

**A q -SUMMATION AND THE ORTHOGONALITY
RELATIONS FOR THE q -HAHN POLYNOMIALS AND THE
BIG q -JACOBI POLYNOMIALS**

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ABSTRACT. Using a general q -summation formula, we derive a generating function for the q -Hahn polynomials, which is used to give a complete proof of the orthogonality relation for the q -Hahn polynomials. A new proof of the orthogonality relation for the big q -Jacobi polynomials is also given. A simple evaluation of the Nassrallah-Rahman integral is derived by using this summation formula. A new q -beta integral formula is established, which includes the Nassrallah-Rahman integral as a special case. The q -summation formula also allows us to recover several strange q -series identities.

1. INTRODUCTION

Throughout this paper we assume that q is a complex number such that $|q| < 1$. For any complex number a , the q -shifted factorials $(a; q)_n$ are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \dots, \text{ or } \infty.$$

For convenience, we also adopt the following compact notation for the multiple q -shifted factorial:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n.$$

The basic hypergeometric series or q -hypergeometric series ${}_r\phi_s(\cdot)$ are defined as

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} \left((-1)^n q^{n(n-1)/2} \right)^{1+s-r} z^n.$$

For any function $f(x)$, the q -derivative of $f(x)$ with respect to x , is defined as

$$\mathcal{D}_{q,x}\{f(x)\} = \frac{f(x) - f(qx)}{x},$$

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and we further define $\mathcal{D}_{q,x}^0\{f\} = f$, and for $n \geq 1$, $\mathcal{D}_{q,x}^n\{f\} = \mathcal{D}_{q,x}\{\mathcal{D}_{q,x}^{n-1}\{f\}\}$.

Using some basic properties of the q -derivative, we [13] proved the following q -expansion formula.

Theorem 1.1. (Liu) *If $f(a)$ is an analytic function of a near $a = 0$, then, we have*

$$f(a) = \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha q/a; q)_n a^n}{(q, a; q)_n} [\mathcal{D}_{q,x}^n\{f(x)(x; q)_{n-1}\}]_{x=\alpha q}.$$

This theorem tells us that if $f(a)$ is analytic at $a = 0$, then, it can be expanded uniquely in terms of $(\alpha q/a; q)_n a^n / (a; q)_n$. Thus, if $f(a)$ has two series expansions in terms of $(\alpha q/a; q)_n a^n / (a; q)_n$, then, the corresponding coefficients of these two series must be equal. This allows us to derive some combinatorial identities by using the method of equating the coefficients.

Using Theorem 1.1, we [18] established the following expansion theorem.

Theorem 1.2. (Liu) *If $f(x)$ is an analytic function near $x = 0$, then, under suitable convergence conditions, we have*

$$\begin{aligned} & \frac{(\alpha q, \alpha ab/q; q)_{\infty}}{(\alpha a, \alpha b; q)_{\infty}} f(\alpha a) \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n}{(1 - \alpha)(q, \alpha a; q)_n} \sum_{k=0}^n \frac{(q^{-n}, \alpha q^n; q)_k q^k}{(q, \alpha b; q)_k} f(\alpha q^{k+1}). \end{aligned}$$

By choosing $f(x) = \prod_{j=1}^m \frac{(b_j x/q; q)_{\infty}}{(c_j x/q; q)_{\infty}}$ in Theorem 1.2, we [19] obtained the following general q -summation formula.

Theorem 1.3. (Liu) *If $\max\{|\alpha a|, |\alpha b|, |\alpha b_1|, |\alpha a c_1/q|, \dots, |\alpha b_m|, |\alpha a c_m/q|\} < 1$, then, we have*

$$\begin{aligned} & \frac{(\alpha q, \alpha ab/q; q)_{\infty}}{(\alpha a, \alpha b; q)_{\infty}} \prod_{j=1}^m \frac{(\alpha ab_j/q, \alpha c_j; q)_{\infty}}{(\alpha a c_j/q, \alpha b_j; q)_{\infty}} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n}{(1 - \alpha)(q, \alpha a; q)_n} {}_{m+2}\phi_{m+1} \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha c_1, \dots, \alpha c_m \\ \alpha b, \alpha b_1, \dots, \alpha b_m \end{matrix}; q, q \right). \end{aligned}$$

This summation formula implies many nontrivial results in q -series as special cases. For example, by setting $b_1 = c$ and $c_1 = bc/q$ in Theorem 1.3 and then using the q -Pfaff-Saalschütz summation formula, we can obtain Rogers' ${}_6\phi_5$ summation formula, which is a q -analogue of Dougall's ${}_5F_4$ summation formula.

Theorem 1.4. *For $|\alpha abc/q^2| < 1$, we have*

$${}_6\phi_5 \left(\begin{matrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, q/a, q/b, q/c \\ \sqrt{\alpha}, -\sqrt{\alpha}, \alpha a, \alpha b, \alpha c \end{matrix}; q, \frac{\alpha abc}{q^2} \right) = \frac{(\alpha q, \alpha ab/q, \alpha ac/q, \alpha bc/q; q)_{\infty}}{(\alpha a, \alpha b, \alpha c, \alpha abc/q^2; q)_{\infty}}.$$

In [19], Theorem 1.3 has been used to derive several important results in number theory, such as a general formula for sums of any number of squares.

It is obvious that [18, Theorem 1.2] is the case $m = 2$ of Theorem 1.3, which has been used to give a surprising proof of the orthogonality relation for the Askey-Wilson polynomials.

In this paper we continue to discuss some amazing application of Theorem 1.3. In particular, this theorem is used to provide new proofs of the orthogonality relations for the q -Hahn polynomials and the q -big Jacobi polynomials, and can also be used to recover some strange q -series identities.

This paper is organized as follows. In Section 2, we will use Theorem 1.1 to prove the following expansion theorem for the two-variable analytic functions. This expansion theorem implies that if a function $f(a, b)$ can be expanded in terms of

$$\frac{(\alpha q/a; q)_n (\alpha q/b; q)_m a^n b^m}{(q, a; q)_n (q, b; q)_m},$$

then, this expansion is unique. This fact enables us to use the method of equating the coefficients to derive some identities.

Theorem 1.5. *If $f(a, b)$ is a two-variable analytic function at $(0, 0) \in \mathbb{C}^2$, then, there exists a unique sequence $\{c_{n,m}\}$ independent of a and b such that*

$$f(a, b) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \frac{(1 - \alpha q^{2n})(1 - \beta q^{2m})(\alpha q/a; q)_n (\alpha q/b; q)_m a^n b^m}{(q, a; q)_n (q, b; q)_m}.$$

In Section 3, we use Theorem 1.3 to give a complete proof of the orthogonality relation for the q -Hahn polynomials.

A new proof of the orthogonality relation for the big q -Jacobi polynomials is derived in Section 4. In Section 5, Theorem 1.3 is used to give a new derivation of the Nassrallah-Rahman integral. Another proof of the Nassrallah-Rahman integral is given in Section 6. The principal result in Section 7 is the following q -beta integral formula, which includes the Nassrallah-Rahman integral formula as a special case.

Theorem 1.6. *Suppose that $q\alpha = a^2bc ds$ and $\max\{|a|, |b|, |c|, |d|, |s|\} < 1$. Then we have the q -beta integral formula*

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d, s)} {}_3\phi_2 \left(\begin{matrix} ae^{i\theta}, ae^{i\theta}, \alpha uv/q \\ \alpha u, \alpha v \end{matrix}; q, bc ds \right) d\theta \\ &= \frac{2\pi(abcd, abcs, abds, acds; q)_\infty}{(q, ab, ac, ad, as, bc, bd, bs, cd, cs, ds, q\alpha; q)_\infty} \\ & \times \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/u, q/v, ab, ac, ad, as; q)_n}{(1 - \alpha)(q, \alpha u, \alpha v, abcd, abcs, abds, acds; q)_n} (-\alpha^2 uv/a^2)^n q^{n(n-1)/2}. \end{aligned}$$

In Section 8, we use Theorem 1.3 to recover some strange q -series identities.

2. THE PROOF OF THEOREM 1.5

To prove Theorem 1.5, we need the following formula of F. H. Jackson [10], which writes the n th q -derivative of $f(x)$ in terms of $f(q^k x)$ for $k = 0, 1, 2, \dots, n$.

Lemma 2.1. (Jackson) *For any function $f(x)$, we have the identity*

$$D_{q,x}^n \{f(x)\} = x^{-n} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^k f(q^k x).$$

Now we begin to prove Theorem 1.5 using Theorem 1.1 and Lemma 2.1.

