A generalized Hardy-Ramanujan formula for the number of restricted integer partitions

Tiefeng Jiang[1] and Ke Wang[2]

University of Minnesota and Hong Kong University of Science and Technology

Abstract

We derive an asymptotic formula for $p_n(N, M)$, the number of partitions of integer n with part size at most N and length at most M. We consider both N and M are comparable to $\sqrt{n}$. This is an extension of the classical Hardy-Ramanujan formula and Szekeres' formula. The proof relies on the saddle point method.

1 Introduction

A partition of integer n is a sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_k > 0$ satisfying

$$\sum_{i=1}^{k} \lambda_i = n,$$

where k is the length and the λ_i's are the parts of the partition. Let p_n be the number of all partitions of n. In a celebrated paper [7], Hardy and Ramanujan proved the asymptotic formula

$$p_n \sim \frac{1}{4\sqrt{3}n} \exp\left(\sqrt{\frac{2n}{3}}\pi\right). \tag{1.1}$$

For two positive sequences $\{a_n\}$ and $\{b_n\}$, we use $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$.

Let $p_n(N)$ be the number of partitions of n with part size at most N. Note that $p_n(n) = p_n$. Szekeres [18, 19] obtained the asymptotic formulas for $p_n(N)$ as n tends to infinity, using substantially different approaches for two distinct though slightly overlapping ranges of N. In particular, Szekeres' formula holds if $\alpha := N/\sqrt{n} \geq C > 0$. The formula was reproduced later by Canfield [3] using recursive equations for $p_n(N)$ and Taylor expansion. It is observed in [3] that Szekeres' formula could be combined into a single form

$$p_n(\alpha\sqrt{n}) = \frac{\rho(\alpha)\exp\left[\left(2\rho(\alpha) - \alpha\log(1 - e^{-\alpha\rho(\alpha)})\right)\sqrt{n}\right]}{2^{3/2}\pi n\sqrt{1 - \left(\frac{\alpha^2}{2} + 1\right)e^{-\alpha\rho(\alpha)}}}\left(1 + O\left(n^{-1/6+\epsilon}\right)\right) \tag{1.2}$$

---

[1]School of Statistics, University of Minnesota, 224 Church Street, S. E., MN55455, USA, jiang040@umn.edu. The research of Tiefeng Jiang is supported in part by NSF Grant DMS-1209166 and DMS-1406279.

[2]Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, kewang@ust.hk. Ke Wang is supported by by HKUST Initiation Grant IGN16SC05.

as $n \to \infty$, where $\rho(\alpha) > 0$ is the unique solution to the implicit equation

$$\rho(\alpha)^2 = \mathrm{Li}_2(1 - e^{-\alpha\rho(\alpha)}) \tag{1.3}$$

and Li_2 is *Spence's function*, or *dilogarithm*, defined for complex number $|z| < 1$ as

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}.$$

Since $\rho(\alpha)$ is an increasing function and satisfies

$$\rho(0) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} \rho(\alpha) = \frac{\pi}{\sqrt{6}},$$

it can be checked that when $\alpha \to \infty$, the right side of (1.2) converges to the right side of (1.1). Several years later, Romik [15] provided another proof of (1.2) using probabilistic methods.

In this paper, we focus on $p_n(N, M)$, the number of partitions of n with part size at most N and length at most M. The $p_n(N, M)$ for $n = 0, \ldots, NM$ are also called the coefficients of the *q-binomial coefficients* or *Gaussian binomial coefficients*. In [16, Theorem 2.4], the asymptotic behavior of $p_n(N, M)$ has been investigated assuming M is fixed and N gets arbitrarily large. Very recently, Richmond [14] derived the asymptotic formula for $p_n(N, M)$ when both N and M are close to their expected values $\frac{\sqrt{6n}}{\pi} \log(\frac{\sqrt{6n}}{\pi})$ (see [4] for the distributions of the length and largest part of a uniform integer partition). When n is close to $NM/2$, an asymptotic formula for $p_n(N, M)$ was obtained before by Takács [20] . More precisely, whenever $N \to \infty$, $M \to \infty$, $|N - M| = O(\sqrt{M + N})$ and $|n - NM/2| = O(\sqrt{NM(N + M)})$, it was shown in [20] (see also [1]) that

$$p_n(N, M) = \binom{N + M}{N} \sqrt{\frac{6}{\pi NM(N + M + 1)}} \exp\left(-\frac{6(n - NM/2)^2}{NM(N + M + 1)}\right) (1 + o(1)). \tag{1.4}$$

In this paper, we aim to complement this result by deriving asymptotic formulas for $p_n(N, M)$ when n is around NM/τ with $\tau \neq 2$. We prove such formulas by imposing some extra requirement on τ (see Theorem 1 and discussion in the end of Section 1.1). Our motivation to derive such formulas lies in studying the limiting distribution of a partition chosen uniformly from the set of restricted partitions $\mathcal{P}_n(N)$, that is, partitions of n with part size at most N, for the entire range $1 \leq N \leq n$. Currently, the uniform distributions on $\mathcal{P}_n(n)$ (see [4, 6, 12]), $\mathcal{P}_n(N)$ for N fixed integer (see [8]) and $\mathcal{P}_n(N)$ for $N = o(n^{1/3})$ (see [9]) have been studied. We believe the asymptotic formula of $p_n(N, M)$ plays an important role in understanding the uniform distribution on $\mathcal{P}_n(N)$ for other values of N. This will be explored in future research.

1.1 Main result

For positive integers n, M, N, we denote

$$\alpha = \frac{N}{\sqrt{n}} \quad \text{and} \quad \beta = \frac{M}{\sqrt{n}}.$$

2

THEOREM 1. *Let $\varepsilon \in (0, 1/8)$ be given. Then uniformly for $N \geq 4\sqrt{n}$ and $M \geq 4\sqrt{n}$, we have, as $n \to \infty$, that*

$$p_n(N, M) = \frac{g(\alpha, \beta)^2}{2\pi\sqrt{L(\alpha, \beta)}}\sqrt{\frac{1 - e^{-(\alpha+\beta)g(\alpha,\beta)}}{(1 - e^{-\alpha g(\alpha,\beta)})(1 - e^{-\beta g(\alpha,\beta)})}} \cdot \frac{e^{\sqrt{n}K(\alpha,\beta)}}{n}\Big(1 + O(n^{-1/4+\epsilon})\Big),$$

where $g(\alpha, \beta) > 0$ is the unique solution to the implicit equation

$$g^2(\alpha, \beta) = Li_2(1 - e^{-\alpha g(\alpha,\beta)}) + Li_2(1 - e^{-\beta g(\alpha,\beta)}) - Li_2(1 - e^{-(\alpha+\beta)g(\alpha,\beta)}),$$

and

$$L(\alpha, \beta) = 2g^2(\alpha, \beta) + \frac{(\alpha+\beta)^2 g^2(\alpha, \beta)}{e^{(\alpha+\beta)g(\alpha,\beta)} - 1} - \frac{\alpha^2 g^2(\alpha, \beta)}{e^{\alpha g(\alpha,\beta)} - 1} - \frac{\beta^2 g^2(\alpha, \beta)}{e^{\alpha g(\alpha,\beta)} - 1},$$

$$K(\alpha, \beta) = g(\alpha, \beta) + \frac{1}{g(\alpha, \beta)}\Big(\frac{\pi^2}{6} + Li_2(e^{-(\alpha+\beta)g(\alpha,\beta)}) - Li_2(e^{-\alpha g(\alpha,\beta)}) - Li_2(e^{-\beta g(\alpha,\beta)})\Big).$$

A few remarks regarding Theorem 1 are in order.

REMARK 1.1. *The uniqueness and existence of $g(\alpha, \beta) > 0$ is guaranteed by Lemma 2.4. .*

REMARK 1.2. *All of $g(\alpha, \beta)$, $L(\alpha, \beta)$, $K(\alpha, \beta)$ and $\sqrt{\frac{1-e^{-(\alpha+\beta)g(\alpha,\beta)}}{(1-e^{-\alpha g(\alpha,\beta)})(1-e^{-\beta g(\alpha,\beta)})}}$ in $p_n(N, M)$ in Theorem 1 are bounded from above and from below by two universal positive constants for all $\alpha, \beta \in [4, \infty]$. The error term $O(n^{-1/4+\epsilon})$ is also uniform in $\alpha, \beta \in [4, \infty]$.*

REMARK 1.3. *If we do not put restrictions on either the part size or the length of the partition, then either α or β is equal to infinity in our setting, respectively. In this case, the asymptotic formula discovered in Theorem 1 is identical to the Szekeres' formula* (1.2).

Let us assume $\beta = \infty$ (by (2.3) the same will apply to the case $\alpha = \infty$). We first notice that $g(\alpha, \infty) = \rho(\alpha)$, where $\rho(\alpha)$ is the one in (1.3), and secondly,

$$K(\alpha, \infty) = \rho(\alpha) + \frac{1}{\rho(\alpha)}\left(\frac{\pi^2}{6} - Li_2(e^{-\alpha\rho(\alpha)})\right)$$

$$= \rho(\alpha) + \frac{1}{\rho(\alpha)}\left[Li_2(1 - e^{-\alpha\rho(\alpha)}) - \alpha\rho(\alpha)\log(1 - e^{-\alpha\rho(\alpha)})\right]$$

$$= 2\rho(\alpha) - \alpha\log(1 - e^{-\alpha\rho(\alpha)}),$$

where we use (2.6) in the second identity. Also, it is easy to check that

$$L(\alpha, \infty) = 2\rho^2(\alpha)\left[1 - \frac{\alpha^2}{2(e^{\alpha\rho(\alpha)} - 1)}\right].$$

Therefore, by plugging these into $p_n(N, M)$ in Theorem 1, we will have (1.2).

We derive the asymptotic formula for $p_n(\alpha\sqrt{n}, \beta\sqrt{n})$ in Theorem 1 assuming α and β are greater than 4. Our proof uses the saddle point method; see [11, Section 12] or [5, Chapter 8] for a detailed introduction to this method. For the case when α, β are both small, we could combine Theorem 1 and $p_n(N, M) = p_{MN-n}(M, N)$ (see (2.4)) to derive the formula, since both

$M/\sqrt{MN-n}$ and $N/\sqrt{MN-n}$ will be large. For instance, this would apply if $\max(\alpha, \beta) < \sqrt{16/15}$.

Finally, we conjecture Theorem 1 holds for arbitrary $\alpha, \beta > 0$ as long as $\alpha\beta = NM/n > 2$. For the remaining case $1 < \alpha\beta < 2$, we can apply the formula for $p_{MN-n}(M, N) = p_n(N, M)$ since $NM/(NM - n) = \alpha\beta/(\alpha\beta - 1) > 2$. Therefore, combining (1.4) with Theorem 1, the asymptotic formula for $p_n(N, M)$ is clear for arbitrary $N/\sqrt{n} \in (0, +\infty]$ and $M/\sqrt{n} \in (0, +\infty]$. The stronger assumption $\min(\alpha, \beta) \geq 4$ in Theorem 1 appears to be only a technical condition (see Lemma 2.5) and is used to control the $O(n^{-1/4+\epsilon})$ error term in the formula. It could be pushed to, say, $\min(\alpha, \beta) \geq 2.5$ with advanced help of Maple. We did not try to optimize the lower bound of $\min(\alpha, \beta)$.

The paper is organized as follows. In Section 2, we will introduce some background, and set up our calculation of obtaining the asymptotic formula in Theorem 1. Some analytic lemmas, which will be used later, are also included. In Section 3, we will carry out the detailed calculations. We will first derive the main term, and estimate the error term afterwards.

