On supercuspidal representations of $SL_n(F)$ associated with tamely ramified extensions

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Abstract

We will give an explicit construction of irreducible suparcuspidal representations of the special linear group over a non-archimedean local field and will speculate its Langlands parameter by means of verifying the Hiraga-Ichino-Ikeda formula of the formal degree of the supercuspidal representations.

1 Introduction

Although the L-packets of the irreducible supercuspidal representations of the special linear group over a non-archimedean local field are well-understood by the works of [6], [2], [7] or of [4], little is known for the explicit determination of the Langlands parameter of an individual explicitly constructed supercuspidal representation.

In this paper, we will construct quite explicitly some supercuspidal representations, associated with tamely ramified extensions, of the special linear group over a non-dyadic non-archimedean local field and will speculate its Langlands parameter by showing the formula of the formal degree established by Hiraga, Ichino and Ikeda [4] (see the subsection 2.4 for the conclusions).

The main results of this paper are Theorem 2.1.1 (a construction of supercuspidal representations associated with a tamely ramified extension of the base field), Theorem 2.2.2 (an explicit formula of the formal degree of the supercuspidal representation) and Theorem 2.3.1 (showing Hiraga-Ichino-Ikeda formula of the formal degree in the form of Gross and Reeder [3]).

The first theorem is proved in Section 3. Basic ideas and arguments are these of Shintani [9] with a small modification to our case of the special linear group (the original Shintani's paper treats the subgroup of the general linear group of unit determinant).

The second theorem is proved in Section 4. Our argument is based upon a general theory, developed by [10], of explicit description of irreducible representations of hyperspecial open compact subgroups associated with regular adjoint orbits.

The third theorem is proved in Section 5 under the assumption that the tamely ramified extension is Galois extension.

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2 Results

2.1 Let F be a non-dyadic non-archimedean local field. The integer ring of F is denoted by O_F with the maximal ideal $\mathfrak{p}=\mathfrak{p}_F$ generated by $\varpi=\varpi_F$. The residue class field $\mathbb{F}=O_F/\mathfrak{p}$ is a finite field of q elements. Fix a continuous unitary additive character $\tau:F\to\mathbb{C}^\times$ such that

$$\{x \in F \mid \tau(xO_F) = 1\} = O_F.$$

Then $\widehat{\tau}(\overline{x}) = \tau(\varpi^{-1}x)$ $(x \in O_F)$ gives a non-trivial unitary additive character $\widehat{\tau} : \mathbb{F} \to \mathbb{C}^{\times}$.

The special linear group $G = SL_n$ of degree n is a smooth connected semisimple group scheme over O_F whose Lie algebra is denoted by $\mathfrak{g} = \mathfrak{sl}_n$. For any commutative O_F -algebra A, the group and the A-Lie algebra of the A-valued points are

$$G(A) = \{ g \in M_n(A) \mid \det g = 1 \}$$

and

$$\mathfrak{g}(A) = \{ X \in M_n(A) \mid \operatorname{tr} X = 0 \}$$

where $M_n(A)$ if the A-algebra of the square matrices of size n with entries in A. Throughout this paper, except the subsection 5.3, we will assume that the characteristic p of \mathbb{F} does not divide n so that the trace form

$$\mathfrak{g}(\mathbb{F}) \times \mathfrak{g}(\mathbb{F}) \to \mathbb{F}$$
 $((X, Y) \mapsto \operatorname{tr}(XY))$

is non-degenerate.

For any integers 0 < l < r, the canonical group homomorphism

$$G(O_F/\mathfrak{p}^r) \to G(O/\mathfrak{p}^l)$$

is surjective, and its kernel is denoted by $G(\mathfrak{p}^l/\mathfrak{p}^r)$. If r = l + l' with $0 < l' \le l$, then we have a group isomorphism

$$\mathfrak{g}(O/\mathfrak{p}^{l'})\,\widetilde{\to}\,G(\mathfrak{p}^l/\mathfrak{p}^r)\quad \left(\overline{X}\mapsto \overline{1_n+\varpi^lX}\right).$$

Fix an integer $r \geq 2$ and put r = l + l' with the minimum integer l such that $0 < l' \leq l$, that is

$$l' = \left[\frac{r}{2}\right] = \begin{cases} l & \text{: if } r = 2l, \\ l - 1 & \text{: if } r = 2l - 1. \end{cases}$$

Let K/F be a field extension of degree n. The ramification index and the inertial degree of K/F are denoted by

$$e = e(K/F)$$
 and $f = f(K/F)$

respectively. Since n is prime to p, the extension K/F is tamely ramified and there exists a prime element $\varpi_K \in O_K$ such that $\varpi_K^e \in O_{K_0}$ where K_0 is the maximal unramified subsextionsion of K/F.

We will identify K with a F-subalgebra of the matrix algebra $M_n(F)$ by means of the regular representation with respect to an O_F -basis of O_K . Take a generator β of O_K as an O_F -algebra. Then Shintani [9] shows that the modulo \mathfrak{p} reduction of the characteristic polynomial $\chi_{\beta}(t) \in O_F[t]$ of $\beta \in M_n(O_F)$ gives

the minimal polynomial of $\beta \pmod{\mathfrak{p}} \in M_n(\mathbb{F})$, that is $\beta \in \mathfrak{gl}_n(O_F)$ is smoothly regular with respect to GL_n in the terminology of [10]. We can assume that $T_{K/F}(\beta) = \operatorname{tr} \beta = 0$ so that $\beta \in \mathfrak{g}(O_F)$ is smoothly regular with respect to $G = SL_n$. Define a character

$$\psi_{\beta}: G(\mathfrak{p}^l/\mathfrak{p}^r) \to \mathbb{C}^{\times}$$

by $\psi_{\beta}(h) = \tau\left(\varpi^{-l'}\operatorname{tr}(X\beta)\right)$ for all $h = \overline{1_n + \varpi^l X} \in G(\mathfrak{p}^l/\mathfrak{p}^r)$. Let δ be an irreducible representation of $G(O_F/\mathfrak{p}^r)$ such that $\delta|_{G(\mathfrak{p}^l/\mathfrak{p}^r)}$ contains the character ψ_{β} . We will regard δ as a representation of $G(O_F)$ via the canonical surjection $G(O_F) \to G(O_F/\mathfrak{p}^r)$. Then our first result is

Theorem 2.1.1 If $r \geq 2e$, then the compactly induced representation $\operatorname{ind}_{G(O_F)}^{G(F)} \delta$ is an irreducible supercuspidal representation of G(F).

2.2 The general theory developed in [10] says that the irreducible representation δ of $G(O_F/\mathfrak{p}^r)$ as above is parametrized by the character

$$\theta: G_{\beta}(O_F/\mathfrak{p}^r) \to \mathbb{C}^{\times}$$

such that $\theta = \psi_{\beta}$ on $G_{\beta}(O_F/\mathfrak{p}^r) \cap G(\mathfrak{p}^l/\mathfrak{p}^r)$. The explicit realization of δ , which is recalled in the subsection 4.1, gives

Proposition 2.2.1

$$\dim \delta = \frac{q^{rn(n-1)/2}}{(O_F^\times: N_{K/F}(O_K^\times))} \cdot \frac{\prod_{k=1}^n (1-q^{-k})}{1-q^{-f}}.$$

Let $d_{G(F)}(x)$ be the Haar measure on G(F) with respect to which the volume of $G(O_F)$ is 1. Then the Euler-Poincaré measure $\mu_{G(F)}$ on G(F) is

$$d\mu_{G(F)}(x) = (-1)^{n-1} q^{n(n-1)/2} \prod_{k=1}^{n-1} (1 - q^{-k}) \cdot d_{G(F)}(x)$$

(see [8, 3.4, Theoréme 7]). Since the formal degree of the supercuspidal representation $\operatorname{ind}_{G(O_F)}^{G(F)}\delta$ (assuming $r \geq 2e$) with respect to $d_{G(F)}(x)$ is the dimension of δ , Proposition 2.2.1 gives our second result

Theorem 2.2.2 Assume $r \geq 2e$. Then the formal degree of the supercuspidal representation $\operatorname{ind}_{G(O_F)}^{G(F)} \delta$ with respect to the Euler-Poincaré measure on G(F) is

$$\frac{q^{(r-1)n(n-1)/2}}{(O_F^\times:N_{K/F}(O_K^\times))}\cdot \frac{1-q^{-n}}{1-q^{-f}}.$$

2.3 Now suppose that the field extension K/F is Galois. Let

$$\delta_K: K^{\times} \stackrel{\sim}{\to} W_K^{\text{ab}} = W_K / \overline{[W_K, W_K]}$$

be the isomorphism of the local class field theory with the Weil group W_K of K. We assume that

$$\delta_K(\varpi_K) \in W_K^{\mathrm{ab}} \subset \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

induces the geometric Frobenius automorphism of K^{ur}/K where K^{ab} and K^{ur} are the maximal abelian and the maximal unramified extension of K respectively. Then the relative Weil group

$$W_{K/F} = W_F/\overline{[W_K, W_K]} \subset \operatorname{Gal}(K^{\operatorname{ab}}/F)$$

sits in a group extension

$$1 \to K^{\times} \xrightarrow{\delta_K} W_{K/F} \xrightarrow{\text{res.}} \operatorname{Gal}(K/F) \to 1$$
 (1)

corresponding to the fundamental class $[\alpha_{K/F}] \in H^2(Gal(K/F), K^{\times})$ of the local class field theory.

