CYLINDERS IN MORI FIBER SPACES: FORMS OF THE QUINTIC DEL PEZZO THREEFOLD

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Abstract. Motivated by the general question of existence of open $\mathbb{A}^1$-cylinders in higher dimensional projective varieties, we consider the case of Mori Fiber Spaces of relative dimension three, whose general closed fibers are isomorphic to the quintic del Pezzo threefold V_5, the smooth Fano threefold of index two and degree five. We show that the total spaces of these Mori Fiber Spaces always contain relative $\mathbb{A}^2$-cylinders, and we characterize those admitting relative $\mathbb{A}^3$-cylinders in terms of the existence of certain special lines in their generic fibers.

Introduction

An $\mathbb{A}^r_k$-*cylinder* in a normal algebraic variety defined over a field k is a Zariski open subset U isomorphic to $Z \times \mathbb{A}^r_k$ for some algebraic variety Z defined over k. In the case where $k = \overline{k}$ is algebraically closed, normal projective varieties V containing $\mathbb{A}^1_k$-cylinders have received a lot of attention recently due to the connection between unipotent group actions on their affine cones and polarized $\mathbb{A}^1_k$-cylinders in them, that is $\mathbb{A}^1_k$-cylinders whose complements are the supports of effective $\mathbb{Q}$-divisors linearly equivalent to an ample divisor on V (cf. [12, 13]). Certainly, the canonical divisor K_V of a normal projective variety containing an $\mathbb{A}^r_k$-cylinder for some $r \geq 1$ is not pseudo-effective. Replacing V if necessary by a birational model with at most $\mathbb{Q}$-factorial terminal singularities, [1] guarantees the existence of a suitable K_V-MMP $V \dashrightarrow X$ whose output X is equipped with a structure of Mori Fiber Space $f : X \to Y$ over some lower dimensional normal projective variety Y. Since an $\mathbb{A}^1_k$-cylinder in X can always be transported back in the initial variety V [4, Lemma 9], total spaces of Mori Fiber Spaces form a natural restricted class in which to search for varieties containing $\mathbb{A}^1_k$-cylinders.

In the case where $\dim Y = 0$, X is a Fano variety of Picard number one. The only smooth Fano surface of Picard number is the projective plane $\mathbb{P}^2_k$ which obviously contains $\mathbb{A}^1_k$-cylinders. Several families of examples of smooth Fano varieties of dimension 3 and 4 and Picard number one containing $\mathbb{A}^1_k$-cylinders have been constructed [14, 21, 22]. The question of existence of $\mathbb{A}^1_k$-cylinders in other possible outputs of MMPs was first considered in [2, 3], in which del Pezzo fibrations $f : X \to Y$, which correspond to the case where $\dim Y = \dim X - 2 > 0$, were extensively studied. In such a relative context, it is natural to shift the focus to cylinders which are compatible with the fibration structure:

Definition. Let $f : X \to Y$ be a morphism between normal algebraic varieties defined over a field k and let $U \simeq Z \times \mathbb{A}^r_k$ be an $\mathbb{A}^r_k$-cylinder inside X. We say that U is *vertical with respect to* f if the restriction $f|_U$ factors as

$$f\,|_U = h \circ \mathrm{pr}_Z : U \simeq Z \times \mathbb{A}^r_k \xrightarrow{\mathrm{pr}_Z} Z \xrightarrow{h} Y$$

for a suitable morphism $h : Z \to Y$.

In the present article, we initiate the study of existence of vertical $\mathbb{A}^1$-cylinders in Mori Fiber Spaces $f : X \to Y$ of relative dimension three, whose general fibers are smooth Fano threefolds. Each smooth Fano threefold of Picard number one can appear as a closed fiber of a Mori Fiber Space, and among these,

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it is natural to first restrict to classes which contain a cylinder of the maximal possible dimension, namely the affine space $\mathbb{A}^3_{\overline{k}}$, and expect that some suitable sub-cylinders of these fiber wise maximal cylinders could arrange into a vertical cylinder with respect to f, possibly of smaller relative dimension. The only four classes of Fano threefold of Picard number one containing $\mathbb{A}^3_{\overline{k}}$ are $\mathbb{P}^3_{\overline{k}}$, the quadric $\mathbb{Q}^3$, the del Pezzo quintic threefold V_5 of index two and degree five [7, 8], and a four dimensional family of prime Fano threefolds V_{22} of genus twelve [6, 20].

The existence inside X of a vertical $\mathbb{A}^r$-cylinder with respect to $f : X \to Y$ translates equivalently into that of an $\mathbb{A}^r_K$-cylinder inside the fiber X_η of f over the generic point η of Y, considered as a variety defined over the function field K of Y [3]. We are therefore led to study the existence of $\mathbb{A}^r_K$-cylinders inside K-forms of the aforementioned Fano threefolds over non-closed fields K of characteristic zero, that is, smooth projective varieties defined over K whose base extensions to an algebraic closure $\overline{K}$ are isomorphic over $\overline{K}$ to one of these Fano threefolds. The case of $\mathbb{P}^3$ is easily dispensed: a K-form V of $\mathbb{P}^3$ contains an $\mathbb{A}^3_K$-cylinder if and only if it has a K-rational point hence if and only if it is the trivial K-form $\mathbb{P}^3_K$. So equivalently, a Mori Fiber Space $f : X \to Y$ whose general closed fibers are isomorphic to $\mathbb{P}^3_{\overline{k}}$ contains a vertical $\mathbb{A}^3_{\overline{k}}$-cylinder if and only if it has a rational section. The case of the quadric $\mathbb{Q}^3$ is already more intricate: one can deduce from [6] that a K-form V of $\mathbb{Q}^3$ contains $\mathbb{A}^3_K$ if and only if it has a hyperplane section defined over K which is a K-rational quadric cone. In this article, we establish a complete characterization of the existence of $\mathbb{A}^r_K$-cylinders in forms of V_5 which can be summarized as follows:

Theorem. *Let K be a field of characteristic zero and let Y be a K-form of V_5. Then Y always contains an $\mathbb{A}^2_K$-cylinder, and it contains an $\mathbb{A}^3_K$-cylinder if and only if it contains a curve $\ell \simeq \mathbb{P}^1_K$ of anticanonical degree $-K_Y \cdot \ell = 2$ and with normal bundle $\mathcal{N}_{\ell/Y} \simeq \mathcal{O}_{\mathbb{P}^1_K}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_K}(1)$.*