We note that shortly after a preprint version of this paper appeared on the arXiv, another preprint [10] appeared on the arXiv which derives an asymptotic formula for $p_n(N, M)$ using a probabilistic approach. In our notations, they obtained an asymptotic formula for $p_n(N, M)$ when $\alpha\beta \geq 2$ and assuming (α, β) is in an arbitrarily fixed compact set $K \subset \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, with an error bound $o(1)$ depending on the compact set K.

Notations: We use standard asymptotic notations o, O as n tends to infinity. We denote $a \lesssim b$ if there is a universal positive constant C such that $a \leq Cb$.

2 Preliminaries

In this section, we will first recall the generating function of $p_n(N, M)$ and Spence's functions, and show some of their properties that will be used. We will see a connection between the generating function of $p_n(N, M)$ and Spence's functions in Lemma 2.3. And then, we express $p_n(N, M)$ as an integral using Cauchy's integral formula, and it is the integral that we are going to estimate. In the end of this section, we prepare some lemmas for the detailed proof of Theorem 1 in Section 3. The usage of each lemma is explained before each of their statements.

2.1 Background materials

We begin with some basic properties of $p_n(N, M)$; see [2, Chapter 3] for a comprehensive introduction. Note that $p_n(N, M) = 0$ if $n > M \times N$ and $p_n(N, M) = 1$ if $n = M \times N$. Since $p_n(N, M) - p_n(N, M - 1)$ counts the number of partitions of n with length exactly M and part size at most N, that is $1 \leq \lambda_i \leq N$ for $1 \leq i \leq M$ and $\sum_{i=1}^{M} \lambda_i = n$, by considering $\tilde{\lambda}_i = \lambda_i - 1$, it follows easily that

$$p_n(N, M) - p_n(N, M - 1) = p_{n-M}(N - 1, M). \tag{2.1}$$

Denote $G(N, M; q)$ the generating function of $p_n(N, M)$ for $n \geq 0$. Thus

$$G(N, M; q) = \sum_{n \geq 0} p_n(N, M)q^n = \sum_{n=0}^{MN} p_n(N, M)q^n.$$

The generating function $G(N, M; q)$ has an explicit expression and is usually referred to as the *Gaussian polynomial*. The following lemma can be found at [2, Theorem 3.1]. We include the proof for the readers' convenience.

LEMMA 2.1. *For $M, N \geq 0$,*

$$G(N, M; q) = \frac{\prod_{j=1}^{N+M}(1 - q^j)}{\prod_{j=1}^{N}(1 - q^j) \prod_{j=1}^{M}(1 - q^j)} = \frac{\prod_{j=M+1}^{N+M}(1 - q^j)}{\prod_{j=1}^{N}(1 - q^j)}. \tag{2.2}$$

Proof. It follows from (2.1) that

$$G(N, M; q) = G(N, M - 1; q) + q^M \cdot G(N - 1, M; q).$$

Denote the RHS of (2.2) by $\widetilde{G}(N, M; q)$. By direct calculation, we see that

$$
\begin{aligned}
\widetilde{G}(N, M; q) &= \frac{\prod_{j=1}^{N+M-1}(1 - q^j)}{\prod_{j=1}^{N}(1 - q^j) \prod_{j=1}^{M-1}(1 - q^j)} \cdot \frac{1 - q^{N+M}}{1 - q^M} \\
&= \widetilde{G}(N, M - 1; q) \cdot \frac{1 - q^M + q^M(1 - q^N)}{1 - q^M} \\
&= \widetilde{G}(N, M - 1; q) + q^M \cdot \widetilde{G}(N - 1, M; q).
\end{aligned}
$$

Besides, it is easy to check that $G(N, M; q)$ and $\widetilde{G}(N, M; q)$ satisfy the same initial conditions

$$
\begin{aligned}
G(N, 0; q) &= G(0, M; q) = 1; \\
\widetilde{G}(N, 0; q) &= \widetilde{G}(0, M; q) = 1.
\end{aligned}
$$

Therefore, $G(N, M; q) = \widetilde{G}(N, M; q)$. □

The following properties of $p_n(N, M)$ can be verified easily from the Ferrers diagram of partitions. We also refer to [2, Theorem 3.10] for a proof.

LEMMA 2.2.

$$p_n(N, M) = p_n(M, N); \tag{2.3}$$
$$p_n(N, M) = p_{MN-n}(M, N). \tag{2.4}$$

It was first proved by Sylvester [17] that $p_n(N, M)$ is unimodal and

$$\max_{1 \leq n \leq NM} p_n(N, M) = p_{[\frac{1}{2}NM]}(N, M). \tag{2.5}$$

Now let us recall Spence's function, or dilogarithm, which is defined for complex numbers $|z| < 1$ as

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}.$$

It has an analytic continuation for $z \in \mathbb{C} \setminus [1, \infty)$ given by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u} \, du, \text{ or equivalently } \quad \mathrm{Li}_2(1-z) = \int_1^z \frac{\log u}{1-u} \, du,$$

where $\log$ is the principal branch of the logarithm function. $\mathrm{Li}_2(z)$ can be continuously extended to $z = 1$, and $\mathrm{Li}_2(1) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6$. Note that by a change of variable $s = -\log(t)$, we have

$$\mathrm{Li}_2(1-v) = \int_1^v \frac{\log t}{1-t} \, dt = \int_0^{\log(1/v)} \frac{s}{e^s - 1} \, ds.$$

Equivalently, we have for $x > 0$,

$$\int_0^x \frac{t}{e^t - 1} \, dt = \mathrm{Li}_2(1 - e^{-x}) = \frac{\pi^2}{6} + x \cdot \log(1 - e^{-x}) - \mathrm{Li}_2(e^{-x}), \tag{2.6}$$

where the last identity follows from the property that

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \lim_{\delta \to 0^+} \left(-\int_0^{1-\delta} \frac{\log(1-u)}{u} \, du - \int_{1-\delta}^z \frac{\log(1-u)}{u} \, du + \int_{1-\delta}^z \frac{\log u}{1-u} \, du \right)$$

$$= \frac{\pi^2}{6} - \lim_{\delta \to 0^+} \int_{1-\delta}^z d \Big(\log(1-u) \log u \Big)$$

$$= \frac{\pi^2}{6} - \log(z) \cdot \log(1-z).$$

2.2 Setting up of the calculation

By Cauchy's integral formula,

$$p_n(N, M) = \frac{1}{2\pi i} \int_{|z|=r} \frac{G(N, M; z)}{z^{n+1}} \, dz$$

for any $0 < r < 1$. We substitute $z = e^{-(v+iw)}$ and for convenience, denote

$$f(v + iw) = f_{N,M}(v + iw) = G(N, M; e^{-(v+iw)}) = \frac{\prod_{j=M+1}^{N+M} (1 - e^{-j(v+iw)})}{\prod_{j=1}^{N} (1 - e^{-j(v+iw)})}. \tag{2.7}$$

Therefore,

$$p_n(N, M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(v + iw) e^{n(v+iw)} \, dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\log f(v+iw)} e^{n(v+iw)} \, dw \tag{2.8}$$

6

for every $v > 0$. In the end, for our purpose, we will choose $v = c \cdot n^{1/2}$ for some constant $c > 0$. In Section 3, we will show that the main term in the above integral is

$$\frac{1}{2\pi} \int_{-w_0}^{w_0} e^{\log f(v+iw)} e^{n(v+iw)} \, dw \tag{2.9}$$

with $w_0 = n^{-3/4+\epsilon/3}$, and we will prove that what is left in the integral is a lower order term.

2.3 Supporting lemmas

To estimate (2.9), we shall first analyze the function $\log f(z)$.

LEMMA 2.3. *Assume $z = v + iw$ with $v > 0$, $|z| < 1$ and assume z stays within some angle in the right half plane, that is, $v/|z| = Re(z)/|z| \geq \lambda$ for some $\lambda > 0$. Recall $f(z)$ in (2.7). Then for $|z| \geq \frac{\bar{c}}{\min(M,N)}$ with some $\bar{c} > 0$, we have*

$$
\begin{aligned}
\log f(z) \; &= \; \frac{1}{z}\left(\frac{\pi^2}{6} + Li_2(e^{-z(N+M)}) - Li_2(e^{-zN}) - Li_2(e^{-zM})\right) \\
&\quad -\frac{1}{2}\log\left(\frac{1}{z}\right) - \frac{1}{2}\log\frac{(1 - e^{-zN})(1 - e^{-zM})}{1 - e^{-z(N+M)}} - \log\sqrt{2\pi} + O(|z|),
\end{aligned}
$$

where $\log$ is the principal branch of the logarithm function.

Proof. Since

$$\log f(z) = \sum_{j=M+1}^{M+N} \log(1 - e^{-jz}) - \sum_{j=1}^{N} \log(1 - e^{-jz}),$$

by Taylor expansion,

$$
\begin{aligned}
\log f(z) \; &= \; -\sum_{j=M+1}^{M+N}\sum_{k=1}^{\infty} \frac{e^{-jkz}}{k} + \sum_{j=1}^{N}\sum_{k=1}^{\infty} \frac{e^{-jkz}}{k} \\
&= \; \sum_{k=1}^{\infty} \frac{1}{k}\left(\sum_{j=1}^{N} e^{-kz\cdot j} - \sum_{j=M+1}^{M+N} e^{-kz\cdot j}\right) \\
&= \; \sum_{k=1}^{\infty} \frac{1}{k}\frac{e^{-kz}(1 - e^{-kz\cdot N})(1 - e^{-kz\cdot M})}{1 - e^{-kz}} \\
&= \; z\cdot\sum_{k=1}^{\infty} \frac{1}{kz}\frac{(1 - e^{-kz\cdot N})(1 - e^{-kz\cdot M})}{e^{kz} - 1}.
\end{aligned}
$$

Now we compare $\log f(z)$ with another summation.

$$
\begin{aligned}
&\log f(z) - z\cdot\sum_{k=1}^{\infty}(1 - e^{-kz\cdot N})(1 - e^{-kz\cdot M})\left(\frac{1}{k^2 z^2} - \frac{e^{-kz}}{2kz}\right) \\
&= \; z\cdot\sum_{k=1}^{\infty}(1 - e^{-kz\cdot N})(1 - e^{-kz\cdot M})\left(\frac{1}{kz(e^{kz} - 1)} - \frac{1}{k^2 z^2} + \frac{e^{-kz}}{2kz}\right).
\end{aligned}
$$

7

Define $\psi(z) = \psi_1(z) \cdot \psi_2(z)$ where

$$\psi_1(z) = (1 - e^{-Nz})(1 - e^{-Mz});$$

$$\psi_2(z) = \frac{1}{z(e^z - 1)} - \frac{1}{z^2} + \frac{e^{-z}}{2z}.$$

It is easy to verify that the Laurent expansion of $\psi_2(z)$ is equal to $-\frac{5}{12} + \lambda(z)$ where $\lambda(z)$ is analytic and $\lambda(0) = 0$. Thus $\psi_2(z)$ and $\psi(z)$ are analytic in the considered region.

