

# DERIVATION BASED DIFFERENTIAL CALCULI FOR NONCOMMUTATIVE ALGEBRAS DEFORMING A CLASS OF THREE DIMENSIONAL SPACES

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ABSTRACT. We equip a family of algebras whose noncommutativity is of Lie type with a derivation based differential calculus obtained, upon suitably using both inner and outer derivations, as a reduction of a redundant calculus over the Moyal four dimensional space.

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## INTRODUCTION

This paper addresses the problem of introducing a differential calculus on the family of (Lie type) non commutative algebras introduced in [20]. They are realised as a deformation of the algebra of smooth functions on a class of three-dimensional Poisson algebras related to coadjoint orbits of Lie algebras. More specifically, we are mainly interested here in deepening the analysis, initiated more than ten years ago in [35], of differential calculi which are obtained from suitable Lie algebras of inner and outer derivations. Such derivations satisfy the standard Leibniz rule, so our approach

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differs from the one developed in [37, 38], where a quantum phase space on a Lie algebra type non commutative space is defined by deforming the coproduct on a suitable Hopf algebra.

Such geometric structures are of interest within the noncommutative formalism for field and gauge theory. In the framework advocated here, these are defined in terms of noncommutative algebras of fields whose dynamics requires a Dirac operator, a Laplacian, and properly defined gauge connections, which can be formulated in terms of the noncommutative analogue of the Koszul gauge connection [46, 5, 10].

Let us mention here that other approaches to noncommutative field theory have been developed. The Connes spectral action [7] (also see [32] and references therein for an updated review) relies on a definition of the differential calculus based on the notion of spectral triples, while a quantum group gauge theory on quantum spaces (see [6, 4, 24]) is based on the notion of covariant calculi (see [47, 48]). We shall not discuss further these approaches in the present article, since the algebras we consider come as subalgebras of the well known Moyal four dimensional one.

Another widely studied approach which goes under the name of twist-deformed field theory concerns noncommutative algebras whose product is obtained by composing the ordinary commutative product with a so-called twist operator (see [2] and references therein). In such a case the differential calculus is obtained as a twist of the standard differential calculus, see [3]. However, Lie algebra based non commutative products cannot be formulated in term of a twist, therefore the above deformation procedure does not apply. We shall not discuss twist-deformed algebra in this paper and refer to the literature for more details.

The paper is organized as follows. In section 1 we review the definition of derivation based differential calculus and the Jordan-Schwinger map which realizes three dimensional Lie algebras as Moyal subalgebras. In section 2 we address the problem of defining a differential calculus for semisimple subalgebras, while in section 3 we consider non semisimple Lie algebras. We conclude in section 4 with a short summary and by pointing out possible new directions.

## 1. THE GENERAL SETTING

In order to provide the setting recalled in the (quite long) title of the paper, we start by describing what a derivation based differential calculus on a unital algebra is. Since the noncommutative algebras we shall consider are realized as subalgebras of the four dimensional Moyal algebra, we present such algebra together with its unitizations, and then the Lie algebra type noncommutativity which deforms the classical three dimensional space.

**1.1. Derivation based differential calculi.** Given an orientable  $N$ -dimensional differentiable manifold  $M$ , it is well known that the differential calculus on it is the differential graded algebra  $(d, \Omega(M) = \bigoplus_{k=0}^N \Omega_k(M))$ , with  $\Omega_k(M)$  the set of  $k$  exterior forms and  $d : \Omega_k(M) \rightarrow \Omega_{k+1}(M)$  the (graded) exterior differential with  $d^2 = 0$ . Notice that  $\mathcal{F}(M) = \Omega_0(M)$  gives the (commutative) algebra of smooth functions on  $M$ . The  $\mathcal{F}(M)$ -bimodule of 1-forms is dual to the set of vector fields  $\mathfrak{X}(M)$ . The set  $\mathfrak{X}(M)$  coincides canonically with the space of all derivations of  $\mathcal{F}(M)$ , the commutator  $[X_1, X_2]f = X_1(X_2f) - X_2(X_1f)$  (with  $X_{1,2} \in \mathfrak{X}(M)$  and  $f \in \mathcal{F}(M)$ ) provides  $\mathfrak{X}(M)$  with an infinite dimensional Lie algebra structure.