An irreducible curve ℓ of anticanonical degree $-K_Y \cdot \ell = 2$ on a K-form Y of V_5 becomes after base extension to an algebraic closure $\overline{K}$ of K a usual line on $Y_{\overline{K}} \simeq V_5$ embedded into $\mathbb{P}^6_{\overline{K}}$ via its half-anticanonical complete linear system. By combining the previous characterization with a closer study of the Hilbert scheme of such curves ℓ on Y (see §2.1), we derive the following result (see Corollary 13):

Corollary. *Let $\overline{k}$ be an algebraically closed field of characteristic zero and let $f : X \to C$ be a Mori Fiber Space over a curve C defined over $\overline{k}$, whose general closed fibers are quintic del Pezzo threefolds V_5. Then X contains a vertical $\mathbb{A}^3_{\overline{k}}$-cylinder with respect to f.*

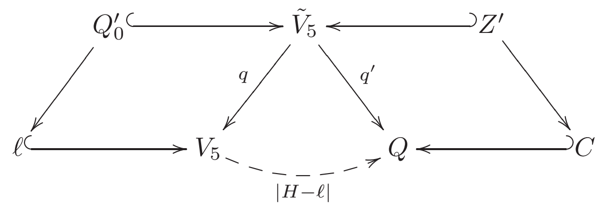
Section 1 contains a brief recollection on the quintic del Pezzo threefold V_5 and its Hilbert scheme of lines. In Section 2, we establish basic geometric properties of forms of V_5 and describe their Hilbert schemes of lines. We also describe an adaptation to non-closed fields of a standard construction of V_5 as the variety of trisecant lines to a Veronese surface in $\mathbb{P}^4$, from which we derive for suitable fields the existence of forms of V_5 which do not contain any line with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. The technical core of the article is then Section 3: we give a new construction of a classical rational map, called the double projection from a rational point of a form of V_5 in the form of a Sarkisov link from a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ explicitly determined by the base locus of a quadratic birational involution of $\mathbb{P}^2$. The main results concerning $\mathbb{A}^r$-cylinders in forms of V_5 are then derived from this construction in Section 4.

1. Geometry of the smooth quintic del Pezzo threefold

In the rest of this article, unless otherwise stated, the notations k and $\overline{k}$ refer respectively to a field of characteristic zero and a fixed algebraic closure of k. In this section, we recall without proof classical descriptions and properties of the quintic del Pezzo threefold V_5 over $\overline{k}$ and of its Hilbert scheme of lines.

1.1. **Two classical descriptions of V_5.** A quintic del Pezzo threefold V_5 over $\overline{k}$ is a smooth projective threefold whose Picard group is isomorphic to $\mathbb{Z}$, generated by an ample class H such that $-K_{V_5} = 2H$ and $H^3 = 5$. In other words, V_5 is a smooth Fano threefold of index two and degree five.

1.1.1. *Sarkisov links to a smooth quadric in $\mathbb{P}^4$.* Let us first recall the classical description due to Iskovskikh [11, Chapter II, §1.6]. Letting H be an ample class such that $-K_{V_5} = 2H$, the complete linear system $|H|$ defines a closed embedding $\Phi_{|H|} : V_5 \hookrightarrow \mathbb{P}^6_{\overline{k}}$. A general hyperplane section of V_5 contains a line ℓ whose normal bundle in V_5 is trivial, and the projection from ℓ induces a birational map $V_5 \dashrightarrow Q$ onto a smooth quadric $Q \subset \mathbb{P}^4_{\overline{k}}$. The inverse map $Q \dashrightarrow V_5$ can be described as the blow-up of Q along a rational normal cubic $C \subset Q$ contained in a smooth hyperplane section $Q_0 \simeq \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ of Q, followed by the contraction of the proper transform Q_0' of Q_0 onto the line ℓ. In sum, the projection from the line ℓ induces a Sarkisov link



where $q : \tilde{V}_5 \to V_5$ is the blow-up of V_5 along ℓ with exceptional divisor Q_0' and $q' : \tilde{V}_5 \to Q$ is the contraction onto the curve C of the proper transform Z' of the surface $Z \subset V_5$ swept out by lines in V_5 intersecting ℓ.

Since the automorphism group $\mathrm{PGL}_5(\overline{k})$ of $\mathbb{P}^4_{\overline{k}}$ acts transitively on the set of flags $C \subset Q_0 \subset Q$, it follows that over an algebraically closed field $\overline{k}$, all smooth Fano threefold of Picard number one, index two and degree five embedded into $\mathbb{P}^6_{\overline{k}}$ by their half-anticanonical complete linear system are projectively equivalent.

1.1.2. *Quasi-homogeneous space of* $\mathrm{PGL}_2(\overline{k})$. We now recall an alternative description of V_5 due to Mukai-Umemura [19] (see also [16, §5.1]). Let $M_d = \mathrm{Sym}^d(\overline{k}^2)^\vee \simeq \overline{k}[x,y]_{(d)}$ be the space of homogeneous polynomials of degree d with coefficients in $\overline{k}$. The natural action of $\mathrm{GL}_2(\overline{k})$ on M_1 induces a linear action on M_6, hence an action of $\mathrm{PGL}_2(\overline{k})$ on $\mathbb{P}(M_6) \simeq \mathbb{P}^6_{\overline{k}}$. We then have the following description:

Proposition 1. *The Fano threefold V_5 is isomorphic to the closure $\overline{\mathrm{PGL}_2(\overline{k}) \cdot [\phi]}$ of the class of the polynomial $\phi = xy(x^4+y^4) \in M_6$. Furthermore, the $\mathrm{PGL}_2(\overline{k})$-orbits on V_5 are described as follows:*

- (*i*) *The open orbit $O = \mathrm{PGL}_2(\overline{k}) \cdot [\phi]$ with stabilizer equal to the binary octahedral group,*
- (*ii*) *The 2-dimensional orbit $S_2 = \mathrm{PGL}_2(\overline{k}) \cdot [xy^5]$, which is neither open nor closed, with stabilizer equal to the diagonal torus $\mathbb{T}$.*
- (*iii*) *The 1-dimensional closed orbit $C_6 = \mathrm{PGL}_2(\overline{k}) \cdot [x^6]$ with stabilizer equal to the Borel subgroup B of upper triangular matrices.*

It follows from this description that the automorphism $\mathrm{Aut}(V_5)$ is isomorphic to $\mathrm{PGL}_2(\overline{k})$. We also observe that C_6 is a normal rational sextic curve and that the closure $\overline{S}_2 = S_2 \cup C_6$ of S_2 is a quadric section of V_5, hence an anti-canonical divisor on V_5, which coincides with the *tangential scroll* of C_6, swept out by the tangent lines to C_6 contained in V_5. It is singular along C_6, and its normalization morphism coincides with the map $\nu : \mathbb{P}(M_1) \times \mathbb{P}(M_1) \to \mathbb{P}(M_6)$, $(f_1, f_2) \mapsto f_1^5 f_2$.