Now we can express

$$\log f(z) - z \cdot \sum_{k=1}^{\infty} (1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M}) \left(\frac{1}{k^2 z^2} - \frac{e^{-kz}}{2kz} \right) = \sum_{k=1}^{\infty} z\psi(kz). \qquad (2.10)$$

Let us compute the summation on the RHS of (2.10). It is natural to use an integral to approximate the summation and we control the difference. Denote γ the ray starting from origin and passing through z. It is clear the integral $\int_\gamma \psi(u)\, du$ is convergent. We are going to show that

$$\left| \sum_{k=1}^{\infty} z\psi(kz) - \int_\gamma \psi(u)\, du \right| = O(|z|). \qquad (2.11)$$

Let γ_k be the line segment on γ from the point $(k-1)z$ to the point kz for $k \geq 1$. Therefore, we have

$$\left| \sum_{k=1}^{\infty} z\psi(kz) - \int_\gamma \psi(u)\, du \right| \leq \left| z\psi(z) - \int_{\gamma_1} \psi(u)\, du \right| + \left| \sum_{k=2}^{\infty} \left(z\psi(kz) - \int_{\gamma_k} \psi(u)\, du \right) \right|.$$

Since $\psi(z)$ is analytic at $z = 0$, we estimate the first term by

$$\left| z\psi(z) - \int_{\gamma_1} \psi(u)\, du \right| = O(|z|).$$

For the other term, by integration by parts, we have

$$\int_{\gamma_k} \psi(u)\, du = kz\psi(kz) - (k-1)z\psi((k-1)z) - \int_{\gamma_k} u\psi'(u)\, du$$

$$= z\psi(kz) + \int_{\gamma_k} \big((k-1)z - u \big)\psi'(u)\, du.$$

Thus

$$\left| z\psi(kz) - \int_{\gamma_k} \psi(u)\, du \right| = \left| \int_{\gamma_k} \big((k-1)z - u \big)\psi'(u)\, du \right|$$

$$\leq |z| \max_{u \in \gamma_k} \big(|u - (k-1)z| \cdot |\psi'(u)| \big)$$

$$\leq |z|^2 \max_{u \in \gamma_k} |\psi'(u)| \leq |z|^2 \max_{u \in \gamma_k} (|\psi_1'(u)\psi_2(u)| + |\psi_1(u)\psi_2'(u)|).$$

8

Since $|\psi_1(u)|$ and $|\psi_2(u)|$ are universally bounded on γ_k for $k \geq 2$, to show (2.11), it is enough to show that for $i = 1, 2$,

$$|z|^2 \cdot \sum_{k=2}^{\infty} \max_{u \in \gamma_k} |\psi_i'(u)| = O(|z|). \qquad (2.12)$$

In the below we will first prove (2.12) for ψ_2, and then for ψ_1.

Note that

$$\psi_2'(u) = -\frac{1}{u^2(e^u - 1)} - \frac{1}{u(e^u - 1)} - \frac{1}{u^2(e^u - 1)^2} + \frac{2}{u^3} - \frac{1}{ue^u} - \frac{1}{u^2 e^u}.$$

Recall $v/|z| = \mathrm{Re}(z)/|z| \geq \lambda$. Since $|\psi_2'(u)|$ is analytic and thus bounded for $u \in \gamma_k$ where $2 \leq k \leq 1/(\lambda|z|) + 1$, it follows that

$$|z|^2 \cdot \sum_{2 \leq k \leq 1/\lambda|z|+1} \max_{u \in \gamma_k} |\psi_2'(u)| = O\left(|z|^2 \cdot \frac{1}{|z|}\right) = O(|z|).$$

For $k > 1/\lambda|z| + 1$, we have $e^{(k-1)v} > e^{(k-1)\lambda|z|} > e$ and thus $e^{(k-1)v} - 1 \geq e^{(k-1)v}/2$. Therefore,

$$|z|^2 \cdot \sum_{k>1/\lambda|z|+1} \max_{u \in \gamma_k} |\psi_2'(u)|$$

$$\leq 2|z|^2 \cdot \sum_{k>1/\lambda|z|+1} \max_{u \in \gamma_k} \frac{1}{|u|^2|e^u - 1|} + |z|^2 \cdot \sum_{k>1/\lambda|z|+1} \max_{u \in \gamma_k} \frac{1}{|u|^2|e^u - 1|^2}$$

$$+ 4|z|^2 \cdot \sum_{k>1/\lambda|z|+1} \max_{u \in \gamma_k} \frac{1}{|u|^3}$$

$$\leq 6|z|^2 \cdot \sum_{k>1/\lambda|z|+1} \frac{1}{(k-1)^2|z|^2 e^{(k-1)v}} + 4|z|^2 \cdot \sum_{k>\lambda|z|+1} \frac{1}{(k-1)^3|z|^3}$$

$$\leq \sum_{k>1/\lambda|z|} \frac{6}{e^{kv}} + \frac{4}{|z|} \sum_{k>1/\lambda|z|} \frac{1}{k^3} = O(|z|),$$

where the last inequality follows from the estimates

$$\sum_{k>1/\lambda|z|} \frac{1}{e^{kv}} \leq \sum_{k=0}^{\infty} \frac{1}{e^{kv}} = \frac{1}{1 - e^{-v}} = O\left(\frac{1}{v}\right) = O\left(\frac{1}{|z|}\right);$$

$$\sum_{k>1/\lambda|z|} \frac{1}{k^3} \leq \int_{1/\lambda|z|}^{\infty} \frac{1}{x^3} \, dx = O(|z|^2).$$

Hence, we proved (2.12) for ψ_2.

Now, let us prove (2.12) for ψ_1. Since $\min_{u \in \gamma_k} |e^{Nu}| = \min_{u \in \gamma_k} e^{N \cdot \mathrm{Re}(u)} \geq e^{(k-1)N \cdot \mathrm{Re}(z)} =$

9

$e^{(k-1)Nv}$, we obtain

$$
\begin{aligned}
|z|^2 \cdot \sum_{k=2}^{\infty} \max_{u \in \gamma_k} |\psi_1'(u)| &= |z|^2 \cdot \sum_{k=2}^{\infty} \max_{u \in \gamma_k} \left(N \frac{|1 - e^{-Mu}|}{|e^{Nu}|} + M \frac{|1 - e^{-Nu}|}{|e^{Mu}|} \right) \\
&\leq 2|z|^2 \cdot \sum_{k=2}^{\infty} \left(\frac{N}{e^{Nv(k-1)}} + \frac{M}{e^{Mv(k-1)}} \right) \\
&\leq \frac{2|z|}{\lambda} \cdot \left(\frac{Nv \cdot e^{-vN}}{1 - e^{-vN}} + \frac{Mv \cdot e^{-vM}}{1 - e^{-vM}} \right) = O(|z|)
\end{aligned}
$$

by the assumption $|z| \leq v/\lambda$, where we used the fact that the function $\frac{x}{e^x - 1}$ is bounded on $(0, \infty)$.

This finishes the proof of (2.12). So, (2.11) is proved.

Since $\psi(u)$ is analytic in the considered region, by Cauchy integral theorem, it is standard to show that

$$
\int_{\gamma} \psi(u) \, du = \int_0^{\infty} \psi(x) \, dx.
$$

Combining (2.11) and (2.10), we obtain

$$
\log f(z) = z \cdot \sum_{k=1}^{\infty} (1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M}) \left(\frac{1}{k^2 z^2} - \frac{e^{-kz}}{2kz} \right) + \int_0^{\infty} \psi(x) \, dx + O(|z|). \quad (2.13)
$$

It remains to estimate the first two terms on the RHS of (2.13).

We split the first term into two parts,

$$
\begin{aligned}
& z \cdot \sum_{k=1}^{\infty} (1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M}) \left(\frac{1}{k^2 z^2} - \frac{e^{-kz}}{2kz} \right) \\
&= \frac{1}{z} \sum_{k=1}^{\infty} \frac{(1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M})}{k^2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{e^{-kz}(1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M})}{k}.
\end{aligned}
$$

By the definition of Spence's function, we have

$$
\begin{aligned}
\frac{1}{z} \sum_{k=1}^{\infty} \frac{(1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M})}{k^2} &= \frac{1}{z} \sum_{k=1}^{\infty} \left(\frac{1}{k^2} - \frac{e^{-k \cdot zN}}{k^2} - \frac{e^{-k \cdot zM}}{k^2} + \frac{e^{-k \cdot z(N+M)}}{k^2} \right) \\
&= \frac{1}{z} \left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM}) \right).
\end{aligned}
$$

Since $\sum_{k=1}^{\infty} e^{-k \cdot z}/k = -\log(1 - e^{-z})$, we have

$$
\begin{aligned}
& \frac{1}{2} \sum_{k=1}^{\infty} \frac{e^{-kz}(1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M})}{k} \\
&= -\frac{1}{2} \log(1 - e^{-z}) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \left(e^{-kz(N+1)} + e^{-kz(M+1)} - e^{-kz(M+N+1)} \right) \\
&= -\frac{1}{2} \log(1 - e^{-z}) + \frac{1}{2} \log \frac{(1 - e^{-z(N+1)})(1 - e^{-z(M+1)})}{1 - e^{-z(M+N+1)}} \\
&= \frac{1}{2} \log \left(\frac{1}{z} \right) + \frac{1}{2} \log \frac{(1 - e^{-zN})(1 - e^{-zM})}{1 - e^{-z(M+N)}} + O(|z|),
\end{aligned}
$$

10

where the last identity follows from the fact that $\lim_{u \to 0} \frac{1}{u} \log(\frac{1-e^{-u}}{u})$ exists, and thus, the function $\frac{1}{u} \log(\frac{1-e^{-u}}{u})$ is bounded around $u = 0$, implying that

$$\frac{1}{2} \log\left(\frac{1}{z}\right) = -\frac{1}{2} \log(1 - e^{-z}) + O(|z|).$$

Hence, we have for the first term on the RHS of (2.13) that

$$
\begin{aligned}
z \cdot \sum_{k=1}^{\infty} (1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M}) &\left(\frac{1}{k^2 z^2} - \frac{e^{-kz}}{2kz}\right) \\
&= \frac{1}{z}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM})\right) \\
&\quad - \frac{1}{2} \log\left(\frac{1}{z}\right) - \frac{1}{2} \log \frac{(1 - e^{-zN})(1 - e^{-zM})}{1 - e^{-z(M+N)}} + O(|z|).
\end{aligned}
\tag{2.14}
$$

Now we calculate the second term on the RHS of (2.13). We have

$$
\begin{aligned}
\int_0^{\infty} \psi(x)\, dx &= \int_0^{\infty} \psi_2(x)\, dx + \int_0^{\infty} (e^{-(N+M)x} - e^{-Nx} - e^{-Mx}) \psi_2(x)\, dx \\
&= -\log\sqrt{2\pi} + \int_0^{\infty} (e^{-(N+M)x} - e^{-Nx} - e^{-Mx}) \psi_2(x)\, dx,
\end{aligned}
$$

where the calculation of $\int_0^{\infty} \psi_2(x)\, dx = -\log\sqrt{2\pi}$ is given in [13, Page 125]. Since $\psi_2(x)$ is bounded, we have

$$
\begin{aligned}
\left|\int_0^{\infty} (e^{-(N+M)x} - e^{-Nx} - e^{-Mx}) \psi_2(x)\, dx\right| &= O\left(\int_0^{\infty} (e^{-(N+M)x} + e^{-Nx} + e^{-Mx})\, dx\right) \\
&= O\left(\frac{1}{M+N} + \frac{1}{N} + \frac{1}{M}\right) = O(|z|).
\end{aligned}
$$

Hence,

$$\int_0^{\infty} \psi(x)\, dx = -\log\sqrt{2\pi} + O(|z|). \tag{2.15}$$

Putting the equations (2.13), (2.14) and (2.15) together, we prove Lemma 2.3. $\qquad\square$

We will use the following lemma to change the variables in our calculations. Instead of using directly the variables α and β in Theorem 1, we will use the variables c_1 and c_2 defined in (2.16) in the next lemma, which will significantly simplify our calculations. So we will derive our asymptotic formula of $p_n(N, M)$ in terms of c_1 and c_2 first, and change back to α and β in the end.

