Analytic structures of unitary RSOS models with integrable boundary conditions

Omar El Deeb¹

¹Department of Physics Modern University of Business and Science Damour, Lebanon odeeb@mubs.edu.lb

Abstract

In this paper, we consider the unitary critical restricted-solid-on-solid (RSOS) lattice $\mathcal{M}(5,6)$ model with integrable boundary conditions. We introduce its commuting double row transfer matrix satisfying the universal functional relations, and we use it in order to study the analytic structure of the transfer matrix eigenvalues and plot representative zero configurations of sample eigenvalues of the transfer matrix. We finally conclude with a comparative analysis with the critical and tricritical Ising models with integrable boundary conditions.

Keywords: $\mathcal{M}(5,6)$ model, conformal field theory, lattice models, Yang-Baxter integrability, unitary minimal models.

2000 Mathematics Subject Classification: 81T25, 81T40.

1 Introduction

Integrable models can be solved in finite volumes due to the infinite number of conservation laws that they have in 1+1 dimensional problems. The energy spectrum can be fully determined in this case, while it is a very difficult task in general. The Thermodynamic Bethe Ansatz (TBA) method allows us to calculate the vacuum polarization effects of the ground state and its energy. (Zamolodchikov, 1990; Zamolodchikov, 1991b; Zamolodchikov, 1991a). Another important and challenging task is to extend this method in order to determine the excited state spectra. The analytic continuation method provides some information about some excited states using the ground state TBA equations as was done in (Dorey and Tateo, 1996), but this method fails in obtaining the full excitation spectrum for many models including the non-unitary $\mathcal{M}(3,5)$ and the scaling Lee-Yang model (Bajnok and El Deeb, 2010; Lee and Yang, 1952).

tions for excited states by solving the functional relations obtained from the Yang-Baxter regularization (Klumper and Pearce, 1991b; Klumper and Pearce, 1991a; Klumper and Pearce, 1992; Baxter, 1982). Their solutions can be used to fully determine the excitation spectrum by exploiting analytic and asymptotic properties. This approach was successfully implemented in solving the tricritical Ising model $\mathcal{M}(4;5)$ with conformal boundary conditions (Pearce, Chim and Ahn, 2001; Pearce, Chim and Ahn, 2003). The lattice regularization approach was also used to solve the Lee-Yang theory (Belavin, Polyakov and Zamolodchikov, 1984; Bajnok, El Deeb and Pearce, 2015; El Deeb, 2015) as well as the $\mathcal{M}(3,5)$ model (El Deeb, 2017).

In this paper we consider the critical unitary $\mathcal{M}(5,6)$ lattice model with integrable boundary conditions. We introduce its commuting double row transfer matrix satisfying the universal functional relations, and we use it in order to study the analytic structure of the transfer matrix eigenvalues and plot representative zero configurations of sample eigenvalues of the transfer matrix. We finally conclude with a comparative analysis with related unitary models with integrable boundary conditions.

The paper is structured as follows: in Section 2, the conformal model of the A_5 RSOS lattice model of Forrester-Baxter (Riggs, 1989; Andrews, Baxter and Forrester, 1984; Baxter and Forrester, 1985) is explored in Regime III with crossing parameter $\lambda = \frac{\pi}{6}$. It introduces the commuting double row transfer matrices with integrable boundaries. Section 3 analyzes the conformal spectra of its transfer matrices. We investigate the analytic structure of the transfer matrix eigenvalues, classify their excited states in the (m, n) system and plot sample zero configurations of representative eigenvalues. We then compare the analytic structure and the zero configuration with corresponding configurations of the related unitary models like the critical and tricritical Ising models. Section 4 concludes the paper with discussions and future work.

2 The $\mathcal{M}(5,6)$ Lattice Model

We analyze the Restricted Solid-on-Solid (RSOS) $\mathcal{M}(5,6)$ lattice model defined on a square lattice built on an A_5 Dynkin diagram, with heights differing by ± 1 at nearest neighbor sites. It is one of the A_L Forrester-Baxter models developed by (Andrews et al., 1984; Baxter and Forrester, 1985; Feverati, Pearce and Ravanini, 2003), with L = 5 in our case.