When, following the approach of the Gelfand duality, the algebra  $\mathcal{F}(M)$  is replaced by a suitable non commutative algebra  $\mathcal{A}$ , the problem of defining a differential calculus for it has been widely studied following different approaches, as we already mentioned in the Introduction. The approach we adopt in this paper is to explore how, given a (finite dimensional) Lie algebra of derivations acting  $\mathcal{A}$ , it is possible to dually define a  $\mathcal{A}$ -bimodule of forms and a whole differential graded algebra that we interpret as a differential calculus on  $\mathcal{A}$ .

Assume indeed (see [41, 42, 28, 13]) that  $\mathfrak{l}$  is a Lie algebra acting upon a unital associative algebra  $\mathcal{A}$  by derivations, i.e.  $\rho : \mathfrak{l} \rightarrow \text{End}(\mathcal{A})$  is a linear map with  $[\rho(X_a), \rho(X_b)] = \rho([X_a, X_b])$

and  $\rho(X)(a_1a_2) = (\rho(X)a_1)a_2 + a_1(\rho(X)a_2)$  for any  $X, X_a, X_b \in \mathfrak{l}$  and  $a_1, a_2$  in  $\mathcal{A}$ . Denote by  $C_\wedge^n(\mathfrak{l}, \mathcal{A})$  the set<sup>1</sup> of  $Z(\mathcal{A})$ -multilinear alternating mappings  $\omega : X_1 \wedge \cdots \wedge X_n \mapsto \omega(X_1, \dots, X_n)$  from  $\mathfrak{l}^{\otimes n}$  to  $\mathcal{A}$ . On the graded vector space  $C_\wedge(\mathfrak{l}, \mathcal{A}) = \bigoplus_{j=0}^{\dim \mathfrak{l}} C_\wedge^j(\mathfrak{l}, \mathcal{A})$ , with  $C_\wedge^0(\mathfrak{l}, \mathcal{A}) = \mathcal{A}$ , one can define a wedge product by (with  $\omega \in C_\wedge^k(\mathfrak{l}, \mathcal{A})$ ,  $\omega' \in C_\wedge^s(\mathfrak{l}, \mathcal{A})$  and  $X_j \in \mathfrak{l}$ )

$$(1.1) \quad (\omega \wedge \omega')(X_1, \dots, X_{k+s}) = \frac{1}{k!s!} \sum_{\sigma \in \mathcal{S}_{k+s}} (\text{sign}(\sigma)) \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \omega'(X_{\sigma(k+1)}, \dots, X_{\sigma(k+s)})$$

(where  $\mathcal{S}_{k+s}$  is the set of permutations of  $k+s$  elements) and the operator  $d : C_\wedge^n(\mathfrak{l}, \mathcal{A}) \rightarrow C_\wedge^{n+1}(\mathfrak{l}, \mathcal{A})$  by

$$(1.2) \quad \begin{aligned} (d\omega)(X_0, X_1, \dots, X_n) &= \sum_{k=0}^n (-1)^k \rho(X_k) (\omega(X_0, \dots, \hat{X}_k, \dots, X_n)) \\ &+ \frac{1}{2} \sum_{r,s} (-1)^{k+s} \omega([X_r, X_s], X_0, \dots, \hat{X}_r, \dots, \hat{X}_s, \dots, X_n) \end{aligned}$$