LEMMA 2.4. *For every $\alpha, \beta \in (0, \infty)$ such that $\alpha\beta > 2$, there exist unique $c_1, c_2 > 0$ satisfying the equations:*

$$c_1 = \alpha \cdot A(c_1, c_2), \quad c_2 = \beta \cdot A(c_1, c_2), \tag{2.16}$$

where

$$A(c_1, c_2) = \sqrt{\int_0^{c_1} \frac{x}{e^x - 1}\, dx - \int_{c_2}^{c_1+c_2} \frac{x}{e^x - 1}\, dx}. \tag{2.17}$$

11

Moreover, if $\min(\alpha, \beta) \geq \sqrt{2} + \delta$ *for some* $\delta > 0$, *then there exist two constants* $\overline{A} > 0$ *and* $\overline{c} > 0$ *both depending only on* δ *such that*

$$\min(c_1, c_2) \geq \overline{c} \quad and \quad A(c_1, c_2) \geq \overline{A}. \tag{2.18}$$

In particular, if $\min(\alpha, \beta) \geq 4$, *then* $\min(c_1, c_2) \geq 13/5$.

Proof. Note that the quantity inside of the square root of (2.17) is positive since the function $\frac{x}{e^x - 1}$ is strictly decreasing for $x \geq 0$. Denote $\lambda = \beta/\alpha$ for brevity. Consider

$$b(t) = A(t, \lambda t)^2 - (t/\alpha)^2, \quad t > 0.$$

We have

$$\begin{aligned} b'(t) &= \frac{t}{e^t - 1} - \frac{\lambda^2 t}{e^{\lambda t} - 1} - \frac{(1+\lambda)^2 t}{e^{(1+\lambda)t} - 1} - \frac{2t}{\alpha^2} \\ &= \phi(t) + \lambda\phi(\lambda t) - (1+\lambda)\phi((1+\lambda)t), \end{aligned}$$

where

$$\phi(t) = \frac{t}{e^t - 1} + \frac{t}{\alpha\beta}.$$

It is elementary to check that ϕ is convex for $t \geq 0$, and thus, ϕ' is strictly increasing. Moreover, one can also verify that $\phi'(0) = -\frac{1}{2} + \frac{1}{\alpha\beta} < 0$ since $\alpha\beta > 2$, and $\phi'(t) > 0$ if t is large. Hence, there exists $t_0 > 0$ such that $\phi'(t) < 0$ in $(0, t_0)$ and $\phi'(t) > 0$ in (t_0, ∞). That is, $\phi(t)$ is strictly decreasing in $(0, t_0)$. Therefore, if $0 < (1+\lambda)t \leq t_0$, then

$$b'(t) = \phi(t) + \lambda\phi(\lambda t) - (1+\lambda)\phi((1+\lambda)t) > 0.$$

If $(1 + \lambda)t > t_0$, since ϕ' is strictly increasing, we have

$$\begin{aligned} b''(t) &= \phi'(t) + \lambda^2\phi'(\lambda t) - (1+\lambda)^2\phi'((1+\lambda)t) \\ &< \phi'((1+\lambda)t) + \lambda^2\phi'((1+\lambda)t) - (1+\lambda)^2\phi'((1+\lambda)t) < 0. \end{aligned}$$

Hence, we have that $b'(t)$ is positive when $0 < t \leq \frac{t_0}{1+\lambda}$, and then is decreasing for $t > \frac{t_0}{1+\lambda}$. Since $b'(t) \to -\infty$ as $t \to \infty$, then we know that there exists $\overline{t}$ such that $b'(t) > 0$ if $t \in (0, \overline{t})$ and $b'(t) < 0$ if $t \in (\overline{t}, \infty)$. That is, $b(t)$ is strictly increasing when $t \in (0, \overline{t})$ and $b(t)$ is strictly decreasing when $t \in (\overline{t}, \infty)$. Since $b(0) = 0$ and $\lim_{t \to +\infty} b(t) = -\infty$, there exists a unique $c_1 > 0$ such that $b(c_1) = 0$. This proves the existence and uniqueness of $c_1, c_2 > 0$, which are solutions to (2.16).

To prove (2.18), without loss of generality, we assume $\alpha \leq \beta$ and thus $c_1 \leq c_2$. Since the function $\frac{x}{e^x - 1}$ is strictly decreasing, we observe that

$$c_1^2 = \alpha^2 A(c_1, c_2) \geq \alpha^2 \left(\int_0^{c_1} \frac{x}{e^x - 1} \, dx - \int_{c_1}^{2c_1} \frac{x}{e^x - 1} \, dx \right).$$

That is

$$\alpha^2 \leq \frac{c_1^2}{\int_0^{c_1} \frac{x}{e^x-1} \, dx - \int_{c_1}^{2c_1} \frac{x}{e^x-1} \, dx} \to 2 \quad as \ c_1 \to 0^+.$$

Therefore, if $\alpha \geq \sqrt{2} + \delta$ for some $\delta > 0$, then $c_1 \geq \bar{c}$ for some $\bar{c} > 0$ depending only on δ. Moreover, for

$$B(c_1) := \int_0^{c_1} \frac{x}{e^x - 1}\, dx - \int_{c_1}^{2c_1} \frac{x}{e^x - 1}\, dx,$$

we have

$$\frac{d}{dc_1} B(c_1) = \frac{2c_1}{e^{c_1} + 1} > 0.$$

Therefore, $B(c_1)$ is increasing, and thus, for all $c_2 \geq c_1 \geq \bar{c}$, we have

$$A(c_1, c_2) \geq \sqrt{B(c_1)} \geq \sqrt{\int_0^{\bar{c}} \frac{x}{e^x - 1}\, dx - \int_{\bar{c}}^{2\bar{c}} \frac{x}{e^x - 1}\, dx} =: \overline{A} > 0.$$

Now we will show that $\min(c_1, c_2) \geq 13/5$ if we suppose $\min(\alpha, \beta) \geq 4$. Without loss of generality, we assume $\alpha \leq \beta$. Hence, $\lambda = \beta/\alpha \geq 1$. Since $\frac{x}{e^x - 1}$ is decreasing,

$$
\begin{aligned}
b(t) \geq A(t, t)^2 - (t/\alpha)^2 &= \int_0^t \frac{x}{e^x - 1}\, dx - \int_t^{2t} \frac{x}{e^x - 1}\, dx - \frac{t^2}{\alpha^2} \\
&= \frac{t^2}{e^t - 1} - \frac{2t^2}{e^{2t} - 1} + \int_0^t \frac{x^2 e^x}{(e^x - 1)^2}\, dx - \int_t^{2t} \frac{x^2 e^x}{(e^x - 1)^2}\, dx - \frac{t^2}{\alpha^2} \\
&\geq \frac{t^2}{e^t - 1} - \frac{2t^2}{e^{2t} - 1} - \frac{t^2}{\alpha^2},
\end{aligned}
$$

where we used integration by parts in the second equality, and the fact that $\frac{x^2 e^x}{(e^x - 1)^2}$ is decreasing in x in the last inequality. From this, one can verify that if $\alpha \geq 4$, then $b(13/5) > 0$. It is clear that $b(0) = 0$ and $\lim_{t \to -\infty} b(t) = -\infty$. Moreover, we know from the proof of Lemma 2.4 that $b(t)$ (which is the same as the $b(t)$ there) is strictly increasing in $(0, \bar{t})$ and strictly decreasing in $(\bar{t}, \infty)$ for some $\bar{t} > 0$. Therefore, we know that the unique solution c_1 of $b(c_1) = 0$ satisfies that $(c_2 >)c_1 > 13/5$. $\square$

By (2.6), we also can express

$$
\begin{aligned}
A(c_1, c_2)^2 &= \mathrm{Li}_2(1 - e^{-c_1}) + \mathrm{Li}_2(1 - e^{-c_2}) - \mathrm{Li}_2(1 - e^{-c_1 - c_2}) \\
&= \frac{\pi^2}{6} + \mathrm{Li}_2(e^{-c_1 - c_2}) - \mathrm{Li}_2(e^{-c_1}) - \mathrm{Li}_2(e^{-c_2}) \\
&\quad - (c_1 + c_2) \log(1 - e^{-c_1 - c_2}) + c_1 \log(1 - e^{-c_1}) + c_2 \log(1 - e^{-c_2}). \quad (2.19)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-c_1 - c_2}) - \mathrm{Li}_2(e^{-c_1}) - \mathrm{Li}_2(e^{-c_2}) \\
&= A(c_1, c_2)^2 + c_1 \log\left(\frac{1 - e^{-c_1 - c_2}}{1 - e^{-c_1}}\right) + c_2 \log\left(\frac{1 - e^{-c_1 - c_2}}{1 - e^{-c_2}}\right) \geq A(c_1, c_2)^2.
\end{aligned}
\quad (2.20)
$$

These two formulae (2.19) and (2.20) will be used later.

The next technical lemma will be used to estimate the error term in Section 3.2, which we include here for convenience. For a complex number z, we denote $\mathrm{Re}(z)$ as the real part of z, and $\mathrm{Im}(z)$ as the imaginary part of z.

13

LEMMA 2.5. *Suppose* $\min(\alpha, \beta) \geq 4$*, and let* c_1, c_2 *be as in Lemma 2.4. Define two functions:*

$$\eta_c(x) = Re(Li_2(e^{-c-icx})) + x \cdot Im(Li_2(e^{-c-icx})), \tag{2.21}$$

$$H(x) = \frac{1}{1+x^2}\left(\frac{\pi^2}{6} + \eta_{c_1+c_2}(x) - \eta_{c_1}(x) - \eta_{c_2}(x)\right). \tag{2.22}$$

Then there exists a universal constant $\delta > 0$ *such that*

$$H(x) \leq H(0) - \frac{\delta x^2}{1+x^2} \quad \text{for all } x > 0.$$

Proof. For convenience, denote

$$F(x) := \frac{\pi^2}{6} + \eta_{c_1+c_2}(x) - \eta_{c_1}(x) - \eta_{c_2}(x).$$

Then $F(0) = H(0)$. We are going to show there exists a universal constant $\delta > 0$ such that

$$F'(x) \leq 2F(0)x - 2\delta x \quad \text{for all } x > 0, \tag{2.23}$$

from which it follows by integrating both sides that

$$H(x) \leq H(0) - \frac{\delta x^2}{1+x^2} \quad \text{for all } x > 0.$$

We have

$$F(x) = \frac{\pi^2}{6} + \sum_{k=1}^{\infty} \frac{e^{-(c_1+c_2)k}}{k^2}\left(\cos(c_1+c_2)kx - x\sin(c_1+c_2)kx\right)$$

$$- \sum_{k=1}^{\infty} \frac{e^{-c_1 k}}{k^2}\left(\cos(c_1 kx) - x\sin(c_1 kx)\right) - \sum_{k=1}^{\infty} \frac{e^{-c_2 k}}{k^2}\left(\cos(c_2 kx) - x\sin(c_2 kx)\right).$$

Therefore,

$$F'(x)$$

$$= \sum_{k=1}^{\infty} \frac{e^{-(c_1+c_2)k}}{k^2}\left(-(c_1+c_2)k\sin(c_1+c_2)kx - \sin(c_1+c_2)kx - (c_1+c_2)kx\cos(c_1+c_2)kx\right)$$

$$+ \sum_{k=1}^{\infty} \frac{e^{-c_1 k}}{k^2}\left(c_1 k\sin(c_1 kx) - \sin(c_1 kx) + c_1 kx\cos(c_1 kx)\right)$$

$$+ \sum_{k=1}^{\infty} \frac{e^{-c_2 k}}{k^2}\left(c_2 k\sin(c_2 kx) - \sin(c_2 kx) + c_2 kx\cos(c_2 kx)\right)$$

$$\leq \sum_{k=1}^{\infty} \frac{e^{-(c_1+c_2)k}}{k^2}\left((c_1+c_2)^2 k^2 x + 2(c_1+c_2)kx\right) + \sum_{k=1}^{\infty} \frac{e^{-c_1 k}}{k^2}\left(c_1^2 k^2 x + 2c_1 kx\right)$$

$$+ \sum_{k=1}^{\infty} \frac{e^{-c_2 k}}{k^2}\left(c_2^2 k^2 x + 2c_2 kx\right)$$

$$= x\left(\frac{(c_1+c_2)^2}{e^{c_1+c_2} - 1} - 2(c_1+c_2)\log(1 - e^{-(c_1+c_2)})\right.$$

$$+ \frac{c_1^2}{e^{c_1} - 1} - 2c_1\log(1 - e^{-c_1}) + \frac{c_2^2}{e^{c_2} - 1} - 2c_2\log(1 - e^{-c_2})\bigg).$$