The Boltzmann weights of the general A_L Forrester-Baxter models are as follows:

$$W\begin{pmatrix} a \pm 1 & a \\ a & a \mp 1 \end{pmatrix} = \frac{s(\lambda - u)}{s(\lambda)}$$
$$W\begin{pmatrix} a & a \pm 1 \\ a \mp 1 & a \end{pmatrix} = \frac{g_{a \mp 1}}{g_{a \pm 1}} \frac{s((a \pm 1)\lambda)}{s(a\lambda)} \frac{s(u)}{s(\lambda)}$$
$$W\begin{pmatrix} a & a \pm 1 \\ a \pm 1 & a \end{pmatrix} = \frac{s(a\lambda \pm u)}{s(a\lambda)}$$
(2.1)

where a = 1, ..., L, while u is the spectral parameter. At criticality, s(u) = sin(u) and corresponds to the conformal massless model. λ is the crossing parameter and it is given by

$$\lambda = \frac{(p'-p)\pi}{p'} \tag{2.2}$$

where p' = L + 1 and p, p' are coprime integers with p < p'.

The local face weights satisfy the Yang-Baxter equation and this ensures that the model is integrable. The gauge factors g_a are arbitrary and here they are all set to be equal to 1. The critical Forrester-Baxter models in Regime III in the continuum scaling limit

Regime III:
$$0 < u < \lambda, \quad 0 < q < 1$$
 (2.3)

correspond to the minimal models $\mathcal{M}(p,p')$ whose central charge is

$$c = 1 - \frac{6(p - p')^2}{pp'}$$
(2.4)

In this paper we consider the $\mathcal{M}(5,6)$ model having $\lambda = \frac{\pi}{6}$ and $c = \frac{4}{5}$. A minimal $\mathcal{M}(p,p')$ model has $\frac{(p-1)(p'-1)}{2}$ scaling fields hence the $\mathcal{M}(5,6)$ has ten independent scaling fields.

Transfer matrices

The local face weights are used to construct the transfer matrices. Since the local face weights satisfy the Yang-Baxter equations, we can show that they form commuting families $[\mathbf{D}(u), \mathbf{D}(v)] = 0$. This model satisfies the same functional relation satisfied by the tricritical hard squares, hard hexagon models and the Lee-Yang model and the $\mathcal{M}(3,5)$ model (Baxter, 1982; Baxter, 1980; Baxter and Pearce, 1982; Baxter and Pearce, 1983; Bajnok et al., 2015; El Deeb, 2015; El Deeb, 2017) but with spectral parameter $\lambda = \frac{\pi}{6}$. However, this model, with its new crossing parameter, has its own analytic structure with three analyticity strips. From the Yang-Baxter equations, we can show that the double row transfer matrices satisfy the functional relation given by

$$\mathbf{D}(u)\mathbf{D}(u+\lambda) = 1 + \mathbf{Y}.\mathbf{D}(u+3\lambda)$$
(2.5)

where \mathbf{Y} in (2.5) is the \mathbb{Z}_2 height reversal symmetry.

 E_n , the conformal spectrum of energies of the $\mathcal{M}(5,6)$ model can be obtained through finite size corrections from the logarithm of the double row transfer matrix eigenvalue. The finite size corrections in the boundary case are given by

$$-\log T(u) = N f_{\text{bulk}}(u) + f_{\text{boundary}}(u,\xi) + \frac{2\pi}{N} E_n \sin \vartheta$$

where T(u) are the eigenvalues of D(u), N is the number of face weights and

$$\vartheta = \frac{\pi u}{\lambda} = 6u \tag{2.6}$$

is the anisotropy angle. f_{bulk} and f_{boundary} are the bulk free energy and the boundary free energy respectively. N is even in the boundary case.

2.0.1 Boundary weights

Commuting row transfer matrices and triangle boundary conditions that satisfy the left and right boundary Yang Baxter equations guarantee the integrability of this model. We label the conformal boundary conditions by the Kac labels (r, s) where $1 \le r \le 4$ and $1 \le s \le 5$. We limit

our study to the (1,1) boundary, as it is a good representative of the other boundary conditions, with minor differences in their analytic structures. The (1,1) triangle boundary weights are arbitrary and they are given by

$$K_L \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \frac{s(2\lambda)}{s(\lambda)}, \quad K_R \begin{pmatrix} 2 & 1 \\ 1 & u \end{pmatrix} = 1$$
(2.7)

Other integrable boundary conditions can be constructed by the repeated action of a seam on the integrable (1,1) boundary (Behrend and Pearce, 2001), and can be derived automatically. The fact that the boundary weights satisfy the left and right boundary Yang-Baxter equations ensures the integrability of the model in presence of those boundaries.