Let $\theta: G_{\beta}(O_F/\mathfrak{p}^r) \to \mathbb{C}^{\times}$ be the character which parametrizes the irreducible representation δ of $G(O_F/\mathfrak{p}^r)$. Since we have

$$G_{\beta}(O_F/\mathfrak{p}^r) = \left\{ \overline{x} \in (O_K/\mathfrak{p}_K^{er})^{\times} \mid N_{K/F}(x) \equiv 1 \pmod{\mathfrak{p}^r} \right\},$$

take an extension of θ to a character of $(O_K/\mathfrak{p}_K^{er})^{\times}$ and consider it as a character of O_K^{\times} via the canonical surjection $O_K^{\times} \to (O_K/\mathfrak{p}_K^{er})^{\times}$ and extend it to a character of K^{\times} , which is denoted also by θ , by fixing any value $\theta(\varpi_K) \in \mathbb{C}^{\times}$. Then the group homomorphism

$$\Theta: W_{F/K} \xrightarrow{\operatorname{Ind}_{K^{\times}}^{W_{K/F}} \theta} GL_{\mathbb{C}}(V) \xrightarrow{\operatorname{canonical}} PGL_{\mathbb{C}}(V)$$

(V is the representation space of the induced representation $\operatorname{Ind}_{K^{\times}}^{W_{K/F}}\theta$) is independent, up to the conjugate in $PGL_{\mathbb{C}}(V)$, of the choice of the extension θ (see Proposition 5.4.1). Note that

$$\dim_{\mathbb{C}} V = (W_{K/F} : K^{\times}) = n$$

and that $PGL_{\mathbb{C}}(V)$ is the dual group of $G = SL_n$. Put

$$\rho: W_F \xrightarrow{\operatorname{canonical}} W_{K/F} \xrightarrow{\Theta} PGL_{\mathbb{C}}(V)$$

and define a representation of the Weil-Deligne group

$$\varphi: W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{projection}} W_F \xrightarrow{\rho} PGL_{\mathbb{C}}(V).$$
 (2)

Let us denote by A_{φ} the centralizer of the image of φ in $PGL_{\mathbb{C}}(V)$. Then we will show that

$$|A_{\varphi}| = (O_F^{\times} : N_{K/F}(O_K^{\times})) \cdot f \tag{3}$$

(see Proposition 5.4.3). Our third result is

Theorem 2.3.1

$$\frac{1}{|A_{\varphi}|} \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| = \frac{q^{(r-1)n(n-1)/2}}{(O_F^{\times}: N_{K/F}(O_K^{\times}))} \cdot \frac{1-q^{-n}}{1-q^{-f}}.$$

Here $\gamma(\varphi) = \gamma(\varphi, Ad, 0)$ is the special value of the gamma factor

$$\gamma(\varphi, \mathrm{Ad}, s) = \varepsilon(\varphi, \mathrm{Ad}, s) \cdot \frac{L(\varphi, \mathrm{Ad}, 1 - s)}{L(\varphi, \mathrm{Ad}, s)}$$

associated with the representation

$$\operatorname{Ad} \circ \varphi : W_F \times SL_2(\mathbb{C}) \xrightarrow{\varphi} PGL_{\mathbb{C}}(V) \xrightarrow{\operatorname{Ad}} GL_{\mathbb{C}}(\widehat{\mathfrak{g}})$$

of the Weil-Deligne group, where $\widehat{\mathfrak{g}} = \mathfrak{pgl}_{\mathbb{C}}(V) = \mathfrak{sl}_{\mathbb{C}}(V)$ is the Lie algebra of $PGL_{\mathbb{C}}(V)$. Another representation

$$\varphi_0: W_F \times SL_2(\mathbb{C}) \xrightarrow{\operatorname{proj.}} SL_2(\mathbb{C}) \xrightarrow{\operatorname{Sym}_{n-1}} GL_n(\mathbb{C}) \xrightarrow{\operatorname{can.}} PGL_n(\mathbb{C})$$

of the Weil-Deligne group, with the symmetric tensor representation Sym_{n-1} , is the *principal parameter* (that is, corresponding to the Steinberg representation, see the subsection 5.2).

2.4 Since $G = SL_n$ is split over F, the L-group of G is $PGL_n(\mathbb{C})$. Theorem 2.2.2 and Theorem 2.3.1 shows that the formula due to Hiraga-Ichino-Ikeda [5] of the formal degree of the suparcuspidal representation $\operatorname{ind}_{G(O_F)}^{G(F)} \delta$ is valid with the representation of Weil-Deligne group given by (2). So we can speculate that the representation (2) is the Langlands (or Arthur) parameter of the supercuspidal representation $\operatorname{ind}_{G(O_F)}^{G(F)} \delta$.

3 Construction of supercuspidal representation

3.1 In this section we will prove Theorem 2.1.1. It is sufficient to show the following two propositions on the compactly induced representation $\operatorname{ind}_{G(O_F)}^{G(F)} \delta$;

Proposition 3.1.1 If K/F is unramified or $r \geq 4$, then $\operatorname{ind}_{G(O_F)}^{G(F)} \delta$ is admissible representation of G(F).

Proposition 3.1.2 If $r \geq 2e$, then $\operatorname{ind}_{G(O_F)}^{G(F)} \delta$ is irreducible representation of G(F).

We will prove these two propositions in the following subsections.

3.2 The field extension K/F is tamely ramified. Fix a prime element $\varpi_K \in O_K$ such that $\varpi_K^e \in O_{K_0}$ where e is the ramification index of K/F and K_0 is the maximal unramified subextension of K/F. The field K is identified with a F-subalgebra of the matrix algebra $M_n(F)$ by means of the regular representation with respect to an O_F -basis of O_K .

Take a generator β of O_K as an O_F -algebra, and let us denote by $\chi_{\beta}(t) \in O_F[t]$ the characteristic polynomial of $\beta \in O_K \subset M_n(O_F)$. Then Shintani [9] shows the following proposition

Proposition 3.2.1 1) $\chi_{\beta}(t) \pmod{\mathfrak{p}} \in \mathbb{F}[t]$ is the minimal polynomial of $\beta \pmod{\mathfrak{p}} \in M_n(\mathbb{F})$,

- 2) $\chi_{\beta}(t) \pmod{\mathfrak{p}} = p(t)^e$ with a polynomial $p(t) \in \mathbb{F}[t]$ irreducible over \mathbb{F} ,
- 3) $\chi_{\beta}(t) \pmod{\mathfrak{p}^2}$ is an irreducible polynomial over the ring O_F/\mathfrak{p}^2 .

Remark 3.2.2 The first statement of Proposition 3.2.1 implies that

- 1) for any m > 0 and $X \in M_n(O_F)$, if $X\beta \equiv \beta X \pmod{\mathfrak{p}^m}$, then there exists a polynomial $f(t) \in O_F[t]$ such that $X \equiv f(\beta) \pmod{\mathfrak{p}^m}$,
- 2) $\{X \in M_n(O_F) \mid X\beta = \beta X\} = O_K$,
- 3) for any $X \in M_n(O_F)$, if $\chi_X(t) = \chi_\beta(t)$, then there exists $g \in GL_n(O_F)$ such that $X = g\beta g^{-1}$.

We have the Cartan decomposition

$$G(F) = \bigsqcup_{m \in \mathbb{M}} G(O_F) \varpi^m G(O_F)$$
(4)

where

$$\mathbb{M} = \left\{ m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n \mid \begin{array}{c} m_1 \ge m_2 \ge \dots \ge m_n, \\ m_1 + m_2 + \dots + m_n = 0 \end{array} \right\}$$

and

$$\varpi^m = \begin{bmatrix} \varpi^{m_1} & & \\ & \ddots & \\ & & \varpi^{m_n} \end{bmatrix} \text{ for } m = (m_1, \dots, m_n) \in \mathbb{M}.$$

Since the restriction to $G(\mathfrak{p}^l/\mathfrak{p}^r)$ of our irreducible representation δ of $G(O_F/\mathfrak{p}^r)$ contains the character ψ_{β} , we have

$$\delta|_{G(\mathfrak{p}^l/\mathfrak{p}^r)} = \left(\bigoplus_{\dot{g}} g * \psi_{\beta}\right)^b \tag{5}$$

with some integer b > 0. Here $(g * \psi_{\beta})(x) = \psi_{\beta}(g^{-1}xg)$ is the conjugate of ψ_{β} and $\bigoplus_{\dot{g}}$ is the direct sum over $\dot{g} \in G(O_F/\mathfrak{p}^r)/G(O_F/\mathfrak{p}^r,\psi_{\beta})$ where

$$G(O_F/\mathfrak{p}^r, \psi_{\beta}) = \{ g \in G(O_F/\mathfrak{p}^r) \mid g * \psi_{\beta} = \psi_{\beta} \}$$
$$= \left\{ \overline{g} \in G(O_F/\mathfrak{p}^r) \mid \operatorname{Ad}(g)\beta \equiv \beta \pmod{\mathfrak{p}^{l'}} \right\}$$
(6)

is the isotropy subgroup of ψ_{β} . The second equality is due to that fact $\overline{g} * \psi_{\beta} = \psi_{\mathrm{Ad}(g)\beta}$ for $g \in G(O_F)$.

3.3 The proof of Proposition 3.1.1. For any integer a > 0, put $G(\mathfrak{p}^a) = G(O_F) \cap (1_n + \varpi^a M_n(O_F))$. We will prove that the dimension of the space of the $G(\mathfrak{p}^a)$ -fixed vectors is finite. The Cartan decomposition (4) gives

$$G(F) = \bigsqcup_{s \in S} G(\mathfrak{p}^a) s G(O_F)$$

with

$$S = \{k\varpi^m \mid \dot{k} \in G(\mathfrak{p}^a) \backslash G(O_F), m \in \mathbb{M}\}.$$

Then we have

$$\operatorname{ind}_{G(O_F)}^{G(F)} \delta \Big|_{G(\mathfrak{p}^a)} = \bigoplus_{s \in S} \operatorname{ind}_{G(\mathfrak{p}^a) \cap sG(O_F)s^{-1}}^{G(\mathfrak{p}^a)} \delta^s$$

with $\delta^s(h) = \delta(s^{-1}hs)$ $(h \in G(\mathfrak{p}^a) \cap sG(O_F)s^{-1})$. The Frobenius reciprocity gives

$$\operatorname{Hom}_{G(\mathfrak{p}^a)}\left(\mathbf{1},\operatorname{ind}_{G(O_F)}^{G(F)}\delta\right) = \bigoplus_{s \in S} \operatorname{Hom}_{s^{-1}G(\mathfrak{p}^a)s \cap G(O_F)}(\mathbf{1},\delta).$$