Let
$$\zeta(c) = \frac{c^2}{e^c - 1} - 2c\log(1 - e^{-c}) + 2\mathrm{Li}_2(e^{-c}).$$

Then $\zeta(0) = \pi^2/3$ and $\lim_{c\to\infty} \zeta(c) = 0$. Also, ζ is strictly decreasing since

$$\zeta'(c) = -\frac{c^2 e^c}{(e^c - 1)^2} < 0.$$

Therefore, there exists $\bar{c}$ to be the unique solution of $\zeta(\bar{c}) = \frac{111\pi^2}{1000}$. Then we have

$$\frac{F'(x)}{x} - 2F(0) \le \zeta(c_1 + c_2) + \zeta(c_1) + \zeta(c_2) - \frac{\pi^2}{3} - 4\mathrm{Li}_2(e^{-(c_1+c_2)})$$

$$\le 3\zeta(\bar{c}) - \frac{\pi^2}{3}$$

$$= -\frac{\pi^2}{3000} =: -2\delta \tag{2.24}$$

as long as $\min(c_1, c_2) \ge \bar{c}$.

Let us estimate the value of $\bar{c}$. Notice that

$$\mathrm{Li}_2(e^{-c}) = \sum_{k=1}^{\infty} \frac{e^{-ck}}{k^2} < e^{-c} + \sum_{k=2}^{\infty} \frac{e^{-ck}}{2^2} = e^{-c} + \frac{e^{-2c}}{4(1 - e^{-c})} = \frac{4e^c - 3}{4(e^{2c} - e^c)}.$$

Hence

$$\zeta(c) < \frac{c^2}{e^c - 1} - 2c\log(1 - e^{-c}) + \frac{4e^c - 3}{2(e^{2c} - e^c)}.$$

Using the above estimate, we can verify that $\zeta(13/5) < 111\pi^2/1000$. Since ζ is decreasing, we have $\bar{c} < 13/5$.

Since we assume that $\min(\alpha, \beta) \ge 4$, we know from Lemma 2.4 that $\min(c_1, c_2) \ge 13/5$. Therefore, $\min(c_1, c_2) \ge \bar{c}$, and thus, (2.24) holds. This proves Lemma 2.5. □

3 Proof of the main result

Now we will start to prove our main result.

Proof of Theorem 1. Since $p_n(N, M) = p_n(M, N)$, without loss of generality we can assume $N \le M$. Recall

$$\alpha = \frac{N}{\sqrt{n}}, \quad \beta = \frac{M}{\sqrt{n}},$$

and also recall our assumption that $\min(\alpha, \beta) \ge 4$. Let c_1, c_2 be the solutions of (2.16) with $A(c_1, c_2)$ defined in (2.17). Set

$$v = \frac{A(c_1, c_2)}{\sqrt{n}} \quad \text{and} \quad w_0 = \frac{1}{n^{\frac{3}{4} - \frac{\epsilon}{3}}}.$$

15

Then $N \cdot v = c_1$ and $M \cdot v = c_2$. First, we observe from the definition of $A(c_1, c_2)$ in (2.17) that

$$A(c_1, c_2) \leq \sqrt{\int_0^\infty \frac{x}{e^x - 1} \, dx} < \infty. \tag{3.1}$$

Secondly, since we have assumed that $\alpha, \beta \geq 4$, we know from Lemma 2.4 that there exist two positive universal constants $\bar{c}$ and $\overline{A}$ such that

$$\min(c_1, c_2) \geq \bar{c} > 0 \quad \text{and} \quad A(c_1, c_2) \geq \overline{A} > 0. \tag{3.2}$$

These two universal lower bounds are important to prove the uniform estimate in the next subsection. Since we assumed that $N \leq M$ at the beginning of this proof, it follows that $c_1 \leq c_2$.

Recall the integral (2.8) that

$$
\begin{aligned}
p_n(N, M) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\log f(v+iw)} e^{n(v+iw)} \, dw \\
&= \frac{1}{2\pi} \int_{-w_0}^{w_0} e^{\log f(v+iw)} e^{n(v+iw)} \, dw + \frac{1}{2\pi} \int_{w_0}^{\pi} e^{\log f(v+iw)} e^{n(v+iw)} \, dw \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{-w_0} e^{\log f(v+iw)} e^{n(v+iw)} \, dw \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.3}
$$

We will show that I_1 is the main term, and both I_2 and I_3 are lower order terms.

3.1 Estimate the main term I_1.

Let $z = v + iw$ for $-w_0 \leq w \leq w_0$. Since $v = A(c_1, c_2) n^{-1/2}$ and $w_0 = n^{-3/4 + \epsilon/3}$, by recalling (3.2), we have $\min(N, M) \cdot |z| \geq \min(c_1, c_2) \geq \bar{c} > 0$ and $|w/v| \leq |w_0/v| = o(1)$. Then we can apply Lemma 2.3 to obtain

$$
\begin{aligned}
\log f(z) &= \frac{1}{z} \left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM}) \right) \\
&\quad - \frac{1}{2} \log\left(\frac{1}{z}\right) - \frac{1}{2} \log \frac{(1 - e^{-zN})(1 - e^{-zM})}{1 - e^{-z(N+M)}} - \log\sqrt{2\pi} + O(|z|).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\log f(v) &= \frac{1}{v} \left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-v(N+M)}) - \mathrm{Li}_2(e^{-vN}) - \mathrm{Li}_2(e^{-vM}) \right) \\
&\quad - \frac{1}{2} \log\left(\frac{1}{v}\right) - \frac{1}{2} \log \frac{(1 - e^{-vN})(1 - e^{-vM})}{1 - e^{-v(N+M)}} - \log\sqrt{2\pi} + O(v). \tag{3.4}
\end{aligned}
$$

Now we compare the above two terms,

$$
\begin{aligned}
& \log f(v+iw) - \log f(v) \\
= \quad & \frac{\pi^2}{6}\left(\frac{1}{z}-\frac{1}{v}\right) + \left(\frac{\mathrm{Li}_2(e^{-z(N+M)})}{z} - \frac{\mathrm{Li}_2(e^{-v(N+M)})}{v}\right) - \left(\frac{\mathrm{Li}_2(e^{-zN})}{z} - \frac{\mathrm{Li}_2(e^{-vN})}{v}\right) \\
& - \left(\frac{\mathrm{Li}_2(e^{-zM})}{z} - \frac{\mathrm{Li}_2(e^{-vM})}{v}\right) - \frac{1}{2}\left(\log\left(\frac{1}{z}\right) - \log\left(\frac{1}{v}\right)\right) + \frac{1}{2}\log\frac{1-e^{-z(N+M)}}{1-e^{-v(N+M)}} \\
& - \frac{1}{2}\log\frac{1-e^{-zN}}{1-e^{-vN}} - \frac{1}{2}\log\frac{1-e^{-zM}}{1-e^{-vM}} + O(|z|) + O(v) \\
=: \quad & J_1 + J_2 - J_3 - J_4 - H + K_1 - K_2 - K_3 + O(v). \quad (3.5)
\end{aligned}
$$

We are going to estimate each term on the right hand side of (3.5).

Let us calculate J_1 first. Since

$$
\begin{aligned}
\frac{1}{z} - \frac{1}{v} &= \frac{1}{v+iw} - \frac{1}{v} = \frac{1}{v}\left(\frac{1}{1+i\frac{w}{v}} - 1\right) \\
&= -i\frac{w}{v^2} - \frac{w^2}{v^3} + O\left(\frac{w_0^3}{v^4}\right),
\end{aligned}
$$

we obtain

$$
\frac{1}{z} = \frac{1}{v} - i\frac{w}{v^2} - \frac{w^2}{v^3} + O\left(\frac{w_0^3}{v^4}\right), \quad (3.6)
$$

and thus,

$$
J_1 = \frac{\pi^2}{6}\left(-i\frac{w}{v^2} - \frac{w^2}{v^3}\right) + O\left(\frac{w_0^3}{v^4}\right).
$$

Now let us calculate J_3 in (3.5). Notice that J_2 and J_4 are in the same form as J_3, so they can be estimated in the same way. Using the property of Spence's function $\frac{\partial}{\partial u}\mathrm{Li}_2(z) = -\frac{1}{z}\log(1-z)$ for $z \in \mathbb{C} \setminus [1, \infty)$, one has

$$
\begin{aligned}
\frac{\partial}{\partial u}\mathrm{Li}_2(e^{-u}) &= e^u \log(1-e^{-u})e^{-u} = \log(1-e^{-u}); \\
\frac{\partial^2}{\partial u^2}\mathrm{Li}_2(e^{-u}) &= \frac{e^{-u}}{1-e^{-u}} = \frac{1}{e^u - 1}.
\end{aligned}
$$

Therefore, with noting that $N \cdot v = c_1$,

$$
\mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-vN}) = iwN\log(1-e^{-vN}) - \frac{1}{2}\frac{1}{e^{vN}-1}w^2 N^2 + O\left(\frac{w_0^3}{v^3}\right),
$$

where in the last term $O(\cdot)$ we used that $c_1 \geq \bar{c} > 0$ independent of n. Thus

$$
\begin{aligned}
J_3 &= \frac{\text{Li}_2(e^{-zN})}{z} - \frac{\text{Li}_2(e^{-vN})}{v} \\
&= \frac{1}{z}\left(\text{Li}_2(e^{-zN}) - \text{Li}_2(e^{-vN})\right) + \left(\frac{1}{z} - \frac{1}{v}\right)\text{Li}_2(e^{-vN}) \\
&= \left(\frac{1}{v} - i\frac{w}{v^2} - \frac{w^2}{v^3} + O\left(\frac{w_0^3}{v^4}\right)\right) \cdot \left(iwN\log(1 - e^{-vN}) - \frac{1}{2}\frac{1}{e^{vN}-1}w^2N^2 + O\left(\frac{w_0^3}{v^3}\right)\right) \\
&\quad + \left(-i\frac{w}{v^2} - \frac{w^2}{v^3} + O\left(\frac{w_0^3}{v^4}\right)\right)\text{Li}_2(e^{-vN}).
\end{aligned}
$$

Since by our assumption $N \cdot v = c_1$, $v = A(c_1, c_2) \cdot n^{-1/2}$ and $w_0/v = o(1)$, it follows that

$$
\begin{aligned}
J_3 &= i\frac{wc_1}{v^2}\log(1 - e^{-c_1}) - \frac{w^2}{2v^3}\frac{c_1^2}{e^{c_1}-1} + \frac{w^2c_1}{v^3}\log(1 - e^{-c_1}) - i\frac{w}{v^2}\text{Li}_2(e^{-c_1}) \\
&\quad - \frac{w^2}{v^3}\text{Li}_2(e^{-c_1}) + O\left(\frac{w_0^3}{v^4}\right) \\
&= -iwn \cdot \frac{-c_1\log(1 - e^{-c_1}) + \text{Li}_2(e^{-c_1})}{A(c_1, c_2)^2} \\
&\quad - w^2 \cdot \frac{n^{3/2}}{A(c_1, c_2)^3}\left(-c_1\log(1 - e^{-c_1}) + \text{Li}_2(e^{-c_1}) + \frac{1}{2}\frac{c_1^2}{e^{c_1}-1}\right) + O\left(\frac{w_0^3}{v^4}\right).
\end{aligned}
$$