Proof. Since $f(a, b)$ is analytic at $(a, b) = (0, 0)$, $f(a, b)$ is analytic at $a = 0$, regarding b as constant. From Theorem 1.1, we have that

$$(2.1) \quad f(a, b) = \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha q/a; q)_n a^n}{(q, a; q)_n} [\mathcal{D}_{q,x}^n \{f(x, b)(x; q)_{n-1}\}]_{x=\alpha q}.$$

Appealing to the Jackson formula in Lemma 2.1, we easily deduce that

$$\begin{aligned} & [\mathcal{D}_{q,x}^n \{f(x, b)(x; q)_{n-1}\}]_{x=\alpha q} \\ &= (q\alpha)^{-n} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} q^k (q^{k+1}\alpha; q)_{n-1} f(q^{k+1}\alpha, b). \end{aligned}$$

Since $f(a, b)$ is analytic at $(a, b) = (0, 0)$, we know that, for each $k \in \{0, 1, 2, \dots, n\}$, $f(q^{k+1}\alpha, b)$ is analytic at $b = 0$. It follows that the left-hand side of the above equation is also an analytic function of b near $b = 0$. Using Theorem 1.1 again, we find that

$$(2.2) \quad \begin{aligned} & [\mathcal{D}_{q,x}^n \{f(x, b)(x; q)_{n-1}\}]_{x=\alpha q} \\ &= \sum_{m=0}^{\infty} \frac{(1 - \beta q^{2m})(q\beta/b; q)_m b^m}{(q, b; q)_m} [\mathcal{D}_{q,y}^m \mathcal{D}_{q,x}^n \{f(x, y)(x; q)_{n-1}(y; q)_{m-1}\}]_{(x=\alpha q, y=\beta q)}. \end{aligned}$$

Letting $c_{n,m} = [\mathcal{D}_{q,y}^m \mathcal{D}_{q,x}^n \{f(x, y)(x; q)_{n-1}(y; q)_{m-1}\}]_{(x=\alpha q, y=\beta q)}$, it is obvious that $c_{m,n}$ are uniquely determined by $f(x, y)$. Combining (2.1) with (2.2), we complete the proof of Theorem 1.5. \square

3. THE ORTHOGONALITY RELATION FOR THE q -HAHN POLYNOMIALS

The q -Hahn polynomials are defined as (see, for example [11])

$$(3.1) \quad H_n(a, b, c, d; z) = \frac{(ac, ad; q)_n}{a^n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, az \\ ac, ad \end{matrix}; q, q \right).$$

For simplicity, in this section we denote $H_n(a, b, c, d; z) := H_n(z)$ and

$$(3.2) \quad A_n(a, b) = \frac{(1 - abcdq^{2n-1})(abcdq^{-1}; q)_n a^n}{(1 - abcdq^{-1})(q, ac, ad; q)_n}.$$

Using Theorems 1.3, we can obtain the following generating function of $H_n(z)$.

Proposition 3.1. For $\max\{|ac|, |ad|, |asz|, |abcds|\} < 1$, we have

$$\sum_{n=0}^{\infty} \frac{s^n (s^{-1}; q)_n}{(abcds; q)_n} A_n(a, b) H_n(z) = \frac{(abcd, acs, ads, az; q)_{\infty}}{(abcds, ac, ad, asz; q)_{\infty}}.$$

Proof. Taking $m = 1$ in Theorems 1.3 and then setting $a = qs$ and $b = t$, we deduce that

$$\begin{aligned} & \frac{(q\alpha, \alpha st, \alpha sb_1, \alpha c_1; q)_{\infty}}{(q\alpha s, \alpha t, \alpha sc_1, \alpha b_1; q)_{\infty}} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, s^{-1}; q)_n s^n}{(1 - \alpha)(q, q\alpha s; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha c_1 \\ \alpha t, \alpha b_1 \end{matrix}; q, q \right). \end{aligned}$$

Replacing (t, b_1, c_1) by $(ac/\alpha, ad/\alpha, az/\alpha)$ in the above equation, we obtain

$$\begin{aligned} & \frac{(q\alpha, acs, ads, az; q)_{\infty}}{(q\alpha s, ac, asz, ad; q)_{\infty}} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, s^{-1}; q)_n s^n}{(1 - \alpha)(q, q\alpha s; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, \alpha q^n, az \\ ac, ad \end{matrix}; q, q \right). \end{aligned}$$

Putting $\alpha = abcdq^{-1}$ in the above equation and noticing the definition of $A_n(a)$ and $H_n(z)$, we complete the proof of Proposition 3.1. \square

It is well-known that $H_n(a, b, c, d; z)$ is symmetric in a and b (see, for example [11, Eq. (2. 18)], [14, Theorem 2]). Thus, by interchanging a and b in Proposition 3.1 and replacing s by r , we can obtain the following proposition.

Proposition 3.2. For $\max\{|bc|, |bd|, |brz|, |abcdr|\} < 1$, we have

$$\sum_{m=0}^{\infty} \frac{r^m (r^{-1}; q)_m}{(abcdr; q)_m} A_m(b, a) H_m(z) = \frac{(abcd, bcr, bdr, bz; q)_{\infty}}{(abcdr, bc, bd, brz; q)_{\infty}}.$$

Proposition 3.3. For $\max\{|q\alpha s|, |q\alpha r|\} < 1$, we have

$$\sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, s^{-1}, r^{-1}; q)_n (-\alpha r s)^n q^{n(n+1)/2}}{(1 - \alpha)(q, q\alpha s, q\alpha r; q)_n} = \frac{(q\alpha, q\alpha r s; q)_{\infty}}{(q\alpha s, q\alpha r; q)_{\infty}}.$$

Proof. Setting $c = 0$ in the Rogers ${}_6\phi_5$ summation in Theorem 1.4 and then replacing (a, b) by (qs, qr) , we complete the proof of the proposition. \square

Askey and Roy [6, Eq. (2. 8)] used the Ramanujan ${}_1\psi_1$ summation to obtain the following interesting integral formula.

Proposition 3.4. For $\max\{|a|, |b|, |c|, |d|\} < 1$ and $cd\rho \neq 0$, we have

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, pce^{-i\theta}, qe^{i\theta}/c\rho; q)_{\infty}}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_{\infty}} d\theta \\ &= \frac{(abcd, \rho, q/\rho, c\rho/d, qd/c\rho; q)_{\infty}}{(q, ac, ad, bc, bd; q)_{\infty}}. \end{aligned}$$

Using the above four propositions we can derive the orthogonality relation for the q -Hahn polynomials.

For brevity, we now introduce L_0, L_n and $K(\theta)$ as follows

$$(3.3) \quad \begin{aligned} L_0 &= \frac{(abcd, \rho, q/\rho, c\rho/d, qd/c\rho; q)_\infty}{(q, ac, ad, bc, bd; q)_\infty}, \\ L_n &= \frac{(1 - abcdq^{-1})(q, ac, ad, bc, bd; q)_n q^{n(n-1)} (-cd)^n}{(1 - abcdq^{2n-1})(abcdq^{-1}; q)_n} L_0, \\ K(\theta) &= \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_\infty}. \end{aligned}$$

The orthogonality relation for the q -Hahn polynomials can be stated in the following theorem. Kalnins and Miller [11, Eq. (2. 8)] proved the $m \neq n$ case of the theorem, and also obtained a recurrence relation to evaluate the $m = n$ case. The value in this case is given below, and it is what Kalnins and Miller could have stated using their recurrence relation.

Theorem 3.5. *Let $H_n(z)$ be the q -Hahn polynomials and $L_n, K(\theta)$ be defined by (3.3). Then we have the orthogonality relation*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K(\theta) H_n(e^{i\theta}) H_m(e^{i\theta}) d\theta = L_n \delta_{m,n}.$$

Proof. Replacing a by as and b by br in the Askey-Roy integral, we easily find that

$$(3.4) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\theta) \frac{(ae^{i\theta}, be^{i\theta}; q)_\infty}{(ase^{i\theta}, bre^{i\theta}; q)_\infty} d\theta = \frac{(abcdrs, \rho, q/\rho, c\rho/d, qd/c\rho; q)_\infty}{(q, acs, ads, bcr, bdr; q)_\infty}.$$

Letting $z = e^{i\theta}$ in Propositions 3.1 and 3.2 and then multiplying the two resulting equations together, we obtain

$$\begin{aligned} & \sum_{n,m=0}^{\infty} \frac{(s^{-1}; q)_n (r^{-1}; q)_m s^n r^m}{(abcds; q)_n (abcdr; q)_m} A_n(a, b) A_m(b, a) H_n(e^{i\theta}) H_m(e^{i\theta}) \\ &= \frac{(abcd, abcd, acs, ads, bcr, bdr, ae^{i\theta}, be^{i\theta}; q)_\infty}{(abcds, abcdr, ad, ac, bc, bd, ase^{i\theta}, bre^{i\theta}; q)_\infty}. \end{aligned}$$

