A NEW PROOF OF A VANISHING RESULT DUE TO BERTHELOT, ESNAULT, AND RÜLLING

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Abstract. The goal of this small note is to give a more concise proof of a result due to Berthelot, Esnault, and Rülling in [5]. For a regular, proper, and flat scheme X over a discrete valuation ring of mixed characteristic $(0, p)$, it relates the vanishing of the cohomology of the structure sheaf of the generic fibre of X with the vanishing of the Witt vector cohomology of its special fibre. We use as a critical ingredient results and constructions by Beilinson [2] and Nekovář–Nizioł [16] related to the h-topos over a p-adic field.

Résumé. Le but de cette brève note est de donner une démonstration plus courte d'un résultat de Berthelot, Esnault et Rülling dans [5]. Pour un schéma régulier, propre et plat X sur un anneau de valuation discrète de caractéristique $(0, p)$, il lie la disparition de la cohomologie du faisceau structural de la fibre générique de X à la disparition de la cohomologie de Witt de sa fibre spéciale. On utilise de manière critique des résultats et des constructions de Beilinson [2] et Nekovář–Nizioł [16] concernant le h-topos sur un corps p-adique.

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According to [1] schemes of semistable reduction over a complete discrete valuation ring $\mathcal{O}_K$ with perfect residue field form a basis of the h-topology on the category $\mathsf{Var}(K)$ of varieties over the fraction field K of $\mathcal{O}_K$. As a consequence, h-sheafification makes it sometimes possible to generalise constructions or results from schemes of semistable reduction to varieties over a p-adic field.

In this small note we want to illustrate the advantages of this technique and give a shorter and, as we hope, more conceptual proof of the following vanishing result due to Berthelot, Esnault, and Rülling in [5, Thm. 1.3].

Theorem (P. Berthelot, H. Esnault, K. Rülling)**.** *Let R be a discrete valuation ring of mixed characteristic with fraction field K and perfect residue field k, and let X be a regular proper flat scheme over R. Assume that $H^q(X_K, \mathcal{O}) = 0$ for some $q \geqslant 0$. Then $H^q(X_0, W\mathcal{O})_{\mathbf{Q}} = 0$ as well.*

As a consequence of this result the authors obtain, under the additional assumption that k is finite, a congruence on the number of rational points of X with values in finite extensions of k. As explained in [5] this fits into the general analogy between the vanishing of Hodge numbers for varieties over a field of characteristic 0 and congruences on the number of rational points with values in finite extensions for varieties over a finite field.

The above theorem itself is an application of p-adic Hodge theory. In [5, Thm. 2.1] the semistable case is discussed which we recall here briefly as it provides a guideline for our proof.

Thus in the situation of the theorem, let X/R be of semistable reduction. Without loss of generality one can assume that R is a complete discrete valuation ring. Endow R and X with the canonical log structure denoted by $R^\times$ and $X^\times$, the special fibre X_0 with the pull-back log structure denoted by $X_0^\times$,

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and the Witt vectors $W(k)$ with the log-structure associated to $(1 \mapsto 0)$ denoted by $W(k)^0$. Consider the log crystalline cohomology groups $H^q_{\mathrm{cris}}(X_0^\times/W(k)^0))$. This cohomology is sometimes called the Hyodo–Kato cohomology. In the case at hand, they can be computed by the logarithmic de Rham–Witt complex $W\omega^\bullet$ which leads to a spectral sequence

$$E_1^{ij} = H^j(X_0^\times, W\omega^i_{\mathbf{Q}}) \Rightarrow H^{i+j}_{\mathrm{cris}}(X_0^\times/W(k)^0)_{\mathbf{Q}}$$

endowed with a Frobenius action. On the left this Frobenius action is induced by p^iF, where F is the Witt vector Frobenius, whereas on the right the Frobenius action φ is induced by the absolute Frobenii of X_0 and $W(k)$. Similarly to the classical case, it follows that this spectral sequence degenerates at E_1 and that $H^j(X_0^\times, W\omega^i_{\mathbf{Q}})$ corresponds to the part of $H^{i+j}_{\mathrm{cris}}(X_0^\times/W(k)^0)_{\mathbf{Q}}$ where Frobenius has slope in $[i, i+1[$. Hence one obtains a canonical quasi-isomorphism

$$H^q_{\mathrm{cris}}(X_0^\times/W(k)^0)^{<1}_{\mathbf{Q}} \xrightarrow{\sim} H^q(X_0, W\mathcal{O})_{\mathbf{Q}}.$$

In other words, this means that Witt vector cohomology, which we want to study, corresponds to the part of crystalline cohomology of Frobenius slope < 1. The cohomology group $H^q_{\mathrm{cris}}(X_0^\times/W(k)^0)_{\mathbf{Q}}$ is also equipped with a monodromy operator N and a Hyodo–Kato isomorphism

$$\iota_{\mathrm{dR},\pi} : H^q_{\mathrm{cris}}(X_0^\times/W(k)^0) \otimes_{W(k)} K \xrightarrow{\sim} H^q_{\mathrm{dR}}(X_K),$$

where the de Rham cohomology on the right hand side is equipped with the Hodge filtration. In particular, $H^q_{\mathrm{cris}}(X_0^\times/W(k)^0)_{\mathbf{Q}}$ can be regarded as an admissible filtered (φ, N)-module which implies that its Newton polygon lies above its Hodge polygon. But by assumption $H^q(X_K, \mathcal{O}) = 0$, which means that the part where the Hodge slope is < 1 vanishes. Hence the same is true for the part where the Newton slope is < 1, which is as we have seen isomorphic to $H^q(X_0, W\mathcal{O})_{\mathbf{Q}}$. This concludes the proof.

The philosophy behind this proof is that the cohomology groups $H^q(X_K, \mathcal{O})$ and $H^q(X_0, W\mathcal{O})_{\mathbf{Q}}$, which are mathematical invariants associated to the generic and the special fibre, respectively, are in a certain sense part of a more comprehensive theory, namely absolute p-adic Hodge cohomology, which is realised in the category of admissible filtered (φ, N)-modules. The inherent structure provides intricate relations between invariants of the special and generic fibre.