2.0.2 Double row transfer matrix

We construct a family of commuting double row transfer matrices D(u) from the face and triangle boundary weights defined before. For a lattice of width N, transfer matrix D(u) is given by

$$\mathbf{D}(u)_{\mathbf{a}}^{\mathbf{b}} = \sum_{c_0,\dots,c_N} K_L \begin{pmatrix} r & c_0 | \lambda - u \end{pmatrix} W \begin{pmatrix} r & b_1 & | \lambda - u \end{pmatrix} W \begin{pmatrix} b_1 & b_2 & | \lambda - u \end{pmatrix} \dots W \begin{pmatrix} b_{N-1} & s & | \lambda - u \end{pmatrix} \times W \begin{pmatrix} c_0 & c_1 & | u \end{pmatrix} W \begin{pmatrix} c_1 & c_2 & | u \end{pmatrix} \dots W \begin{pmatrix} c_{N-1} & c_N & | u \end{pmatrix} K_R \begin{pmatrix} c_N & s & | u \end{pmatrix} \dots W \begin{pmatrix} c_{N-1} & c_N & | u \end{pmatrix} K_R \begin{pmatrix} c_N & s & | u \end{pmatrix}$$
(2.8)

It satisfies periodicity $\mathbf{D}(u + \pi) = \mathbf{D}(u)$, commutativity $[\mathbf{D}(u), \mathbf{D}(v)] = 0$ and the crossing symmetry property $\mathbf{D}(u) = \mathbf{D}(\lambda - u)$. In the general case, $\mathbf{D}(u)$ is not symmetric or normal, but it can be diagonalized because $\tilde{\mathbf{D}}(u) = \mathbf{G}\mathbf{D}(\mathbf{u}) = \tilde{\mathbf{D}}(u)^T$ is symmetric where the diagonal matrix \mathbf{G} is given by

$$\mathbf{G}_{\mathbf{a}}^{\mathbf{b}} = \prod_{j=1}^{N-1} G(a_j, a_{j+1}) \delta(a_j, b_j) \quad \text{with} \quad G(a, b) = \begin{cases} \frac{s(\lambda)}{s(2\lambda)}, & b = 1, 4\\ 1 & \text{otherwise} \end{cases}$$
(2.9)

We introduce the normalized transfer matrix

$$\mathbf{D}(u) = S_b(u) \frac{s^2(2u-\lambda)}{s(2u+\lambda)s(2u-3\lambda)} \left(\frac{s(\lambda)s(u+2\lambda)}{s(u+\lambda)s(u+3\lambda)}\right)^N \mathbf{T}(u)$$
(2.10)

In the following analysis we discuss (1,1) left and right boundary weights corresponding to the (r,s) = (1,1) boundary. The eigenvalues of the normalized double row transfer matrix $\mathbf{T}(u)$ satisfy the functional equation

$$t(u)t(u+\lambda) = 1 + t(u+3\lambda)$$
 (2.11)

3 Conformal Spectra

In this section, we analyze the complex zero distributions of the eigenvalues of the double row transfer matrix with emphasis on the behavior of finite excitations above the ground state.

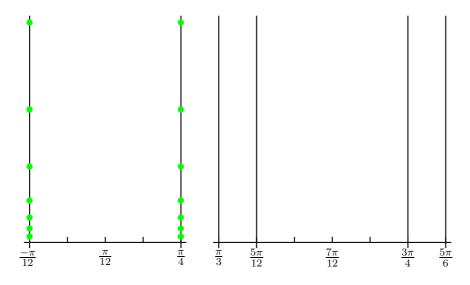


Figure 1: The zero configuration of the eigenvalue of the transfer matrix corresponding to the ground state. All the zeros are distributed as 2-strings in the first analyticity strip.

(m, n) systems and zero patterns

This lattice model corresponds to the conformal field theory model with central charge $c = \frac{4}{5}$. The face weights and the triangle boundary weights are expressed in terms of the trigonometric functions $s(u) = \sin(u)$. We characterize its eigenvalues by the locations and patterns of the zeros in the complex u- plane. The elements of the unrenormalized transfer matrix are Laurent polynomials in the variables $z = e^{iu}$ and $z^{-1} = e^{-iu}$. The transfer matrices are commuting families with a common set of u-independent eigenvectors. Consequently, the eigenvalues are also Laurent polynomials of the same degree. We numerically diagonalize those eigenvalues and obtain their zeros. They are characterized by the location and the pattern of the zeros in the complex u-plane that are analyzed in terms of the (m, n) systems.

