lambda_1 : x_i \mapsto y_{-i}, y_i \mapsto x_i, \quad \lambda_2 : x_i \leftrightarrow y_i, \\ &\lambda_3 : x_i \mapsto x_{-i}, \quad \lambda_4 : y_i \mapsto y_{-i}, \quad \lambda_5 : x_i \mapsto x_{2i}, \qquad \lambda_6 : y_i \mapsto y_{2i}. \end{aligned}$$

Note that

$$\lambda_3 = \rho_1 \text{ for } a = -1, \quad \lambda_4 = \rho_2 \text{ for } a = -1, \quad \lambda_5 = \rho_1 \text{ for } a = 2, \quad \lambda_6 = \rho_2 \text{ for } a = 2.$$

The first six cases $G = G(i)$ where $i = 12, 13, 14, 15, 22, 23$ are covered by Theorem 1.7. More specifically,

$$\begin{cases} G(12) \text{ corresponds the first (and the second) group for } (p,a) = (7,-1), \\ G(13) \text{ corresponds the third (and the fourth) group for } (p,a) = (7,-1), \\ G(14) \text{ corresponds the third group for } (p,a) = (7,2), \\ G(15) \text{ corresponds the fourth group for } (p,a) = (7,2), \\ G(22) \text{ corresponds the second group for } (p,a) = (7,2), \\ G(23) \text{ corresponds the first group for } (p,a) = (7,2). \end{cases}$$

For the remaining cases, i.e., $G = G(i)$ where $i = 24, 25, 31, 32, 36, 37$, although we cannot apply Theorem 1.7 directly, the proofs are almost the same because these groups are defined by the similar fashion as before. Here are the indications:

$$\begin{cases} G(24) = \langle \sigma_1, \sigma_2, \lambda_2, \rho_1\rho_2 \text{ for } a = -1, \rho_1\rho_2 \text{ for } a = 2 \rangle, \\ G(25) = \left\langle \sigma_1, \sigma_2, \lambda_2, \rho_1\rho_2 \text{ for } a = -1, \rho_1^{-1}\rho_2 \text{ for } a = 2 \right\rangle, \\ G(31) = \left\langle \sigma_1, \sigma_2, \lambda_2, \rho_1 \text{ for } a = -1, \rho_2 \text{ for } a = -1, \rho_1^{-1}\rho_2 \text{ for } a = 2 \right\rangle, \\ G(32) = \langle \sigma_1, \sigma_2, \lambda_2, \rho_1 \text{ for } a = -1, \rho_2 \text{ for } a = -1, \rho_1\rho_2 \text{ for } a = 2 \rangle, \\ G(36) = \langle \sigma_1, \sigma_2, \lambda_1, \rho_1 \text{ for } a = 2, \rho_2 \text{ for } a = 2 \rangle, \\ G(37) = \langle \sigma_1, \sigma_2, \lambda_2, \rho_1\rho_2 \text{ for } a = -1, \rho_1 \text{ for } a = 2, \rho_2 \text{ for } a = 2 \rangle. \end{cases}$$

■

5.4 $G = G(i)$ belonging to Classes 4, 5, 6, and 9

Proof. We will work on the field k with $\mathrm{char}(k) \neq 2$.

Write $\sigma := \sigma_1\sigma_2$. Define $y_i := x_{2i+1} - x_{2i+2}$ for $0 \leq i \leq 6$.

To show that $k(x_1, \cdots, x_{14})^G$ is k-rational, we apply Theorem 2.1 again. Since G is faithful on $k(y_0, y_1, \cdots, y_6)$, it suffices to show that $k(y_0, y_1, \cdots, y_6)^G(s)$ is k-rational (note that we add an extra variable s).