(with  $\hat{X}_r$  denoting that the  $r$ -th term is omitted). Such operator is easily proven to be a graded antiderivation with  $d^2 = 0$ , so  $(C_\wedge(\mathfrak{l}, \mathcal{A}), d)$  is a graded differential algebra. Although the relations (1.1) and (1.2) are valid for both commutative and non commutative algebras, when the algebra  $\mathcal{A}$  is not commutative one easily see that it is in general  $f_1 df_2 \neq (df_2) f_1$  and  $\omega \wedge \omega' \neq -\omega' \wedge \omega$ . One has indeed  $(f_1 df_2)(X) = f_1(\rho(X)f_2)$  and  $((df_2)f_1)(X) = (\rho(X)f_2)f_1$ , while  $(\omega \wedge \omega')(X_1, X_2) = \omega(X_1)\omega'(X_2) - \omega(X_2)\omega'(X_1)$  and  $(\omega' \wedge \omega)(X_1, X_2) = \omega'(X_1)\omega(X_2) - \omega'(X_2)\omega(X_1)$ . This exterior algebra is an example of a derivation based calculus, where the derivations come from the action of the Lie algebra  $\mathfrak{l}$  upon the algebra  $\mathcal{A}$ . Such exterior algebra is often denoted by  $\underline{\Omega}_\mathfrak{l}(\mathcal{A})$ , while its subset  $\Omega_\mathfrak{l}(\mathcal{A}) \subset \underline{\Omega}_\mathfrak{l}(\mathcal{A})$  is defined as the smallest differential graded subalgebra of  $\underline{\Omega}_\mathfrak{l}(\mathcal{A})$  generated in degree 0 by  $\mathcal{A}$ . By construction, every element in  $\Omega_\mathfrak{l}^n(\mathcal{A})$  can be written as a sum of  $a_0 da_1 \wedge \cdots \wedge da_n$  terms with  $a_j \in \mathcal{A}$ , while this is not necessary for elements in  $\underline{\Omega}_\mathfrak{l}(\mathcal{A})$ . This difference will be seen in some of the examples we shall describe in the following sections. It is easy to prove that, if the algebra  $\mathcal{A}$  is the set of smooth functions on a paracompact manifold, then  $\underline{\Omega}_\mathfrak{l}(\mathcal{A}) = \Omega_\mathfrak{l}(\mathcal{A})$ .

Upon the graded differential algebra  $C_\wedge(\mathfrak{l}, \mathcal{A}) = \underline{\Omega}_\mathfrak{l}(\mathcal{A})$  a contraction operator can be defined. If  $X \in \mathfrak{l}$ , then

$$(1.3) \quad (i_X \omega)(X_1, \dots, X_n) = \omega(X, X_1, \dots, X_n)$$

(with  $X_j \in \mathfrak{l}$ ) gives a degree  $(-1)$  antiderivation from  $C_\wedge^{n+1}(\mathfrak{l}, \mathcal{A}) \rightarrow C_\wedge^n(\mathfrak{l}, \mathcal{A})$ . The operator defined by  $L_X = i_X d + di_X$  is the degree zero Lie derivative along  $X$ , the set  $(C_\wedge(\mathfrak{l}, \mathcal{A}) = \underline{\Omega}_\mathfrak{l}(\mathcal{A}), d, i_X, L_X = i_X d + di_X)$  gives a Cartan calculus on  $\mathcal{A}$  depending on the Lie algebra  $\mathfrak{l}$  of derivations.

**1.2. The Moyal algebra.** Consider the symplectic vector space  $(\mathbb{R}^{2N}, \omega)$ , where  $\omega$  is the non degenerate closed (i.e. symplectic) 2-form on  $\mathbb{R}^{2N}$  which is translationally invariant, that we write as  $\omega = dq_a \wedge dp_a$  along a global Darboux coordinate system given by  $(q_a, p_a)_{a=1, \dots, N}$ . The corresponding Moyal product (see [16, 21, 44]) reads (with  $\theta > 0$ )

$$(1.4) \quad (f * g)(x) = \frac{1}{(\pi\theta)^{2N}} \int \int dudv f(x+u)g(x+v) e^{-2i\omega^{-1}(u,v)/\theta}, \quad \omega = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}$$

for  $f, g \in \mathcal{S}(\mathbb{R}^{2N})$ , i.e. the Schwartz space in  $\mathbb{R}^{2N}$ . The set

$$(1.5) \quad \mathcal{A}_\theta = (\mathcal{S}(\mathbb{R}^{2N}), *)$$

<sup>1</sup>Notice that the space of derivations for a given algebra  $\mathcal{A}$  is a module only with respect to the centre  $Z(\mathcal{A})$  of the algebra.