1.2. **Lines on** V_5**.** The family of lines on V_5 is very well-studied [11, 7, 10]. We list below some of its properties which will be useful later on for the study of cylinders on forms of V_5.

First, by a *line* on V_5, we mean an integral curve $\ell \subset V_5$ of anticanonical degree $-K_{V_5} \cdot \ell = 2$. It thus corresponds through the half-anticanonical embedding $\Phi_{|H|} : V_5 \hookrightarrow \mathbb{P}^6_{\overline{k}}$ to a usual line in $\mathbb{P}^6_{\overline{k}}$ which is contained in the image of V_5. A general line ℓ in V_5 has trivial normal bundle, whereas there is a one-dimensional subfamily of lines with normal bundle $\mathcal{N}_{\ell/V_5} \simeq \mathcal{O}_{\mathbb{P}^1_{\overline{k}}}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_{\overline{k}}}(1)$, which we call *special lines*.

The Hilbert scheme $\mathcal{H}(V_5)$ of lines in V_5 is isomorphic to $\mathbb{P}^2_{\overline{k}}$, and the evaluation map $\upsilon : \mathcal{U} \to V_5$ from the universal family $\mathcal{U} \to \mathcal{H}(V_5)$ is a finite morphism of degree 3. There are thus precisely three lines counted with multiplicities passing through a given closed point of V_5.

The curve in $\mathcal{H}(V_5) \simeq \mathbb{P}^2_{\overline{k}}$ that parametrizes special lines in V_5 is a smooth conic C. The restriction of υ to $\mathcal{U}|_C$ is injective, and in the description of V_5 as a quasi-homogeneous space of $\mathrm{PGL}_2(\overline{k})$ given in § 1.1.2 above, $\upsilon(\mathcal{U}|_C)$ is the tangential scroll $\overline{S}_2$ to the rational normal sextic C_6, while C_6 itself coincides with the image of the intersection of $\mathcal{U}|_C$ with the ramification locus of υ. In particular, special lines on V_5 never intersect each others. Furthermore, there are three lines with trivial normal bundle through any point in $V_5 \setminus \overline{S}_2$, a line with trivial normal bundle and a special line through any point of S_2, and a unique special line through every point of C_6.

2. Forms of V_5 over non-closed fields

A *k-form of* V_5 is a smooth projective variety Y defined over k such that $Y_{\overline{k}}$ is isomorphic to V_5. In this subsection, we establish basic properties of these forms and their Hilbert schemes of lines.

2.1. **Hilbert scheme of lines on a k-form of V_5.** Let Y be a k-form of V_5, let $\mathcal{H}(Y)$ be the Hilbert scheme of irreducible curves of $-K_Y$-degree equal to 2 on Y and let $\upsilon : \mathcal{U} \to Y$ be the evaluation map from the universal family $\mathcal{U} \to \mathcal{H}(Y)$. By § 1.2, $\mathcal{H}(Y)_{\overline{k}} = \mathcal{H}(Y_{\overline{k}})$ is isomorphic to $\mathbb{P}^2_{\overline{k}}$, i.e. $\mathcal{H}(Y)$ is a k-form of $\mathbb{P}^2_{\overline{k}}$. Furthermore, since the smooth conic parametrizing special lines on $Y_{\overline{k}}$ is invariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$, it corresponds to a smooth curve $C \subset \mathcal{H}(Y)$ defined over k, and such that $C_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}}$.

Lemma 2. *Let Y be a k-form of V_5.*

a) The Hilbert scheme $\mathcal{H}(Y)$ of lines on Y is isomorphic to $\mathbb{P}^2_k$, in particular Y always contains lines defined over k with trivial normal bundles.

b) The following assertions are equivalent:

(i) Y contains a special line defined over k,
(ii) The conic C has a k-rational point,
(iii) The surface $\upsilon(\mathcal{U}|_C)$ has a k-rational point,
(iv) The image of the intersection of $\mathcal{U}|_C$ with the ramification locus of υ has a k-rational point.

Proof. Since $\mathcal{H}(Y)$ is a k-form of $\mathbb{P}^2_{\overline{k}}$, to prove the first assertion it is enough to show that $\mathcal{H}(Y)$ has a k-rational point. Since $C_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}}$, there exists a quadratic extension $k \subset k'$ such that $C(k')$ is nonempty, so that in particular $\mathcal{H}(Y)_{k'} \simeq \mathbb{P}^2_{k'}$. Let p be a k'-rational point of $C_{k'}$. If p is invariant under the action of the Galois group $\mathrm{Gal}(k'/k)$, then it corresponds to a k-rational point of C, hence of $\mathcal{H}(Y)$, and we are done. Otherwise, its Galois conjugate $\overline{p}$ is a k'-rational point of $C_{k'}$ distinct from p, and then the tangent lines $T_pC_{k'}$ and $T_{\overline{p}}C_{k'}$ to $C_{k'}$ at p and $\overline{p}$ respectively intersect each other at unique point. The latter is thus $\mathrm{Gal}(k'/k)$-invariant, hence corresponds to a k-rational point of $\mathcal{H}(Y) \setminus C$. The existence of lines with trivial normal bundles defined over k then follows from the description of $\mathcal{H}(V_5)$ given in § 1.2.