We realised that it is possible to use a very similar argument to obtain the more general statement of the theorem. In fact, it allows us to prove the theorem in a slightly more general case, namely for a proper, reduced and flat scheme over a discrete valuation ring R of mixed characteristic, such that the generic fibre has at most Du Bois singularities.

Let X be such a scheme. Again, we want to interpret the cohomology groups $H^q(X_K, \mathcal{O})$ and $H^q(X_0, W\mathcal{O})_{\mathbf{Q}}$ in terms of absolute p-adic Hodge cohomology. The "right" realisation category in this more general case is the category of admissible filtered (φ, N, G_K)-modules. In fact, Déglise and Nizioł identified absolute p-adic Hodge cohomology for K-varieties (even without integral model) in [8], where they look at the h-sheafification on K-varieties of a certain presentation of Hyodo–Kato cohomology introduced by Beilinson. According to their results, the associated cohomology groups $H^q_{\mathrm{HK,h}}(X_K)$ of the generic fibre X_K can be seen as admissible filtered (φ, N, G_K)-module. For these the Newton and Hodge polygons are related in the same way as for (φ, N)-modules, i.e. their Newton polygon lies above their Hodge polygon.

We need now descent for the h-topology to finish the proof. On the side of de Rham cohomology a descent result due to Huber and Jörder in [11] shows that in the case of Du Bois singularities the part where the Hodge slope is < 1 is exactly $H^q(X_K, \mathcal{O})$, which vanishes by assumption. This means that the part where the Newton slope of $H^q_{\mathrm{HK,h}}(X_K)$ is < 1 vanishes as well. We use a descent result for Witt vector cohomology due to Bhatt and Scholze in [6] to show that this is in fact $H^q(X_0, W\mathcal{O})_{\mathbf{Q}}$, which allows us to conclude.

Note that in our proof, as well as in the original one, descent results play an important role. In [5], the authors work very explicitly with proper hypercovers over a discrete valuation ring to reduce to the semistable case. However, as they remark, the special fibre of such a hypercovering might not be a proper hypercover of the special fibre of the original scheme. This is why they have to show an injectivity theorem [5, Thm. 1.5] which turns out to be very subtle.

Our proof relies on the very sophisticated work of Déglise–Nizioł and Nekovář–Nizioł which allows us to abstractly identify $H^q(X_K, \mathcal{O})$ and $H^q(X_0, W\mathcal{O})_{\mathbf{Q}}$ as part of a bigger picture. Consequently, we don't have to reduce to the semistable case, but nevertheless we have to invoke Bhatt–Scholze's descent theorem to extract the desired information about the special fibre.

Notation and conventions. All schemes considered are separated and of finite type over a base S. For a fixed base scheme S we denote this category by $\mathsf{Sch}(S)$, and by $\mathsf{Var}(S)$ the category of separated reduced schemes of finite type. When $S = \operatorname{Spec} R$ is affine, we write $\mathsf{Sch}(R)$ for $\mathsf{Sch}(S)$ and $\mathsf{Var}(R)$ for $\mathsf{Var}(S)$.

The general set-up throughout this note is as follows. Let $\mathcal{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0, p)$, with fraction field K and perfect residue field k. Furthermore, we assume the valuation on K normalised so that $v(p) = 1$. As usual denote by $\overline{K}$ an algebraic closure of K and by $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of k, K_0 the fraction field of $W(k)$, and K_0^{nr} its maximal unramified extension. Let σ be the absolute Frobenius on $W(\overline{k})$. For a scheme $X/\mathcal{O}_K$ denote by X_n, for $n \in \mathbf{N}$, its reduction modulo p^n and let X_0 and X_K be its special and generic fibre respectively.

By abuse of notation, we denote by $\mathcal{O}_K$, $\mathcal{O}_K^\times$, and $\mathcal{O}_K^0$ the scheme $\operatorname{Spec} \mathcal{O}_K$ regarded as log scheme with the trivial, canonical (i.e. associated to the closed point) and $(\mathbf{N} \to \mathcal{O}_K, 1 \mapsto 0)$-log structure respectively, and similarly for $W(k)$ and k.

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1. The logarithmic de Rham–Witt complex

It is common when one studies non-smooth objects to consider log versions of the usual complexes appearing in the different cohomology theories. In [12, Sec. 2] Hyodo and Kato describe the log crystalline site for schemes with fine log structure as a generalisation of the usual crystalline site. One case which is relatively well studied, is the semistable case in positive characteristic p, and the case of semistable reduction in mixed characteristic $(0, p)$. This is a special case of fine log schemes of Cartier type over k^0 or $\mathcal{O}_K^\times$.

The Hyodo–Kato complex computes log crystalline cohomology over $W(k)^0$. There are different quasi-isomorphic presentations of this complex. One that is particularly useful to us is the de Rham–Witt presentation233 from [12, Sec. 4] (see also [14, Sec. 1]).

For a fine log scheme Y of Cartier type over k^0, let $(Y/W_n(k)^0)_{\mathrm{cris}}$ be its log crystalline site over $W_n(k)$ endowed with the $(1 \mapsto 0)$-log structure, let $\mathcal{O}_{Y/W_n(k)^0}$ be its crystalline structure sheaf, and let $u_{Y/W_n(k)^0} : (Y/W_n(k)^0)_{\mathrm{cris}} \to Y_{\text{ét}}$ be the canonical morphism of sites.