is a non unital pre -  $C^*$ -algebra. This algebra has a tracial property, with

$$(1.6) \quad \langle f | g \rangle = \frac{1}{(\pi\theta)^N} \int dx f(x) * g(x) = \frac{1}{(\pi\theta)^N} \int dx f(x)g(x),$$

so it is possible to define a Moyal product on a larger set than  $\mathcal{S}(\mathbb{R}^{2N})$ . If  $T \in \mathcal{S}'(\mathbb{R}^{2N})$  (with  $\mathcal{S}'(\mathbb{R}^{2N})$  the space of continuous linear functional on  $\mathcal{S}(\mathbb{R}^{2N})$ , i.e. the space of tempered distributions), its action (evaluation) upon  $g \in \mathcal{S}(\mathbb{R}^{2N})$  is denoted by  $\langle T | g \rangle \in \mathbb{C}$ . With  $f \in \mathcal{S}(\mathbb{R}^{2N})$  one defines  $T * f$  and  $f * T$  in  $\mathcal{S}'(\mathbb{R}^{2N})$  by

$$(1.7) \quad \begin{aligned} \langle T * f | g \rangle &= \langle T | f * g \rangle, \\ \langle f * T | g \rangle &= \langle T | g * f \rangle. \end{aligned}$$

One considers the space of left and right multipliers

$$(1.8) \quad \begin{aligned} \mathcal{M}_L^\theta &= \{T \in \mathcal{S}'(\mathbb{R}^{2N}) : T * f \in \mathcal{S}(\mathbb{R}^{2N}) \forall f \in \mathcal{S}(\mathbb{R}^{2N})\} \\ \mathcal{M}_R^\theta &= \{T \in \mathcal{S}'(\mathbb{R}^{2N}) : f * T \in \mathcal{S}(\mathbb{R}^{2N}) \forall f \in \mathcal{S}(\mathbb{R}^{2N})\}; \end{aligned}$$

the set  $\mathcal{M}^\theta = \mathcal{M}_L^\theta \cap \mathcal{M}_R^\theta$  is a unital  $*$ -algebra<sup>2</sup>. Such a unitization is not unique: it turns to be indeed the maximal compactification of  $\mathcal{A}_\theta$  defined by duality. It contains polynomials, plane waves, Dirac's  $\delta$  and its derivatives. Its classical limit

$$(1.9) \quad \lim_{\theta \rightarrow 0} \mathcal{M}^\theta = \mathcal{O}_M$$

is the set of smooth functions of polynomial growth on  $\mathbb{R}^{2N}$  in all derivatives. If  $\{ , \}$  denotes the Poisson bracket structure on  $\mathbb{R}^{2N}$  corresponding to the symplectic form  $\omega$ , the Moyal product has, on a suitable subset of  $\mathcal{M}^\theta$ , the asymptotic expansion in  $\theta$  given by

$$(1.10) \quad f * g \sim fg + \frac{i\theta}{2} \{f, g\} + \sum_{k=2}^{\infty} \left(\frac{i\theta}{2}\right)^k \frac{1}{k!} D_k(f, g) \quad \text{as } \theta \rightarrow 0$$

with  $D_k$  the  $k$ -th order bidifferential operator which, in the easiest case ( $\mathbb{R}^2, \omega = dq \wedge dp$ ), is written as

$$(1.11) \quad D_k(f, g) = \frac{\partial^k f}{\partial q^k} \frac{\partial^k g}{\partial p^k} - \binom{k}{1} \frac{\partial^k f}{\partial^{k-1} q \partial p} \frac{\partial^k g}{\partial^{k-1} p \partial q} + \dots + (-1)^k \frac{\partial^k f}{\partial p^k} \frac{\partial^k g}{\partial q^k}.$$