In the boundary case, it is enough to study the eigenvalue zero distributions on the upper half plane as the transfer matrix is symmetric under complex conjugation. The zeros form strings and the excitations are described by their string content in the analyticity strips. In this paper we only consider the boundary case with (r, s) = (1, 1). There are three different analyticity strips in the complex *u*-plane but the third is a subset of the second. They are given by

$$\frac{-\pi}{12} < \text{Re } u < \frac{\pi}{4} , \ \frac{\pi}{3} < \text{Re } u < \frac{5\pi}{6} , \ \frac{5\pi}{12} < \text{Re } u < \frac{3\pi}{4}$$
(3.1)

In terms of λ , the analyticity strips in the complex *u*-plane could be written as:

$$\frac{-\lambda}{2} < \operatorname{Re} u < \frac{3\lambda}{2} , \ 2\lambda < \operatorname{Re} u < 5\lambda , \ \frac{5\lambda}{2} < \operatorname{Re} u < \frac{9\lambda}{2}$$
(3.2)

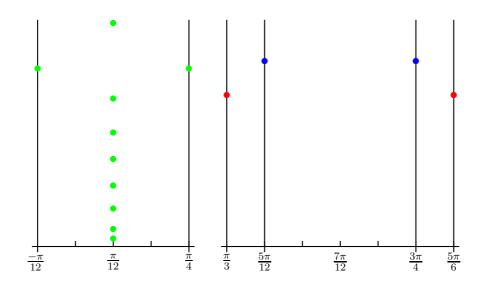


Figure 2: A typical configuration of zeros of an eigenvalue of the transfer matrix corresponding to an excited state. The zeros of the first strip are in green, the second in red, and the third in blue.

We notice the occurrence of zeros in all analyticity strips. In the first strip we assign those patterns as "1-strings" and "2-strings" formed by single zeroes and pairs of zeroes respectively. In the second, only "2-strings" appear while in the third we obtain again "1-strings" and "2-strings". The second and the third strips could be treated as one analyticity strip with a pattern of long and short 2-strings. However, we follow here the general classification of RSOS models with more than one analyticity strips for unitary $\mathcal{M}(L, L+1)$ models. Figure 1 gives the zero configuration content for the ground state eigenvalue of the boundary $\mathcal{M}(5,6)$ model while figures 2 and 3 display sample configurations for eigenvalues corresponding to excited states .

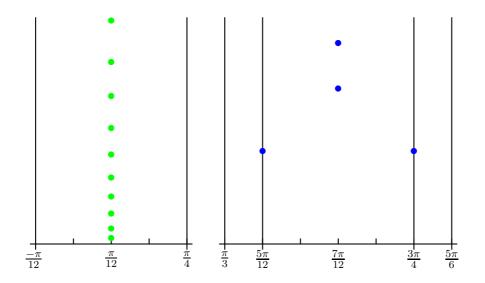


Figure 3: Another configuration of zeros of an eigenvalue of the transfer matrix corresponding to an excited state. Here we see the 1-strings of the first analyticity strip in green together with 1-strings and 2-strings of the third analyticity strips in blue.

In the first strip, a 1-string $u_j = \frac{\pi}{12} + iv_j$ whose real part is $\frac{\pi}{12}$ lies in the middle of the analyticity strip. Each 2-string consists of a pair of zeros whose real parts are at $\frac{-\pi}{12}$ and $\frac{\pi}{4}$, with equal imaginary parts, thus $u_j = \frac{-\pi}{12} + iy_j$, $\frac{\pi}{4} + iy_j$. In the second strip, the 2-string lies at $u_j = \frac{\pi}{3} + iy_j$, $\frac{5\pi}{6} + iy_j$ with equal imaginary parts and with real parts $\frac{\pi}{3}$ and $\frac{5\pi}{6}$. Finally, the third strip contains a pattern of a 1-string occurring at $u_j = \frac{7\pi}{12} + iv_j$ whose real part is $\frac{7\pi}{12}$ and 2-strings occurring at $u_j = \frac{5\pi}{12} + iy_j$, $\frac{3\pi}{4} + iy_j$ with real parts $\frac{5\pi}{12}$ and $\frac{3\pi}{4}$.

The string contents are described by (m, n) systems (Berkovich, 1994). The (1, 1) sector, in unitary minimal models $\mathcal{M}(L, L-1)$ satisfies the relation:

$$\boldsymbol{m} + \boldsymbol{n} = \frac{1}{2}(N\boldsymbol{e_1} + A\boldsymbol{m}) \tag{3.3}$$

where A is the adjacency matrix of the A_{L-2} model, $e_1 = (1, 0, .., 0)$, $m = (m_1, m_2, ..., m_{L-2})$, and $n = (n_1, n_2, ..., n_{L-2})$.