Definition 1.1. The logarithmic Witt differentials of Y of degree i and level $n \geqslant 1$ are defined as

$$W_n\omega_Y^i := R^i u_{Y/W_n(k)^0,*}\mathcal{O}_{Y/W_n(k)^0}.$$

According to [14, Cor. 1.17] $W_n\omega_Y^i$ is a coherent $W_n\mathcal{O}_Y$-module. For n fixed $W_n\omega_Y^\bullet$ is a complex called the logarithmic de Rham–Witt complex of level n. By [12, Prop. (4.6)] there is a canonical isomorphism of $W_n(k)$-algebras

$$W_n\omega_Y^0 \cong W_n\mathcal{O}_Y.$$

The Frobenius F, Verschiebung V and projection operators extend from the Witt vectors to the de Rham–Witt complex, and are subject to certain relations. In particular, $W_n\omega_Y^\bullet$ is a differentially graded $W_n\mathcal{O}_Y$-algebra, which computes log crystalline cohomology, i.e. there is by [12, Thm. (4.19)] a canonical quasi-isomorphism

$$W_n\omega_Y^\bullet \xrightarrow{\sim} Ru_{Y/W_n(k)^0,*}\mathcal{O}_{Y/W_n(k)^0} \tag{1}$$

compatible with Frobenius and the canonical projections as n varies. Here the Frobenius on the left hand side is given by p^iF in degree i, while on the right hand side, it is induced by the absolute Frobenius of Y

and the endomorphism σ of $W_n(k)$. The compatibility of this morphism with the canonical projections was treated properly in [15, § 8].

Finally, for n varying we obtain a projective system $\{W_n\omega_Y^\bullet\}_{n\geqslant 1}$ differentially graded algebras.

Definition 1.2. We call the limit

$$W\omega_Y^\bullet := \varprojlim W_n\omega_Y^\bullet$$

the logarithmic de Rham–Witt complex of Y over $W(k)^0$. It is equipped with operators called Frobenius and Verschiebung induced by F and V which satisfy the usual relations.

The log crystalline cohomology of Y over $W(k)^0$ is

$$R\Gamma_{\mathrm{cris}}(Y/W(k)^0) := \mathrm{holim} R\Gamma_{\text{ét}}(Y, Ru_{Y/W_n(k)^0,*}\mathcal{O}_{Y/W_n(k)^0}).$$

and we have by [14, Cor. 1.23] the following statement.

Lemma 1.3 (P. Lorenzon)**.** *If Y is a proper fine log scheme of Cartier type over k^0, then $R\Gamma_{ét}(Y, W\omega_Y^\bullet)$ is a perfect complex and the canonical morphism (1) induces a natural isomorphism*

$$\lambda : R\Gamma_{ét}(Y, W\omega^\bullet) \xrightarrow{\sim} R\Gamma_{\mathrm{cris}}(Y/W(k)^0).$$

In case $Y = X_0$ is the special fibre of a fine log scheme X of Cartier type over $\mathcal{O}_K^\times$, we write $R\Gamma_{\mathrm{HK}}(X) := R\Gamma_{\mathrm{cris}}(Y/W(k)^0)$ and call it the Hyodo–Kato complex of X. In this case, there is yet another presentation of the rational Hyodo–Kato complex due to Beilinson [2, 1.16.1] denoted by $R\Gamma^B_{\mathrm{HK}}(X)$. Without going into details we remark that there is a natural quasi-isomorphism [16, (28)]

$$\kappa : R\Gamma^B_{\mathrm{HK}}(X) \xrightarrow{\sim} R\Gamma_{\mathrm{HK}}(X)_{\mathbf{Q}} \tag{2}$$

compatible with Frobenius. The advantage of Beilinson's definition is that it admits a *nilpotent* monodromy N as explained in [16, § 3.1]. We will see later why this is of interest for us.

2. Hyodo–Kato complexes for K-varieties

In [16] Nekovář and Nizioł describe how to extend several p-adic cohomology theories for log schemes over k or $\mathcal{O}_K$ to K-varieties. This technique is based on observations due to Beilinson in [1] and we start by recalling some of the relevant notions.

Definition 2.1. For a field K of characteristic 0 a geometric pair is an open embedding $i : U \hookrightarrow \overline{U}$ of K-varieties such that U is dense in $\overline{U}$ and $\overline{U}$ is proper. A geometric pair is called a normal crossings pair, if $\overline{U}$ is regular and $\overline{U}\backslash U$ is a divisor with normal crossings. It is said to be strict, if the irreducible components of $\overline{U}\backslash U$ are regular.

This definition can be adjusted to the arithmetic setting as follows.

Definition 2.2. An arithmetic K-pair is an open embedding $i : U \hookrightarrow \overline{U}$ of a K-variety U with dense image in a reduced proper flat $\mathcal{O}_K$-scheme $\overline{U}$. Such a pair is called a semistable pair if $\overline{U}$ is regular, $\overline{U}\backslash U$ is a divisor with normal crossings, and the closed fibre $\overline{U}_0$ is reduced. It is said to be strict, if the irreducible components of $\overline{U}\backslash U$ are regular.

As explained in [16, 2.3] one can regard a semistable pair $(U, \overline{U})$ as a log scheme $\overline{U}$ over $\mathcal{O}_K^\times$ equipped with the log structure associated to the divisor $\overline{U}\backslash U$. As such it is a proper log smooth fine log scheme of Cartier type over $\mathcal{O}_K^\times$. In particular, its special fibre $\overline{U}_0$ equipped with the pull back log structure is a proper log smooth fine log scheme of Cartier type over k^0.

Following [16] we denote by $\mathcal{P}_K^{ar}$, $\mathcal{P}_K^{ss}$ and $\mathcal{P}_K^{nc}$ respectively the category of arithmetic, semistable and normal crossings pairs over K respectively. Moreover, we denote by $\mathcal{P}_K^{log}$ the subcategory of $\mathcal{P}_K^{ar}$ of log schemes $(U, \overline{U})$ which are log smooth over $\mathcal{O}_{\overline{U}}(\overline{U})^\times$.