Substituting the above equation into (3.4), we easily obtain the series expansion

$$\begin{aligned} & \sum_{n,m=0}^{\infty} \frac{(s^{-1}; q)_n (r^{-1}; q)_m A_n(a, b) A_m(b, a) s^n r^m}{(abcds; q)_n (abcdr; q)_m} \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\theta) H_n(e^{i\theta}) H_m(e^{i\theta}) d\theta \\ &= A_0 \frac{(abcd, abcdrs; q)_\infty}{(abcds, abcdr; q)_\infty}. \end{aligned}$$

Now we will use another method to expand the right-hand side member of the above equation in terms of

$$\frac{(s^{-1}; q)_n (r^{-1}; q)_m s^n r^m}{(abcds; q)_n (abcdr; q)_m}.$$

In fact, by taking $\alpha = abcd/q$ in Proposition 3.3, we immediately find that

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1 - abcdq^{2n-1})(abcdq^{-1}, s^{-1}, r^{-1}; q)_n (-abcdrs)^n q^{n(n-1)/2}}{(1 - abcdq^{-1})(q, abcds, abcdr; q)_n} \\ &= \frac{(abcd, abcdrs; q)_{\infty}}{(abcds, abcdr; q)_{\infty}}. \end{aligned}$$

Combining the above two equations, we are led to the series identity

$$\begin{aligned} & \sum_{n,m=0}^{\infty} \frac{(s^{-1}; q)_n (r^{-1}; q)_m A_n(a, b) A_m(b, a) s^n r^m}{(abcds; q)_n (abcdr; q)_m} \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\theta) H_n(e^{i\theta}) H_m(e^{i\theta}) d\theta \\ &= A_0 \sum_{n=0}^{\infty} \frac{(1 - abcdq^{2n-1})(abcdq^{-1}, s^{-1}, r^{-1}; q)_n (-abcdrs)^n q^{n(n-1)/2}}{(1 - abcdq^{-1})(q, abcds, abcdr; q)_n}. \end{aligned}$$

Using Theorem 1.5, we can in the above equation equate the coefficients of

$$\frac{(s^{-1}; q)_n (r^{-1}; q)_m s^n r^m}{(abcds; q)_n (abcdr; q)_m}$$

to arrive at the integral formula in Theorem 3.5. This completes the proof of the theorem. \square

4. THE ORTHOGONALITY RELATION FOR THE BIG q -JACOBI POLYNOMIALS

We begin this section with the following transformation formula for q -series.

Proposition 4.1. *For $\lambda = q\alpha^2/bcd$ and $|q\alpha/cd| < 1$, we have*

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} c, d, \alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix}; q, \frac{q\alpha}{cd} \right) = \frac{(q\alpha/c, q\alpha/d, q\lambda/a; q)_{\infty}}{(\alpha q/a, q\alpha/cd, q\lambda; q)_{\infty}} \\ & \quad \times \sum_{n=0}^{\infty} \frac{(1 - \lambda q^{2n})(\lambda, a, \lambda b/\alpha, \lambda c/\alpha, \lambda d/\alpha, q)_n}{(1 - \lambda)(q, q\lambda/a, q\alpha/b, q\alpha/c, q\alpha/d; q)_n} \left(-\frac{q\alpha}{a} \right)^n q^{n(n-1)/2}. \end{aligned}$$

Proof. Recall Watson's q -analogue of Whipple's theorem (see, for example [9, p. 43, Eq. (2.5.1)], [20, Proposition 10.3])

$$\begin{aligned} & {}_8\phi_7 \left(\begin{matrix} \alpha, q\alpha^{1/2}, -q\alpha^{1/2}, a, b, c, d, q^{-n} \\ \alpha^{1/2}, -\alpha^{1/2}, q\alpha/a, q\alpha/b, q\alpha/c, q\alpha/d, \alpha q^{n+1} \end{matrix}; q, \frac{\alpha^2 q^{2+n}}{abcd} \right) \\ &= \frac{(q\alpha, q\alpha/cd; q)_n}{(q\alpha/c, q\alpha/d; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, c, d, q\alpha/ab \\ q\alpha/a, q\alpha/b, cdq^{-n}/\alpha \end{matrix}; q, q \right). \end{aligned}$$

Setting $\lambda = q\alpha^2/bcd$ and using the transformation formula [9, p. 49, Eq.(2.10.3)], we have that

$$\begin{aligned} & {}_8\phi_7 \left(\begin{matrix} \alpha, q\alpha^{1/2}, -q\alpha^{1/2}, a, b, c, d, q^{-n} \\ \alpha^{1/2}, -\alpha^{1/2}, q\alpha/a, q\alpha/b, q\alpha/c, q\alpha/d, \alpha q^{n+1} \end{matrix}; q, \frac{\alpha^2 q^{2+n}}{abcd} \right) \\ &= \frac{(q\alpha, q\lambda/a; q)_n}{(q\alpha/a, q\lambda; q)_n} {}_8\phi_7 \left(\begin{matrix} \lambda, q\lambda^{1/2}, -q\lambda^{1/2}, \lambda b/\alpha, \lambda c/\alpha, \lambda d/\alpha, a, q^{-n} \\ \lambda^{1/2}, -\lambda^{1/2}, q\lambda/a, q\alpha/b, q\alpha/c, q\alpha/d, \lambda q^{n+1} \end{matrix}; q, \frac{\alpha q^{1+n}}{a} \right). \end{aligned}$$

Combining these two equations, we are led to the q -transformation formula

$$\begin{aligned} & {}_4\phi_3 \left(\begin{matrix} q^{-n}, c, d, q\alpha/ab \\ q\alpha/a, q\alpha/b, cdq^{-n}/\alpha \end{matrix}; q, q \right) = \frac{(q\alpha/c, q\alpha/d, q\lambda/a; q)_n}{(\alpha q/a, q\alpha/cd, q\lambda; q)_n} \\ & \quad \times {}_8\phi_7 \left(\begin{matrix} \lambda, q\lambda^{1/2}, -q\lambda^{1/2}, \lambda b/\alpha, \lambda c/\alpha, \lambda d/\alpha, a, q^{-n} \\ \lambda^{1/2}, -\lambda^{1/2}, q\lambda/a, q\alpha/b, q\alpha/c, q\alpha/d, \lambda q^{n+1} \end{matrix}; q, \frac{\alpha q^{1+n}}{a} \right). \end{aligned}$$

Letting $n \rightarrow \infty$ in the both sides of the above equation, we complete the proof of Proposition 4.1. \square

Proposition 4.2. *If there are no zero factors in the denominator of the integral and $\lambda = rhuv/q$, then, we have*

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v, hx; q)_\infty}{(rx, sx, tx; q)_\infty} d_q x = \frac{(1-q)v(q, u/v, qv/u, hu, hv, rsuv, rtuv; q)_\infty}{(rhuv, ru, rv, su, sv, tu, tv; q)_\infty} \\ & \quad \times \sum_{n=0}^{\infty} \frac{(1-\lambda q^{2n})(\lambda, ru, rv, h/s, h/t; q)_n}{(1-\lambda)(q, hu, hv, rsuv, rtuv; q)_n} (-stuv)^n q^{n(n-1)/2}. \end{aligned}$$

Proof. In [16, Theorem 9], we have proved the q -integral formula

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v, hx; q)_\infty d_q x}{(rx, sx, tx; q)_\infty} = \frac{(1-q)v(q, u/v, qv/u, hv, stuv; q)_\infty}{(rv, su, sv, tu, tv; q)_\infty} \\ & \quad \times {}_3\phi_2 \left(\begin{matrix} h/r, sv, tv \\ stuv, hv \end{matrix}; q, ru \right). \end{aligned}$$

If we replace (α, a, b, c, d) by $(rstuv^2/q, rv, rstuv/h, sv, tv)$ in Proposition 4.1, then, we have $q\lambda = ruvh$ and

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} h/r, sv, tv \\ stuv, hv \end{matrix}; q, ru \right) = \frac{(rsuv, rtuv, hu; q)_\infty}{(stuv, rhuv, ru; q)_\infty} \\ & \quad \times \sum_{n=0}^{\infty} \frac{(1-\lambda q^{2n})(\lambda, ru, rv, h/s, h/t; q)_n}{(1-\lambda)(q, hu, hv, rsuv, rtuv; q)_n} (-stuv)^n q^{n(n-1)/2}. \end{aligned}$$

Combining the above two equations, we complete the proof of Proposition 4.2. \square

The big q -Jacobi polynomials are defined as (see, for example [12, p. 438])

$$(4.1) \quad P_n(a, b, c; x) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ qa, qc \end{matrix}; q, q \right).$$

For simplicity, in this section we use $P_n(x)$ to denote the big q -Jacobi polynomials. Using Theorems 1.3, we can obtain the following generating function of the big q -Jacobi polynomials.