It is routine to check that

$$\begin{aligned} &\mu_i : y_i \mapsto -y_i, && \sigma : y_i \mapsto y_{i+1}, \\ &\tau_1 : y_i \mapsto -y_{3i}, && \tau_2 : y_i \mapsto y_{3i}, \\ &\nu_1 : y_0 \leftrightarrow -y_1, y_i \mapsto -y_i, \text{ for } 2 \leq i \leq 6, && \nu_2 : y_0 \leftrightarrow y_1, y_i \mapsto y_i, \text{ for } 2 \leq i \leq 6, \end{aligned}$$

where the indices of y_i are taken modulo 7.

We claim that

$$\begin{aligned} k(y_0, y_1, \cdots, y_6)^{N_8} &= k(\prod_{i=0}^{6} y_i, y_j y_{j+3} y_{j+5} y_{j+6} : 1 \leq j \leq 6), \\ k(y_0, y_1, \cdots, y_6)^{N_{16}} &= k(\prod_{i=0}^{6} y_i^2, y_j y_{j+3} y_{j+5} y_{j+6} : 1 \leq j \leq 6), \\ k(y_0, y_1, \cdots, y_6)^{N_{64}} &= k(\prod_{i=0}^{6} y_i, y_j^2 : 1 \leq j \leq 6), \end{aligned}$$

because all of the generators on the right-hand side are fixed by the corresponding group and the determinants of the coefficient matrices are given by

$$\begin{vmatrix}1&1&1&1&1&1&1\\1&1&0&0&1&0&1\\1&1&1&0&0&1&0\\0&1&1&1&0&0&1\\1&0&1&1&1&0&0\\0&1&0&1&1&1&0\\0&0&1&0&1&1&1\end{vmatrix}=8,\begin{vmatrix}2&2&2&2&2&2&2\\1&1&0&0&1&0&1\\1&1&1&0&0&1&0\\0&1&1&1&0&0&1\\1&0&1&1&1&0&0\\0&1&0&1&1&1&0\\0&0&1&0&1&1&1\end{vmatrix}=16,\begin{vmatrix}1&1&1&1&1&1&1\\0&2&0&0&0&0&0\\0&0&2&0&0&0&0\\0&0&0&2&0&0&0\\0&0&0&0&2&0&0\\0&0&0&0&0&2&0\\0&0&0&0&0&0&2\end{vmatrix}=64.$$

- $G = G(i)$ belonging to Class 4: Define $t := \prod_{i=0}^{6} y_i$ and $z_i := y_i y_{i+3} y_{i+5} y_{i+6}$ for $0 \le i \le 6$. We find that $t^4 = \prod_{i=0}^{6} z_i$, and

  $$k(y_0, y_1, \cdots, y_6)^G(s) = k(t, s, z_0, z_1, \cdots, z_6)^{G/N_8}, \quad \sigma : t \mapsto t, z_i \mapsto z_{i+1}, \quad \tau_2^2 : t \mapsto t, z_i \mapsto z_{2i},$$

  where the indices of z_i are taken modulo 7.

- $G = G(i)$ belonging to Class 5: Define $t := \prod_{i=0}^{6} y_i^2$ and $z_i := y_i y_{i+3} y_{i+5} y_{i+6}$ for $0 \le i \le 6$. We find that $t^2 = \prod_{i=0}^{6} z_i$, and

  $$k(y_0, y_1, \cdots, y_6)^G(s) = k(t, s, z_0, z_1, \cdots, z_6)^{G/N_{16}}, \quad \sigma : t \mapsto t, z_i \mapsto z_{i+1}, \quad \tau_2^2 : t \mapsto t, z_i \mapsto z_{2i},$$

  where the indices of z_i are taken modulo 7.