A key point in Beilinson's work is that the categories $\mathcal{P}_K^{ar}$, $\mathcal{P}_K^{ss}$, $\mathcal{P}_K^{log}$, and $\mathcal{P}_K^{nc}$ respectively form a base for the h-site $(Var/K)_{\mathrm{h}}$ of K-varieties in the sense that there is an equivalence between the associated h-topoi [1, 2.5 Prop.]. Keeping in mind that alterations are h-morphisms, Beilinson uses the following formulation of de Jong's alteration theorem [1, 2.3 Thm.].

Theorem 2.3 (A.J. de Jong). *Every geometric pair admits a strict normal crossings alteration. Every arithmetic pair over K or $\overline{K}$ admits a strict semistable alteration. Alterations can be chosen in such a way that $\overline{V}$ is projective.*

Nekovář and Nizioł explain how to h-sheafify the rational Hyodo–Kato and the Beilinson–Hyodo–Kato complexes on $\mathsf{Var}(K)$ [16, Sec. 3.3]. Denote the resulting sheaves by $\mathscr{A}_{\mathrm{HK}}$ and $\mathscr{A}^B_{\mathrm{HK}}$. The same procedure can be done to the logarithmic de Rham–Witt complex $W\omega^\bullet$ defined above.

Definition 2.4. Let $\mathscr{A}_{\mathrm{dRW}}$ be the h-sheafification of the presheaf

$$(U,\overline{U}) \mapsto R\Gamma_{\text{ét}}((U,\overline{U})_0, W\omega^\bullet)_{\mathbf{Q}}$$

on the category $\mathscr{P}^{log}_K$ of proper log smooth fine log schemes of Cartier type over $\mathscr{O}_K^\times$. This results in an h-sheaf of commutative dg K_0-algebras on $\mathsf{Var}(K)$.

The canonical maps $\kappa : R\Gamma^B_{\mathrm{HK}}(U,\overline{U}) \to R\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbf{Q}}$ [16, Sec. 3.3] and $\lambda : R\Gamma_{\text{ét}}((U,\overline{U})_0, W\omega^\bullet)_{\mathbf{Q}} \to R\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbf{Q}}$ [12, Thm. (4.19)] h-sheafify and we obtain functorial quasi-isomorphisms of h-sheaves

$$\mathscr{A}_{\mathrm{dRW}} \cong \mathscr{A}_{\mathrm{HK}} \cong \mathscr{A}^B_{\mathrm{HK}}. \tag{3}$$

Lemma 2.5. *For any proper log smooth fine log scheme of Cartier type $(U,\overline{U}) \in \mathscr{P}^{log}_K$ over $\mathscr{O}_K^\times$, the canonical map*

$$R\Gamma_{\acute{e}t}((U,\overline{U})_0, W\omega^\bullet_{\mathbf{Q}}) \to R\Gamma_{\mathrm{h}}(U, \mathscr{A}_{\mathrm{dRW}})$$

is a quasi-isomorphism.

Proof. According to [16, Prop. 3.18] analogous statements are true for the Hyodo–Kato and Beilinson–Hyodo–Kato complexes. Because of the canonical quasi-isomorphism (3) the statement follows. □

3. Admissible filtered (φ, N, G_K)-modules

The Beilinson–Hyodo–Kato complex has additional structure and as such fits into the theory of p-adic Galois representations [10]. Indeed, it's cohomology groups are admissible filtered (φ, N, G_K)-modules. Let us explain this.

Definition 3.1.
(i) A filtered φ-module is a triple $(M_0, \varphi, \mathrm{Fil}^\bullet)$, where M_0 is a finite dimensional K_0-vector space, with a σ-semi-linear isomorphism $\varphi : M_0 \to M_0$, called the Frobenius map and a decreasing, separated, exhaustive filtration $\mathrm{Fil}^\bullet$ on $M = M_0 \otimes_{K_0} K$ called the Hodge filtration.
(ii) A filtered (φ, N)-module $(M_0, \varphi, N, \mathrm{Fil}^\bullet)$ consists of a filtered φ-module $(M_0, \varphi, \mathrm{Fil}^\bullet)$ together with a K_0-linear monodromy operator N on M_0 satisfying the relation $N\varphi = p\varphi N$.
(iii) A filtered (φ, N, G_K)-module is a tuple $(M_0, \varphi, N, \rho, \mathrm{Fil}^\bullet)$ where M_0 is a finite dimensional K_0^{nr}-vector space, $\varphi : M_0 \to M_0$ is a Frobenius map, $N : M_0 \to M_0$ is a K_0^{nr}-linear monodromy operator such that $N\varphi = p\varphi N$, ρ is a K_0^{nr}-semi-linear action of G_K on M_0 factoring through a quotient of the inertia group and commuting with φ and N, and $\mathrm{Fil}^\bullet$ is a decreasing, separated, exhaustive filtration of $M = (M_0 \otimes_{K_0^{\mathrm{nr}}} \overline{K})^{G_K}$.

Denote by

$$\mathrm{MF}^{\mathrm{ad}}_K(\varphi) \subset \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N) \subset \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N, G_K)$$

the categories of admissible filtered φ-modules, (φ, N)-modules and (φ, N, G_K)-modules, where admissible is meant in the sense of [10]. They are equivalent to crystalline, semistable, and potentially semistable Galois representations [7, 3]. The categories of admissible filtered φ-, (φ, N), and (φ, N, G_K)-modules are known to be Tannakian (c.f [10, § 4.3.4]). Thus it makes sense to consider their respective bounded derived dg categories denoted by $\mathscr{D}^b(\mathrm{MF}^{\mathrm{ad}}_K(\varphi))$, $\mathscr{D}^b(\mathrm{MF}^{\mathrm{ad}}_K(\varphi, N))$ and $\mathscr{D}^b(\mathrm{MF}^{\mathrm{ad}}_K(\varphi, N, G_K))$ respectively.

A filtered φ-, (φ, N)-, or (φ, N, G_K)-module M_0 has both, a Newton polygon, associated to the eigenvalues of the Frobenius morphism φ on M_0, and a Hodge polygon, associated to the filtration $\mathrm{Fil}^\bullet$ on M. If the Newton polygon lies above the Hodge polygon, M_0 is weakly admissible (c.f. [10, 4.4.6 Rem.]).