Proposition 4.3. *For $\max\{|qa|, |qb|, |tx|, |qabt|\} < 1$, we have*

$$\frac{(qab, qat, qct, x; q)_\infty}{(q^2abt, qa, qc, tx; q)_\infty} = \sum_{n=0}^{\infty} \frac{(1 - abq^{2n+1})(qab, 1/t; q)_n t^n}{(q, q^2abt; q)_n} P_n(x).$$

Proof. Setting $c_1 = \lambda\alpha^{-1}e^{i\theta}$, $c_2 = \lambda\alpha^{-1}e^{-i\theta}$ and $b = 0$ in Theorems 1.3, we deduce that

$$\begin{aligned} & \frac{(\alpha q, \alpha ab_1/q, \alpha ab_2/q; q)_\infty}{(\alpha a, ab_1, ab_2; q)_\infty} \prod_{n=0}^{\infty} \frac{(1 - 2\lambda q^n \cos \theta + \lambda^2 q^{2n})}{(1 - 2\lambda \alpha q^{n-1} \cos \theta + \lambda^2 \alpha^2 q^{2n-2})} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n}{(1 - \alpha)(q, \alpha a; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, \alpha q^n, \lambda e^{i\theta}, \lambda e^{-i\theta} \\ 0, ab_1, ab_2 \end{matrix}; q, q \right). \end{aligned}$$

Taking $\cos \theta = x/(2\lambda)$ in the above equation, letting $\lambda \rightarrow 0$, and replacing a by qt , we find that

$$\begin{aligned} & \frac{(\alpha q, \alpha tb_1, \alpha tb_2, x; q)_\infty}{(q\alpha t, ab_1, ab_2, tx; q)_\infty} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, 1/t; q)_n t^n}{(1 - \alpha)(q, q\alpha t; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, \alpha q^n, x \\ ab_1, ab_2 \end{matrix}; q, q \right). \end{aligned}$$

Replacing $\alpha b_1 = qa$, $\alpha b_2 = qc$ and $\alpha = qab$ in the above equation, we complete the proof of Proposition 4.3. \square

If the q -integral of the function $f(x)$ from a to b is defined as

$$\int_a^b f(x) d_q x = (1 - q) \sum_{n=0}^{\infty} [bf(bq^n) - af(aq^n)] q^n,$$

then, the orthogonality relation for the big q -Jacobi polynomials can be stated in the following theorem (see, for example [9, p. 182], [12, p. 438]).

Theorem 4.4. *The orthogonality relation for the big q -Jacobi polynomials is*

$$\begin{aligned} & \int_{cq}^{aq} \frac{(x/a, x/c; q)_\infty}{(x, bx/c; q)_\infty} P_m(x) P_n(x) d_q x = aq(1 - q) \frac{(q, abq^2, a/c, qc/a; q)_\infty}{(aq, bq, cq, abq/c; q)_\infty} \\ & \quad \times \frac{(1 - abq)(q, qb, abq/c; q)_n}{(1 - abq^{2n+1})(aq, abq, cq; q)_n} (-acq^2)^n q^{n(n-1)/2} \delta_{mn}. \end{aligned}$$

Proof. Replacing t by s in Proposition 4.3, we immediately deduce that

$$\frac{(qab, qas, qcs, x; q)_\infty}{(q^2abs, qa, qc, sx; q)_\infty} = \sum_{m=0}^{\infty} \frac{(1 - abq^{2m+1})(qab, 1/s; q)_m s^m}{(q, q^2abs; q)_m} P_m(x).$$

If we multiply this equation with the equation in Proposition 4.3 together, we obtain

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(1-abq^{2m+1})(1-abq^{2n+1})(qab;q)_m(qab;q)_n(1/s;q)_m(1/t;q)_n s^m t^n}{(q;q)_m(q;q)_n(q^2abs;q)_m(q^2abt;q)_n} \\ & \quad \times P_m(x)P_n(x) \\ &= \frac{(qab, x; q)_{\infty}^2 (qas, qcs, qat, qct; q)_{\infty}}{(qa, qc; q)_{\infty}^2 (q^2abs, q^2abt, sx, tx; q)_{\infty}}. \end{aligned}$$

Multiplying the above equation by $(x/a, x/c; q)_{\infty}/(x, bx/c; q)_{\infty}$ and then taking the q -integral over $[cq, aq]$, we find that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(1-abq^{2m+1})(1-abq^{2n+1})(qab;q)_m(qab;q)_n(1/s;q)_m(1/t;q)_n s^m t^n}{(q;q)_m(q;q)_n(q^2abs;q)_m(q^2abt;q)_n} \\ & \quad \times \int_{cq}^{aq} \frac{(x/a, x/c; q)_{\infty}}{(x, bx/c; q)_{\infty}} P_m(x)P_n(x) d_q x \\ &= \frac{(qab; q)_{\infty}^2 (qas, qcs, qat, qct; q)_{\infty}}{(qa, qc; q)_{\infty}^2 (q^2abs, q^2abt; q)_{\infty}} \int_{cq}^{aq} \frac{(x/a, x/c, x; q)_{\infty} d_q x}{(bx/c, sx, tx; q)_{\infty}}. \end{aligned}$$

Setting $(h, r, u, v) = (1, b/c, qc, qa)$ in Proposition 4.2 and simplifying, we obtain

$$\begin{aligned} & \int_{cq}^{aq} \frac{(x/a, x/c, x; q)_{\infty} d_q x}{(bx/c, sx, tx; q)_{\infty}} = \frac{aq(1-q)(q, c/a, qa/c, qa, qc, absq^2, abtq^2; q)_{\infty}}{(abq^2, qb, abq/c, aqs, aqt, cqs, cqt; q)_{\infty}} \\ & \quad \times \sum_{n=0}^{\infty} \frac{(1-abq^{2n+1})(abq, bq, abq/c, 1/s, 1/t; q)_n}{(1-abq)(q, aq, cq, absq^2, abtq^2; q)_n} (-acstq^2)^n q^{n(n-1)/2}. \end{aligned}$$

Combining the above equations, we deduce that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(1-abq^{2m+1})(1-abq^{2n+1})(qab;q)_m(qab;q)_n(1/s;q)_m(1/t;q)_n s^m t^n}{(q;q)_m(q;q)_n(q^2abs;q)_m(q^2abt;q)_n} \\ & \quad \times \int_{cq}^{aq} \frac{(x/a, x/c; q)_{\infty}}{(x, bx/c; q)_{\infty}} P_m(x)P_n(x) d_q x \\ &= \frac{aq(1-q)(q, qab, c/a, qa/c; q)_{\infty}}{(qa, qb, qc, qab/c; q)_{\infty}} \\ & \quad \times \sum_{n=0}^{\infty} \frac{(1-abq^{2n+1})(abq, bq, abq/c, 1/s, 1/t; q)_n}{(q, aq, cq, absq^2, abtq^2; q)_n} (-acstq^2)^n q^{n(n-1)/2}. \end{aligned}$$

Using Theorem 1.5, we can obtain the orthogonality relation for the big q -Jacobi polynomials by equating the coefficients of

$$\frac{(1/s; q)_m(1/t; q)_n s^m t^n}{(q^2abs; q)_m(q^2abt; q)_n}$$

in the above equation. This completes the proof of Theorem 4.4. \square

5. ON THE NASSRALLAH-RAHMAN INTEGRAL

Analogous to the hypergeometric case, we call the q -hypergeometric series

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right)$$

well-poised if the parameters satisfy the relations $qa_1 = a_2b_1 = a_3b_2 = \dots = a_{r+1}b_r$; very-well-poised if, in addition, $a_2 = q\sqrt{a_1}$, $a_3 = -q\sqrt{a_1}$.

For simplicity, we sometimes use ${}_{r+1}W_r(a_1; a_4, a_5, \dots, a_{r+1}; q, z)$ to denote

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, q\sqrt{a_1}, -q\sqrt{a_1}, a_4, \dots, a_{r+1} \\ \sqrt{a_1}, -\sqrt{a_1}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{matrix}; q, z \right)$$

Definition 5.1. For $x = \cos \theta$, we define the notation $h(x; a)$ and $h(x; a_1, a_2, \dots, a_m)$ as follows

$$h(x; a) = (ae^{i\theta}, ae^{-i\theta}; q)_\infty = \prod_{k=0}^{\infty} (1 - 2q^k ax + q^{2k} a^2)$$

$$h(x; a_1, a_2, \dots, a_m) = h(x; a_1)h(x; a_2) \cdots h(x; a_m).$$

The following important integral evaluation is due to Askey and Wilson, which can be used to obtain an elegant proof of the orthogonality relation for these Askey-Wilson polynomials.