- $G = G(i)$ belonging to Classes 6 and 9: Define $t := \prod_{i=0}^{6} y_i$ and $z_i := y_i^2$ for $0 \le i \le 6$. We find that $t^2 = \prod_{i=0}^{6} z_i$, and

  $$\begin{aligned} &k(y_0, y_1, \cdots, y_6)^G(s) = k(t, s, z_0, z_1, \cdots, z_6)^{G/N_{64}}, && \sigma : t \mapsto t, z_i \mapsto z_{i+1}, \\ &\tau_1 : t \mapsto -t, z_i \mapsto z_{3i}, && \tau_2 : t \mapsto t, z_i \mapsto z_{3i}, \\ &\nu_1 : t \mapsto -t, z_0 \leftrightarrow z_1, z_i \mapsto z_i, \text{ for } 2 \le i \le 6, && \nu_2 : t \mapsto t, z_0 \leftrightarrow z_1, z_i \mapsto z_i, \text{ for } 2 \le i \le 6, \end{aligned}$$

  where the indices of z_i are taken modulo 7.

The rationality of $k(y_0, y_1, \cdots, y_6)^G(s) = k(t, s, z_0, z_1, \cdots, z_6)^{G/N}$ (where $N = N_8, N_{16}, N_{64}$) follows from Theorem 1.5. ■

5.5 A New Proof of Theorem 2.4

In the following, we will give a new proof of Theorem 2.4 other than that in [KW, Theorem 3.2], because similar techniques will be useful in many situations when $\mathrm{char}(k) = p > 0$.

Proof of Theorem 2.4. Recall that $(\mathbb{Z}/p\mathbb{Z})^\times = \langle \bar{a} \rangle$, $\sigma : x_i \mapsto x_{i+1}$, $\tau : x_i \mapsto x_{ai}$, $p-1 = de$, and $G_{pd} := \langle \sigma, \tau^e \rangle$, where the indices of x_i are taken modulo p. We will show that $k(x_0, x_1, \cdots, x_{p-1})^{G_{pd}}$ is k-rational.

Define

$$u := \sum_{i=0}^{d-1} a^{ei} \tau^{ei} \cdot x_0.$$

We find that

$$\tau^e \cdot u = a^{-e} u.$$

Define

$$y := \sum_{i=0}^{p-1} \sigma^i \cdot u \qquad \text{and} \qquad z := \sum_{i=0}^{p-1} i \sigma^i \cdot u.$$

Then we find that $\sigma : y \mapsto y, z \mapsto z - y$, and $\tau^e : y \mapsto a^{-e}y, z \mapsto a^{-2e}z$.

To show that $k(x_0, x_1, \cdots, x_{p-1})^{G_{pd}}$ is k-rational, we apply Theorem 2.1 because G_{pd} is faithful on $k(y, z)$. It remains to prove that $k(y, z)^{G_{pd}}$ is k-rational. By applying Theorem 2.2 to $k(y, z) = k(y)(z)$, the problem is reduced to the rationality of $k(y)^{G_{pd}}$, which is the case because of Lüroth's theorem. Explicitly, we know that $k(y)^{G_{pd}} = k(y^d)$ because $\sigma \cdot y = y$ and $\tau^e \cdot y = a^{-e}y$. ■

For the groups $G(2)$ and $G(4)$ in Class 2, if we assume that $\mathrm{char}(k) = 7$, here is another proof of the rationality by using the idea in the above proof.

For simplicity, write $\sigma := \sigma_1\sigma_2$ and $\tau := \tau_1$, and the field we consider is $k(x_1, \cdots, x_{14})$.

- $G = G(2)$: Define $u := x_1 - \tau^3 \cdot x_1$, $y := \sum_{i=0}^{6} \sigma^i \cdot u$, and $z := \sum_{i=0}^{6} i\sigma^i \cdot u$. Then $\sigma : y \mapsto y, z \mapsto z - y$, and $\tau^3 : y \mapsto -y, z \mapsto z$.
- $G = G(4)$: Define $u := \sum_{i=0}^{5} 5^i\tau^i \cdot x_1$, $y := \sum_{i=0}^{6} \sigma^i \cdot u$, and $z := \sum_{i=0}^{6} i\sigma^i \cdot u$. Then $\sigma : y \mapsto y, z \mapsto z - y$, and $\tau : y \mapsto 3y, z \mapsto z$.