For the unitary $\mathcal{M}(5,6)$ model, we obtain the relations

$$m_1 + n_1 = \frac{N + m_2}{2}$$
, $m_2 + n_2 = \frac{m_1 + m_3}{2}$, $m_3 + n_3 = \frac{m_2}{2}$

where *m* is the number of short 2-strings, *n* is the number of long 2-strings and *N* is even. We can verify that the number of zeros $N = 2n_1 + 2n_2 + 2n_3 + m_1 + m_3$ hence the 1-strings corresponding to the second analyticity strip do not contribute to the zero configuration plot. In all of the sectors, the 1-string contributes to one zero. In addition, 2-string contributes two zeroes. Hence, the (m, n) system expresses the conservation of the 2N zeroes in the periodicity strip. The ground state occurs when all zeros occur as 2-strings in the first sector solely. The appearance of 1-strings in this sector and all other strings in the other sectors expresses excited states. The first excited states are expressed by the 1-strings of the first sector and 2-strings from the second strip. The appearance of zero patterns from the 1-string and 2-string content of the third strip represents the next higher excited states. Note that $n \to N$ as $N \to \infty$ while m_i and n_2 and n_3 are finite for finite excitations.

Other unitary models Several other unitary models were analyzed including the $\mathcal{M}(3,4)$ critical Ising model and the $\mathcal{M}(4,5)$ tricritical Ising model. We can notice that the analytic structure consists of a single strip for the critical Ising model with the real part given by

$$\frac{-\pi}{8} < \operatorname{Re} \, u < \frac{3\pi}{8}$$

In terms of the spectral parameter λ ,

$$\frac{-\lambda}{2} < \operatorname{Re} u < \frac{3\lambda}{2}$$

and symmetric with respect to Re $u = \frac{3\pi}{8}$ with $\lambda = \frac{\pi}{4}$.

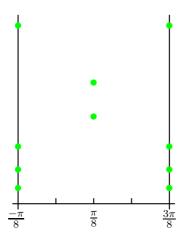


Figure 4: A configuration of zeros of an eigenvalue of the transfer matrix corresponding to an excited state of the Ising model.

The tricritical Ising model consists of two analyticity strips given by

$$rac{-\pi}{10} < {\rm Re}\; u {<} rac{3\pi}{10}$$
 , $\; rac{2\pi}{5} < {\rm Re}\; u {<} rac{4\pi}{5}$

corresponding to

$$rac{-\lambda}{2} < {\rm Re}\; u {<} rac{3\lambda}{2}$$
 , $\; 2\lambda < {\rm Re}\; u {<} 4\lambda$

with $\lambda = \frac{\pi}{5}$. Sample configurations of zeroes of eigenvalues representing excited states of those models are given in figure 4 and figure 5.

In this respect, the $\mathcal{M}(5,6)$ model has a similar structure to those unitary models, with three analyticity strips as discussed before. This is a general feature of the $\mathcal{M}(L, L+1)$ unitary models with $\lambda = \frac{\pi}{L+1}$ with L-2 analyticity strips.

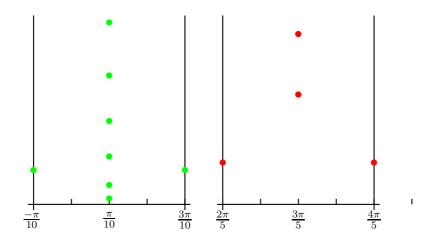


Figure 5: A configuration of zeros of an eigenvalue of the transfer matrix corresponding to an excited state of the tricritical Ising model. Here we see the 1-strings of the first analyticity strip in green together with 1-strings and 2-strings of the second analyticity strips in red.

4 Conclusion

In this paper, the $\mathcal{M}(5,6)$ relativistic integrable theory was partially analyzed from the lattice point of view, in the (r = 1, s = 1) sector. We described the patterns of zeros of the corresponding double row transfer matrix eigenvalues and their (m, n) systems. We adopted a similar approach to analyze this model as was used before in (Bajnok et al., 2015; El Deeb, 2015; El Deeb, 2017). Other sectors of the boundary case are similar in their patterns of zeros. The only difference is that some analytic strips would contain a fixed zeroes at their centers. Future work should extend the scope and exploit the lattice description of the integrable scattering theory in order to fully solve the TBA equations of the system and determine the spectrum of the model. The massive $\mathcal{M}(3,5)$ and $\mathcal{M}(5,6)$ models must be studied in following papers. It also remains essential to study the same models using the bootstrap methods.