Theorem 5.2. (Askey–Wilson) *With $h(x; a)$ being defined in Definition 5.1 and $\max\{|a|, |b|, |c|, |d|\} < 1$, we have*

$$(5.1) \quad I(a, b, c, d) := \int_0^\pi \frac{h(\cos 2\theta; 1)d\theta}{h(\cos \theta; a, b, c, d)} = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.$$

For $x = \cos \theta$, the Askey-Wilson polynomials $p_n(a, b, c, d; \cos \theta)$ are defined as [5], [9, p. 188]

$$(5.2) \quad (ab, ac, ad)_n a^{-n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q \right).$$

Using Theorem 1.3, we can obtain the following generating function for the Askey-Wilson polynomials.

Proposition 5.3. *For $\max\{|abcds|, |ab|, |ac|, |ad|, |sae^{i\theta}|, |sae^{-i\theta}|\} < 1$, we have*

$$\frac{(abcd, abs, acs, ads, ae^{i\theta}, ae^{-i\theta}; q)_\infty}{(abcds, ab, ac, ad, sae^{i\theta}, sae^{-i\theta}; q)_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{(1 - abcdq^{2n-1})(abcdq^{-1}, s^{-1}; q)_n (sa)^n}{(1 - abcdq^{-1})(q, ab, ac, ad, abcds; q)_n} p_n(a, b, c, d; \cos \theta).$$

Proof. Taking $m = 2$ in Theorem 1.3 and then setting $a = qs$ and $b = t$, we conclude that

$$\begin{aligned} & \frac{(q\alpha, \alpha st, \alpha b_1 s, \alpha b_2 s, \alpha c_1, \alpha c_2; q)_\infty}{(q\alpha s, \alpha t, \alpha c_1 s, \alpha c_2 s, \alpha b_1, \alpha b_2; q)_\infty} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, s^{-1}; q)_n s^n}{(1 - \alpha)(q, q\alpha s; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha c_1, \alpha c_2 \\ \alpha t, \alpha b_1, \alpha b_2 \end{matrix}; q, q \right). \end{aligned}$$

Replacing $(\alpha t, \alpha b_1, \alpha b_2, \alpha c_1, \alpha c_2)$ by $(ab, ac, ad, ae^{i\theta}, ae^{-i\theta})$ in the above equation and then taking $\alpha = abcdq^{-1}$, we complete the proof of the theorem. \square

Nassrallah and Rahman [21] used the integral representation of the sum of two non-terminating ${}_3\phi_2$ series and the Askey-Wilson integral to find the following q -beta integral formula. In this section, we will use Proposition 5.3 to give a new proof of the Nassrallah-Rahman integral formula [9, Eq. (6.3.7)].

Theorem 5.4. (Nassrallah-Rahman) *For $\max\{|a|, |b|, |c|, |d|, |s|\} < 1$, we have*

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; r)d\theta}{h(\cos \theta; a, b, c, d, s)} \\ &= \frac{2\pi(r/s, rs, abcs, bc ds, ac ds, ab ds; q)_\infty}{(q, ab, ac, ad, as, bc, bd, bs, cd, cs, ds, abcds^2; q)_\infty} \\ & \quad \times {}_8W_7(abcds^2/q; as, bs, cs, ds, abcds/r; q, r/s). \end{aligned}$$

Proof. Replacing a by r in Proposition 5.3, we immediately have

$$\begin{aligned} & \frac{(r b c d, r b s, r c s, r d s; q)_\infty h(\cos \theta; r)}{(r b c d s, r b, r c, r d; q)_\infty h(\cos \theta; r s)} \\ &= \sum_{n=0}^{\infty} \frac{(1 - r b c d q^{2n-1})(r b c d q^{-1}, s^{-1}; q)_n (r s)^n}{(1 - r b c d q^{-1})(q, r b, r c, r d, r b c d s; q)_n} p_n(r, b, c, d; \cos \theta). \end{aligned}$$

It is well-know that the Askey-Wilson polynomials $p_n(r, b, c, d; \cos \theta)$ is symmetric in r, b, c and d (see, for example [17, Corollary 4]). Thus, we have

$$\begin{aligned} & \frac{(r b c d, r b s, r c s, r d s; q)_\infty h(\cos \theta; r)}{(r b c d s, r b, r c, r d; q)_\infty h(\cos \theta; r s)} \\ &= \sum_{n=0}^{\infty} \frac{(1 - r b c d q^{2n-1})(r b c d q^{-1}, s^{-1}; q)_n (r s)^n}{(1 - r b c d q^{-1})(q, r b, r c, r d, r b c d s; q)_n} p_n(d, b, c, r; \cos \theta) \\ &= \sum_{n=0}^{\infty} \frac{(1 - r b c d q^{2n-1})(b d, c d, r b c d q^{-1}, s^{-1}; q)_n (r s/d)^n}{(1 - r b c d q^{-1})(q, r b, r c, r b c d s; q)_n} \\ & \quad \times {}_4\phi_3 \left(\begin{matrix} q^{-n}, r b c d q^{n-1}, d e^{i\theta}, d e^{-i\theta} \\ b d, c d, r d \end{matrix}; q, q \right). \end{aligned}$$

Multiplying the above equation by $h(\cos 2\theta; 1)/h(\cos \theta; a, b, c, d)$, and then taking the definite integral over $0 \leq \theta \leq \pi$, we find that

$$\begin{aligned} & \frac{(rbcd, rbs, rcs, rds; q)_\infty}{(rbcds, rb, rc, rd; q)_\infty} \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; r)d\theta}{h(\cos \theta; a, b, c, d, sr)} \\ &= \sum_{n=0}^\infty \frac{(1 - rbcdq^{2n-1})(bd, cd, rbcdq^{-1}, s^{-1}; q)_n (rs/d)^n}{(1 - rbcdq^{-1})(q, rb, rc, rbc ds; q)_n} \\ & \quad \times \sum_{k=0}^n \frac{(q^{-n}, rbcdq^{n-1}; q)_k q^k}{(q, bd, cd, rd)_k} \int_0^\pi \frac{h(\cos 2\theta; 1)d\theta}{h(\cos \theta; a, b, c, dq^k)}. \end{aligned}$$

With the help of the Askey-Wilson integral in Theorem 5.2, we find that

$$\int_0^\pi \frac{h(\cos 2\theta; 1)d\theta}{h(\cos \theta; a, b, c, dq^k)} = \frac{2\pi(ad, bd, cd; q)_k (abcd; q)_\infty}{(abcd; q)_k (q, ab, ac, ad, bc, bd, cd; q)_\infty}.$$

It follows that

$$\begin{aligned} & \sum_{k=0}^n \frac{(q^{-n}, rbcdq^{n-1}; q)_k q^k}{(q, bd, cd, rd)_k} \int_0^\pi \frac{h(\cos 2\theta; 1)d\theta}{h(\cos \theta; a, b, c, dq^k)} \\ &= \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} {}_3\phi_2 \left(\begin{matrix} q^{-n}, rbcdq^{n-1}, ad \\ abcd, rd \end{matrix}; q, q \right) \end{aligned}$$

We can apply the q -Pfaff-Saalschütz formula to sum the ${}_3\phi_2$ series on the right hand side of the above equation to obtain

$${}_3\phi_2 \left(\begin{matrix} q^{-n}, rbcdq^{n-1}, ad \\ abcd, rd \end{matrix}; q, q \right) = \frac{(bc, r/a; q)_n (ad)^n}{(rd, abcd; q)_n}.$$

Combining the above equations, we finally conclude that

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; r)d\theta}{h(\cos \theta; a, b, c, d, sr)} \\ &= \frac{2\pi(abcd, rbc ds, rb, rc, rd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, rbc d, rbs, rcs, rds; q)_\infty} \\ & \quad \times {}_8W_7(rbcdq^{-1}; bc, bd, cd, s^{-1}, r/a; q, ars). \end{aligned}$$

Replacing s by s/r in the above equation and then interchanging a and d , we deduce that

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; r)d\theta}{h(\cos \theta; a, b, c, d, s)} \\ (5.3) \quad &= \frac{2\pi(abcd, abcs, ra, rb, rc; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, rabc, as, bs, cs; q)_\infty} \\ & \quad \times {}_8W_7(rabcq^{-1}; r/s, ab, ac, bc, r/d; q, ds). \end{aligned}$$

Applying the transformation formula for ${}_8\phi_7$ in [9, III. 24] to the right-hand of the above equation, we complete the proof of Theorem 5.4. \square

Taking $r = 0$ in (5.3), we immediately find, for $\max\{|a|, |b|, |c|, |d|, |s|\} < 1$, that

$$(5.4) \quad \int_0^\pi \frac{h(\cos 2\theta; 1)d\theta}{h(\cos \theta; a, b, c, d, s)} \\ = \frac{2\pi(abcd, abcs; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, as, bs, cs; q)_\infty} {}_3\phi_2 \left(\begin{matrix} ab, ac, bc \\ abcd, abcs \end{matrix}; q, ds \right).$$

6. ANOTHER PROOF OF THE NASSRALLAH-RAHMAN INTEGRAL FORMULA

The Al-Salam and Verma q -integral formula [1, Eq. (1.3)] can be stated in the following proposition.