Here 1 is the one-dimensional trivial representation of $G(\mathfrak{p}^a)$. If

$$\operatorname{Hom}_{G(\mathfrak{p}^a)}\left(\mathbf{1},\operatorname{ind}_{G(O_F)}^{G(F)}\delta\right) \neq 0$$

then there exists a

$$s = k \varpi^m \in S \quad (k \in G(O_F), m = (m_1, \cdots, m_n) \in \mathbb{M})$$

such that $\operatorname{Hom}_{s^{-1}G(\mathfrak{p}^a)s\cap G(O_F)}(\mathbf{1},\delta)\neq 0$. If

$$\max\{m_i - m_{i+1} \mid 1 \le i < n\} = m_i - m_{i+1} \ge a$$

then $\varpi^m U_i(O_F)\varpi^{-m} \subset G(\mathfrak{p}^a)$ where

$$U_i = \left\{ \begin{bmatrix} 1_i & B \\ 0 & 1_{n-1} \end{bmatrix} \mid B \in M_{i,n-i} \right\}$$

is the unipotent part of the maximal parabolic subgroup

$$P_i = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in G \mid A \in GL_i, D \in GL_{n-1} \right\}.$$

So we have $U_i(O_F) \subset s^{-1}G(\mathfrak{p}^a)s \cap G(O_F)$ so that

$$\operatorname{Hom}_{U_i(\mathfrak{p}^l)}(\mathbf{1},\delta) \supset \operatorname{Hom}_{s^{-1}G(\mathfrak{p}^a)s \cap G(O_F)}(\mathbf{1},\delta) \neq 0$$

where $U_i(\mathfrak{p}^l) = U_i(O_F) \cap G(\mathfrak{p}^l)$. Then the decomposition (5) implies that there exists a $g \in G(O_F)$ such that $\psi_{\mathrm{Ad}(g)\beta}(h) = 1$ for all $h \in U_i(\mathfrak{p}^l)$, that is

$$\tau \left(\varpi^{-l'} \operatorname{tr}(g\beta g^{-1} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}) \right) = 0$$

for all $B \in M_{i,n-1}(O_F)$. This means

$$g\beta g^{-1} \equiv \begin{bmatrix} A & * \\ 0 & D \end{bmatrix} \pmod{\mathfrak{p}^{l'}}$$

with $A \in M_i(O_F), D \in M_{n-i}(O_F)$, that is

$$\chi_{\beta}(t) \equiv \det(t1_i - A) \cdot \det(t1_{n-i} - D) \pmod{\mathfrak{p}^{l'}}.$$

If K/F is unramified, this is contradict against 2) of Proposition 3.2.1. If K/F is ramified, then $r \geq 4$ and $l' \geq 2$ and a contradiction to 3) of the proposition. So we have

$$\max\{m_i - m_{i+1} \mid 1 \le i < n\} < a.$$

This implies that the number of $s \in S$ such that $\operatorname{Hom}_{s^{-1}G(\mathfrak{p}^a)s \cap G(O_F)}(\mathbf{1}, \delta) \neq 0$ is finite, and then

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G(\mathfrak{p}^a)} \left(\mathbf{1}, \operatorname{ind}_{G(O_F)}^{G(F)} \delta \right) < \infty.$$

This proves Proposition 3.1.1.

3.4 In this subsection, we will prove Proposition 3.1.2. To begin with, we will prove the following proposition

Proposition 3.4.1 *If* $r \ge 2e$, then

- 1) $\dim_{\mathbb{C}} \operatorname{Hom}_{G(O_F)} \left(\delta, \operatorname{ind}_{G(O_F)}^{G(F)} \delta \right) = 1,$
- 2) if Δ is an irreducible representation of $G(O_F)$ which factors through the canonical surjection $G(O_F) \to G(O_F/\mathfrak{p}^r)$ such that

$$\operatorname{Hom}_{G(O_F)}\left(\Delta, \operatorname{ind}_{G(O_F)}^{G(F)}\delta\right) \neq 0,$$

then $\Delta = \delta$.

[Proof] Cartan decomposition (4) gives

$$\operatorname{ind}_{G(O_F)}^{G(F)}\delta\bigg|_{G(O_F)}=\bigoplus_{m\in\mathbb{M}}\operatorname{ind}_{G(O_F)\cap\varpi^mG(O_F)\varpi^{-m}}^{G(O_F)}\delta^{\varpi^m}.$$

Then Frobenius reciprocity gives

$$\operatorname{Hom}_{G(O_F)}\left(\Delta,\operatorname{ind}_{G(O_F)}^{G(F)}\delta\right) = \bigoplus_{m \in \mathbb{M}} \operatorname{Hom}_{\varpi^{-m}G(O_F)\varpi^m \cap G(O_F)}\left(\Delta^{\varpi^{-m}},\delta\right).$$

Now take a $m = (m_1, \dots, m_n) \in \mathbb{M}$ such that

$$\operatorname{Hom}_{\varpi^{-m}G(O_F)\varpi^m\cap G(O_F)}\left(\Delta^{\varpi^{-m}},\delta\right)\neq 0.$$

Suppose $\max\{m_i - m_{i+1} \mid 1 \le i < n\} = m_i - m_{i+1} \ge 2$. Then

$$U_i(O_F) = \varpi^{-m} U_i(O_F) \varpi^m \cap U_i(O_F) \subset \varpi^{-m} G(O_F) \varpi^m \cap G(O_F)$$

and we have

$$\operatorname{Hom}_{U_{i}(\mathfrak{p}^{r-2})}\left(\Delta^{\varpi^{-m}},\delta\right) \supset \operatorname{Hom}_{\varpi^{-m}G(O_{F})\varpi^{m}\cap G(O_{F})}\left(\Delta^{\varpi^{-m}},\delta\right) \neq 0.$$

Because $G(\mathfrak{p}^r) \subset \operatorname{Ker} \Delta$ and $\varpi^m U_i(\mathfrak{p}^{r-2}) \varpi^{-m} \subset U_i(\mathfrak{p}^r)$, we have

$$\operatorname{Hom}_{U_i(\mathfrak{p}^{r-k})}(\mathbf{1},\delta) \supset \operatorname{Hom}_{U_i(\mathfrak{p}^{r-2})}(\mathbf{1},\delta) \neq 0.$$

Here **1** is the trivial one-dimensional representation of $U_i(\mathfrak{p}^{r-2})$. On the other hand $r \geq 2e$ implies $r-e \geq l$ and $U_i(\mathfrak{p}^{r-2}) \subset G(\mathfrak{p}^l)$. Then due to the decomposition (5), there exists a $g \in G(O_F)$ such that $\psi_{\mathrm{Ad}(g)\beta}(h) = 1$ for all $h \in U_i(\mathfrak{p}^{r-k})$ with

$$k = \begin{cases} 1 & : e = 1, \\ 2 & : e > 1. \end{cases}$$

This means

$$g\beta g^{-1} \equiv \begin{bmatrix} A & * \\ 0 & D \end{bmatrix} \pmod{\mathfrak{p}^k}$$

with some $A \in M_i(O_F)$ and $D \in M_{n-i}(O_F)$. Then

$$\chi_{\beta}(t) \equiv \det(t1_i - A) \cdot \det(t1_{n-i} - D) \pmod{\mathfrak{p}^k}$$

which contradicts to Proposition 3.2.1. So, if

$$\operatorname{Hom}_{\varpi^{-m}G(O_F)\varpi^m\cap G(O_F)}\left(\Delta^{\varpi^{-m}},\delta\right)\neq 0$$

then $0 \le m_i - m_{i+1} \le 1$ for all $1 \le i < n$. If there exists $1 \le i < n$ such that $m_i - m_{i+1} = 1$, then we have as above

$$g\beta g^{-1} \equiv \begin{bmatrix} A & * \\ 0 & D \end{bmatrix} \pmod{\mathfrak{p}}$$

with some $g \in G(O_F)$, $A \in M_i(O_F)$ and $D \in M_{n-i}(O_F)$, and hence

$$\chi_{\beta}(t) \equiv \det(t1_i - A) \cdot \det(t1_{n-i} = D) \pmod{\mathfrak{p}}.$$

Then 2) of Proposition 3.2.1 implies that $i = \deg \det(t1_i - A)$ is a multiple of f. So we have $m_1 - m_n < e \le l'$. We can take a $\gamma \in \mathfrak{g}(O_F)$ such that

$$\Delta|_{G(\mathfrak{p}^l/\mathfrak{p}^r)} = \left(\bigoplus_{\dot{h}} h * \psi_{\gamma}\right)^c$$

with some integer c > 0. Here $\bigoplus_{\dot{h}}$ is the direct sum over $\dot{h} \in G(O_F/\mathfrak{p}^r)/G(O_F/\mathfrak{p}^r, \psi_{\gamma})$.

Then we have

$$\bigoplus_{\dot{h}} \bigoplus_{\dot{g}} \operatorname{Hom}_{G(\mathfrak{p}^{l+m_1-m_n})} \left((h * \psi_{\gamma})^{\varpi^{-m}}, g * \psi_{\beta} \right) \neq 0$$

because $G(\mathfrak{p}^{l+m_1-m_n}) \subset \varpi^{-m}G(O_F)\varpi^m \cap G(O_F)$ and $\varpi^mG(\mathfrak{p}^{l+m_1-m_n})\varpi^{-m} \subset G(\mathfrak{p}^l)$. Hence there exist $g, h \in G(O_F)$ such that

$$\psi_{\mathrm{Ad}(g)\beta}(x) = \psi_{\mathrm{Ad}(h)\gamma}(\varpi^m x \varpi^{-m})$$

for all $x \in G(\mathfrak{p}^{l+m_1-m_n})$. This means

$$q\beta q^{-1} \equiv \varpi^{-m} h \gamma h^{-1} \varpi^m \pmod{\mathfrak{p}},$$

that is $\varpi^m g \beta g^{-1} \varpi^{-m} \in M_n(O_F)$. Then there exists a $g' \in GL_n(O_F)$ such that $\varpi^m g \beta g^{-1} \varpi^{-m} = g' \beta g'^{-1}$ and hence $g'^{-1} \varpi^m g \in K$ due to 2) and 3) of Remark 3.2.2. On the other hand we have

$$N_{K/F}(g'^{-1}\varpi^m g) = \det(g'^{-1}\varpi^m g) \in O_F^{\times}$$

so that $g'^{-1}\varpi^m g\in O_K^\times\subset GL_n(O_F)$. Hence $m=(0,\cdots,0)$. Now we have proved

$$\operatorname{Hom}_{G(O_F)}\left(\Delta,\operatorname{ind}_{G(O_F)}^{G(F)}\delta\right) = \operatorname{Hom}_{G(O_F)}(\Delta,\delta)$$

which implies the two statements of the proposition.