It turns out that this is the critical piece of information that will allow us to relate $H^q(X_K, \mathscr{O}_{X_K})$ and $H^q(X_0, W\mathscr{O})_{\mathbf{Q}} = 0$. This is possible because the Beilinson–Hyodo–Kato complex $R\Gamma_{\mathrm{h}}(Z_{\overline{K}}, \mathscr{A}^B_{\mathrm{HK}})$ of a

K-variety is an object in $\mathcal{D}^\flat(\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N, G_K))$. By definition it is equipped with a Frobenius φ, and a nilpotent monodromy operator N. Note that this is a difference from the usual Hyodo–Kato complex, where the monodromy is at best homotopically nilpotent. This is crucial to h-sheafify the Hyodo–Kato morphism, relating the (Beilinson–)Hyodo–Kato complex to the de Rham complex, which provides the filtration.

For a proper log smooth fine log scheme of Cartier type $(U, \overline{U}) \in \mathcal{P}_K^{log}$ over $\mathcal{O}_K^\times$ there is a morphism (c.f. [16, (22)], [12, Sec. 5])

$$\iota_{\mathrm{dR},\pi} : R\Gamma_{\mathrm{HK}}(U, \overline{U})_{\mathbf{Q}} \to R\Gamma_{\mathrm{dR}}(U, \overline{U}_K)$$

called the Hyodo–Kato morphism which becomes a K-linear functorial quasi-isomorphism after tensoring with K. However, it depends on the choice of a uniformiser π of $\mathcal{O}_K$ and is therefore not suitable for h-sheafification. By contrast, there is a morphism [2, 1.16.3]

$$\iota_{\mathrm{dR}}^B : R\Gamma_{\mathrm{HK}}^B(U, \overline{U}) \to R\Gamma_{\mathrm{dR}}(U, \overline{U}_K)$$

independent of the choice of a uniformiser which is also a K-linear functorial quasi-isomorphism after tensoring with K. For more details see [16, Ex. 3.5(1)]. The two morphisms are compatible with the comparison map (2)

$$\begin{tikzcd} R\Gamma_{\mathrm{HK}}(U, \overline{U})_{\mathbf{Q}} \arrow[r, "\iota_{\mathrm{dR},\pi}"] & R\Gamma_{\mathrm{dR}}(U, \overline{U}_K) \\ R\Gamma_{\mathrm{HK}}^B(U, \overline{U}) \arrow[u, "\kappa", "\sim"'] \arrow[ur, "\iota_{\mathrm{dR}}^B"'] & \end{tikzcd}$$

as explained in [16] after Ex. 3.5.

The map ι_{dR}^B can be h-sheafified. For this we also have to h-sheafify the de Rham complex on $\mathsf{Var}(K)$. Thus, consider the presheaf

$$(U, \overline{U}) \mapsto R\Gamma((U, \overline{U}), \Omega^\bullet)$$

of filtered K-algebras on $\mathcal{P}_K^{nc}$ and let $\mathcal{A}_{\mathrm{dR}}$ be its h-sheafification. This results in an h-sheaf of commutative filtered dg K-algebras on $\mathsf{Var}(K)$. It can be identified with Deligne's de Rham complex equipped with Deligne's Hodge filtration $\mathrm{Fil}^\bullet$ (c.f. [11, Prop. 7.4 and Thm. 7.7]). Moreover, Beilinson showed in [1, 2.4] that for $(U, \overline{U}) \in \mathcal{P}_K^{nc}$ the canonical map

$$R\Gamma_{\mathrm{dR}}(U, \overline{U}) \xrightarrow{\sim} R\Gamma_{\mathrm{h}}(U, \mathcal{A}_{\mathrm{dR}})$$

is a quasi-isomorphism.

There are analogous statements over $\overline{K}$, and for $Z \in \mathsf{Var}(K)$ the projection $\varepsilon : Z_{\overline{K},\mathrm{h}} \to Z_{\mathrm{h}}$ of sites induces pull-back maps

$$\varepsilon^* : R\Gamma_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{HK}}^B) \to R\Gamma_{\mathrm{h}}(Z_{\overline{K}}, \mathcal{A}_{\mathrm{HK}}^B)^{G_K} \tag{4}$$

$$\varepsilon^* : R\Gamma_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dR}}) \to R\Gamma_{\mathrm{h}}(Z_{\overline{K}}, \mathcal{A}_{\mathrm{dR}})^{G_K} \tag{5}$$

where by [16, Prop. 3.22] the first one is a quasi-isomorphism and the second one is a filtered quasi-isomorphism. The Beilinson–Hyodo–Kato map extends to $R\Gamma_{\mathrm{h}}(Z_{\overline{K}}, \mathcal{A}_{\mathrm{HK}}^B) \to R\Gamma_{\mathrm{h}}(Z_{\overline{K}}, \mathcal{A}_{\mathrm{dR}})$, which induces a quasi-isomorphism

$$R\Gamma_{\mathrm{h}}(Z_{\overline{K}}, \mathcal{A}_{\mathrm{HK}}^B) \otimes_{K_0^{\mathrm{nr}}} \overline{K} \to R\Gamma_{\mathrm{h}}(Z_{\overline{K}}, \mathcal{A}_{\mathrm{dR}}).$$

This identification provides the last piece of data necessary and by [8, 2.20] we have the following statement.

Lemma 3.2 (F. Déglise, W. Nizioł)**.** *Let $Z \in \mathsf{Var}(K)$. Then $R\Gamma_{\mathrm{h}}(Z_{\overline{K}}, \mathcal{A}_{\mathrm{HK}}^B)$ with the Frobenius φ, the monodromy operator N, the canonical G_K-action, and the Hodge filtration on $R\Gamma_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dR}})$ is an object in $\mathcal{D}^\flat(\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N, G_K))$.*

As a consequence we obtain the promised statement which relates the Hodge and the Newton polygon of $H_{\mathrm{h}}^q(Z_{\overline{K}}, \mathcal{A}_{\mathrm{HK}}^B)$.