Proposition 6.1. *If there are no zero factors in the denominator of the integral, then, we have*

$$\int_d^s \frac{(qx/d, qx/s, abcdsx; q)_\infty}{(ax, bx, cx; q)_\infty} d_q x = \frac{(1-q)s(q, d/s, qs/d, abds, acds, bc ds; q)}{(ad, as, bd, bs, cd, cs; q)_\infty}.$$

The following q -integral formula is a special case of [9, Eq. (2.10.19)]. Now we will use Theorem 5.4 to give a derivation of this q -integral formula.

Theorem 6.2. *If there are no zero factors in the denominator of the integral and $|r/s| < 1$, then, we have*

$$\int_d^s \frac{(abcx, qx/d, qx/s, rx; q)_\infty}{(ax, bx, cx, rx/ds; q)_\infty} d_q x \\ = \frac{(1-q)s(q, d/s, qs/d, rs, abcs, acds, abds, bc ds; q)_\infty}{(r/d, ad, bd, cd, as, bs, cs, abcds^2; q)_\infty} \\ \times {}_8W_7(abcds^2/q; as, bs, cs, ds, abcds/r; q, r/s).$$

Proof. Letting $d = x$ in the Askey-Wilson integral in Theorem 5.2 and then multiplying both sides of the resulting equation by $(qx/d, qx/s, drsx; q)_\infty / (rx; q)_\infty$, we obtain

$$\int_0^\pi \frac{h(\cos 2\theta; 1)(qx/d, qx/s, drsx; q)_\infty d\theta}{h(\cos \theta; a, b, c, x)(rx; q)_\infty} = \frac{2\pi(abcx, qx/d, qx/s, drsx; q)_\infty}{(q, ab, ac, bc, ax, bx, cx, rx; q)_\infty}.$$

Taking the q -integral over $d \leq x \leq s$ in the both side of the above equation, we obtain

$$\int_0^\pi \frac{h(\cos 2\theta; 1)d\theta}{h(\cos \theta; a, b, c)} \int_d^s \frac{(qx/d, qx/s, drsx; q)_\infty d_q x}{(xe^{i\theta}, xe^{-i\theta}, rx; q)_\infty} \\ = \frac{2\pi}{(q, ab, ac, bc; q)_\infty} \int_d^s \frac{(abcx, qx/d, qx/s, drsx; q)_\infty d_q x}{(ax, bx, cx, rx; q)_\infty}.$$

Using the Al-Salam and Verma q -integral formula, we immediately find that

$$\int_d^s \frac{(qx/d, qx/s, drsx; q)_\infty d_q x}{(xe^{i\theta}, xe^{-i\theta}, rx; q)_\infty} = \frac{(1-q)s(q, d/s, qs/d, ds; q)_\infty h(\cos \theta; drs)}{(rd, rs; q)_\infty h(\cos \theta; d, s)}$$

Combining the above two equations, we arrive at

$$\begin{aligned} & \int_d^s \frac{(abcx, qx/d, qx/s, drsx; q)_\infty d_q x}{(ax, bx, cx, rx; q)_\infty} \\ &= \frac{(1-q)s(q, q, ab, ac, bc, d/s, qs/d, ds; q)_\infty}{2\pi(rd, rs; q)_\infty} \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; drs)d\theta}{h(\cos \theta; a, b, c, d, s)}. \end{aligned}$$

Replacing r by r/ds in the above equation, we find that

$$(6.1) \quad \begin{aligned} & \int_d^s \frac{(abcx, qx/d, qx/s, rx; q)_\infty d_q x}{(ax, bx, cx, rx/ds; q)_\infty} \\ &= \frac{(1-q)s(q, q, ab, ac, bc, d/s, qs/d, ds; q)_\infty}{2\pi(r/d, r/s; q)_\infty} \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; r)d\theta}{h(\cos \theta; a, b, c, d, s)}. \end{aligned}$$

Applying Theorem 5.4 to the above equation, we complete the proof of Theorem 6.2. \square

Next we will use (6.1) to give another proof of the Nassrallah-Rahman integral formula.

Proof. If we choose $r = abcds$ in (6.1), we immediately deduce that

$$\begin{aligned} & \int_d^s \frac{(abcdsx, qx/d, qx/s; q)_\infty d_q x}{(ax, bx, cx; q)_\infty} \\ &= \frac{(1-q)s(q, q, ab, ac, bc, d/s, qs/d, ds; q)_\infty}{2\pi(abcs, abcd; q)_\infty} \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; abcds)d\theta}{h(\cos \theta; a, b, c, d, s)}. \end{aligned}$$

Applying the Al-Salam and Verma q -integral formula to the left-hand of the above equation and simplifying, we find, for $\max\{|a|, |b|, |c|, |d|, |s|\} < 1$, that [9, Eq. (6.4.1)]

$$(6.2) \quad \begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; abcds)d\theta}{h(\cos \theta; a, b, c, d, s)} \\ &= \frac{2\pi(abcd, abcs, abds, acds, bc ds; q)_\infty}{(q, ab, ac, ad, as, bc, bd, bs, cd, cs, ds; q)_\infty}. \end{aligned}$$

Using $T(\theta)$ to denote the integrand of the above integral and the value of this integral by I , then, we have

$$\int_0^\pi T(\theta)d\theta = I.$$

Replacing s by sq^n in the above equation and simplifying, we easily obtain

$$\int_0^\pi \frac{T(\theta)(se^{i\theta}, se^{-i\theta}; q)_n}{(abcdse^{i\theta}, abcdse^{-i\theta}; q)_n} d\theta = \frac{(as, bs, cs, ds; q)_n}{(abcs, abds, acds, bc ds; q)_n} I.$$

If we multiply both sides of the above equation by the following factor:

$$\frac{(1 - abcds^2 q^{2n-1})(abcds^2/q, abcds/r; q)_n (r/s)^n}{(1 - abcds^2/q)(q, rs, r/s; q)_n},$$

and then summing the resulting the equation over, $0 \leq n \leq \infty$, we find that

$$\begin{aligned} & \int_0^\pi T(\theta) {}_6W_5(abcds^2/q; abcds/r, se^{i\theta}, se^{-i\theta}; q, r/s) d\theta \\ &= {}_8W_7(abcds^2/q; as, bs, cs, ds, abcds/r; q, r/s) I. \end{aligned}$$

Using the q -Dougall summation, we immediately find that

$${}_6W_5(abcds^2/q; abcds/r, se^{i\theta}, se^{-i\theta}; q, r/s) = \frac{(abcds^2, abcd; q)_\infty h(\cos \theta; r)}{(rs, r/s; q)_\infty h(\cos \theta; abcds)}.$$

Combining the above two equations, we complete the proof of the theorem. \square

7. A NEW q -BETA INTEGRAL FORMULA

We first recall the following well-known q -formula (see, for example [9, p. 62], [18, Theorem 1.8]).

Proposition 7.1. *For $|\alpha xy/q| < 1$, we have the q -transformation formula*

$$\begin{aligned} & \frac{(\alpha q, \alpha xy/q; q)_\infty} {(\alpha x, \alpha y; q)_\infty} {}_3\phi_2 \left(\begin{matrix} q/x, q/y, \alpha uv/q \\ \alpha u, \alpha v \end{matrix}; q, \frac{\alpha xy}{q} \right) \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/x, q/y, q/u, q/v; q)_n (-\alpha^2 xyuv/q^2)^n q^{n(n-1)/2}}{(1 - \alpha)(q, \alpha x, \alpha y, \alpha u, \alpha v; q)_n}. \end{aligned}$$

Now we begin to prove Theorem 1.6 by using the above proposition and the q -beta integral formula in (6.2).