The proof of Proposition 3.1.2. Let $W \subset \operatorname{ind}_{G(O_F)}^{G(F)} \delta$ be a non-trivial G(F)-stable subspace. Then we have by Frobanius reciprocity

$$0 \neq \operatorname{Hom}_{G(F)} \left(W, \operatorname{ind}_{G(O_F)}^{G(F)} \delta \right) \subset \operatorname{Hom}_{G(O_F)} \left(W, \operatorname{ind}_{G(O_F)}^{G(F)} \delta \right)$$
$$= \operatorname{Hom}_{G(O_F)} (W, \delta),$$

and hence W contains δ as $G(O_F)$ -module. On the other hand, also by Frobenius reciprocity, we have

$$0 \neq \operatorname{Hom}_{G(F)}(V, V/W) = \operatorname{Hom}_{G(O_F)}(\delta, V/W)$$

with $V=\operatorname{ind}_{G(O_F)}^{G(F)}\delta$, and hence V/W contains δ as a $G(O_F)$ -module. Since the admissible representations of G(F) are semi-simple over the compact subgroup $G(O_F)$, this means that $\operatorname{ind}_{G(O_F)}^{G(F)}\delta$ contains δ with multiplicity as least two, which contradicts to 1) of Proposition 3.4.1. So the G(F)-stable subspace of $\operatorname{ind}_{G(O_F)}^{G(F)}\delta$ is trivial.

4 Formal degree of supercuspidal representations

In this section, we will prove Proposition 2.2.1.

- **4.1** We assume that n is prime to p so that
 - I) the trace form

$$\mathfrak{g}(\mathbb{F}) \times \mathfrak{g}(\mathbb{F}) \to \mathbb{F} \quad ((X, Y) \mapsto \operatorname{tr}(XY))$$

is non-degenerate.

For any $X \in M_n(O_F)$, we have

$$\det(1_n + \varpi^l X) \equiv 1 + \varpi^l \operatorname{tr} X \pmod{\mathfrak{p}^{2l}}.$$

On the other hand we have

$$\det g \equiv 1 + \varpi^{l-1} \operatorname{tr} X + 2^{-1} \varpi^{2l-2} (\operatorname{tr} X)^2 \pmod{\mathfrak{p}^{2l-1}}$$

for
$$g = 1_n + \varpi^{l-1}X + 2^{-1}\varpi^{2l-2}X^2 \in GL_n(O_F)$$
 so that

II) for any r = l + l' with $0 < l' \le l$, we have a group isomorphism

$$\mathfrak{g}(O_F/\mathfrak{p}^{l'}) \tilde{\to} G(\mathfrak{p}^l/\mathfrak{p}^r) \quad (X \pmod{\mathfrak{p}^{l'}}) \mapsto 1_n + \varpi^l X \pmod{\mathfrak{p}^r}),$$

III) if r = 2l - 1 > 1 is odd, then we have a map

$$\mathfrak{g}(O_F) \to G(\mathfrak{p}^{l-1}/\mathfrak{p}^r) \quad (X \mapsto 1_n + \varpi^{l-1}X + 2^{-1}\varpi^{2l-2}X^2 \pmod{\mathfrak{p}^r}).$$

Then the general theory developed by [10] is applicable to our case. Let us recall the general theory by writing down explicitly the irreducible representation δ of $G(O_F/\mathfrak{p}^r)$ corresponding to the character

$$\theta: G_{\beta}(O_F/\mathfrak{p}^r) \to \mathbb{C}^{\times}$$

such that $\theta = \psi_{\beta}$ on $G_{\beta}(O_F/\mathfrak{p}^r) \cap G(\mathfrak{p}^l/\mathfrak{p}^r)$. The general theory of [10] says that

$$\delta = \operatorname{Ind}_{G(O_F/\mathfrak{p}^r, \psi_{\beta})}^{G(O_F/\mathfrak{p}^r)} \sigma_{\beta, \theta} \tag{7}$$

with an irreducible representation $\sigma_{\beta,\theta}$ of the isotropy subgroup $G(O_F/\mathfrak{p}^r,\psi_\beta)$ of ψ_β such that

$$\dim \sigma_{\beta,\theta} = \begin{cases} 1 & : r \text{ is even,} \\ q^{n(n-1)/2} & : r \text{ is odd.} \end{cases}$$
 (8)

Because the canonical group homomorphism $G(O_F/\mathfrak{p}^r) \to G(O_F/\mathfrak{p}^{l'})$ is surjective, (6) implies

$$G(O_F/\mathfrak{p}^r, \psi_\beta) = G_\beta(O_F/\mathfrak{p}^r) \cdot G(\mathfrak{p}^{l'}/\mathfrak{p}^r). \tag{9}$$

If r=2l is even, then $\sigma_{\beta,\theta}$ is defined by

$$\sigma_{\beta,\theta}(gh) = \theta(g) \cdot \psi_{\beta}(h)$$

for $g \in G_{\beta}(O_F/\mathfrak{p}^r)$ and $h \in G(\mathfrak{p}^l/\mathfrak{p}^r)$.

If r = 2l - 1 is odd, then $\sigma_{\beta,\theta}$ is realized on the complex vector space $L^2(\mathbb{W}')$ of the complex valued functions on \mathbb{W}' where we fix a polarization $\mathbb{V}_{\beta} = \mathbb{W} \oplus \mathbb{W}'$ of the symplectic space \mathbb{V}_{β} over \mathbb{F} . The first statement of Remark 3.2.2 implies

$$\begin{split} \mathfrak{g}_{\beta}(\mathbb{F}) &= \{X \in \mathbb{F}[\overline{\beta}] \subset M_n(\mathbb{F}) \mid \operatorname{tr} X = 0\} \\ &= \{\overline{X} \in O_K/\mathfrak{p}_K^e \mid T_{K/F}(X) \equiv 0 \pmod{\mathfrak{p}}\} \end{split}$$

with $\overline{\beta} = \beta \pmod{\mathfrak{p}} \in M_n(\mathbb{F})$. Then we have

$$\dim_{\mathbb{F}} \mathbb{V}_{\beta} = (n^2 - 1) - (n - 1) = n(n - 1)$$

so that $\dim_{\mathbb{C}} L^2(\mathbb{W}') = q^{n(n-1)/2}$. Let H_{β} be the Heisenberg group associated with the symplec space \mathbb{V}_{β} over \mathbb{F} , that is $H_{\beta} = \mathbb{V}_{\beta} \times \mathbb{C}^{\times}$ with a group operation

$$(u,s)\cdot(v,t)=(u+v,st\widehat{\tau}\left(2^{-1}\langle u,v\rangle_{\beta}\right))$$

and π_{β} the Schrödinger representation of H_{β} on $L^{2}(\mathbb{W}')$, that is

$$(\pi_{\beta}(u,s)f)(w) = s \cdot \widehat{\tau} \left(2^{-1}\langle u_{-}, u_{+}\rangle_{\beta} + \langle w, u_{+}\rangle_{\beta}\right) \cdot f(w+u_{-})$$

for $(u,s) \in H_{\beta}$ and $f \in L^{2}(\mathbb{W}')$ with $u = u_{-} + u_{+}$ $(u_{-} \in \mathbb{W}', u_{+} \in \mathbb{W})$. A representation $\pi_{\beta,\theta}$ of $G(\mathfrak{p}^{l-1}/\mathfrak{p}^{r})$ on $L^{2}(\mathbb{W}')$ is defined as follows. Take a $h = 1_{n} + \varpi^{l-1}T \pmod{\mathfrak{p}^{r}} \in G(\mathfrak{p}^{l-1}/\mathfrak{p}^{r})$ with $T \in M_{n}(O_{F})$. Then $T \pmod{\mathfrak{p}^{l-1}} \in \mathfrak{g}(O_{F}/\mathfrak{p}^{l-1})$ and the image of it under the canonical surjection $\mathfrak{g}(O_{F}/\mathfrak{p}^{l-1}) \to \mathfrak{g}(O_{F}/\mathfrak{p}^{l-1})$ $\mathfrak{g}(\mathbb{F})$ is denoted by \widehat{T} . Put $v = \widehat{T} \pmod{\mathfrak{g}_{\beta}(\mathbb{F})} \in \mathbb{V}_{\beta}$ and $Y = \widehat{T} - [v] \in \mathfrak{g}_{\beta}(\mathbb{F})$.

$$\pi_{\beta,\theta}(h) = \tau \left(\varpi^{-l} \operatorname{tr}(T\beta) - 2^{-1} \varpi^{-1} \operatorname{tr}(T^2\beta) \right) \cdot \rho(Y) \cdot \pi_{\beta}(v,1).$$

Here an additive character $\rho: \mathfrak{g}_{\beta}(\mathbb{F}) \to \mathbb{C}^{\times}$ is defined as follows. Since the O_F -group scheme G_{β} is smooth, the canonical map $\mathfrak{g}_{\beta}(O_F) \to \mathfrak{g}_{\beta}(\mathbb{F})$ is surjective. So for any $Y \in \mathfrak{g}_{\beta}(\mathbb{F})$, we can take a $X \in \mathfrak{g}_{\beta}(O_F)$ such that $Y = X \pmod{\mathfrak{p}}$. Then

$$g = 1_n + \varpi^{l-1}X + 2^{-1}\varpi^{2l-2}X^2 \pmod{\mathfrak{p}^r} \in G_\beta(\mathfrak{p}^{l-1}/\mathfrak{p}^r)$$

and we will define

$$\rho(Y) = \tau \left(-\varpi^{-l} \operatorname{tr}(X\beta) \right) \cdot \theta(g).$$

There exists a group homomorphism $U: G_{\beta}(O_F/\mathfrak{p}^r) \to GL_{\mathbb{C}}(L^2(\mathbb{W}'))$ such that

$$\pi_{\beta,\theta}(g^{-1}hg) = U(g)^{-1} \circ \pi_{\beta,\theta}(h) \circ U(g)$$

for all $a \in G_{\beta}(O_F/\mathfrak{p}^r)$ and $h \in G(\mathfrak{p}^{l-1}/\mathfrak{p}^r)$. Now the representation $\sigma_{\beta,\theta}$ of

$$G(O_F/\mathfrak{p}^r, \psi_\beta) = G_\beta(O_F/\mathfrak{p}^r) \cdot G(\mathfrak{p}^{l-1}/\mathfrak{p}^r)$$

is defined by

$$\sigma_{\beta,\theta}(gh) = \theta(g) \cdot U(g) \circ \pi_{\beta,\theta}(h)$$

for $g \in G_{\beta}(O_F/\mathfrak{p}^r)$ and $h \in G(\mathfrak{p}^{l-1}/\mathfrak{p}^r)$.