References

- Andrews, G., Baxter, R. and Forrester, P. 1984. Eight-vertex SOS model and generalized Rogers-Rmanujan-type identities, *J. Stat. Phys.* **35**: 193–266.
- Bajnok, Z. and El Deeb, O. 2010. Form factors in the presence of integrable defects, *Nucl. Phys. B* **832**: 500–519.
- Bajnok, Z., El Deeb, O. and Pearce, P. 2015. Finite-volume spectra of the Lee-Yang model, *JHEP* 04: 1–43.
- Baxter, R. 1980. Hard hexagons: exact solution, J. Phys. A 13: L61.
- Baxter, R. 1982. Exactly Solved Models in Statistical Mechanics, Academic Press, London.
- Baxter, R. and Forrester, P. 1985. Further exact solutions of the eight-vertex SOS model and generalizations of the Rogers-Ramanujan identities, *J. Stat. Phys.* **38**: 435–472.
- Baxter, R. and Pearce, P. 1982. Hard hexagons: interfacial tension and correlation length, *J. Phys. A* **15**: 897–910.
- Baxter, R. and Pearce, P. 1983. Hard squares with diagonal attractions, *J. Phys. A* **16**: 2239–2255.
- Behrend, R. and Pearce, P. 2001. Integrable and conformal boundary conditions for s'(2) A-D-E lattice models and unitary minimal conformal field theories, *J. Stat. Phys.* **102**: 577–640.
- Belavin, A., Polyakov, A. and Zamolodchikov, A. 1984. Infinite conformal symmetry in twodimensional quantum field theory, *Nucl. Phys. B* **241**: 333–380.
- Berkovich, A. 1994. Fermionic counting of RSOS states and virasoro character formulas for the unitary minimal series $\mathcal{M}(v, v + 1)$: Exact results, *Nucl. Phys. B* **431**: 315–348.
- Dorey, P. and Tateo, R. 1996. Excited states by analytic continuation of TBA equations, *Nucl. Phys. B* **482**: 639–659.

- El Deeb, O. 2015. *Lee-Yang Model in Presence of Defects*, PhD thesis. URL: http://inspirehep.net/record/1344797/files/arXiv:1502.03976.pdf
- El Deeb, O. 2017. The critical boundary RSOS $\mathcal{M}(3,5)$ model, *Theoret. and Math. Phys.* **193:3**: 1811–1825.
- Feverati, G., Pearce, P. and Ravanini, F. 2003. Exact $\varphi_{1,3}$ boundary flows of the tricritical Ising model, *Nucl. Phys. B* **675**: 469–515.
- Klumper, A. and Pearce, P. 1991a. Analytic calculation of scaling dimensions: Tricritical hard squares and critical hard hexagons, *J. Stat. Phys.* **64**: 13–76.
- Klumper, A. and Pearce, P. 1991b. Finite-size corrections and scaling dimensions of solvable lattice models: An analytic method, *Phys. Rev. Lett.* **66**: 974–977.
- Klumper, A. and Pearce, P. 1992. Conformal weights of RSOS lattice models and their fusion hierarchies, *Physica A* **183**: 304.
- Lee, T. and Yang, C. 1952. Statistical theory of equations of state and phase transitions. I. Theory of condensation, *Phys. Rev.* 87: 404–409.
- Pearce, P., Chim, L. and Ahn, C. 2001. Excited TBA equations I: massive tricritical Ising model, *Nucl. Phys. B* **601**: 539–568.
- Pearce, P., Chim, L. and Ahn, C. 2003. Excited TBA equations II: massless flow from tricritical to critical Ising model, *Nucl. Phys. B* 660: 579–606.
- Riggs, H. 1989. Solvable lattice models with minimal and non unitary critical behavior in two dimensions, *Nucl. Phys. B* **326**: 673–688.
- Zamolodchikov, A. 1990. Thermodynamic bethe ansatz in relativistic models: Scaling 3-state Potts and Lee-Yang models, *Nucl. Phys. B* **342**: 695–720.
- Zamolodchikov, A. 1991a. From tricritical Ising to critical Ising by thermodynamic Bethe ansatz, *Nucl. Phys. B* **358**: 524–546.
- Zamolodchikov, A. 1991b. Thermodynamic Bethe ansatz for RSOS scattering theories, *Nucl. Phys. B* **358**: 497–523.