We estimate the other two terms J_2 and J_4 in the same way. Therefore

$$
\begin{aligned}
&J_1 + J_2 - J_3 - J_4 \\
&= -iwn \cdot \frac{1}{A(c_1, c_2)^2}\left(\frac{\pi^2}{6} + \text{Li}_2(e^{-c_1-c_2}) - \text{Li}_2(e^{-c_1}) - \text{Li}_2(e^{-c_2})\right. \\
&\qquad \left. -(c_1 + c_2)\log(1 - e^{-c_1-c_2}) + c_1\log(1 - e^{-c_1}) + c_2\log(1 - e^{-c_2})\right) \\
&\quad - w^2 \cdot \frac{n^{3/2}}{A(c_1, c_2)^3}\left(\frac{\pi^2}{6} + \text{Li}_2(e^{-c_1-c_2}) - \text{Li}_2(e^{-c_1}) - \text{Li}_2(e^{-c_2})\right. \\
&\qquad -(c_1 + c_2)\log(1 - e^{-c_1-c_2}) + c_1\log(1 - e^{-c_1}) + c_2\log(1 - e^{-c_2}) \\
&\qquad \left. + \frac{1}{2}\frac{(c_1 + c_2)^2}{e^{c_1+c_2}-1} - \frac{1}{2}\frac{c_1^2}{e^{c_1}-1} - \frac{1}{2}\frac{c_2^2}{e^{c_2}-1}\right) + O\left(\frac{w_0^3}{v^4}\right).
\end{aligned}
$$

By recalling (2.19), we eventually get

$$
\begin{aligned}
&J_1 + J_2 - J_3 - J_4 \\
&= -iwn - w^2 \cdot \frac{n^{3/2}}{A(c_1, c_2)^3}\left(A(c_1, c_2)^2 + \frac{1}{2}\frac{(c_1 + c_2)^2}{e^{c_1+c_2}-1} - \frac{1}{2}\frac{c_1^2}{e^{c_1}-1} - \frac{1}{2}\frac{c_2^2}{e^{c_2}-1}\right) \\
&\quad + O\left(\frac{w_0^3}{v^4}\right).
\end{aligned} \tag{3.7}
$$

Now let us estimate H in (3.5). Since for n sufficiently large, $|w/v| < 1/2$, we have that

$$
\left|\log\left(\frac{1}{z}\right) - \log\left(\frac{1}{v}\right)\right| = \left|\log\left(1 + i\frac{w}{v}\right)\right| \leq \sum_{k=1}^{\infty}\frac{1}{k}\left|\frac{w}{v}\right|^k \leq \frac{|w/v|}{1 - |w/v|} = O\left(\frac{w_0}{v}\right),
$$

18

and thus

$$H = O\left(\frac{w_0}{v}\right). \tag{3.8}$$

Now let us estimate K_1, K_2 and K_3 in (3.5). These three quantities are in the same form, so we just show the details for K_2. Since

$$\frac{\partial}{\partial u}\log(1 - e^{-u}) = \frac{e^{-u}}{1 - e^{-u}} = \frac{1}{e^u - 1};$$

$$\frac{\partial^2}{\partial u^2}\log(1 - e^{-u}) = -\frac{e^u}{(e^u - 1)^2},$$

we obtain

$$\begin{aligned}
2K_2 = \log\frac{1 - e^{-zN}}{1 - e^{-vN}} &= \log(1 - e^{-vN-iwN}) - \log(1 - e^{-vN}) \\
&= \frac{iwN}{e^{vN} - 1} + \frac{1}{2}\frac{e^{vN}}{(e^{vN} - 1)^2}w^2N^2 + O\left(\frac{w_0^3}{v^3}\right).
\end{aligned}$$

Since $N \cdot v = c_1$, $v = A(c_1, c_2) \cdot n^{-1/2}$ and $w_0/v = o(1)$,

$$K_2 = O\left(\frac{w_0}{v}\right).$$

Similarly, $K_i = O(\frac{w_0}{v})$ for $i = 1, 3$ and thus,

$$K_1 - K_2 - K_3 = O\left(\frac{w_0}{v}\right). \tag{3.9}$$

Plugging (3.7), (3.8) and (3.9) into (3.5), we arrive at

$$\begin{aligned}
&\log f(v + iw) - \log f(v) \\
&= -iwn - w^2 \cdot \frac{n^{3/2}}{A(c_1, c_2)^3}\left(A(c_1, c_2)^2 + \frac{1}{2}\frac{(c_1 + c_2)^2}{e^{c_1+c_2} - 1} - \frac{1}{2}\frac{c_1^2}{e^{c_1} - 1} - \frac{1}{2}\frac{c_2^2}{e^{c_2} - 1}\right) \\
&\quad + O\left(\frac{w_0^3}{v^4}\right) + O(v) + O\left(\frac{w_0}{v}\right) \\
&= -iwn - \frac{1}{2}w^2 n^{3/2}S(c_1, c_2) + O(n^{-1/4+\epsilon}), \tag{3.10}
\end{aligned}$$

where for convenience we denoted

$$S(c_1, c_2) = \frac{1}{A(c_1, c_2)^3}\left(2A(c_1, c_2)^2 + \frac{(c_1 + c_2)^2}{e^{c_1+c_2} - 1} - \frac{c_1^2}{e^{c_1} - 1} - \frac{c_2^2}{e^{c_2} - 1}\right).$$

In obtaining the term of $O(n^{-1/4+\epsilon})$ in (3.10), we used that $A(c_1, c_2)$ is bounded from above and below by two universal positive constants (see (3.1) and (3.2)).

We now show that $S(c_1, c_2)$ is bounded from above and from below by two universal positive constants. Recall that

$$A(c_1, c_2)^2 = \int_0^{c_1} \frac{x}{e^x - 1}\,dx - \int_{c_2}^{c_1+c_2} \frac{x}{e^x - 1}\,dx.$$

19

By integration by parts (noticing that we assumed $\alpha \le \beta$, so that $c_1 \le c_2$), we have

$$
\begin{aligned}
L(c_1, c_2) &:= 2A(c_1, c_2)^2 + \frac{(c_1 + c_2)^2}{e^{c_1 + c_2} - 1} - \frac{c_1^2}{e^{c_1} - 1} - \frac{c_2^2}{e^{c_2} - 1} \\
&= \int_0^{c_1} \frac{x^2 e^x}{(e^x - 1)^2}\, dx - \int_{c_2}^{c_1 + c_2} \frac{x^2 e^x}{(e^x - 1)^2}\, dx \\
&\ge \int_0^{c_1} \frac{x^2 e^x}{(e^x - 1)^2}\, dx - \int_{c_1}^{2c_1} \frac{x^2 e^x}{(e^x - 1)^2}\, dx =: \widetilde{B}(c_1).
\end{aligned}
\tag{3.11}
$$

as the function $x^2 e^x / (e^x - 1)^2$ is strictly decreasing for $x \ge 0$ (which can be elementarily verified by showing that its derivative is negative). Since

$$
\frac{d}{dc} \widetilde{B}(c) = \frac{2c^2 e^c}{(e^c + 1)^2} > 0,
$$

$\widetilde{B}(c)$ is increasing in c. Thus, continuing from (3.11) and using $\min(c_1, c_2) \ge \bar{c}$ in (3.2), we obtain

$$
0 < \widetilde{B}(\bar{c}) \le \widetilde{B}(c_1) \le L(c_1, c_2) \le \int_0^\infty \frac{x^2 e^x}{(e^x - 1)^2}\, dx < \infty.
\tag{3.12}
$$

Using (3.1) and (3.2), we conclude that $S(c_1, c_2) = L(c_1, c_2) A(c_1, c_2)^{-3}$ is bounded from above and from below by two universal positive constants.

Using (3.10), we can estimate the term

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_{-w_0}^{w_0} e^{\log f(v + iw)} e^{n(v + iw)}\, dw \\
&= \frac{1}{2\pi} \int_{-w_0}^{w_0} e^{\log f(v) - iwn - \frac{1}{2} w^2 n^{3/2} S(c_1, c_2) + O(n^{-1/4 + \epsilon})} e^{n(v + iw)}\, dw \\
&= \exp(\log f(v) + nv + O(n^{-1/4 + \epsilon})) \cdot \frac{1}{2\pi} \int_{-w_0}^{w_0} e^{-\frac{1}{2} w^2 n^{3/2} S(c_1, c_2)}\, dw.
\end{aligned}
\tag{3.13}
$$

For the first term, since we have from (3.4) that

$$
\begin{aligned}
\log f(v) = \; & \frac{n^{1/2}}{A(c_1, c_2)} \left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-c_1 - c_2}) - \mathrm{Li}_2(e^{-c_1}) - \mathrm{Li}_2(e^{-c_2}) \right) \\
& - \frac{1}{2} \log \left(\frac{\sqrt{n}}{A(c_1, c_2)} \right) - \frac{1}{2} \log \frac{(1 - e^{-c_1})(1 - e^{-c_2})}{1 - e^{-c_1 - c_2}} - \log \sqrt{2\pi} + O(v),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
& \exp(\log f(v) + nv + O(n^{-1/4 + \epsilon})) \\
&= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{A(c_1, c_2)}}{n^{1/4}} \sqrt{\frac{1 - e^{-(c_1 + c_2)}}{(1 - e^{-c_1})(1 - e^{-c_2})}} \cdot e^{\sqrt{n} K(c_1, c_2)} \left(1 + O(n^{-1/4 + \epsilon}) \right),
\end{aligned}
\tag{3.14}
$$

where

$$
K(c_1, c_2) = A(c_1, c_2) + \frac{1}{A(c_1, c_2)} \left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-(c_1 + c_2)}) - \mathrm{Li}_2(e^{-c_1}) - \mathrm{Li}_2(e^{-c_2}) \right).
$$

20

For the other term, by setting $t = \sqrt{S(c_1, c_2)}n^{3/4} \cdot w$, we get

$$\int_{-w_0}^{w_0} e^{-\frac{1}{2}w^2 n^{3/2} S(c_1, c_2)}\, dw = \frac{1}{\sqrt{S(c_1,c_2)}n^{3/4}} \int_{-\sqrt{S(c_1,c_2)}n^{\epsilon/3}}^{\sqrt{S(c_1,c_2)}n^{\epsilon/3}} e^{-t^2/2}\, dt$$

$$= \frac{1}{\sqrt{S(c_1,c_2)}n^{3/4}} \left(\int_{-\infty}^{\infty} e^{-t^2/2}\, dt - 2\int_{\sqrt{S(c_1,c_2)}n^{\epsilon/3}}^{\infty} e^{-t^2/2}\, dt \right)$$

$$= \frac{1}{\sqrt{S(c_1,c_2)}n^{3/4}} \left(\sqrt{2\pi} + O\Big(\frac{2}{\sqrt{S(c_1,c_2)}n^{\epsilon/3}} e^{-\frac{S(c_1,c_2)}{2}n^{2\epsilon/3}} \Big) \right)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{S(c_1,c_2)}n^{3/4}} \left(1 + O(n^{-1/4+\epsilon}) \right), \tag{3.15}$$

where in the last equality we used the fact that $S(c_1, c_2)$ bounded from above and from below by two universal positive constants.