The proof of Proposition 2.2.1. By (7), we have

$$\dim \delta = (G(O_F/\mathfrak{p}^r) : G(O_F/\mathfrak{p}^r, \psi_\beta)) \cdot \dim \sigma_{\beta,\theta}. \tag{10}$$

Because of (9), we have

$$|G(O_F/\mathfrak{p}^r,\psi_\beta)| = \frac{|G_\beta(O_F/\mathfrak{p})||G(\mathfrak{p}^{l'}/\mathfrak{p}^r)|}{|G_\beta(O_F/\mathfrak{p}^r) \cap G(\mathfrak{p}^{l-1}/\mathfrak{p}^r)|}.$$

 $G(\mathfrak{p}^{l'}/\mathfrak{p}^r)$ is the kernel of the canonical surjection $G(O_F/\mathfrak{p}^r) \to G(O_F/\mathfrak{p}^{l'})$, and $G_{\beta}(O_F/\mathfrak{p}^r) \cap G(\mathfrak{p}^{l'}/\mathfrak{p}^r)$ is the kernel of the canonical surjection $G_{\beta}(O_F/\mathfrak{p}^r) \to$ $G_{\beta}(O_F/\mathfrak{p}^{l'})$. Hence we have

$$(G(O_F/\mathfrak{p}^r): G(O_F/\mathfrak{p}^r, \psi_\beta)) = \frac{|G(O_F/\mathfrak{p}^{l'})|}{|G_\beta(O_F/\mathfrak{p}^{l'})|}.$$
 (11)

We have

$$|G(O_F/\mathfrak{p}^{l'})| = q^{l'n(n-1)/2} \prod_{k=2}^n (1 - q^{-k}).$$
 (12)

On the other hand $G_{\beta}(O_F/\mathfrak{p}^{l'})$ is the kernel of

$$\left(O_K/\mathfrak{p}_K^{el'}\right)^{\times} \to \left(O_F/\mathfrak{p}^{l'}\right) \quad (\varepsilon \mapsto N_{K/F}(\varepsilon)),$$

and $1 + \mathfrak{p} = N_{K/F}(1 + \mathfrak{p}_K)$ (see [1, p.32, Prop.2]). Then we have

$$|G_{\beta}(O_F/\mathfrak{p}^{l'})| = (O_F^{\times} : N_{K/F}(O_K^{\times})) \cdot q^{l'(n-1)} \cdot \frac{1 - q^{-f}}{1 - q^{-1}}.$$
 (13)

Combining the equations (8), (10), (11), (12) and (13), the proof of Proposition 2.2.1 is completed.

5 Induced representations of Weil group

In this section, we will assume that K/F is a tamely ramified Galois extension and will prove Theorem 2.3.1.

The algebraic extensions of F are taken within a fixed algebraic closure \overline{F} of F. Define a group homomorphism

$$\nu_F: \overline{F}^{\times} \to \mathbb{Q}$$

by $\nu_F(x) = (F(x):F)^{-1} \operatorname{ord}_F(N_{F(x)/F}(x)) \ (0 \neq x \in \overline{F})$ and put $\nu_F(0) = \infty$. For an algebraic extension L/F, put

$$O_L = \{ x \in L \mid \nu_F(x) \ge 0 \}, \quad \mathfrak{p}_L = \{ x \in L \mid \nu_F(x) > 0 \}.$$

Then $\mathbb{L} = O_L/\mathfrak{p}_L$ is an algebraic extension of $\mathbb{F} = O_F/\mathfrak{p}$. Let us denote by F^{sep} the separable closure of F.

5.1 To begin with, we will recall the definition of the L-factor, the ε -factor and the γ -factor of a representation of a Weil group (or Weil-Deligne group, more precisely). See [3] for the details.

The Weil group W_F of F is the inverse image by the canonical restriction mapping

$$\operatorname{Gal}(F^{\operatorname{sep}}/F) \to \operatorname{Gal}(F^{\operatorname{ur}}/F)$$

of the cyclic subgroup $\langle \operatorname{Fr} \rangle \subset \operatorname{Gal}(F^{\operatorname{ur}}/F)$ generated by the geometric Frobanius automorphism Fr which induces the inverse of the Frobanius automorphism in $\operatorname{Gal}(\overline{\mathbb{F}}/\mathbb{F})$. Fix an extension $\operatorname{\widetilde{Fr}} \in \operatorname{Gal}(F^{\operatorname{sep}}/F)$ of Fr. Put

$$I_F = \operatorname{Gal}(F^{\operatorname{sep}}/F^{\operatorname{ur}})$$

which is a normal subgroup of W_F and we have $W_F = \langle \widetilde{Fr} \rangle \ltimes I_F$. Weil group W_F is endowed with the topology such that I_F , with the usual Krull topology, is an open compact subgroup of W_F .

Take a complex linear algebraic group \mathcal{G} such that its connected component \mathcal{G}^o is reductive. There is a bijective correspondence between the conjugacy classes of the triplets (ρ, \mathcal{G}, N) such that

- 1) $\rho: W_F \to \mathcal{G}$ is a group homomorphism which is continuous on I_F ,
- 2) $\rho(\widetilde{Fr}) \in \mathcal{G}$ is a semi-simple element,
- 3) $N \in \text{Lie}(\mathcal{G})$ is a nilpotent element such that $\rho(\sigma)N = |\sigma|_F N$ for all $\sigma \in W_F$

and the conjugacy classes of the continuous group homomorphisms

$$\varphi: W_F \times SL_2(\mathbb{C}) \to \mathcal{G}$$

such that

- 1) $\operatorname{Ker}(\varphi) \cap I_F$ is an open subgroup of I_F ,
- 2) $\varphi(\widetilde{Fr}) \in \mathcal{G}$ is semi-simple element,
- 3) $\varphi|_{SL_2(\mathbb{C})}$ is a morphism of complex algebraic group

defined by

$$\rho|_{I_F} = \varphi|_{I_F}, \quad \rho(\widetilde{\operatorname{Fr}}) = \varphi(\widetilde{\operatorname{Fr}}) \cdot \varphi \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}, \quad N = d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where $d\varphi : \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{Lie}(\mathcal{G})$ is the differential of $\varphi|_{SL_2(\mathbb{C})}$. The triplet (ρ, \mathcal{G}, N) or the group homomorphism φ is called a representation of Weil-Deligne group on \mathcal{G} .

Take an algebraic complex representation (r, V) of \mathcal{G} , that is V is a finite dimensional complex vector space and $r: \mathcal{G} \to GL_{\mathbb{C}}(V)$ is a morphism of complex algebraic group. Since the kernel V_N of $dr(N) \in \text{Lie}(GL_{\mathbb{C}}(V)) = \text{End}_{\mathbb{C}}(V)$ is $r \circ \rho(I_F)$ -stable, we will put

$$L(\varphi, r, s) = \det\left(1 - q^{-s}r \circ \rho(\widetilde{Fr})|_{V_N^{I_F}}\right)^{-1}$$

where $V_N^{I_F}$ is the subspace of V_N of the I_F -fixed vectors. The ε -factor is defined by

$$\varepsilon(\varphi, r, s) = \varepsilon_0(r \circ \rho, s) \cdot \det\left(-q^{-s}r \circ \rho(\widetilde{Fr})|_{V^{I_F}/V_N^{I_F}}\right)$$

where V^{I_F} is the subspace of V of the I_F -fixed vectors and

$$\varepsilon_0(r \circ \rho, s) = w(r \circ \rho) \cdot q^{-a(r \circ \rho)(s-1/2)}$$

is the ε -factor of the representation

$$r \circ \rho : W_F \to GL_{\mathbb{C}}(V)$$

of Weil group. Here $w(r \circ \rho)$ is a complex number of absolute value one (the root number) and $a(r \circ \rho)$ is the Artin conductor defined as follows. There exists a finite extension L/F^{ur} such that $\mathrm{Gal}(F^{\mathrm{sep}}/L) \subset I_F \cap \mathrm{Ker}(\rho)$. Put

$$D_t = D_t(L/F^{ur})$$

$$= \left\{ \sigma \in \operatorname{Gal}(L/F^{ur}) \mid x^{\sigma} \equiv x \pmod{\mathfrak{p}_L^{t+1}} \text{ for } \forall x \in O_L \right\}$$

for $t = 0, 1, 2, \cdots$. Then $a(r \circ \rho)$ is defined by

$$a(r \circ \rho) = \sum_{t=0}^{\infty} (D_0 : D_t)^{-1} \dim_{\mathbb{C}} \left(V/V^{D_t} \right)$$

where V^{D_t} is the subspace of V of the $r \circ \rho(\widetilde{D}_t)$ -fixed vectors with the inverse image \widetilde{D}_t of D_t by the restriction mapping $I_F \to \operatorname{Gal}(L/F^{\operatorname{ur}})$. Finally the γ -factor is defined by

$$\gamma(\varphi, r, s) = \varepsilon(\varphi, r, s) \cdot \frac{L(\varphi, r^{\vee}, 1 - s)}{L(\varphi, r, s)}$$

where r^{\vee} is the dual of r.