Proof. Setting $x = (q/a)e^{i\theta}$, $y = (q/a)e^{-i\theta}$ and $\alpha = a^2 bc ds/q$ in Proposition 7.1, we obtain

$$\begin{aligned} & \frac{(q\alpha, bc ds; q)_\infty} {h(\cos \theta; abcds)} {}_3\phi_2 \left(\begin{matrix} ae^{i\theta}, ae^{-i\theta}, \alpha uv/q \\ \alpha u, \alpha v \end{matrix}; q, bc ds \right) \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, ae^{i\theta}, ae^{-i\theta}, q/u, q/v; q)_n (-\alpha^2 uv/a^2)^n q^{n(n-1)/2}}{(1 - \alpha)(q, abcdse^{i\theta}, abcdse^{-i\theta}, \alpha u, \alpha v; q)_n}. \end{aligned}$$

If we multiply both sides of the above equation by the factor

$$\frac{h(\cos 2\theta; 1)h(\cos \theta; abcds)}{h(\cos \theta; a, b, c, d, s)},$$

and then take the definite integral over $0 \leq \theta \leq \pi$ in the resulting equation, we deduce that

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d, s)} {}_3\phi_2 \left(\begin{matrix} ae^{i\theta}, ae^{-i\theta}, \alpha uv/q \\ \alpha u, \alpha v \end{matrix}; q, bc ds \right) d\theta \\ &= \frac{1}{(q\alpha, bc ds; q)_\infty} \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/u, q/v; q)_n}{(1 - \alpha)(q, \alpha u, \alpha v; q)_n} (-\alpha^2 uv/a^2)^n q^{n(n-1)/2} \\ & \quad \times \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; abcdsq^n) d\theta}{h(\cos \theta; aq^n, b, c, d, s)}. \end{aligned}$$

If a is replaced by aq^n in (6.2), then, we immediately conclude that

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; abcds q^n) d\theta}{h(\cos \theta; aq^n, b, c, d, s)} \\ &= \frac{2\pi(abcd, abcs, abds, acds, bc ds; q)_\infty (ab, ac, ad, as; q)_n}{(q, ab, ac, ad, as, bc, bd, bs, cd, cs, ds; q)_\infty (abcd, abcs, abds, acds; q)_n}. \end{aligned}$$

Combining the above two equations, we complete the proof of Theorem 1.6. \square

When $s = 0$, it is obvious that Theorem 1.6 immediately becomes the Askey-Wilson integral.

When $u = q$, the series in the right-hand side of the equation in Theorem 1.6 immediately reduces to 1, and the ${}_3\phi_2$ series becomes a ${}_2\phi_1$ series which can be summed by the q -Gauss summation formula,

$${}_3\phi_2 \left(\begin{matrix} ae^{i\theta}, ae^{i\theta} \\ q\alpha \end{matrix}; q, bc ds \right) = \frac{h(\cos \theta; abcds)}{(q\alpha, bc ds; q)_\infty}.$$

Hence, in this case, the integral formula in Theorem 1.6 becomes the q -integral formula (6.2).

Letting $v \rightarrow \infty$ in Theorem 1.6 and by a direct computation, we easily deduce that

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d, s)} {}_2\phi_1 \left(\begin{matrix} ae^{i\theta}, ae^{i\theta} \\ \alpha u \end{matrix}; q, \frac{\alpha u}{a^2} \right) d\theta \\ &= \frac{2\pi(abcd, abcs, abds, acds; q)_\infty}{(q, ab, ac, ad, as, bc, bd, bs, cd, cs, ds, q\alpha; q)_\infty} \\ & \quad \times {}_8W_7(\alpha; q/u, ab, ac, ad, as; q, \alpha uv/a^2). \end{aligned}$$

The ${}_2\phi_1$ series in the above equation can be summed by the q -Gauss summation,

$${}_2\phi_1 \left(\begin{matrix} ae^{i\theta}, ae^{i\theta} \\ \alpha u \end{matrix}; q, \frac{\alpha u}{a^2} \right) = \frac{h(\cos \theta; \alpha u/a)}{(\alpha u, \alpha u/a^2; q)_\infty}.$$

Combining the above two equations, we are led to the q -beta integral formula

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; \alpha u/a)}{h(\cos \theta; a, b, c, d, s)} d\theta \\ &= \frac{2\pi(abcd, abcs, abds, acds, \alpha u, \alpha u/a^2; q)_\infty}{(q, ab, ac, ad, as, bc, bd, bs, cd, cs, ds, q\alpha; q)_\infty} \\ & \quad \times {}_8W_7(\alpha; q/u, ab, ac, ad, as; q, \alpha uv/a^2). \end{aligned}$$

Setting $u = qr/abcds$ in the above equation and noting that $q\alpha = a^2bc ds$, we find that for $\max\{|a|, |b|, |c|, |d|, |s|\} < 1$,

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; r)}{h(\cos \theta; a, b, c, d, s)} d\theta \\ &= \frac{2\pi(abcd, abcs, abds, acds, ar, r/a; q)_\infty}{(q, ab, ac, ad, as, bc, bd, bs, cd, cs, ds, a^2bc ds; q)_\infty} \\ & \quad \times {}_8W_7(a^2bc ds/q; abcds/r, ab, ac, ad, as; q, r/a), \end{aligned}$$

which is the same as the integral formula in Theorem 5.4 if we interchanging a and s .

8. STRANGE EVALUATIONS OF BASIC HYPERGEOMETRIC SERIES

Theorem 1.3 can be used to provide new proofs of some strange q -series identity. We begin by proving the following strange q -series identity due to Andrews [3, Eq. (4.5)] (see also, [8, Eq. (4.26)] and [7, Eq. (4.5d)]).

Proposition 8.1. (Andrews) *We have the summation formula*

$$\begin{aligned} & {}_5\phi_4 \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha^{1/3} q^{1/3}, \alpha^{1/3} q^{2/3}, \alpha^{1/3} q \\ \alpha^{1/2} q, -\alpha^{1/2} q, \alpha^{1/2} q^{1/2}, -\alpha^{1/2} q^{1/2}; q, q \end{matrix} \right) \\ &= \frac{(1-\alpha)(1-\alpha^{1/3} q^{2n/3})(q; q)_n (\alpha^{1/3}; q^{1/3})_n (q\alpha)^{n/3}}{(1-\alpha^{1/3})(1-\alpha q^{2n})(\alpha; q)_n (q^{1/3}; q^{1/3})_n}. \end{aligned}$$

Proof. Let ω be the primitive cube root of unity given by $\omega = \exp(2\pi i/3)$. Then we have

$$(1-x)(1-x\omega)(1-x\omega^2) = 1-x^3.$$

If we replace (q, α, a, b, c) by $(q^{1/3}, \alpha^{1/3}, a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2)$ in Theorem 1.4 and then use the above identity in the resulting equation, we find that

$$\begin{aligned} (8.1) \quad & \frac{(\alpha a^2/q; q)_\infty (\alpha^{1/3} q^{1/3}; q^{1/3})_\infty}{(\alpha a; q)_\infty (\alpha^{1/3} a/q^{2/3}; q^{1/3})_\infty} \\ &= \sum_{n=0}^{\infty} \frac{(1-\alpha^{1/3} q^{2n/3})(\alpha^{1/3}; q^{1/3})_n (q/a; q)_n (a/q)^n (q\alpha)^{n/3}}{(1-\alpha^{1/3})(\alpha a; q)_n (q^{1/3}; q^{1/3})_n}. \end{aligned}$$

On the other hand, taking $m = 3$ in Theorem 1.3, and then letting $(\alpha b, \alpha b_1, \alpha b_2, \alpha b_3) = (\alpha^{1/2} q, -\alpha^{1/2} q, \alpha^{1/2} q^{1/2}, -\alpha^{1/2} q^{1/2})$ and $(\alpha c_1, \alpha c_2, \alpha c_3) = (\alpha^{1/3} q^{1/3}, \alpha^{1/3} q^{2/3}, \alpha^{1/3} q)$, we are led to derive the identity

$$\begin{aligned} & \frac{(\alpha a^2/q; q)_\infty (\alpha^{1/3} q^{1/3}; q^{1/3})_\infty}{(\alpha a; q)_\infty (\alpha^{1/3} a/q^{2/3}; q^{1/3})_\infty} \\ &= \sum_{n=0}^{\infty} \frac{(1-\alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n} {(1-\alpha)(q, \alpha a; q)_n} {}_5\phi_4 \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha^{1/3} q^{1/3}, \alpha^{1/3} q^{2/3}, \alpha^{1/3} q \\ \alpha^{1/2} q, -\alpha^{1/2} q, \alpha^{1/2} q^{1/2}, -\alpha^{1/2} q^{1/2}; q, q \end{matrix} \right), \end{aligned}$$

Combining the above two equations, we conclude that

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1 - \alpha^{1/3} q^{2n/3})(\alpha^{1/3}; q^{1/3})_n (q/a; q)_n (a/q)^n (q\alpha)^{n/3}}{(1 - \alpha^{1/3})(\alpha a; q)_n (q^{1/3}; q^{1/3})_n} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n}{(1 - \alpha)(q, \alpha a; q)_n} {}_5\phi_4 \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha^{1/3} q^{1/3}, \alpha^{1/3} q^{2/3}, \alpha^{1/3} q \\ \alpha^{1/2} q, -\alpha^{1/2} q, \alpha^{1/2} q^{1/2}, -\alpha^{1/2} q^{1/2} \end{matrix}; q, q \right). \end{aligned}$$

