MONOCHROMATIC SOLUTIONS TO $x+y=z^2$ IN THE INTERVAL $[N, cN^4]$

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Abstract. Green and Lindqvist proved that for any 2-colouring of $\mathbb{N}$, there are infinitely many monochromatic solutions to $x+y=z^2$. In fact, they showed the existence of a monochromatic solution in every interval $[N, cN^8]$ with large enough N. In this short note we give a different proof for their theorem and prove that a monochromatic solution exists in every interval $[N, 10^4N^4]$ with large enough N. A 2-colouring of $[N, (1/27)N^4]$ avoiding monochromatic solutions to $x+y=z^2$ is also presented, which shows that in 10^4N^4 only the constant factor can be reduced.

1. Introduction

Csikvári, Gyarmati and Sárközy [1] proved that the equation $x+y=z^2$ is not partition regular. Indeed, they gave a 16-colouring of $\mathbb{N}$ with no monochromatic solutions to $x+y=z^2$ other than the trivial one $x=y=z=2$. Recently, Green and Lindqvist [2] have shown that such a colouring also exists with only 3 colours, but for any 2-colouring of $\mathbb{N}$ there are infinitely many monochromatic solutions to $x+y=z^2$. In fact, from their proof it also follows that for every large enough N there is a monochromatic solution in the interval $[N, cN^8]$ (where c is a huge constant that can be explicitly given). The proof of Green and Lindqvist uses several tools from additive combinatorics.

In this paper we give another, shorter proof for this theorem. The proof is elementary and involves several combinatorial ideas. Our result is the following:

Theorem 1. *Every 2-colouring of $\mathbb{N}$ has infinitely many monochromatic solutions to $x+y=z^2$. Moreover, for every large enough N there is a monochromatic solution in $[N, 10^4N^4]$.*

This strengthens the result of Green and Lindqvist, in the sense that it verifies the existence of a monochromatic solution in a much shorter interval, $[N, 10^4N^4]$ instead of

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$[N, cN^8]$. Note that the exponent 4 in 10^4N^4 can not be further improved, since colouring $[N, N^2/3]$ with the first colour and $(N^2/3, N^4/27]$ with the second colour avoids any monochromatic solution for $x + y = z^2$.

The proof of Theorem 1 is given in Section 2. Throughout the paper we use the notion $[n] := \{1, 2, \dots, n\}$ and by an interval $[a, b]$ we mean the set of integers that are at least a and at most b.

Finally, we shall mention two related works. Khalfallah and Szemerédi [3] have shown that for any finite colouring of $\mathbb{N}$ there is a monochromatic solution to $x + y = z^2$ with x and y having the same colour (but not necessarily z).

Lindqvist [4] considered the modular version and showed that if $p > p_0(k)$ is a prime and if $\mathbb{Z}/p\mathbb{Z}$ is k-coloured, then there are $\gg_k p^2$ monochromatic solutions to $x + y = z^2$.

2. Proof

It suffices to prove the second statement of Theorem 1.

Let $c : [N, 10^4N^4] \to \{-1, 1\}$ be an arbitrary 2-colouring.

If all elements of $[9N, 80N^2]$ are coloured with the same colour, then $x = N^2, y = 80N^2, z = 9N$ is a monochromatic solution.

Otherwise, let $k \in [9N, 80N^2]$ be such that $c(k) \neq c(k + 1)$. Without loss of generality we shall assume that $c(k) = 1$ and $c(k + 1) = -1$.

If there exists some $i \in [N, k^2-N]$ with $c(i) = c(k^2-i) = 1$, then $x = i, y = k^2-i, z = k$ is a monochromatic solution. Therefore, we can assume that

$$c(i) + c(k^2 - i) \leq 0 \tag{1}$$

holds for every $i \in [N, k^2 - N]$.

Similarly, if there exists some $i \in [N, (k + 1)^2 - N]$ with $c(i) = c((k + 1)^2 - i) = -1$, then $x = i, y = (k + 1)^2 - i, z = k + 1$ is a monochromatic solution. Therefore, we can also assume that

$$c(i) + c((k + 1)^2 - i) \geq 0 \tag{2}$$

holds for every $i \in [N, (k + 1)^2 - N]$.

Now, let $j \in [N, k^2 - N]$. By taking $i = j$ in (1) and $i = j + 2k + 1$ in (2) we obtain

$$c(j) + c(k^2 - j) \leq 0 \tag{3}$$

and

$$c(j + 2k + 1) + c(k^2 - j) \geq 0. \tag{4}$$

Inequalities (3) and (4) yield that $c(j) \leq c(j + 2k + 1)$ holds for every $j \in [N, k^2 - N]$. That is, for every $j \in [N, N + 2k]$ we have

$$c(j) \leq c(j + (2k + 1)) \leq c(j + 2(2k + 1)) \leq \cdots \leq c(j + m_j(2k + 1)), \tag{5}$$

where m_j is the largest integer such that $j + (m_j - 1)(2k + 1) \le k^2 - N$. Note that $j+m_j(2k+1) > k^2-N$. For $j \in [N, N+2k]$ let $H_j := \{j, j+(2k+1), \dots, j+m_j(2k+1)\}$.