In the following discussions, the complex algebraic group \mathcal{G} is the L-group of $G = SL_n$ over F, that is $\mathcal{G} = PGL_n(\mathbb{C})$ since SL_n is split over F,

5.2 Let Sym_{n-1} be the symmetric tensor representation of $SL_2(\mathbb{C})$ on the space of the complex coefficient homogeneous polynomials of X, Y of degree n-1, which gives the group homomorphism

$$\operatorname{Sym}_{n-1}: SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$$

with respect to the \mathbb{C} -basis

$${X^{n-1}, X^{n-2}Y, \cdots, XY^{n-2}, Y^{n-1}}.$$

Then

$$d\mathrm{Sym}_{n-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N_0 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$$

is the nilpotent element in $\mathfrak{pgl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C})$ associated with the standard épinglage of the standard root system of $\mathfrak{sl}_n(\mathbb{C})$. Then

$$\varphi_0: W_F \times SL_2(\mathbb{C}) \xrightarrow{\operatorname{proj.}} SL_2(\mathbb{C}) \xrightarrow{\operatorname{Sym}_{n-1}} GL_n(\mathbb{C}) \xrightarrow{\operatorname{canonical}} PGL_n(\mathbb{C})$$

is a representation of Weil-Deligne group with the associated triplet $(\rho_0, PGL_n(\mathbb{C}), N_0)$ such that $\rho_0|_{I_F}$ is trivial and

$$\rho_0(\widetilde{\mathrm{Fr}}) = \overline{\begin{bmatrix} q^{-(n-1)/2} & & & & \\ & q^{-(n-3)/2} & & & \\ & & \ddots & & \\ & & & q^{(n-3)/2} & \\ & & & & q^{(n-1)/2} \end{bmatrix}} \in PGL_n(\mathbb{C}).$$

Let Ad : $PGL_n(\mathbb{C}) \to GL_{\mathbb{C}}(\widehat{\mathfrak{g}})$ be the adjoint representation of $PGL_n(\mathbb{C})$ on $\widehat{\mathfrak{g}} = \mathfrak{sl}_n(\mathbb{C})$. Then

$$\{N_0^k \mid k = 1, 2, \cdots, n - 1\}$$
 (14)

is the \mathbb{C} -basis of $\widehat{\mathfrak{g}}_{N_0}$. The representation matrix of $\operatorname{Ad} \circ \rho_0(\widetilde{\operatorname{Fr}})$ on $\widehat{\mathfrak{g}}_{N_0}$ with respect to the \mathbb{C} -basis (14) is

$$\begin{bmatrix} q^{-1} & & & & & \\ & q^{-2} & & & & \\ & & \ddots & & & \\ & & & q^{-(n-1)} \end{bmatrix}$$

so that we have

$$L(\varphi_0, \mathrm{Ad}, s) = \prod_{k=1}^{n-1} \left(1 - q^{-(s+k)}\right)^{-1}.$$

On the other hand [3, p.448] shows

$$\varepsilon(\varphi_0, \operatorname{Ad}, 0) = q^{n(n-1)/2}$$

Since the symmetric tensor representation Sym_{n-1} is self-dual, we have

$$\gamma(\varphi_0) = \gamma(\varphi_0, \text{Ad}, 0) = q^{n(n-1)/2} \cdot \frac{1 - q^{-1}}{1 - q^{-n}}.$$
 (15)

5.3 In this subsection, we will compute $L(\varphi, \operatorname{Ad}, s)$ for a representation φ induced from a character of K^{\times} in the situation a little more general than that of subsection 2.3.

Let K/F be a tamely ramified Galois extension of degree n and

$$\theta:K^{\times}\to\mathbb{C}^{\times}$$

is a continuous character such that $x \mapsto \theta(x^{\sigma-1})$ is the trivial character of K^{\times} only if $\sigma \in \operatorname{Gal}(K/F)$ is 1. Based upon the group extension (1), we have an identification $W_{K/F} = \operatorname{Gal}(K/F) \times K^{\times}$ with group operation

$$(\sigma, x) \cdot (\tau, y) = (\sigma \tau, x^{\tau} \cdot y \cdot \alpha_{K/F}(\sigma, \tau))$$

with the fundamental class $[\alpha_{K/F}] \in H^2(\mathrm{Gal}(K/F), K^{\times})$. Then the induced representation $\mathrm{Ind}_{K^{\times}}^{W_{K/F}} \theta$ is realized on the complex vector space V of the complex valued functions on $\mathrm{Gal}(K/F)$ with the action of $W_{K/F}$ defined by

$$(\sigma \cdot \psi)(\tau) = \theta \left(\alpha_{K/F}(\sigma, \sigma^{-1}\tau) \right) \cdot \psi(\sigma^{-1}\tau), \qquad \alpha \cdot \psi = \theta_{\alpha} \cdot \psi$$

for $(\sigma, \alpha) \in W_{K/F}$ and $\psi \in V$ with $\theta_{\alpha} \in V$ defined by $\theta_{\alpha}(\tau) = \theta(\alpha^{\tau})$. Take the standard basis $\{\psi_{\sigma}\}_{\sigma \in \operatorname{Gal}(K/F)}$ of V where

$$\psi_{\sigma}(\tau) = \begin{cases} 1 & : \tau = \sigma, \\ 0 & : \tau \neq \sigma. \end{cases}$$

Then

$$\rho \cdot \psi_{\sigma} = \theta \left(\alpha_{K/F}(\rho, \sigma) \right) \cdot \psi_{\rho \sigma}$$

for $\rho, \sigma \in \operatorname{Gal}(K/F)$. In particular $\sigma \cdot \psi_1 = \psi_{\sigma}$. We have

Proposition 5.3.1 $\operatorname{Ind}_{K^{\times}}^{W_{K/F}} \theta$ is an irreducible representation of $W_{K/F}$.

[**Proof**] Take a $T \in \operatorname{End}_{W_{K/F}}(V)$. If $(T\psi_1)(\tau) \neq 0$ with $\tau \in \operatorname{Gal}(K/F)$, then, for any $\alpha \in K^{\times}$, $\alpha \cdot T\psi_1 = T(\alpha \cdot \psi_1)$ implies $\theta(\alpha^{\tau}) = \theta(\alpha)$, and hence $\tau = 1$. This means $T\psi_1 = c\psi_1$ with a $c \in \mathbb{C}$. Then we have $T\psi_{\sigma} = c\psi_{\sigma}$ for all $\sigma \in \operatorname{Gal}(K/F)$ and hence $T = c \cdot \operatorname{id}_V$.

Put

$$\Theta: W_{K/F} \xrightarrow{\operatorname{Ind}_{K^{\times}}^{W_{K/F}} \theta} GL_{\mathbb{C}}(V) \xrightarrow{\operatorname{canonical}} PGL_{\mathbb{C}}(V)$$

and

$$\rho: W_F \xrightarrow{\text{canonical}} W_{K/F} \xrightarrow{\Theta} PGL_{\mathbb{C}}(V).$$

Then the representation of Weil-Deligne group corresponding to the triplet $(\rho, PGL_{\mathbb{C}}(V), 0)$ is

$$\varphi: W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{projection}} W_F \xrightarrow{\rho} PGL_{\mathbb{C}}(V).$$

Let A_{φ} be the centralizer of $\operatorname{Im}(\varphi)$ in $PGL_{\mathbb{C}}(V)$.

Let us denote by \mathcal{A}_{θ} the set of the group homomorphism $\lambda: W_{K/F} \to \mathbb{C}^{\times}$ whose restriction to K^{\times} is a character $\alpha \mapsto \theta(\alpha^{\tau-1})$ with some $\tau \in \operatorname{Gal}(K/F)$ which is uniquely determined by λ . Let us call it associated with λ . If $\tau \in \operatorname{Gal}(K/F)$ is associated with $\lambda \in \mathcal{A}_{\theta}$, then we have

$$\theta(\alpha^{\sigma(\tau-1)}) = \theta(\alpha^{\tau-1})$$

for all $\alpha \in K^{\times}$ and $\sigma, \tau \in \operatorname{Gal}(K/F)$, because

$$\lambda(\alpha^{\sigma}) = \lambda((\sigma, 1)^{-1}(1, \alpha)(\sigma, 1)) = \lambda(\alpha).$$

This implies that \mathcal{A}_{θ} is in fact a subgroup of the character group of $W_{K/F}$. Take a $\overline{T} \in A_{\varphi}$ with $T \in GL_{\mathbb{C}}(V)$. Then we have a character

$$\lambda:W_{K/F}\to\mathbb{C}^{\times}$$

such that $gT = \lambda(g)Tg$ for all $g \in W_{K/F}$. If $(T\psi_1)(\tau) \neq 0$ with $\tau \in \operatorname{Gal}(K/F)$ then $\alpha \cdot T(\psi_1) = \lambda(\alpha)T(\alpha \cdot \psi_1)$ for $\alpha \in K^{\times}$ implies $\theta(\alpha^{\tau}) = \lambda(\alpha)\theta(\alpha)$ for all $\alpha \in K^{\times}$. Hence we have $T\psi_1 = c\psi_{\tau}$ with $c \in \mathbb{C}^{\times}$. Then we have

$$T\psi_{\sigma} = c \cdot \lambda(\sigma)^{-1} \sigma \cdot \psi_{\tau} = c \cdot \lambda(\sigma)^{-1} \theta \left(\alpha_{K/F}(\sigma, \tau) \right) \cdot \psi_{\sigma\tau}$$

for all $\sigma \in \operatorname{Gal}(K/F)$. We have

Proposition 5.3.2 $\overline{T} \mapsto \lambda$ gives a group isomorphism of A_{φ} onto A_{θ} .

[**Proof**] It is clear that $\overline{T}\mapsto \lambda$ is injective group homomorphism, because $\operatorname{Ind}_{K^\times}^{W_{K/F}}\theta$ is irreducible. Take any $\lambda\in\mathcal{A}_\theta$ and the $\tau\in\operatorname{Gal}(K/F)$ associated with it. Define a $T\in GL_{\mathbb{C}}(V)$ by

$$T\psi_{\sigma} = \lambda(\sigma)^{-1}\theta\left(\alpha_{K/F}(\sigma,\tau)\right)\cdot\psi_{\sigma\tau}$$

for all $\sigma \in \operatorname{Gal}(K/F)$. Then we have $gT = \lambda(g) \cdot Tg$ for all $g \in W_{K/F}$.