The second assertion is an immediate consequence of the facts that special lines in $Y_{\overline{k}}$ are in one-to-one correspondence with closed points of the image D of the intersection of $(\mathcal{U}|_C)_{\overline{k}}$ with the ramification locus of υ, and that $\upsilon(\mathcal{U}|_C)_{\overline{k}}$ coincides with the surface swept out by the tangent lines to D. □

Corollary 3. *A k-form Y of V_5 is k-rational and the natural map $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_{\overline{k}})$ is an isomorphism.*

Proof. By Lemma 2 a), Y contains a line $\ell \simeq \mathbb{P}^1_k$ with trivial normal bundle. The surface Z in $Y_{\overline{k}}$ swept out by lines intersecting $\ell_{\overline{k}}$ is defined over k. As in § 1.1.1, the composition of the blow-up $q : \tilde{Y} \to Y$ of ℓ with exceptional divisor $Q'_0 \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$ followed by the contraction $q' : \tilde{Y} \to Q$ of the proper transform of Z' of Z yields a birational map $Y \dashrightarrow Q$ defined over k onto a smooth quadric $Q \subset \mathbb{P}^4_k$, which maps Q'_0 onto a hyperplane section Q_0 of Q. Since Q contains k-rational points, it is k-rational. The natural map $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_{\overline{k}})$ is an isomorphism if and only $-K_Y$ is divisible in $\mathrm{Pic}(Y)$. But since $-K_Q \sim 3Q_0$ and $q'^*Q_0 = Q'_0 + Z'$, we deduce from the ramification formula for q and q' that

$$-K_Y \sim q_*(-K_{\tilde{Y}}) \sim q_*(-q'^*K_Q - Z') \sim q_*(3Q'_0 + 2Z') \sim 2Z.$$

□

2.2. **Varieties of trisecant lines to Veronese surfaces in $\mathbb{P}^4$.** In this subsection, we review a third classical construction of V_5 as the variety of trisecant line to the Veronese surface in $\mathbb{P}^4_k$ which, when performed over k gives rise, depending on the choices made, to nontrivial k-forms of V_5 .

Let V be a k-vector space of dimension 3, let $f \in \mathrm{Sym}^2 V^*$ be a homogeneous form defining a smooth conic $Q \subset \mathbb{P}(V)$, and let $W = \mathrm{Sym}^2(V^*/\langle f \rangle)$. Recall that a closed subscheme $Z \subset \mathbb{P}(V^*)$ defined over k is called *apolar* to Q if the class of $[f] \in \mathbb{P}(\mathrm{Sym}^2 V^*)$ lies in the linear span of the image of Z by the second Veronese embedding $v_2 : \mathbb{P}(V^*) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 V^*)$. When $Z = \{[\ell_1], [\ell_2], [\ell_3]\} \subset \mathbb{P}(V^*)$ consists of three distinct k-rational points, this says equivalently that there are scalars $\lambda_i \in k$, $i = 1, 2, 3$, such that $f = \lambda_1\ell_1^2 + \lambda_2\ell_2^2 + \lambda_3\ell_3^2$. We denote by $\mathrm{VSP}(f)$ the variety of closed subschemes of length 3 of $\mathbb{P}(V^*)$ which are apolar to Q.

Let $\pi_{[f]} : \mathbb{P}(\mathrm{Sym}^2 V^*) \dashrightarrow \mathbb{P}(W)$ be the projection from $[f]$. Since Q is smooth, the composition $\pi_{[f]} \circ v_2$ is a closed embedding of $\mathbb{P}(V^*)$, whose image is a Veronese surface X in $\mathbb{P}(W)$. Since X is defined over k, the closed subscheme Y of the Grassmannian $G(1,\mathbb{P}(W_{\overline{k}}))$ of lines in $\mathbb{P}(W_{\overline{k}})$ consisting of trisecant lines to $X_{\overline{k}}$, that is, lines ℓ in $\mathbb{P}(W_{\overline{k}})$ such that $\ell\cdot X_{\overline{k}}$ is a closed subscheme of length 3 of ℓ, is defined over k, and we obtain an identification between $\mathrm{VSP}(f)$ and Y.

Proposition 4. *(see [9, §2.3]) With the notation above, $Y=\mathrm{VSP}(f)$ is a k-form of V_5. Furthermore, the stratification of $Y_{\overline{k}}$ by the types of the corresponding closed subschemes of length 3 of $\mathbb{P}(V^*)$ is related to that of V_5 as a quasi-homogenous space of $\mathrm{PGL}_2(\overline{k})$ given in § 1.1.2 as follows:*

- (i) *Points of the open orbit O correspond to reduced subschemes $\{[\ell_1],[\ell_2],[\ell_3]\}$ apolar to $Q_{\overline{k}}$ such that none of the $[\ell_i]$ belongs to the dual conic $Q^*_{\overline{k}}\subset\mathbb{P}(V^*_{\overline{k}})$ of $Q_{\overline{k}}$.*
- (ii) *Points in the 2-dimensional orbit S_2 correspond to non-reduced subschemes $\{2[\ell_1],[\ell_2]\}$ where $[\ell_1]$ belongs to $Q^*_{\overline{k}}$ and $[\ell_2]$ is a point of the tangent line $T_{[\ell_1]}Q^*_{\overline{k}}$ to $Q^*_{\overline{k}}$ at $[\ell_1]$ distinct from $[\ell_1]$.*
- (iii) *Points in the 1-dimensional orbit C_6 correspond to non-reduced subschemes $\{3[\ell]\}$ where $[\ell]$ is a point of $Q^*_{\overline{k}}$.*

The Hilbert scheme $\mathcal{H}(Y)$ of lines on a k-form $Y=\mathrm{VSP}(f)\subset G(1,\mathbb{P}(W))$ can be explicitly described as follows. Viewing $G(1,\mathbb{P}(W))$ as a closed subscheme of $\mathbb{P}(\Lambda^2W)$ via the Plücker embedding, $\mathcal{H}(Y)$ is a closed subscheme of the Hilbert scheme $\mathcal{H}(G(1,\mathbb{P}(W)))$ of lines on $G(1,\mathbb{P}(W))$. The latter is isomorphic to the flag variety $\mathcal{F}(0,2,\mathbb{P}(W))\subset\mathbb{P}(W)\times G(2,\mathbb{P}(W))$ of pairs (x,P) consisting of a point $x\in\mathbb{P}(W)$ and a 2-dimensional linear space $P\subset\mathbb{P}(W)$ containing it.