Finally, putting (3.13), (3.14) and (3.15) together, we have

$$\begin{aligned}
I_1 &= \frac{1}{2\pi} \frac{\sqrt{A(c_1,c_2)}}{\sqrt{S(c_1,c_2)}} \sqrt{\frac{1 - e^{-(c_1+c_2)}}{(1-e^{-c_1})(1-e^{-c_2})}} \cdot \frac{e^{\sqrt{n}K(c_1,c_2)}}{n}(1 + O(n^{-1/4+\epsilon})) \\
&= \frac{A(c_1,c_2)^2}{2\pi\sqrt{L(c_1,c_2)}} \sqrt{\frac{1 - e^{-(c_1+c_2)}}{(1-e^{-c_1})(1-e^{-c_2})}} \cdot \frac{e^{\sqrt{n}K(c_1,c_2)}}{n}(1 + O(n^{-1/4+\epsilon})), \quad (3.16)
\end{aligned}$$

where

$$K(c_1, c_2) = A(c_1, c_2) + \frac{1}{A(c_1,c_2)}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-(c_1+c_2)}) - \mathrm{Li}_2(e^{-c_1}) - \mathrm{Li}_2(e^{-c_2}) \right);$$

$$L(c_1, c_2) = 2A(c_1,c_2)^2 + \frac{(c_1+c_2)^2}{e^{c_1+c_2}-1} - \frac{c_1^2}{e^{c_1}-1} - \frac{c_2^2}{e^{c_2}-1}.$$

We know from (3.1), (3.2) and (3.12) that $A(c_1, c_2)$ and $L(c_1, c_2)$ are bounded from above and from below by two universal positive constants. Since $\min(c_1, c_2) \geq \bar{c} > 0$, we have

$$\sqrt{1 - e^{-2\bar{c}}} \leq \sqrt{\frac{1 - e^{-(c_1+c_2)}}{(1-e^{-c_1})(1-e^{-c_2})}} \leq \frac{1}{1 - e^{-\bar{c}}}. \tag{3.17}$$

Using (2.20) and $\mathrm{Li}_2(1) = \frac{\pi^2}{6}$, we have

$$2A(c_1, c_2) \leq K(c_1, c_2) \leq A(c_1, c_2) + \frac{\pi^2}{3A(c_1,c_2)}. \tag{3.18}$$

This implies that $K(c_1, c_2)$ is also bounded from above and from below by two universal positive constants.

Hence, all of $A(c_1, c_2)$, $L(_1, c_2)$, $K(c_2, c_2)$ and $\sqrt{\frac{1-e^{-(c_1+c_2)}}{(1-e^{-c_1})(1-e^{-c_2})}}$ in (3.16) are bounded from above and from below by two universal positive constants. To prove that I_2 and I_3 in (3.3) are lower order terms compared to I_1, it suffices to show that

$$|I_2| + |I_3| = o\left(\frac{\exp\{\sqrt{n}K(c_1, c_2)\}}{n^{\frac{5}{4}-\varepsilon}} \right).$$

We will prove the above estimate in the next subsection.

21

3.2 Estimate the remainder terms I_2 and I_3.

It is enough to estimate I_2 in (3.3). The proof for I_3 is identical. Let $w_1 = C \cdot v$ where C is a universally large constant to be fixed in *Step 2* below (one will see later that $C = 20\pi$ would suffice). We further split I_2 into two parts.

$$
\begin{aligned}
I_2 &= \frac{1}{2\pi} \int_{w_0}^{\pi} f(v+iw)e^{n(v+iw)}\, dw \\
&= \frac{1}{2\pi} \int_{w_0}^{w_1} f(v+iw)e^{n(v+iw)}\, dw + \frac{1}{2\pi} \int_{w_1}^{\pi} f(v+iw)e^{n(v+iw)}\, dw \\
&=: I_2' + I_2''.
\end{aligned}
\tag{3.19}
$$

Step 1. Estimate the term I_2'. Note that for $z = v + iw$ with $w_0 \le w \le w_1$, the hypothesis of Lemma 2.3 is satisfied. Thus

$$
\begin{aligned}
\log f(z) &= \frac{1}{z}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM}) \right) \\
&\quad - \frac{1}{2}\log\left(\frac{1}{z}\right) - \frac{1}{2}\log\frac{(1-e^{-zN})(1-e^{-zM})}{1-e^{-z(N+M)}} + O(1)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2'| &= \left| \frac{1}{2\pi} \int_{w_0}^{w_1} e^{\log f(v+iw)+n(v+iw)}\, dw \right| \\
&= \left| e^{O(1)} \int_{w_0}^{w_1} \sqrt{z} \cdot \sqrt{\frac{1-e^{-z(N+M)}}{(1-e^{-zN})(1-e^{-zM})}} \right. \\
&\quad \left. \cdot \exp\left\{ \frac{1}{z}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM}) \right) + nv + inw \right\} dw \right| \\
&\lesssim \int_{w_0}^{w_1} \sqrt{|z|} \cdot \sqrt{\frac{|1-e^{-z(N+M)}|}{|1-e^{-zN}| \cdot |1-e^{-zM}|}} \\
&\quad \cdot \exp\left\{ \mathrm{Re}\frac{1}{z}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM}) \right) + nv \right\} dw.
\end{aligned}
$$

Since $|z| = O(v)$, and

$$
2 \ge |1-e^{-zN}| = |1-e^{-vN-iwN}| \ge 1 - |e^{-vN-iwN}| = 1 - e^{-vN} = 1 - e^{-c_1},
$$

we conclude that (recalling $v = A(c_1,c_2)n^{-1/2}$),

$$
\begin{aligned}
|I_2'| &\lesssim \sqrt{v} \int_{w_0}^{w_1} \exp\left\{ \mathrm{Re}\frac{1}{z}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM}) \right) + nv \right\} dw \\
&= \sqrt{v} \cdot e^{\sqrt{n}K(c_1,c_2)} \int_{w_0}^{w_1} \exp\left\{ \mathrm{Re}\frac{1}{z}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM}) \right) \right. \\
&\quad \left. - \frac{1}{v}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-v(N+M)}) - \mathrm{Li}_2(e^{-vN}) - \mathrm{Li}_2(e^{-vM}) \right) \right\} dw. \tag{3.20}
\end{aligned}
$$

Let us denote

$$\mathcal{D} := \frac{1}{z}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-z(N+M)}) - \mathrm{Li}_2(e^{-zN}) - \mathrm{Li}_2(e^{-zM})\right)$$

$$-\frac{1}{v}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-v(N+M)}) - \mathrm{Li}_2(e^{-vN}) - \mathrm{Li}_2(e^{-vM})\right). \qquad (3.21)$$

Next we are going to show that

$$\mathrm{Re}(\mathcal{D}) \leq -\frac{w^2}{v^3} \cdot T \qquad (3.22)$$

for some universal constant $T > 0$, where $\mathrm{Re}(\mathcal{D})$ is the real part of $\mathcal{D}$.

We first get

$$\mathrm{Re}\left(\frac{\mathrm{Li}_2(e^{-c_1-iwN})}{z}\right) = \mathrm{Re}\left(\frac{v-iw}{v^2+w^2}\mathrm{Li}_2(e^{-c_1-iwN})\right)$$

$$= \frac{1}{v^2+w^2}\left\{v\mathrm{Re}(\mathrm{Li}_2(e^{-c_1-iwN})) + w\mathrm{Im}(\mathrm{Li}_2(e^{-c_1-iwN}))\right\}$$

$$= \frac{v}{v^2+w^2}\left\{\mathrm{Re}(\mathrm{Li}_2(e^{-c_1-ic_1\frac{w}{v}})) + \frac{w}{v}\cdot\mathrm{Im}(\mathrm{Li}_2(e^{-c_1-ic_1\frac{w}{v}}))\right\}$$

$$= \frac{1}{v}\cdot\frac{1}{1+(w/v)^2}\left\{\mathrm{Re}(\mathrm{Li}_2(e^{-c_1-ic_1\frac{w}{v}})) + \frac{w}{v}\cdot\mathrm{Im}(\mathrm{Li}_2(e^{-c_1-ic_1\frac{w}{v}}))\right\}.$$

Therefore,

$$v\cdot\mathrm{Re}(\mathcal{D}) = \frac{1}{1+(\frac{w}{v})^2}\left(\frac{\pi^2}{6} + \eta_{c_1+c_2}\left(\frac{w}{v}\right) - \eta_{c_1}\left(\frac{w}{v}\right) - \eta_{c_2}\left(\frac{w}{v}\right)\right)$$

$$-\left(\frac{\pi^2}{6} + \eta_{c_1+c_2}(0) - \eta_{c_1}(0) - \eta_{c_2}(0)\right)$$

$$= H\left(\frac{w}{v}\right) - H(0),$$

where η_c is defined in (2.21) and H is defined in (2.22). By Lemma 2.5, we have for $0 < w/v \leq C$ that

$$v\cdot\mathrm{Re}(\mathcal{D}) = H\left(\frac{w}{v}\right) - H(0) \leq -\frac{\delta}{1+C^2}\frac{w^2}{v^2} =: -T\frac{w^2}{v^2}.$$

This proves (3.22).

Therefore, continuing from (3.20) with the help of (3.22), we have that

$$|I_2'| \lesssim \sqrt{v}\cdot e^{\sqrt{n}K(c_1,c_2)}\int_{w_0}^{w_1}\exp\{-w^2T/v^3\}dw$$

$$\lesssim \sqrt{v}\cdot e^{\sqrt{n}K(c_1,c_2)}\frac{1}{w_0}\int_{w_0}^{w_1}w\exp\{-w^2T/v^3\}dw$$

$$\lesssim \frac{A(c_1,c_2)^{\frac{7}{2}}}{2Tn^{1+\frac{\epsilon}{3}}}\exp\left\{\sqrt{n}K(c_1,c_2) - \frac{Tn^{\frac{2\varepsilon}{3}}}{A(c_1,c_2)^3}\right\}$$

$$= o\left(\frac{\exp\{\sqrt{n}K(c_1,c_2)\}}{n^{\frac{5}{4}-\varepsilon}}\right). \qquad (3.23)$$

This finishes the estimate of I'_2, which is indeed a lower order term compared to I_1.

Step 2. Estimate the term I''_2. Recall that

$$\log f(z) = \sum_{k=1}^{\infty} \frac{1}{k} \frac{(1 - e^{-kz \cdot N})(1 - e^{-kz \cdot M})}{e^{kz} - 1}$$

and

$$I''_2 = \frac{1}{2\pi} \int_{w_1}^{\pi} e^{\log f(v+iw) + n(v+iw)} \, dw, \tag{3.24}$$

where $w_1 = C \cdot v$ with C being a large constant (to be determined below).

First, note that $|e^{k(v+iw)} - 1| \geq e^{kv} - 1 \geq kv$. Secondly, we have

$$\begin{aligned}
|e^z - 1| &= |e^{v+iw} - 1| = \sqrt{(e^v \cos w - 1)^2 + e^{2v} \sin^2 w} \\
&= \sqrt{(e^v - 1)^2 + 4e^v \sin^2(w/2)},
\end{aligned}$$

and thus,

$$|e^{v+iw} - 1| \geq 2\sin(w/2) \geq 2w/\pi$$

which follows from the inequality that $\sin\theta \geq \frac{2}{\pi}\theta$ for $0 \leq \theta \leq \pi/2$.