We obtained that restricting the colouring to any mod $2k + 1$ residue class H_j it is monotonic, in the sense that the pattern of colours looks like $-1, -1, \dots, -1, 1, 1, \dots, 1$. Let us introduce a function which tells us where the breaking point is.

For $j \in [N, N + 2k]$ let $f(j) = \infty$, if all elements of H_j are coloured -1, otherwise let $f(j)$ be the smallest element of H_j which is coloured 1. That is, $f(j) = j + l(2k + 1)$, if $c(j) = \dots = c(j+(l-1)(2k+1)) = -1$ and $c(j+l(2k+1)) = \dots = c(j+m_j(2k+1)) = 1$. (If $c(j) = \dots = c(j + m_j(2k + 1)) = -1$, then $f(j) = \infty$.)

The function f is defined on a complete residue system modulo $2k + 1$. Let us extend it to the set of integers, in such a way that for an integer j' let $f(j') = f(j)$ with the unique $j \in [N, N + 2k]$ which is congruent with j' modulo $2k + 1$. Similarly, $H_{j'} := H_j$ for the unique $j \in [N, N + 2k]$ congruent with j' modulo $2k + 1$.

Assume now that for some j we have $f(j) + f(k^2 - j) \le k^2$. By taking the sum of an element from H_j and an element from H_{k^2-j} we always get a number which is congruent with k^2 modulo $2k + 1$. Note that those numbers in H_j which are coloured 1 are next to each other, and so do those numbers in H_{k^2-j} that are coloured 1. Therefore, the set of such possible sums is also an interval (in the mod $2k + 1$ residue class of k^2). The smallest such sum is $f(j) + f(k^2 - j)$ and the largest one is at least $2(k^2 - N)$. As $k^2 \in [f(j) + f(k^2 - j), 2(k^2 - N)]$, from the classes H_j and H_{k^2-j} we can choose two elements coloured 1 whose sum is k^2, and we obtain a monochromatic solution, since $c(k) = 1$.

From now on, let us assume that $f(j)+f(k^2-j) > k^2$ for every j. Note that this implies that for every j we have $f(j) + f(k^2 - j) \ge k^2 + 2k + 1$. Let $A = \{j \in [N, N + 2k] : f(j) \ge (k^2 + 2k + 1)/2\}$. Since $f(j) + f(k^2 - j) \ge k^2 + 2k + 1$ for every j, we have $|A| \ge k+1$. By the pigeon-hole principle $A+A$ contains elements from all of the residue classes modulo $2k + 1$.

Let $m \in [0.2k, 0.8k]$ be an integer. (Note that $m \ge 0.2k \ge N$.) We can choose $j_1, j_2 \in A$ such that $m^2 \equiv j_1 + j_2 \pmod{2k + 1}$.

For $j \in A$ let $g(j)$ denote the largest element from H_j which is coloured -1. That is, if $f(j) < \infty$, then $g(j) = f(j) - (2k+1)$ and if $f(j) = \infty$, then $g(j) = \max(H_j)$. Note that according to $j \in A$, we have $g(j) \ge (k^2 - 2k - 1)/2$, since either $g(j) = f(j) - (2k+1) \ge (k^2 - 2k - 1)/2$ or $g(j) = \max(H_j) > k^2 - N > k^2 - k$.

From the residue class of m^2 we can write each element between j_1+j_2 and $g(j_1)+g(j_2)$ as a sum of two numbers coloured -1 (taken from H_{j_1} and H_{j_2}). Since $j_1 + j_2 \le 2 \cdot (N + 2k) < 6k < (0.2k)^2$ and $g(j_1) + g(j_2) \ge k^2 - 2k - 1 > (0.8k)^2$, the number m^2 is also the sum of two integers which are coloured -1. If $c(m) = -1$, then this results a monochromatic solution with $z = m$. Therefore, we can assume that all integers from $[0.2k, 0.8k]$ are coloured 1, and so do all the elements from their mod $2k+1$ residue classes up to $k^2 - N > k^2 - k$, according to (5). Note that $k^2 \equiv 1.5k + 1$ or $0.5k + 0.5$ (mod $2k + 1$) (depending on the parity of k). Since in both cases the modulo $2k + 1$ residue of k^2 can be expressed as a sum of two residues taken from $[0.2k, 0.8k]$ we obtain a monochromatic solution with $z = k$.

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