Proposition 5. *(see e.g. [10, §1.2]) Let $f\in\mathrm{Sym}^2V^*$ be a homogeneous form defining a smooth conic $Q\subset\mathbb{P}(V)$, let $W=\mathrm{Sym}^2(V^*/\langle f\rangle)$, and let $\mathrm{VSP}(f)\subset G(1,\mathbb{P}(W))$ be the variety of trisecant lines to the Veronese surface $X=\pi_{[f]}\circ v_2(\mathbb{P}(V^*))$.*

- (i) *The projection $\mathrm{pr}_1:\mathcal{F}(0,2,\mathbb{P}(W))\to\mathbb{P}(W)$ restricts to a closed embedding $\mathcal{H}(\mathrm{VSP}(f))\hookrightarrow\mathbb{P}(W)$ whose image is equal to $X\simeq\mathbb{P}(V^*)$.*
- (ii) *The image of the smooth conic $C\subset\mathcal{H}(\mathrm{VSP}(f))$ parametrizing special lines in $\mathrm{VSP}(f)$ coincides with the conic $Q^*\subset\mathbb{P}(V^*)$ dual to $Q\subset\mathbb{P}(V)$.*
- (iii) *For every line ℓ in $\mathrm{VSP}(f)_{\overline{k}}$ defined by a point $x\in\mathbb{P}(V^*_{\overline{k}})$, the set of lines in $\mathrm{VSP}(f)_{\overline{k}}$ which intersect ℓ is parametrized by the conjugate line of x with respect to $Q^*_{\overline{k}}$.*

As a consequence, we obtain the following characterization of which k-forms $Y=\mathrm{VSP}(f)$ of V_5 contain special lines defined over k.

Corollary 6. *The k-form $Y=\mathrm{VSP}(f)$ of V_5 contains a special line defined over k if and only if the conic $Q=V(f)\subset\mathbb{P}(V)$ has a k-rational point.*

Proof. By combining Lemma 2 b) and Proposition 5 (ii), we deduce that Y contains a special line defined over k if and only if the conic $Q^*\subset\mathbb{P}(V^*)$ dual to Q has a k-rational point, hence if and only if Q has a k-rational point. □

Example 7. Let $k=\mathbb{C}(s,t)$ with s,t algebraically independent over $\mathbb{C}$, and let $f=x^2+sy^2+tz^2\in k[x,y,z]$. Since the conic $Q=V(f)\subset\mathbb{P}^2_k=\mathrm{Proj}_k(k[x,y,z])$ has no k-rational point, the variety $Y=\mathrm{VSP}(f)$ is a k-form of V_5 which does not contain any special line defined over k.

3. Double projection from a rational point

Let $V_5\hookrightarrow\mathbb{P}^6_k$ be embedded by its half-anticanonical complete linear system. For every closed point $y\in V_5$, the linear system $|\mathcal{O}_{V_5}(1)\otimes\mathfrak{m}_y^2|$ of hyperplane sections of V_5 which are singular at y defines a rational map $\psi_y:V_5\dashrightarrow\mathbb{P}^2_k$ called the *double projection from y*, whose description is classical knowledge in the birational geometry of threefolds [8, 15]. In this section, inspired by [24], we give an explicit "reverse construction" of these maps, valid over any field k of characteristic zero, formulated in terms of Sarkisov links performed from certain locally trivial $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ associated to standard birational quadratic involutions of $\mathbb{P}^2$.

3.1. **Recollection on standard quadratic involutions of $\mathbb{P}^2$.** Let $\sigma:\mathbb{P}^2_k\dashrightarrow\mathbb{P}^2_k$ be a birational quadratic involution of $\mathbb{P}^2_k$ and let $\Gamma_\sigma\subset\mathbb{P}^2_k\times\mathbb{P}^2_k$ be its graph. Via the two projections $\tau=\mathrm{pr}_1:\Gamma_\sigma\to\mathbb{P}^2_k$ and $\tau'=\mathrm{pr}_2:\Gamma_\sigma\to\mathbb{P}^2_k$, Γ_σ is canonically identified with the blow-up of the scheme-theoretic base loci $Z=\mathrm{Bs}(\sigma)$ and $Z'=\mathrm{Bs}(\sigma^{-1})$ respectively. We denote by $e=\tau^*Z$ and $e'=\tau'^*Z'$ the scheme-theoretic inverse images of Z and Z' respectively in Γ_σ. It is well-known that Z and Z' are local complete intersection 0-dimensional closed sub-scheme of length 3 of $\mathbb{P}^2_k$ with one of the following possible structure:

(Type I) Z (resp. Z') is smooth and its base extension to $\overline{k}$ consists of three non-collinear points p_1, p_2, p_3 (resp. p'_1, p'_2, p'_3) whose union is defined over k. The surface Γ_σ is smooth, $e_{\overline{k}} = e_{p_1} + e_{p_2} + e_{p_3}$, $e'_{\overline{k}} = e'_{p'_1} + e'_{p'_2} + e'_{p'_3}$ where the e_{p_i} and $e'_{p'_i}$ are (-1)-curves.

(Type II) Z (resp. Z') consists of the disjoint union $(p_1, \mathbf{p}_2)$ (resp. $(p'_1, \mathbf{p}'_2)$) of a smooth k-rational point p_1 (resp. p'_1) and a 0-dimensional sub-scheme $\mathbf{p}_2$ (resp. $\mathbf{p}'_2$) of length 2 locally isomorphic to $V(x, y^2)$ and supported at a k-rational point p_2 (resp. p'_2). We have $e = e_{p_1} + 2e_{p_2}$, $e' = e'_{p'_1} + 2e'_{p'_2}$ where e_{p_1} and e_{p_2} (resp. $e'_{p'_1}$ and $e'_{p'_2}$) are smooth k-rational curves with self-intersections -1 and $-\frac{1}{2}$ respectively, and Γ_σ has a unique A_1-singularity at the k-rational point $e_{p_2} \cap e'_{p'_2}$.

(Type III) Z (resp. Z') is supported on a unique k-rational point p (resp. p') and is locally isomorphic to $V(y^3, x - y^2)$. We have $e = 3e_p$ and $e' = 3e'_{p'}$ where e_p and $e'_{p'}$ are smooth k-rational curves with self-intersection $-\frac{1}{3}$, and Γ_σ has a unique A_2-singularity at the k-rational point $e_p \cap e'_{p'}$.