From now on, we will suppose that $x \mapsto \theta(x^{\sigma-1})$ is a trivial character of O_K^{\times} only if $\sigma \in \operatorname{Gal}(K/F)$ is 1.

The 2-cocycle $\alpha_{K/F}$ can be chosen so that $\alpha_{K/F}(\sigma,\tau) \in O_K^{\times}$ for all $\sigma,\tau \in \operatorname{Gal}(K/K_0)$ where $K_0 = K \cap F^{\operatorname{ur}}$ is the maximal unramified subextension of K/F. Then the image of $I_F \subset W_F$ by the canonical surjection $W_F \to W_{K/F}$ is $\operatorname{Gal}(K/K_0) \times O_K^{\times} \subset W_{K/F}$. Since K/F is tamely ramified, the Galois group $\operatorname{Gal}(K/K_0) = \langle \tau_0 \rangle$ is a cyclic group of order e = e(K/F). Then $\operatorname{Gal}(K/F)$ is generated by $\sigma_0 = \widetilde{\operatorname{Fr}}|_K$ and τ_0 . We have

$$\sigma_0 \tau_0 \sigma_0^{-1} = \tau_0^m \tag{16}$$

with some 0 < m < e such that $GCD\{e, m\} = 1$. Put

$$\psi_{ij} = \psi_{\tau_0^i \sigma_0^j} \in V \text{ with } 0 \le i < e, \ 0 \le j < f.$$

Then the C-basis $\{\psi_{ij}\}_{i,j}$ gives the identification

$$GL_{\mathbb{C}}(V) = GL_n(\mathbb{C})$$
 and $PGL_{\mathbb{C}}(V) = PGL_n(\mathbb{C})$.

We have, for all $\alpha \in K^{\times}$

$$\alpha \cdot \psi_{ij} = \theta \left(\alpha^{\tau_0^i \sigma_0^j} \right) \cdot \psi_{ij} \quad (0 \le i < e, 0 \le j < f)$$
 (17)

so that $\Theta(\alpha) \in PGL_n(\mathbb{C})$ is diagonal. On the other hand we have

$$\tau_0 \cdot \psi_{ij} = \begin{cases} \theta \left(\alpha_{K/F}(\tau_0, \tau_0^i \sigma_0^j) \right) \cdot \psi_{i+1,j} &: 0 \le i < e - 1, \\ \theta \left(\alpha_{K/F}(\tau_0, \tau_0^{-1} \sigma_0^j) \right) \cdot \psi_{0,j} &: i = e - 1 \end{cases}$$

hence

$$\Theta(\tau_0) = \boxed{ \begin{bmatrix} J_0 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_{f-1} \end{bmatrix}} \in PGL_n(\mathbb{C})$$
 (18)

with

 $(a_{ij} = \theta \left(\alpha_{K/F}(\tau_0, \tau_0^i \sigma_0^j) \right))$. So the space of the Ad $\circ \varphi(I_F)$ -fixed vectors in $\widehat{\mathfrak{g}} = \mathfrak{pgl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C})$ is

$$\widehat{\mathfrak{g}}^{\mathrm{Ado}\varphi(I_F)} = \left\{ \begin{bmatrix} a_1 1_e & & & \\ & a_2 1_e & & \\ & & \ddots & \\ & & & a_f 1_e \end{bmatrix} \middle| \begin{array}{l} a_i \in \mathbb{C}, \\ a_1 + a_2 + \dots + a_f = 0 \end{array} \right\}.$$

A C-basis of it is given by

$$X_{1} = \begin{bmatrix} P & & & & \\ & 0_{e} & & & \\ & & \ddots & \\ & & & 0_{e} \end{bmatrix}, X_{2} = \begin{bmatrix} 0_{e} & & & & \\ & P & & & \\ & & \ddots & & \\ & & & 0_{e} \end{bmatrix}, \dots, X_{f-1} = \begin{bmatrix} 0_{e} & & & & \\ & \ddots & & & \\ & & 0_{e} & & \\ & & & P \end{bmatrix}$$

$$(19)$$

with $P = \begin{bmatrix} 1_e \\ -1_e \end{bmatrix}$. The relation (16) gives

$$\sigma_0 \tau_0^i \sigma_0^j = \begin{cases} \tau_0^{i+m} \sigma_0^{j+1} &: 0 \le j < f-1, \\ \tau_0^{i+m} &: j = f-1. \end{cases}.$$

Put $i+m \equiv i' \pmod{e}$ for $0 \leq i, i' < e$ and let $[m]_e \in GL_e(\mathbb{Z})$ be the permutation matrix associated with the element

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & e-1 \\ 0' & 1' & 2' & \cdots & (e-1)' \end{pmatrix}$$

of the symmetric group of degree e. Then we have

$$\Theta(\sigma_0) = \begin{bmatrix} 0 & 0 & \cdots & 0 & I_{f-1} \\ I_0 & 0 & & & 0 \\ & I_1 & \ddots & & \vdots \\ & & \ddots & 0 & 0 \\ & & & I_{f-2} & 0 \end{bmatrix}$$

with

$$I_j = [m]_e \begin{bmatrix} b_{0j} & & & & \\ & b_{1j} & & & \\ & & \ddots & & \\ & & & b_{e-1,j} \end{bmatrix}$$

 $(b_{ij} = \theta \left(\alpha_{K/F}(\sigma_0, \tau_0^i \sigma_0^j)\right))$. So the representation matrix of Ad $\circ \varphi(\widetilde{Fr})$ on $\widehat{\mathfrak{g}}^{\mathrm{Ad}\circ\varphi(I_F)}$ with respect to the basis (19) is

$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Hence we have

$$L(\varphi, \operatorname{Ad}, s) = \det \left(1 - q^{-s} \operatorname{Ad} \circ \varphi(\widetilde{\operatorname{Fr}}) \big|_{\mathfrak{g}^{\operatorname{Ad} \circ \varphi(\widetilde{\operatorname{Fr}})}} \right)^{-1}$$
$$= \left(1 + q^{-s} + q^{-2s} + \dots + q^{-(f-1)s} \right)^{-1}$$

and

$$\frac{L(\varphi, \operatorname{Ad}, 1)}{L(\varphi, \operatorname{Ad}, 0)} = f \cdot \frac{1 - q^{-1}}{1 - q^{-f}}.$$
(20)

5.4 In this subsection, we will prove Theorem 2.3.1. To begin with, we will prove

Proposition 5.4.1 The group homomorphism

$$\Theta: W_{K/F} \xrightarrow{\operatorname{Ind}_{K^{\times}}^{W_{K/F}} \theta} GL_{\mathbb{C}}(V) \xrightarrow{\operatorname{canonical}} PGL_{\mathbb{C}}(V)$$

is independent, up to the conjugate in $PGL_{\mathbb{C}}(V)$, of the choice of the extension $\theta: K^{\times} \to \mathbb{C}^{\times}$ from the character θ of $G_{\beta}(O_F/\mathfrak{p}^r)$.

[**Proof**] Take another extension $\theta': K^{\times} \to \mathbb{C}^{\times}$ from the character θ of

$$G_{\beta}(O_F/\mathfrak{p}^r) = \left\{ \overline{\varepsilon} \in (O_K/\mathfrak{p}_K^{er})^{\times} \mid N_{K/F}(\varepsilon) \equiv 1 \pmod{\mathfrak{p}^r} \right\}$$

Then $\theta = \theta'$ on the subgroup

$$U_{K/F} = \{ \varepsilon \in O_K^{\times} \mid N_{K/F}(\varepsilon) = 1 \}$$

of K^{\times} because the canonical group homomorphism $U_{K/F} \to G_{\beta}(O_F/\mathfrak{p}^r)$ is surjective. So there exists a character $\chi: O_F^{\times} \to \mathbb{C}^{\times}$ such that

$$\theta'(\varepsilon) = \theta(\varepsilon) \cdot \chi\left(N_{K/F}(\varepsilon)\right)$$

for all $\varepsilon \in O_K^{\times}$. We can extend χ to a character of F^{\times} so that

$$\theta'(x) = \theta(x) \cdot \chi \left(N_{K/F}(x) \right)$$

for all $x \in K^{\times}$. The induced representations $\operatorname{Ind}_{K^{\times}}^{W_{K/F}}\theta$ and $\operatorname{Ind}_{K^{\times}}^{W_{K/F}}\theta'$ are realized on the complex vector space of the complex valued functions on $\operatorname{Gal}(K/F)$. For any $\psi' \in \operatorname{Ind}_{K^{\times}}^{W_{K/F}}\theta'$, put $\psi(\tau) = \chi(\gamma(\tau)) \cdot \psi'(\tau)$ $(\tau \in \operatorname{Gal}(K/F))$, where

$$\gamma(\tau) = \prod_{\sigma \in \operatorname{Gal}(K/F)} \alpha_{K/F}(\tau, \sigma) \in F^{\times}.$$

Note that we have

$$N_{K/F}(\alpha_{K/F}(\sigma, \tau)) = \gamma(\sigma)\gamma(\sigma\tau)^{-1}\gamma(\tau)$$

for all $\sigma, \tau \in \operatorname{Gal}(K/F)$. Then the direct calculations show that the \mathbb{C} -linear map $T: \psi' \mapsto \psi$ satisfies the relations

$$\sigma T = \chi(\gamma(\sigma))^{-1} \cdot T\sigma, \qquad \alpha T = \chi(N_{K/F}(\alpha)) \cdot T\alpha$$

for all $\sigma \in \operatorname{Gal}(K/F)$ and $\alpha \in K^{\times}$.