The remaining proof is the same and therefore omitted.

References

[AHK] H. Ahmad, M. Hajja, and M. Kang, Rationality of some projective linear actions. J. Algebra 228 (2000), 643–658.

[Br] S. Breuer, Metazyklische Minimalbasis und komplexe Primzahlen. J. Reine Angew. Math. 156 (1927), 13–42.

[CHM] J. Conway, A. Hulpke, and J. McKay, On transitive permutation groups. LMS J. Comput. Math. 1 (1998), 1–8.

[EM] S. Endo and T. Miyata, Invariants of finite abelian groups. J. Math. Soc. Japan 25 (1973), 7–26.

[GAP] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.10; 2018. (https://www.gap-system.org)

[Ga] W. Gaschütz, Fixkörper von p-Automorphismengruppen rein-transzendener Körpererweiterungen von p-Charakteristik. Math. Z. 71 (1959), 466–468.

[Ha1] M. Hajja, A note on monomial automorphisms. J. Algebra 85 (1983), 243–250.

[Ha2] M. Hajja, A note on a result of Kuniyoshi. J. Algebra 92 (1985), 171–175.

[HK1] M. Hajja and M. Kang, Three-dimensional purely monomial actions. J. Algebra 170 (1994), 805–860.

[HK2] M. Hajja and M. Kang, Some actions of symmetric groups. J. Algebra 177 (1995), 511–535.

[HKY] A. Hoshi, M. Kang, and A. Yamasaki, Rationality problems for relation modules of dihedral groups. J. Algebra 530 (2019), 368–401.

[KP] M. Kang and B. Plans, Reduction theorems for Noether's problem. Proc. Amer. Math. Soc. 137 (2009), 1867–1874.

[KW] M. Kang and B. Wang, Rational invariants for subgroups of S_5 and S_7. J. Algebra 413 (2014), 345–363.

[KWZ] M. Kang, B. Wang, and J. Zhou, Invariants of wreath products and subgroups of S_6. Kyoto J. Math. 55 (2015), 257–279.

[Ku1] H. Kuniyoshi, On purely-transcendency of a certain field. Tôhoku Math. J. (2) 6 (1954), 101–108.

[Ku2] H. Kuniyoshi, On a problem of Chevalley. Nagoya Math. J. 8 (1955), 65–67.

[Ku3] H. Kuniyoshi, Certain subfields of rational function fields. Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 241–243. Science Council of Japan, Tokyo, 1956.

[MT] Yu. I. Manin and M. A. Tsfasman, Rational varieties: algebra, geometry, arithmetic. Russian Math. Surveys 41 (1986), 51–116.

[Mi] G. A. Miller, On the transitive substitution groups of degree thirteen and fourteen, Quart. J. Pure Appl. Math. 29 (1898), 224–249.

[Sa] D. J. Saltman, Generic Galois extensions and problems in field theory. Adv. in Math. 43 (1982), 250–283.

[Sw] R. G. Swan, Noether's problem in Galois theory. Emmy Noether in Bryn Mawr, 21–40, Springer, New York-Berlin, 1983.

[WW] B. Wang and G. Wang, Rationality problem for some transitive subgroups of S_{14}. Comm. Algebra 48 (2020), 917–929.

[WZ] B. Wang and J. Zhou, Rationality problem for transitive subgroups of S_8. J. Algebra 518 (2019), 272–303.

Hang Fu (fu@ncts.ntu.edu.tw)
National Center for Theoretical Sciences, National Taiwan University, Taipei, Taiwan

Ming-chang Kang (kang@math.ntu.edu.tw)
Department of Mathematics, National Taiwan University, Taipei, Taiwan

Baoshan Wang (bwang@buaa.edu.cn)
School of Mathematics and System Sciences, Beihang University, Beijing, China

Jian Zhou (zhjn@math.pku.edu.cn)
School of Mathematical Sciences, Peking University, Beijing, China