It follows that

$$\begin{aligned}
&|\log f(v+iw)| \\
&= \left| \sum_{k=1}^{\infty} \frac{1}{k} \frac{(1 - e^{-k(v+iw)N})(1 - e^{-k(v+iw)M})}{e^{k(v+iw)-1}} \right| \\
&\leq \frac{2\pi}{Cv} + \frac{1}{v} \sum_{k=2}^{\infty} \frac{(1 + e^{-kc_1})(1 + e^{-kc_2})}{k^2} \\
&\leq \frac{2\pi}{Cv} + \frac{1}{v} \sum_{k=1}^{\infty} \frac{(1 + e^{-kc_1})(1 + e^{-kc_2})}{k^2} - \frac{1}{v}(1 + e^{-c_1})(1 + e^{-c_2}) \\
&= \frac{1}{v} \left(\frac{2\pi}{C} + \frac{\pi^2}{6} + \text{Li}_2(e^{-(c_1+c_2)}) + \text{Li}_2(e^{-c_1}) + \text{Li}_2(e^{-c_2}) - (1 + e^{-c_1})(1 + e^{-c_2}) \right).
\end{aligned}$$

We claim that for C sufficiently large, we have

$$\begin{aligned}
&\frac{4\pi}{C} + \frac{\pi^2}{6} + \text{Li}_2(e^{-(c_1+c_2)}) + \text{Li}_2(e^{-c_1}) + \text{Li}_2(e^{-c_2}) - (1 + e^{-c_1})(1 + e^{-c_2}) \\
&\leq \frac{\pi^2}{6} + \text{Li}_2(e^{-(c_1+c_2)}) - \text{Li}_2(e^{-c_1}) - \text{Li}_2(e^{-c_2}). \tag{3.25}
\end{aligned}$$

Suppose the above inequality (3.25) holds. Then continuing from (3.24), we have

$$\begin{aligned}
|I''_2| &\leq \frac{1}{2\pi} \int_{w_1}^{\pi} e^{|\log f(v+iw)| + nv} \, dw \\
&\leq \frac{1}{2\pi} \int_{w_1}^{\pi} e^{\sqrt{n}(K(c_1,c_2) - 2\pi/C)} \, dw = o\left(\frac{\exp\{\sqrt{n}K(c_1,c_2)\}}{n^{\frac{5}{4} - \varepsilon}} \right). \tag{3.26}
\end{aligned}$$

24

Now we are left to prove (3.25), which is equivalent to

$$\frac{2\pi}{C} + \mathrm{Li}_2(e^{-c_1}) + \mathrm{Li}_2(e^{-c_2}) \leq \frac{(1 + e^{-c_1})(1 + e^{-c_2})}{2}.$$

Since we assume that $\min(\alpha, \beta) \geq 4$, we know from Lemma 2.4 that $\min(c_1, c_2) \geq 13/5 > 2$. Hence,

$$\mathrm{Li}_2(e^{-c_1}) \leq \frac{e^{-c_1}}{1 - e^{-c_1}} = \frac{1}{e^{c_1} - 1} \leq \frac{1}{e^2 - 1} < \frac{1}{5},$$

and similarly $\mathrm{Li}_2(e^{-c_2}) \leq 1/5$. Hence, we only need to choose C large so that

$$\frac{2\pi}{C} + \frac{2}{5} \leq \frac{1}{2}.$$

Therefore, $C = 20\pi$ would suffice.

From (3.19), (3.23) and (3.26), we obtained that

$$|I_2| \leq o\left(\frac{\exp\{\sqrt{n}K(c_1, c_2)\}}{n^{\frac{5}{4} - \varepsilon}}\right). \tag{3.27}$$

Since I_3 is in the same form as I_2 defined in (3.3), one can show similarly that

$$|I_3| \leq o\left(\frac{\exp\{\sqrt{n}K(c_1, c_2)\}}{n^{\frac{5}{4} - \varepsilon}}\right). \tag{3.28}$$

3.3 Asymptotic formula

Combining (3.3), (3.16), (3.27) and (3.28), we obtain

$$p_n(N, M) = \frac{A(c_1, c_2)^2}{2\pi\sqrt{L(c_1, c_2)}}\sqrt{\frac{1 - e^{-(c_1 + c_2)}}{(1 - e^{-c_1})(1 - e^{-c_2})}} \cdot \frac{e^{\sqrt{n}K(c_1, c_2)}}{n}\left(1 + O(n^{-1/4 + \epsilon})\right), \tag{3.29}$$

where

$$K(c_1, c_2) = A(c_1, c_2) + \frac{1}{A(c_1, c_2)}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-(c_1 + c_2)}) - \mathrm{Li}_2(e^{-c_1}) - \mathrm{Li}_2(e^{-c_2})\right);$$

$$L(c_1, c_2) = 2A(c_1, c_2)^2 + \frac{(c_1 + c_2)^2}{e^{c_1 + c_2} - 1} - \frac{c_1^2}{e^{c_1} - 1} - \frac{c_2^2}{e^{c_2} - 1}.$$

By recalling (2.17) and (3.11), we have the explicit integral representation for $A(c_1, c_2)$ and $L(c_1, c_2)$:

$$A(c_1, c_2) = \sqrt{\int_0^{c_1} \frac{x}{e^x - 1}\, dx - \int_{c_2}^{c_1 + c_2} \frac{x}{e^x - 1}\, dx},$$

$$L(c_1, c_2) = \int_0^{c_1} \frac{x^2 e^x}{(e^x - 1)^2}\, dx - \int_{c_2}^{c_1 + c_2} \frac{x^2 e^x}{(e^x - 1)^2}\, dx.$$

Therefore, all the terms in the equation (3.29) are explicit.

The last step is to express the asymptotic formula in terms of α, β instead of c_1, c_2. Recall that $c_1 = \alpha A(c_1, c_2)$ and $c_2 = \beta A(c_1, c_2)$. We denote $g(\alpha, \beta) = A(c_1, c_2) > 0$ and $g(\alpha, \beta)$ satisfies

$$g^2(\alpha, \beta) = \int_0^{\alpha g(\alpha, \beta)} \frac{x}{e^x - 1}\, dx - \int_{\beta g(\alpha, \beta)}^{(\alpha+\beta)g(\alpha, \beta)} \frac{x}{e^x - 1}\, dx$$

or equivalently,

$$g^2(\alpha, \beta) = \mathrm{Li}_2(1 - e^{-\alpha g(\alpha, \beta)}) + \mathrm{Li}_2(1 - e^{-\beta g(\alpha, \beta)}) - \mathrm{Li}_2(1 - e^{-(\alpha+\beta)g(\alpha, \beta)}).$$

The uniqueness and existence of $g(\alpha, \beta) > 0$ is guaranteed by Lemma 2.4. Then by recalling (3.29), we have

$$p_n(N, M)$$
$$= \frac{g(\alpha, \beta)^2}{2\pi\sqrt{L(\alpha, \beta)}} \sqrt{\frac{1 - e^{-(\alpha+\beta)g(\alpha, \beta)}}{(1 - e^{-\alpha g(\alpha, \beta)})(1 - e^{-\beta g(\alpha, \beta)})}} \cdot \frac{e^{\sqrt{n}K(\alpha, \beta)}}{n}(1 + O(n^{-1/4+\epsilon})), \quad (3.30)$$

where

$$K(\alpha, \beta) = g(\alpha, \beta) + \frac{1}{g(\alpha, \beta)}\left(\frac{\pi^2}{6} + \mathrm{Li}_2(e^{-(\alpha+\beta)g(\alpha, \beta)}) - \mathrm{Li}_2(e^{-\alpha g(\alpha, \beta)}) - \mathrm{Li}_2(e^{-\beta g(\alpha, \beta)})\right)$$

and

$$L(\alpha, \beta) = 2g^2(\alpha, \beta) + \frac{(\alpha+\beta)^2 g^2(\alpha, \beta)}{e^{(\alpha+\beta)g(\alpha, \beta)} - 1} - \frac{\alpha^2 g^2(\alpha, \beta)}{e^{\alpha g(\alpha, \beta)} - 1} - \frac{\beta^2 g^2(\alpha, \beta)}{e^{\alpha g(\alpha, \beta)} - 1}.$$

Since $\min(\alpha, \beta) \geq 4$, it follows from Lemma 2.4 that $\min(c_1, c_2) \geq 13/5$. By (3.1), (3.2), (3.12), (3.17) and (3.18), we know all of $A(c_1, c_2)$, $L(_1, c_2)$, $K(c_2, c_2)$ and $\sqrt{\frac{1 - e^{-(c_1+c_2)}}{(1-e^{-c_1})(1-e^{-c_2})}}$ in (3.29) are bounded from above and from below by two universal positive constants. Hence, all the functions $g(\alpha, \beta)$, $L(\alpha, \beta)$, $K(\alpha, \beta)$ and $\sqrt{\frac{1 - e^{-(\alpha+\beta)g(\alpha, \beta)}}{(1-e^{-\alpha g(\alpha, \beta)})(1-e^{-\beta g(\alpha, \beta)})}}$ in (3.30) are bounded from above and from below by two universal positive constants. The error term $O(n^{-1/4+\epsilon})$ is also uniform in $\alpha, \beta \in [4, \infty]$. $\square$

References

[1] G. Almkvist and G. E. Andrews. A Hardy-Ramanujan formula for restricted partitions. *Journal of Number Theory*, 38(2):135–144, 1991.

[2] G. E. Andrews. *The Theory of Partitions*. Cambridge University Press, 1998.

[3] E. R. Canfield. From recursions to asymptotics: on Szekeres' formula for the number of partitions. *Electron. J. Combin.*, 4(2):19, 1997.

[4] P. Erdős and J. Lehner. The distribution of the number of summands in the partitions of a positive integer. *Duke Math. J.*, 8:335–345, 1941.

[5] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[6] B. Fristedt. The structure of random partitions of large integers. *Trans. Amer. Math. Soc.*, 337(2):703–735, 1993.

[7] G. H. Hardy and S. Ramanujan. Asymptotic formulæ in combinatory analysis [Proc. London Math. Soc. (2) **17** (1918), 75–115]. In *Collected papers of Srinivasa Ramanujan*, pages 276–309. AMS Chelsea Publ., Providence, RI, 1918.

[8] T. Jiang and K. Wang. Statistical properties of eigenvalues of laplace-beltrami operators. *arXiv preprint arXiv:1602.00406*, 2016.

[9] T. Jiang and K. Wang. Random restricted partitions. *arXiv preprint arXiv:1709.08504*, 2017.

[10] S. Melczer, G. Panova, and R. Pemantle. Counting partitions inside a rectangle. *arXiv preprint arXiv:1805.08375*, 05 2018.

[11] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1063–1229. Elsevier Sci. B. V., Amsterdam, 1995.

[12] B. Pittel. On a likely shape of the random Ferrers diagram. *Adv. in Appl. Math.*, 18(4):432–488, 1997.

[13] A. G. Postnikov. *Introduction to analytic number theory*, volume 68 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1988. Translated from the Russian by G. A. Kandall, Translation edited by Ben Silver, With an appendix by P. D. T. A. Elliott.

[14] L. B. Richmond. A George Szekeres formula for restricted partitions. *arXiv preprint arXiv:1803.08548*, 2018.

[15] D. Romik. Partitions of n into $t\sqrt{n}$ parts. *European J. Combin.*, 26(1):1–17, 2005.

[16] R. P. Stanley and F. Zanello. Some asymptotic results on q-binomial coefficients. *Annals of Combinatorics*, 20(3):623–634, 2016.

[17] J. J. Sylvester. XXV. Proof of the hitherto undemonstrated fundamental theorem of invariants. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 5(30):178–188, 1878.

[18] G. Szekeres. An asymptotic formula in the theory of partitions. *Quart. J. Math., Oxford Ser. (2)*, 2:85–108, 1951.

[19] G. Szekeres. Some asymptotic formulae in the theory of partitions. II. *Quart. J. Math., Oxford Ser. (2)*, 4:96–111, 1953.

[20] L. Takács. Some asymptotic formulas for lattice paths. *J. Statist. Plann. Inference*, 14(1):123–142, 1986.