We have also

Proposition 5.4.2 Take a $\tau \in \operatorname{Gal}(K/F)$ and an integer $0 \le k < er$. Then $\theta(\alpha^{\tau}) = \theta(\alpha)$ for all $\alpha \in 1 + \mathfrak{p}_K^{er-k}$ if and only if

$$\tau \in \begin{cases} \operatorname{Gal}(K/F) & : \text{if } k < e, \\ \operatorname{Gal}(K/K_0) & : \text{if } k = e, \\ \{1\} & : \text{if } k > e. \end{cases}$$

[Proof] We can assume that $0 \le k \le el'$. Then we have

$$(1 + \varpi^r \varpi_K^{-k} x)^\tau (1 + \varpi^r \varpi_K^{-k} x)^{-1} \equiv 1 + \varpi^r \left(\varpi_K^{-k\tau} x^\tau - \varpi_K^{-k} x \right) \pmod{\mathfrak{p}_K^{er}}$$

for all $x \in O_K$. So $\theta(\alpha^{\tau}) = \theta(\alpha)$ for all $\alpha \in 1 + \mathfrak{p}_K^{er-k}$ if and only if

$$T_{K/F}(x \cdot \varpi_K^{-k}(\beta^{\tau} - \beta)) \in O_F$$

for all $x \in O_K$. This means that $\varpi_K^{-k}(\beta^{\tau} - \beta) \in \mathcal{D}(K/F)^{-1} = \mathfrak{p}_K^{1-e}$. Here $\mathcal{D}(K/F) = \mathfrak{p}_K^{e-1}$ is the different of K/F because K/F is tamely ramified. Since $O_K = O_F[\beta]$, the condition is $\operatorname{ord}_K(x^{\tau} - x) \geq k - e + 1$ for all $x \in O_K$ which means that

$$\tau \in \begin{cases} D_{-1}(K/F) = \operatorname{Gal}(K/F) & : k - e < 0, \\ D_{0}(K/F) = \operatorname{Gal}(K/K_{0}) & : k - e = 0, \\ D_{1}(K/F) = \{1\} & : k - e > 0 \end{cases}$$

where

$$D_t(K/F) = \{ \sigma \in \operatorname{Gal}(K/F) \mid \operatorname{ord}_K(x^{\sigma} - x) \ge t + 1 \ \forall x \in O_K \}$$

for $-1 \le t \in \mathbb{R}$ is the ramification groups of K/F.

In particular $\theta(\alpha^{\tau}) = \theta(\alpha)$ for all $\alpha \in O_K^{\times}$ only if $\tau \in \operatorname{Gal}(K/F)$ is 1, and hence the results of the preceding subsection are applicable to our case.

Proposition 5.4.3 \mathcal{A}_{θ} is equal to the group of the character λ of $W_{K/F}$ which is trivial on K^{\times} . In particular

$$|A_{\omega}| = |\mathcal{A}_{\theta}| = (O_K : N_{K/F}(O_K^{\times})) \cdot f. \tag{21}$$

[**Proof**] Let $\lambda: W_{K/F} \to \mathbb{C}^{\times}$ be a group homomorphism such that $\lambda(\alpha) = \theta(\alpha^{\tau-1})$ for all $\alpha \in K^{\times}$ with some $\tau \in \operatorname{Gal}(K/F)$. We have $\theta(\alpha^{\sigma(\tau-1)}) = \theta(\alpha^{\tau-1})$ for all $\sigma \in \operatorname{Gal}(K/F)$ and $\alpha \in K^{\times}$, and hence

$$\theta(\alpha^{\tau-1})^n = \prod_{\sigma \in \operatorname{Gal}(K/F)} \theta(\alpha^{\sigma(\tau-1)}) = 1$$

for all $\alpha \in K^{\times}$. This means that the group index $(O_K^{\times} : \operatorname{Ker}(\lambda|_{O_K^{\times}}))$ is finite and a divisor of n, and hence prime to p. $1 + \mathfrak{p}_K^m \subset \operatorname{Ker}(\lambda|_{O_K^{\times}})$ with some integer m > 0. On the other hand we have $O_K^{\times} = \langle w \rangle \times (1 + \mathfrak{p}_K)$ with a primitive $q^f - 1$ -th root of unity $w \in K$ and $(1 + \mathfrak{p}_K : 1 + \mathfrak{p}_K^m) = q^{f(m-1)}$ is a power of p. This implies that m = 1, that is $\theta(\alpha^{\tau}) = \theta(\alpha)$ for all $\alpha \in 1 + \mathfrak{p}_K$. Then $\tau = 1$ by Proposition 5.4.2. So λ is trivial on K^{\times} . Now we have

$$|\mathcal{A}_{\theta}| = (K_1 : F) = (K_1 : K_0) \cdot f$$

where K_1 is the maximal abelian subextension of K/F. On the other hand we have $N_{K/F}(O_K^{\times}) = N_{K_1/F}(O_{K_1}^{\times})$, and hence we have

$$(O_F^{\times}: N_{K/F}(O_K^{\times}) = e(K_1/F) = (K_1:K_0)$$

because K_0 is the maximal unramified subextension of K/F.

The image of $I_F \subset W_F$ under the canonical surjection

$$W_F \to W_F/\overline{[W_K,W_K]} = W_{K/F} \subset \operatorname{Gal}(K^{\operatorname{ab}}/F)$$

is $\operatorname{Gal}(K^{\operatorname{ab}}/F^{\operatorname{ur}})$ which sits in the group extension

$$1 \to O_K^{\times} \xrightarrow{\delta_K} \operatorname{Gal}(K^{\operatorname{ab}}/F^{\operatorname{ur}}) \xrightarrow{\operatorname{res.}} \operatorname{Gal}(K/K_0) \to 1.$$

Let us denote by $K_k = K_{\varpi_K,k}$ $(k = 1, 2, \cdots)$ the field of ϖ_K^k -th division points of Lubin-Tate theory. Then we have an isomorphism

$$\delta_K: 1 + \mathfrak{p}_K^k \tilde{\to} \operatorname{Gal}(K^{\mathrm{ab}}/K_kK^{\mathrm{ur}}).$$

Because the character $\theta: K^{\times} \to \mathbb{C}^{\times}$ comes from a character of

$$G_{\beta}(O_F/\mathfrak{p}^r) \subset (O_K/\mathfrak{p}_K^{er})^{\times},$$

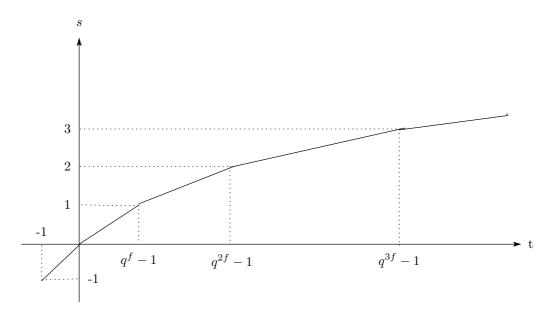
 Θ is trivial on $\operatorname{Gal}(K^{\operatorname{ab}}/K_{er}K^{\operatorname{ur}})$. Note that

$$K_{er}K^{\mathrm{ur}} = K_{er}F^{\mathrm{ur}}$$

is a finite extension of F^{ur} . Let us use the upper numbering

$$D^s = D_t(K_{er}F^{\mathrm{ur}}/F^{\mathrm{ur}})$$

of the higher ramification group, where $t\mapsto s$ is the inverse of Hasse function whose graph is



Then δ_K induces the isomorphism

$$(1 + \mathfrak{p}_K^k)/(1 + \mathfrak{p}_K^{er}) \tilde{\to} \operatorname{Gal}(K_{er}K^{\mathrm{ur}}/K_kK^{\mathrm{ur}}) = D^s$$

for $k-1 < s \le k \ (k=1,2,\cdots)$, and hence

$$|D_t| = \begin{cases} e \cdot q^{nr} (1 - q^{-f}) & : t = 0, \\ q^{nr - fk} & : q^{f(k-1)} - 1 < t \le q^{fk} - 1. \end{cases}$$

The explicit actions (17) and (18) and Proposition 5.4.2 shows that the space of $Ad \circ \Theta(D_t)$ -fixed vectors in $\widehat{\mathfrak{g}}$ is

$$\left\{ \begin{bmatrix} a_1 1_e & & & & \\ & a_2 1_e & & & \\ & & \ddots & & \\ & & & a_f 1_e \end{bmatrix} \middle| \begin{array}{l} a_i \in \mathbb{C}, & & \\ a_1 + a_2 + \dots + a_f = 0 \end{array} \right\}$$

if
$$t = 0$$
,
$$\left\{ \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_1 + a_2 + \dots + a_n = 0 \end{bmatrix} \middle| a_i \in \mathbb{C}, a_1 + a_2 + \dots + a_n = 0 \right\}$$

if $0 < t \le q^{f\{e(r-1)-1\}} - 1$,

$$\left\{ \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_f \end{bmatrix} \middle| \begin{array}{c} A_i \in M_e(\mathbb{C}), \\ \operatorname{tr}(A_1 + A_2 + \dots + A_n) = 0 \end{array} \right\}$$

if $q^{f\{e(r-1)-1\}} - 1 < t \le q^{fe(r-1)} - 1$ and $\widehat{\mathfrak{g}}$ if $q^{fe(r-1)} - 1 < t \le q^{fer} - 1$. So we have

$$\dim_{\mathbb{C}} \widehat{\mathfrak{g}}^{D_t} = \begin{cases} f - 1 & : t = 0, \\ n - 1 & : 0 < t \le q^{f\{e(r-1)-1\}} - 1, \\ fe^2 - 1 & : q^{f\{e(r-1)-1\}} - 1 < t \le q^{fe(r-1)} - 1, \\ n^2 - 1 & : q^{fe(r-1)} - 1 < t \le q^{fer} - 1 \end{cases}$$

Hence we have

$$\sum_{t=0}^{\infty} (D_0 : D_t)^{-1} \dim_{\mathbb{C}} (\widehat{\mathfrak{g}}/\widehat{\mathfrak{g}}^{D_t})$$

$$= n^2 - f + (n^2 - n) \cdot \frac{1}{e} \cdot \{e(r-1) - 1\} + (n^2 - fe^2) \cdot \frac{1}{e}$$

$$= rn(n-1).$$

Combined with (20), we have

$$\gamma(\varpi, \mathrm{Ad}, 0) = q^{rn(n-1)/2} \cdot f \cdot \frac{1 - q^{-n}}{1 - q^{-f}}.$$
(22)

The equations (15), (21) and (22) prove Theorem 2.3.1.

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