Lemma 3.3. *Let $Z \in \mathsf{Var}(K)$. For any $q \in \mathbf{N}_0$, the Newton polygon of $H^q_{\mathrm{h}}(Z_{\overline{K}}, \mathscr{A}^B_{\mathrm{HK}})$ lies above its Hodge polygon.*

Proof. By the previous lemma $H^q_{\mathrm{h}}(Z_{\overline{K}}, \mathscr{A}^B_{\mathrm{HK}})$ is an admissible filtered (φ, N, G_K)-module. Therefore it is weakly admissible in the sense of Fontaine [10, 5.6.7 Thm. (vi)]. This means as remarked in [10, 4.4.6 Rem.] that for each q the Newton polygon of $H^q_{\mathrm{h}}(Z_{\overline{K}}, \mathscr{A}^B_{\mathrm{HK}})$ lies above its Hodge polygon. □

4. Two spectral sequences

In this section we consider two spectral sequences, one for de Rham cohomology and one for Hyodo–Kato cohomology, which are very similar in spirit for they are related to the Hodge and Newton slope of a (φ, N, G_K)-module in a geometric situation. The first one is an h-sheafified version of the Hodge-to-de Rham spectral sequence.

For this we introduce the h-sheafifications $\mathscr{A}^i_{\mathrm{dR}}$ of the differential sheaves Ω^i, $i \geqslant 0$, on $\mathsf{Var}(K)$. To be consistent with the constructions in the previous sections, we can think of $\mathscr{A}^i_{\mathrm{dR}}$ as the h-sheafification of the presheaf

$$(U, \overline{U}) \mapsto \Gamma((U, \overline{U}), \Omega^i)$$

on the category $\mathscr{P}^{nc}_K$ of normal crossing pairs. It gives a coherent h-sheaf on $\mathsf{Var}(K)$.

Now the Hodge filtration of $\mathscr{A}_{\mathrm{dR}}$ induces a spectral sequence for which Huber and Jörder in [11, Thm. 7.7] prove the following.

Lemma 4.1 (A. Huber, C. Jörder)**.** *Let $Z \in \mathsf{Var}(K)$ be proper. Then the Hodge-to-de Rham spectral sequence*

$$E_1^{rs} = H^s_{\mathrm{h}}(Z, \mathscr{A}^r_{\mathrm{dR}}) \Rightarrow H^{r+s}_{\mathrm{h}}(Z, \mathscr{A}_{\mathrm{dR}})$$

degenerates at E_1.

It becomes immediately clear that the Hodge filtration on $\mathscr{A}_{\mathrm{dR}}$ induces Deligne's Hodge filtration as mentioned above.

Corollary 4.2. *Let $Z \in \mathsf{Var}(K)$. Then for any $q \geqslant 0$ the Hodge-to-de Rham spectral sequence yields an isomorphism*

$$H^q_{\mathrm{h}}(Z, \mathscr{A}_{\mathrm{dR}})^{<1} \xrightarrow{\sim} H^q_{\mathrm{h}}(Z, \mathscr{A}^0_{\mathrm{dR}}),$$

where on the left hand side we mean the part of $H^q_{\mathrm{h}}(Z, \mathscr{A}_{\mathrm{dR}})$ with Hodge slope < 1.

We come now to an analogous statement for the Hyodo–Kato cohomology. Thus for $i \geqslant 0$ consider the h-sheafification $\mathscr{A}^i_{\mathrm{dRW}}$ of the presheaf

$$(U, \overline{U}) \mapsto \Gamma((U, \overline{U})_0, W\omega^i)_{\mathbf{Q}}$$

on the category $\mathscr{P}^{log}_K$ of proper log smooth fine log scheme of Cartier type over $\mathscr{O}_K$. This is a quasi-coherent h-sheaf on $\mathsf{Var}(K)$.

Lemma 4.3. *Let $Z \in \mathsf{Var}(K)$. There is a spectral sequence*

$$E_1^{rs} = H^s_{\mathrm{h}}(Z, \mathscr{A}^r_{\mathrm{dRW}}) \Rightarrow H^{r+s}_{\mathrm{h}}(Z, \mathscr{A}_{\mathrm{dRW}})$$

which is Frobenius equivariant and degenerates at E_1.

Proof. The existence of the spectral sequence follows as in the classical case, meaning it is induced from the naive filtration of the complex $\mathscr{A}_{\mathrm{dRW}}$. As mentioned earlier, the Frobenius F induces an endomorphism of the de Rham–Witt complex, which in each degree r is given by p^rF. It h-sheafifies well. Thus we can use the same argumentation as in [13, III.3.1] to see that the spectral sequence is Frobenius equivariant. Namely, the Frobenius endomorphism of $\mathscr{A}_{\mathrm{dRW}}$ fixes $\mathscr{A}^{\geqslant r}_{\mathrm{dRW}}$ for all $r \geqslant 0$ and thus it induces an endomorphism of the spectral sequence. On the abutment $H^{r+s}_{\mathrm{h}}(Z, \mathscr{A}_{\mathrm{dRW}})$, it coincides with the Frobenius action φ. On $H^s_{\mathrm{h}}(Z, \mathscr{A}^r_{\mathrm{dRW}})$ it is given by p^rF.

To see that the spectral sequence degenerates, we will show that $H^s_{\mathrm{h}}(Z, \mathscr{A}^r_{\mathrm{dRW}})$ is a finite rank K_0-vector space for all $r, s \geqslant 0$. More precisely, we will show that $(H^s_{\mathrm{h}}(Z, \mathscr{A}^r_{\mathrm{dRW}}), p^rF)$ is canonically isomorphic to the part of $(H^{r+s}_{\mathrm{h}}(Z, \mathscr{A}_{\mathrm{dRW}}), \varphi)$ which has slopes in $[r, r+1[$, denoted by $(H^{r+s}_{\mathrm{h}}(Z, \mathscr{A}_{\mathrm{dRW}}), \varphi)^{[r,r+1[}$.