The following lemma whose proof is left to the reader records for later use some additional basic properties of the surfaces Γ_σ.

Lemma 8. *With the notation above, the following hold:*

a) The canonical divisor K_{Γ_σ} is linearly equivalent to $-e - e'$.

b) The pull-back by $\tau : \Gamma_\sigma \to \mathbb{P}^2_k$ (resp. $\tau' : \Gamma_\sigma \to \mathbb{P}^2_k$) of a general line $\ell \simeq \mathbb{P}^1_k$ in $\mathbb{P}^2_k$ is $\mathbb{Q}$-linearly equivalent to $\frac{1}{3}(2e + e')$ (resp. $\frac{1}{3}(e + 2e')$).

3.2. **Locally trivial $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ associated to quadratic involutions.** Let $\sigma : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ be a birational quadratic involution of $\mathbb{P}^2_k$ with graph $\Gamma_\sigma \subset \mathbb{P}^2_k \times \mathbb{P}^2_k$, and let $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2_k}$ be the ideal sheaf of its scheme-theoretic base locus Z.

Lemma 9. *There exists a locally free sheaf $\mathcal{E}$ of rank 2 on $\mathbb{P}^2_k$ and an exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^2_k}(-1) \xrightarrow{s} \mathcal{E} \longrightarrow \mathcal{I}_Z \to 0. \tag{3.1}$$

Proof. Since Z is a local complete intersection of codimension 2 in $\mathbb{P}^2_k$, the existence of $\mathcal{E}$ with the required properties follows from Serre correspondence [23]. More precisely, the local-to-global spectral sequence

$$E_2^{p,q} = H^P(\mathbb{P}^2_k, \mathcal{E}xt^q(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}^2_k}(-1)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}^2_k}(-1))$$

degenerates to a long exact sequence

$$0 \to H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-1)) \to \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}^2_k}(-1)) \to H^0(Z, \det \mathcal{N}_{Z/\mathbb{P}^2_k} \otimes \mathcal{O}_Z(-1)) \to H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-1)).$$

Since $H^i(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-1)) = 0$ for $i = 1, 2$ and Z is 0-dimensional, this sequence provides an isomorphism

$$\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}^2_k}(-1)) \simeq H^0(Z, \det \mathcal{N}_{Z/\mathbb{P}^2_k} \otimes \mathcal{O}_Z(-1)) \simeq H^0(Z, \mathcal{O}_Z).$$

The extension corresponding via this isomorphism to the constant section $1 \in H^0(Z, \mathcal{O}_Z)$ has the desired property. □

Let $\pi : \mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}^{\cdot} \mathcal{E}) \to \mathbb{P}^2_k$ be the locally trivial $\mathbb{P}^1$-bundle associated with the locally free sheaf $\mathcal{E}$ of rank 2 as in Lemma 9. Since $\det \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2_k}(-1)$, the canonical sheaf $\omega_{\mathbb{P}(\mathcal{E})}$ of $\mathbb{P}(\mathcal{E})$ is isomorphic to

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2) \otimes \pi^* \det \mathcal{E} \otimes \pi^* \omega_{\mathbb{P}^2_k} \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2_k}(-4). \tag{3.2}$$

The surjection $\mathcal{E} \to \mathcal{I}_Z \to 0$ defines a closed embedding $\mathbb{P}(\mathcal{I}_Z) = \mathrm{Proj}(\mathrm{Sym}^{\cdot} \mathcal{I}_Z) \hookrightarrow \mathbb{P}(\mathcal{E})$. Since Z is a local complete intersection, the canonical homomorphism of graded $\mathcal{O}_{\mathbb{P}^2_k}$-algebras $\mathrm{Sym}^{\cdot} \mathcal{I}_Z \to \mathcal{R}(\mathcal{I}_Z) = \bigoplus_{n \geq 0} \mathcal{I}_Z \cdot t^n$ is an isomorphism [17, Théorème 1], and it follows that the restriction $\pi : \mathbb{P}(\mathcal{I}_Z) \to \mathbb{P}^2_k$ is isomorphic to the blow-up $\tau : \Gamma_\sigma \to \mathbb{P}^2_k$ of Z. We can thus identify from now on Γ_σ with the closed sub-scheme $\mathbb{P}(\mathcal{I}_Z)$ of $\mathbb{P}(\mathcal{E})$. The composition of $\pi^* s : \pi^* \mathcal{O}_{\mathbb{P}^1_k}(-1) \to \pi^* \mathcal{E}$ with the canonical surjection $\pi^* \mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ defines a global section

$$\overline{s} \in \mathrm{Hom}_{\mathbb{P}(\mathcal{E})}(\pi^* \mathcal{O}_{\mathbb{P}^1_k}(-1), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2_k}(1))$$

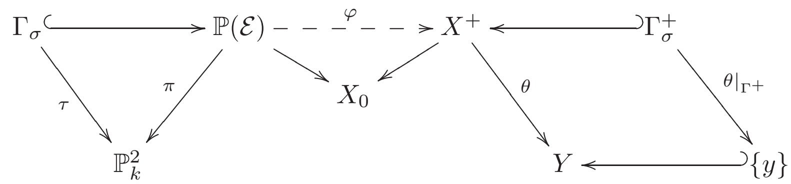
whose zero locus $V(\overline{s})$ coincides with Γ_σ.

Letting ξ and A be the classes in the divisor class group of $\mathbb{P}(\mathcal{E})$ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and of the inverse image of a general line in $\mathbb{P}^2_k$ by π respectively, we have $\Gamma_\sigma = V(\overline{s}) \sim \xi + A$. On the other hand, it follows from (3.2) that the canonical divisor $K_{\mathbb{P}(\mathcal{E})}$ of $\mathbb{P}(\mathcal{E})$ is linearly equivalent to $-2(\Gamma_\sigma + A) \sim -2(\xi + 2A)$. Since $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 3$ by construction, we derive the following numerical information:

$$K^3_{\mathbb{P}(\mathcal{E})} = -32, \qquad K^2_{\mathbb{P}(\mathcal{E})} \cdot \Gamma_\sigma = 4, \qquad K_{\mathbb{P}(\mathcal{E})} \cdot \Gamma^2_\sigma = 2, \qquad \Gamma^3_\sigma = -2. \tag{3.3}$$