Appealing to Theorem 1.1, we can equate the coefficients of $a^n (q/a; q)_n / (\alpha a; q)_n$ on both sides of the above equation to complete the proof of Proposition 8.1. \square

Replacing α by α^3 and q by q^3 in (8.1) and then setting $\alpha = 1$ and $a = -q^3$, we deduce that (see, for example, [4, Eq. (13)])

$$(8.2) \quad \phi(-q)\phi(-q^3) = \frac{(q; q)_{\infty}(q^3; q^3)_{\infty}}{(-q; q)_{\infty}(-q^3; q^3)_{\infty}} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \frac{q^n(1+q^n)}{1+q^{3n}}.$$

Proposition 8.2. (Andrews) *We have the summation formula*

$$\begin{aligned} & {}_5\phi_4 \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha^{1/3}, \alpha^{1/3} e^{2\pi i/3}, \alpha^{1/3} e^{4\pi i/3} \\ \sqrt{\alpha}, -\sqrt{\alpha}, \sqrt{q\alpha}, -\sqrt{q\alpha} \end{matrix}; q, q \right) \\ &= \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod{3} \\ \frac{(\alpha; q^3)_l (q; q)_{3l} \alpha^l}{(\alpha; q)_{3l} (q^3; q^3)_l}, & \text{if } n = 3l. \end{cases} \end{aligned}$$

This identity was first proved by Andrews [3, Eq. (4.7)]. For other proofs, see [7, Eq. (4.4d)] and [8, Eq. (4.32)].

Proof. If we first replace q by q^3 in Theorem 1.4 and then setting $b = qa$ and $c = q^2a$ in the resulting equation, we obtain

$$\sum_{n=0}^{\infty} \frac{(1 - \alpha q^{6n})(q/a; q)_{3n} (\alpha; q^3)_n \alpha^n (a/q)^{3n}}{(\alpha a; q)_{3n} (q^3; q^3)_n} = \frac{(\alpha a^2/q^2; q)_{\infty} (\alpha; q^3)_{\infty}}{(\alpha a, q)_{\infty} (\alpha a^3/q^3; q^3)_{\infty}}.$$

Using the same argument that we used to prove Proposition 8.1, from Theorem 1.3 we can deduce that

$$\begin{aligned} & \frac{(\alpha a^2/q^2; q)_{\infty} (\alpha; q^3)_{\infty}}{(\alpha a, q)_{\infty} (\alpha a^3/q^3; q^3)_{\infty}} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n}{(q, \alpha a; q)_n} {}_5\phi_4 \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha^{1/3}, \alpha^{1/3} e^{2\pi i/3}, \alpha^{1/3} e^{4\pi i/3} \\ \sqrt{\alpha}, -\sqrt{\alpha}, \sqrt{q\alpha}, -\sqrt{q\alpha} \end{matrix}; q, q \right). \end{aligned}$$

Combining the above two equations, we arrive at the q -identity

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{6n})(q/a; q)_{3n} (\alpha; q^3)_n \alpha^n (a/q)^{3n}}{(\alpha a; q)_{3n} (q^3; q^3)_n} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n}{(1 - \alpha)(q, \alpha a; q)_n} {}_5\phi_4 \left(\begin{matrix} q^{-n}, \alpha q^n, \alpha^{1/3}, \alpha^{1/3} e^{2\pi i/3}, \alpha^{1/3} e^{4\pi i/3} \\ \sqrt{\alpha}, -\sqrt{\alpha}, \sqrt{q\alpha}, -\sqrt{q\alpha} \end{matrix}; q, q \right). \end{aligned}$$

Using Theorem 1.1, we compare the coefficients of $a^n(q/a; q)_n/(\alpha a; q)_n$ to complete the proof of Proposition 8.2. \square

The q -Watson formula due to Andrews [2] can be stated in the following proposition.

Proposition 8.3. (Andrews) *We have the q -formula*

$${}_4\phi_3 \left(\begin{matrix} q^{-n}, \alpha q^n, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{q\alpha}, -\sqrt{q\alpha}, \lambda \end{matrix}; q, q \right) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{(q, \alpha q/\lambda; q^2)_{n/2} \lambda^{n/2}}{(q\alpha, q\lambda; q^2)_{n/2}}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Replacing q by q^2 in Theorem 1.4 and then setting $b = aq$ and $c = q\lambda/\alpha$, we find that

$$\frac{(\alpha q, \lambda a/q; q)_\infty (\lambda, \alpha a^2/q; q^2)_\infty}{(\alpha a, \lambda; q)_\infty (q\alpha, \lambda a^2/q; q^2)_\infty} = \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{4n})(\alpha, q/a; q)_{2n} (q, q\alpha/\lambda; q^2)_n (a/q)^{2n} \lambda^n}{(1 - \alpha)(q, \alpha a; q)_{2n} (q\alpha, q\lambda; q^2)_n}$$

Taking $m = 2$ and $(\alpha c_1, \alpha c_2, \alpha b_1, \alpha b_2, \alpha b) = (\sqrt{\lambda}, -\sqrt{\lambda}, \sqrt{q\alpha}, -\sqrt{q\alpha}, \lambda)$ in Theorem 1.3, we find that the left-hand side member of the above equation also equals

$$\sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n}{(1 - \alpha)(q, \alpha a; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, \alpha q^n, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{q\alpha}, -\sqrt{q\alpha}, \lambda \end{matrix}; q, q \right).$$

Thus we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{2n})(\alpha, q/a; q)_n (a/q)^n}{(1 - \alpha)(q, \alpha a; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, \alpha q^n, \sqrt{\lambda}, -\sqrt{\lambda} \\ \sqrt{q\alpha}, -\sqrt{q\alpha}, \lambda \end{matrix}; q, q \right) \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha q^{4n})(\alpha, q/a; q)_{2n} (q, q\alpha/\lambda; q^2)_n (a/q)^{2n} \lambda^n}{(1 - \alpha)(q, \alpha a; q)_{2n} (q\alpha, q\lambda; q^2)_n}. \end{aligned}$$

Using Theorem 1.1, we can equate the coefficients of $a^n(q/a; q)_n/(\alpha a; q)_n$ on both sides of the above equation, to complete the proof of Proposition 8.3. \square

Verma and Jain [22, Eq. (5.4)] proved the following series summation formula.

Proposition 8.4. (Verma and Jain) *We have the summation formula*

$${}_4\phi_3 \left(\begin{matrix} q^{-2n}, \alpha^2 q^{2n}, \lambda, q\lambda \\ q\alpha, q^2\alpha, \lambda^2 \end{matrix}; q^2, q^2 \right) = \frac{\lambda^n (-q, q\alpha/\lambda; q)_n (1 - \alpha)}{(\alpha, -\lambda; q)_n (1 - \alpha q^{2n})}.$$

Proof. Replacing α by $-\alpha$ and then (a, b, c) by $(\sqrt{a}, -\sqrt{a}, \lambda/\alpha)$ in Theorem 1.4, we deduce that

$$\frac{(\lambda^2 a/q^2; q^2)_\infty (-\alpha, \alpha a/q; q)_\infty}{(\alpha^2 a; q^2)_\infty (-\lambda, \lambda a/q^2; q)_\infty} = \sum_{n=0}^{\infty} \frac{(1 + \alpha q^{2n})(q^2/a; q^2)_n (q\alpha/\lambda, -\alpha; q)_n (\lambda a/q^2)^n}{(\alpha^2 a; q^2)_n (q, -\lambda; q)_n}.$$

Taking $m = 2$ in Theorem 1.3, then replacing q by q^2 and α by α^2 , and finally replacing $(\alpha^2 c_1, \alpha^2 c_2, \alpha^2 b, \alpha^2 b_1, \alpha^2 b_2)$ by $(\lambda, q\lambda, \lambda^2, q\alpha, q^2\alpha)$ we find that

$$\begin{aligned} & \frac{(\lambda^2 a/q^2; q^2)_\infty (-\alpha, \alpha a/q; q)_\infty}{(\alpha^2 a; q^2)_\infty (-\lambda, \lambda a/q^2; q)_\infty} \\ &= \sum_{n=0}^{\infty} \frac{(1 - \alpha^2 q^{4n})(\alpha^2, q^2/a; q^2)_n (a/q^2)^n}{(1 - \alpha)(q^2, \alpha^2 a; q^2)_n} {}_4\phi_3 \left(\begin{matrix} q^{-2n}, \alpha^2 q^{2n}, \lambda, q\lambda \\ q\alpha, q^2\alpha, \lambda^2 \end{matrix}; q^2, q^2 \right). \end{aligned}$$

Equating the coefficients of $a^n (q^2/a; q^2)_n / (\alpha^2 a; q^2)_n$ on the right-hand side of the above two equations, we complete the proof of Proposition 8.4. \square

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