The statement then follows from the fact that the cohomology groups $H^{r+s}_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dRW}})$ are finite rank K_0-vector spaces [16, p. 5].

Let $(U_\bullet, \overline{U}_\bullet) \to Z$ by an h-hypercover of Z by semistable pairs over K which exists by de Jong's alteration theorem. For every n, let $K_{U_n} := \Gamma(\overline{U}_{n,K}, \mathcal{O}_{\overline{U}_n})$ which is a finite product of finite extensions of K labelled by the connected components of $\overline{U}_n$, that is $K_{U_n} = \prod K_{n,i}$. As we may assume that all the fields $K_{n,i}$ are Galois over K, we choose a finite Galois extension $(\mathcal{O}_{K'}, K')/(\mathcal{O}_K, K)$ with residue field k' such that K' is Galois over all the fields $K_{n,i}$ for fixed n (c.f. proof of [16, Prop. 3.20]). Then $(U'_n, \overline{U}'_n) := (U_n, \overline{U}_n) \times_{\mathcal{O}_K} \mathcal{O}_{K'}$ is a proper log smooth fine log scheme of Cartier type over $\mathcal{O}^\times_{K'}$. Base change for crystalline cohomology implies that one has isomorphisms

$$H^s((U'_n, \overline{U}'_n)_0, W\omega^r_{\mathbf{Q}}) \cong H^s((U_n, \overline{U}_n)_0, W\omega^r_{\mathbf{Q}}) \otimes_{K_0} K'_0$$

where K'_0 denotes the fraction field of $W(k')$. As the left hand side is a finite dimensional K'_0-vector space, it follows that $H^s((U_n, \overline{U}_n)_0, W\omega^r_{\mathbf{Q}})$ is a finite dimensional K_0-vector space. We can do this for all n separately. It follows, that the spectral sequence

$$E_1^{rs} = H^s((U_n, \overline{U}_n)_0, W\omega^r_{\mathbf{Q}}) \Rightarrow H^{r+s}((U_n, \overline{U}_n)_0, W\omega^\bullet_{\mathbf{Q}}),$$

which is again induced by the naive filtration of the logarithmic de Rham–Witt complex, degenerates at E_1, and that we have canonical isomorphisms

$$\big(H^s((U_n, \overline{U}_n)_0, W\omega^r_{\mathbf{Q}}), p^r F\big) \cong \big(H^{r+s}((U_n, \overline{U}_n)_0, W\omega^\bullet_{\mathbf{Q}}), \varphi\big)^{[r,r+1[}$$

for all n. Accordingly, there is a canonical isomorphism

$$\big(H^s((U_\bullet, \overline{U}_\bullet)_0, W\omega^r_{\mathbf{Q}}), p^r F\big) \cong \big(H^{s+q}((U_\bullet, \overline{U}_\bullet)_0, W\omega^\bullet_{\mathbf{Q}}), \varphi\big)^{[r,r+1[}.$$

of finite dimensional K_0-vector spaces.

But as mentioned before, φ and F sheafify well with respect to the h-topology, so that we may take the limit over all h-hypercovers of Z by semistable pairs over K and obtain a canonical isomorphism

$$(H^s_{\mathrm{h}}(Z, \mathcal{A}^r_{\mathrm{dRW}}), p^r F) \cong (H^{r+s}_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dRW}}), \varphi)^{[r,r+1[}$$

where the right hand side is the part of $(H^{r+s}_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dRW}}), \varphi)$ with slope in the interval $[r, r+1[$. In particular, the $H^s(Z, \mathcal{A}^r_{\mathrm{dRW}})$ $s \geqslant 0$, are finite rank K_0-vector spaces and the spectral sequence from the statement degenerates. □

For obvious reasons this spectral sequence is called the slope spectral sequence.

Corollary 4.4. *Let $Z \in \mathsf{Var}(K)$. For any $q \geqslant 0$ the slope spectral sequence yields an isomorphism*

$$H^q_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dRW}})^{<1} \xrightarrow{\sim} H^q_{\mathrm{h}}(Z, \mathcal{A}^0_{\mathrm{dRW}}),$$

where on the left hand side we mean the part of $H^q_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dRW}})$ where Frobenius acts with slope < 1.

This allows us to transfer the descent statement from Lemma 2.5 to differentials of degree zero.

Corollary 4.5. *For any proper log smooth fine log scheme of Cartier type $(U, \overline{U}) \in \mathcal{P}^{log}_K$ over $\mathcal{O}^\times_K$, the canonical map*

$$R\Gamma_{\acute{e}t}((U, \overline{U})_0, W\omega^0)_{\mathbf{Q}} \to R\Gamma_{\mathrm{h}}(U, \mathcal{A}^0_{\mathrm{dRW}})$$

is a quasi-isomorphism.

Proof. For any $q \geqslant 0$ we have a commutative diagram

$$\begin{tikzcd}
H^q((U,\overline{U})_0, W\omega^\bullet)^{<1}_{\mathbf{Q}} \arrow[r, "\sim"] \arrow[d, "\sim"] & H^q((U,\overline{U})_0, W\omega^0)_{\mathbf{Q}} \arrow[d] \\
H^q_{\mathrm{h}}(U, \mathcal{A}_{\mathrm{dRW}})^{<1} \arrow[r, "\sim"] & H^q_{\mathrm{h}}(U, \mathcal{A}^0_{\mathrm{dRW}})
\end{tikzcd}$$

where the horizontal isomorphisms are induced from the classical and the h-sheafified slope spectral sequence respectively and the vertical maps are the canonical morphisms. Since the left vertical map is an isomorphism by Lemma 2.5, the right one is as well. □

We can take this a bit further.