3.3. Construction of Sarkisov links.

Proposition 10. *Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2_k$ be the $\mathbb{P}^1$-bundle associated to a quadratic involution $\sigma : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ with graph $\Gamma_\sigma \subset \mathbb{P}(\mathcal{E})$ as in §3.2. Then there exists a k-form Y of V_5 and a Sarkisov link*



where:

(*i*) *$\varphi : \mathbb{P}(\mathcal{E}) \dashrightarrow X^+$ is a flop whose flopping locus coincides with the support of $e' \subset \Gamma_\sigma$,*
(*ii*) *$\Gamma^+_\sigma \simeq \mathbb{P}^2_k$ is the proper transform of Γ_σ and $\theta|_{\Gamma_\sigma} : \Gamma_\sigma \dashrightarrow \Gamma^+_\sigma$ is the contraction of e',*
(*iii*) *$\theta : X^+ \to Y$ is the divisorial contraction of Γ^+_σ to a smooth k-rational point $y \in Y$,*
(*iv*) *The support of the image in Y of the flopped locus $e^+ \subset X^+$ of φ coincides with the union of the lines in $Y_{\overline{k}}$ passing through y.*

Proof. We denote $\mathbb{P}(\mathcal{E})$ and $\Gamma_\sigma \subset \mathbb{P}(\mathcal{E})$ simply by X and Γ. Since $-K_X \sim 2\xi + 4A \sim 2\Gamma + 2A$ we see that $-K_X$ is nef and that any irreducible curve $C \subset X_{\overline{k}}$ such that $-K_{X_{\overline{k}}} \cdot C \leq 0$ is contained in $\Gamma_{\overline{k}}$. By the adjunction formula and Lemma 8 a), we have

$$\Gamma^2 = -K_\Gamma - 2A \cdot \Gamma \sim_{\mathbb{Q}} \frac{1}{3}(-e + e') \quad \text{and} \quad \Gamma^2 - K_\Gamma \sim_{\mathbb{Q}} \frac{2}{3}(e + 2e'),$$

which implies by adjunction again that

$$-K_{X_{\overline{k}}} \cdot C = (\Gamma^2_{\overline{k}} - K_{\Gamma_{\overline{k}}}) \cdot C = \frac{2}{3}(e_{\overline{k}} + 2e'_{\overline{k}}) \cdot C.$$

Since $e + 2e'$ is τ-ample and τ'-numerically trivial by virtue of Lemma 8 b), we conclude that the irreducible curves $C \subset X_{\overline{k}}$ such that $-K_{X_{\overline{k}}} \cdot C = 0$ are precisely the irreducible components of the exceptional locus $e'_{\overline{k}}$ of $\tau'_{\overline{k}} : \Gamma_{\overline{k}} \to \mathbb{P}^2_{\overline{k}}$ (see § 3.1 for the notation).

Let $\varphi : X \dashrightarrow X^+$ be the flop of the union of the irreducible components of $e'_{\overline{k}}$. Since the union of these components is defined over k, so is the union e^+ of the flopped curves of φ, and so X^+ is a smooth threefold defined over k and φ is a birational map defined over k restricting to an isomorphism between $X \setminus e'$ and $X^+ \setminus e^+$. Let Γ^+ and A^+ be the proper transforms in X^+ of Γ and A respectively. By construction, the restriction $\varphi|_\Gamma : \Gamma \dashrightarrow \Gamma^+$ coincide with $\tau' : \Gamma \to \mathbb{P}^2_k$. Since $K_{X^+} \sim -2(\Gamma^+ + A^+)$, we deduce from the adjunction formula that

$$-(\Gamma^+)^2 = K_{\Gamma^+} + 2A^+ \cdot \Gamma^+ = \tau'_*(K_\Gamma + 2A \cdot \Gamma) = \frac{1}{3}\tau'_* e,$$

which is linearly equivalent to a line $\ell \simeq \mathbb{P}^1_k$ in $\Gamma^+ \simeq \mathbb{P}^2_k$. The normal bundle $\mathcal{N}_{\Gamma^+/X^+}$ of Γ^+ in X^+ is thus isomorphic to $\mathcal{O}_{\mathbb{P}^2_k}(-1)$.

The divisor class group of X^+ is freely generated by Γ^+ and A^+. The Mori cone $\mathrm{NE}(X^+)$ is spanned by two extremal rays: one R_1 corresponding to the flopped curves of φ and a second one R_2 which is K_{X^+}-negative. Since $\Gamma^+ \cdot C \geq 0$ for any irreducible curve not contained in Γ^+, including thus the irreducible components of e^+, whereas $\Gamma^+ \cdot \ell = -1$ for any line $\ell \simeq \mathbb{P}^1_k$ in $\Gamma^+ \simeq \mathbb{P}^2_k$, it follows that R_2 is generated by the class of ℓ. The extremal contraction associated to R_2 is thus the divisorial contraction $\theta : X^+ \to Y$ of $\Gamma^+ \simeq \mathbb{P}^2_k$ to a smooth k-rational point p of a smooth projective threefold Y.

Since $-K_{X^+} = 2(\Gamma^+ + A^+)$, we conclude that the image H of A^+ by θ is an ample divisor on Y generating the divisor class group of Y and such that $-K_Y = 2H$. Furthermore, we have

$$\begin{aligned} K_Y^3 &= \theta^*(K_Y)^3 = (K_{X^+} - 2\Gamma^+)^3 \\ &= K_{X^+}^3 - 6K_{X^+}^2 \cdot \Gamma^+ + 12K_{X^+} \cdot (\Gamma^+)^2 - 8(\Gamma^+)^3 \\ &= K_X^3 - 6K_X^2 \cdot \Gamma + 12K_X \cdot \Gamma - 8 \end{aligned}$$

so that $K_Y^3 = -40$ by (3.3). Altogether, this shows $Y_{\overline{k}}$ is a smooth Fano threefold of Picard number 1, index 2 and degree $d = H^3 = 5$, hence is isomorphic to V_5.

The fact that the support of the image in Y of e^+ coincides with the union of the lines in $Y_{\overline{k}}$ passing through p is clear by construction. □

By construction, the proper transform by the reverse composition $\psi_y = \pi \circ \varphi^{-1} \circ \theta^{-1} : Y \dashrightarrow \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^2$ of the complete linear system of lines in $\mathbb{P}_k^2$ consists of divisors H on Y singular at y and such that $-K_Y = 2H$. This shows that $\psi_{y,\overline{k}} : Y_{\overline{k}} \simeq V_5 \dashrightarrow \mathbb{P}_{\overline{k}}^2$ coincides with the double projection from the point y. Conversely, given any k-form Y of V_5, Corollary 3 ensures that $-K_Y$ is divisible, equal to $2H$ for some ample divisor H on Y. So given any k-rational point $y \in Y$, the double projection $\psi_y : Y \dashrightarrow \mathbb{P}_k^2$ from y is defined over k, given by the linear system $|\mathcal{O}_Y(H) \otimes \mathfrak{m}_y^2|$, and it coincides with the composition $\pi \circ \varphi^{-1} \circ \theta^{-1} : Y \dashrightarrow \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^2$ for a suitable quadratic birational involution σ of $\mathbb{P}_k^2$.

4. Application : cylinders in forms of the quintic del Pezzo threefold

Theorem 11. *Let Y be a k-form of V_5, let $y \in Y$ be a k-rational point and let C be the union of the lines in $Y_{\overline{k}}$ passing through y. Then $Y \setminus C$ has the structure of a Zariski locally trivial $\mathbb{A}^1$-bundle $\rho : Y \setminus C \to \mathbb{P}_k^2 \setminus Z$ over the complement of a closed sub-scheme $Z \subset \mathbb{P}_k^2$ of length 3 with as many irreducible geometric components as C.*

Proof. Indeed, the birational map $\xi = \theta \circ \varphi : \mathbb{P}(\mathcal{E}) \dashrightarrow Y$ constructed in Proposition 10 restricts to an isomorphism between $Y \setminus C$ and the complement of the proper transform Γ in $\mathbb{P}(\mathcal{E})$ of the exceptional divisor of the blow-up of Y at y. On the other hand, it follows from the construction of $\mathcal{E}$ in § 3.2 that Γ is the graph of a quadratic birational involution σ of $\mathbb{P}_k^2$ and that the restriction of $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}_k^2$ to the complement of Γ is a locally trivial $\mathbb{A}^1$-bundle over the complement $\mathbb{P}_k^2 \setminus Z$ of the base locus Z of σ. □

Theorem 12. *Every k-form Y of V_5 contains a Zariski open $\mathbb{A}_k^2$-cylinder. Furthermore, Y contains $\mathbb{A}_k^3$ as a Zariski open subset if and only if it contains a special line defined over k.*

Proof. By Lemma 2, Y contains a line $\ell \simeq \mathbb{P}_k^1$ with trivial normal bundle. Projecting from ℓ as in the proof of Corollary 3, we obtain a birational map $Y \dashrightarrow Q$ onto a smooth quadric $Q \subset \mathbb{P}_k^4$ defined over k, which restricts to an isomorphism between the complement of the surface $Z \subset Y$ defined over k swept out by the lines in $Y_{\overline{k}}$ intersecting $\ell_{\overline{k}}$ and the complement of a hyperplane section $Q_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$ of Q. The complement $Q \setminus Q_0$ is isomorphic to the smooth affine quadric $\tilde{Q}_0 = \{xv - yu = 1\} \subset \mathbb{A}_k^4$, which contains for instance the principal open $\mathbb{A}_k^2$-cylinder $\tilde{Q}_{0,x} \simeq \mathrm{Spec}(k[x^{\pm 1}][y, u])$.

By Lemma 2 b), Y contains a special line if and only it has a k-rational point p through which there exists at most two lines in $Y_{\overline{k}}$. Letting C be the union of these lines, the previous theorem implies that $\rho : Y \setminus C \to \mathbb{P}_k^2 \setminus Z$ is a Zariski locally trivial $\mathbb{A}^1$-bundle over the complement of a 0-dimensional closed subset Z defined over k whose support consists of at most two points. Letting $L \simeq \mathbb{P}_k^1$ be any line in $\mathbb{P}_k^2$ containing the support of Z, the restriction of ρ over $\mathbb{P}_k^2 \setminus L \simeq \mathbb{A}_k^2$ is then a trivial $\mathbb{A}^1$-bundle, so that $Y \setminus \rho^{-1}(L)$ is a Zariski open subset of Y isomorphic to $\mathbb{A}_k^3$.

Conversely, suppose that Y contains a Zariski open subset $U \simeq \mathbb{A}_k^3$. Since U is affine, $D = Y \setminus U$ has pure codimension 1 in Y. Since $Y_{\overline{k}} \simeq V_5$ and $Y_{\overline{k}} \setminus D_{\overline{k}} \simeq U_{\overline{k}} \simeq \mathbb{A}_{\overline{k}}^3$, according to [7, Corollary 1.2, b] and [8], we have the following alternative for the divisor $D_{\overline{k}}$ in $Y_{\overline{k}}$:

1) $D_{\overline{k}}$ is a normal del Pezzo surface of degree 5 with a unique singular point q of type A_4,

2) $D_{\overline{k}}$ is a non-normal del Pezzo surface of degree 5 whose singular locus is a special line ℓ in $Y_{\overline{k}}$, and $D_{\overline{k}}$ is swept out by lines in V_5 which intersect ℓ.

In the first case, there exists a unique special line ℓ in $Y_{\overline{k}}$ passing through q, which is then automatically contained in $D_{\overline{k}}$. Since D is defined over k and q is the unique singular point of $D_{\overline{k}}$, q corresponds to a

k-rational point of D, so that ℓ is actually a special line in Y defined over k. In the second case, since ℓ is the singular locus of $D_{\overline{k}}$, it is invariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$, hence is defined over k. So in both cases, Y contains a special line defined over k. □

Corollary 13. *Every form of V_5 defined over a C_1-field k contains $\mathbb{A}^3_k$ as a Zariski open subset.*

Proof. This follows from Lemma 2 b) and the previous theorem since every conic over a C_1-field has a rational point. □

Example 14. The $\mathbb{C}(s,t)$-form $Y = \mathrm{VSP}(f)$ of V_5 associated to the quadratic form $f = x^2 + sy^2 + tz^2 \in \mathbb{C}(s,t)[x,y,z]$ considered in Example 7 does not contain any special line defined over $\mathbb{C}(s,t)$, hence does not contain $\mathbb{A}^3_{\mathbb{C}(s,t)}$ as a Zariski open subset.

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