Lemma 4.6. *Let X be a reduced, proper and flat $\mathcal{O}_K$-scheme of finite type. Then there is a quasi-isomorphism*

$$R\Gamma_{\mathrm{h}}(X_K, \mathcal{A}^0_{\mathrm{dRW}}) \cong R\Gamma_{\acute{e}t}(X_0, W\mathcal{O})_{\mathbf{Q}}.$$

Proof. Since (X_K, X) forms an arithmetic pair in the sense of Beilinson (see Definition 2.2), it has by de Jong's theorem a strictly semistable alteration $(U, \overline{U}) \to (X_K, X)$. This gives rise to an h-cover $U \to X_K$ of the generic fibre, and an h-cover $\overline{U}_0 \to X_0$ of the special fibre. Denote by $(U_\bullet, \overline{U}_\bullet) \to (X_K, X)$ its Čech nerve. In particular $U_\bullet \to X_K$ is an h-hypercover of the generic fibre of X and $\overline{U}_{\bullet,0} \to X_0$ is an h-hypercover of its special fibre.

Using Corollary 4.5 we have

$$R\Gamma_{\mathrm{h}}(X_K, \mathcal{A}^0_{\mathrm{dRW}}) \xrightarrow{\sim} R\Gamma_{\mathrm{h}}(U_\bullet, \mathcal{A}^0_{\mathrm{dRW}}) \xleftarrow{\sim} R\Gamma_{\text{ét}}((U_\bullet, \overline{U}_\bullet)_0, W\omega^0)_{\mathbf{Q}} \cong R\Gamma_{\text{ét}}(\overline{U}_{\bullet,0}, W\mathcal{O})_{\mathbf{Q}}.$$

However, by [6, Prop. 11.41] rational Witt cohomology satisfies cohomological h-descent and hence the right most expression is just $R\Gamma_{\text{ét}}(X_0, W\mathcal{O})_{\mathbf{Q}}$. □

Therefore we can rewrite Corollary 4.4.

Corollary 4.7. *Let X be a reduced, proper and flat $\mathcal{O}_K$-scheme of finite type. For any $q \geqslant 0$ the slope spectral sequence yields an isomorphism*

$$H^q_{\mathrm{h}}(X_K, \mathcal{A}_{\mathrm{dRW}})^{<1} \xrightarrow{\sim} H^q(X_0, W\mathcal{O})_{\mathbf{Q}}.$$

5. A vanishing theorem

We can now put the pieces together to give a simplified proof of the vanishing theorem due to Pierre Berthelot, Hélène Esnault and Kay Rülling. We will use the following statement about de Rham cohomology groups. Note that h-sheafifying the de Rham complex results in Deligne's de Rham complex [11, Thm. 7.4], i.e. $H^q_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dR}}) = H^q_{\mathrm{dR}}(Z)$.

Lemma 5.1. *Let $Z \in \mathsf{Var}(K)$ be proper with only Du Bois singularities, and assume that $H^q(Z, \mathcal{O}) = 0$ for some $q \geqslant 0$. Then the smallest Hodge slope of $H^q_{\mathrm{dR}}(Z)$ is at least* 1.

Proof. If Z has only Du Bois singularities $H^q_{\mathrm{h}}(Z, \mathcal{A}^0_{\mathrm{dR}}) = H^q(Z, \mathcal{O})$ by [11, Cor. 7.17]. As the h-sheafified Hodge-to-de Rham spectral sequence from Lemma 4.1 degenerates for a proper K-variety at E_1, the hypothesis $H^q(Z, \mathcal{O}) = 0$ implies that the smallest Hodge slope of $H^q_{\mathrm{h}}(Z, \mathcal{A}_{\mathrm{dR}}) \cong H^q_{\mathrm{dR}}(Z)$ is at least 1. □

Remark 5.2. In general we can say that for a proper K-variety $Z \in \mathsf{Var}(K)$ such that $H^q_{\mathrm{h}}(Z, \mathcal{O}_{\mathrm{h}}) = 0$ for some q the smallest Hodge slope of $H^q_{\mathrm{dR}}(Z)$ is at least 1.

We obtain now the desired vanishing theorem in a slightly more general form than originally stated.

Theorem 5.3. *Let X be a proper, reduced and flat scheme over $\mathcal{O}_K$, such that X_K has at most Du Bois singularities. Fix $q \in \mathbf{N}_0$. If $H^q(X_K, \mathcal{O}) = 0$, then $H^q(X_0, W\mathcal{O})_{\mathbf{Q}} = 0$.*

Proof. Consider the cohomology group $H^q_{\mathrm{h}}(X_{\overline{K}}, \mathcal{A}^B_{\mathrm{HK}})$. By Lemma 3.3 its Newton polygon lies above its Hodge polygon. Because X_K has only Du Bois singularities Lemma 5.1 applies, which means that the smallest Hodge slope of $H^q_{\mathrm{dR}}(X_K)$ is $\geqslant 1$. Therefore the part of $H^q_{\mathrm{h}}(X_{\overline{K}}, \mathcal{A}^B_{\mathrm{HK}})$, where the Newton slope is < 1 vanishes. By definition this is exactly the part where Frobenius acts with slope < 1. Via the first quasi-isomorphism of (4) and the fact that the action of G_K commutes with φ we deduce the same for $H^q_{\mathrm{h}}(X_K, \mathcal{A}^B_{\mathrm{HK}})$. Finally, the quasi-isomorphisms (3) imply $H^q_{\mathrm{h}}(X_K, \mathcal{A}_{\mathrm{dRW}})^{<1} = 0$ and hence, by Corollary 4.4

$$H^q_{\mathrm{h}}(X_K, \mathcal{A}^0_{\mathrm{dRW}}) = 0.$$

But according to Lemma 4.6 this means that $H^q(X_0, W\mathcal{O})_{\mathbf{Q}} = 0$ as well. □

Remark 5.4. As pointed out in [5] one easily generalises this result to a scheme X as in the theorem, but over a discrete valuation ring V which is not necessarily complete.

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