If  $f, g \in \mathcal{M}^\theta$ , their commutator is defined as

$$(1.12) \quad [f, g]_\theta = f * g - g * f.$$

Since from (1.11) we see  $D_s(f, g) = (-1)^s D_s(g, f)$ , we have that

$$(1.13) \quad [f, g]_\theta = i\theta \{f, g\} + \sum_{s=1}^{\infty} \frac{2}{(2s+1)!} \left(\frac{i\theta}{2}\right)^{2s+1} D_{2s+1}(f, g).$$

Upon denoting by

$$\mathcal{P}_k = \{q_1^{a_1} q_2^{a_2} p_1^{b_1} p_2^{b_2} : a_1 + a_2 + b_1 + b_2 = k \in \mathbb{N}\},$$

i.e. the vector space of order  $k$  polynomials in  $\mathbb{R}^4$ , the relation (1.13) allows to immediately see that the elements in  $\mathcal{P}_1$  fulfill the so called canonical commutation relations

$$(1.14) \quad \begin{aligned} [q_a, q_b]_\theta &= 0, \\ [p_a, p_b]_\theta &= 0, \\ [q_a, p_b]_\theta &= i\theta \delta_{ab}, \end{aligned}$$

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<sup>2</sup>In [21] one proves that, if  $T \in \mathcal{S}'$ ,  $R \in \mathcal{M}_L$  and  $S \in \mathcal{M}_R$  then  $\langle R * T | f \rangle = \langle T | f * R \rangle$  and  $\langle T * S | f \rangle = \langle T | S * f \rangle$  for any  $f \in \mathcal{S}$ . These give a meaningful definition of the Moyal product within the multiplier algebra.

while, if at least one among the elements  $f, g$  is in  $S = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2$ , it is

$$(1.15) \quad [f, g]_\theta = i\theta\{f, g\}.$$

This means that  $(S, \{, \})$  is a Poisson subalgebra of  $\mathcal{F}(\mathbb{R}^4)$ , while  $(S, [, ]_\theta)$  is a Lie subalgebra in  $\mathcal{M}^\theta$  with respect to the  $*$ -product commutator (1.12), indeed isomorphic to a one dimensional central extension of the Lie algebra  $\mathfrak{isp}(4, \mathbb{R})$  providing the infinitesimal generators of the inhomogeneous symplectic linear group  $\text{ISP}(4, \mathbb{R})$ . Moreover,  $S$  is the maximal Lie algebra acting upon both  $\mathcal{F}(\mathbb{R}^4)$  and  $\mathcal{M}^\theta$  in terms of derivations, via the operators

$$(1.16) \quad X_f : g \mapsto i\theta\{f, g\} = [f, g]_\theta.$$

**1.3. Lie algebra type noncommutative spaces.** We know from [19] that any three dimensional Lie algebra  $\mathfrak{g}$  with basis  $(x_1, x_2, x_3)$  and commutator structure

$$(1.17) \quad [x_a, x_b] = f_{ab}^c x_c$$

can be cast in the form

$$(1.18) \quad \begin{aligned} [x_1, x_2] &= cx_3 + hx_2, \\ [x_2, x_3] &= ax_1, \\ [x_3, x_1] &= bx_2 - hx_3 \end{aligned}$$

with real parameters  $a, b, c, h$  such that  $ah = 0$ , while from [34] we know that a (classical) Jordan – Schwinger map  $\pi_{\mathfrak{g}} : \mathbb{R}^4 \rightarrow \mathfrak{g}^* \sim \mathbb{R}^3$  can be defined such that

$$(1.19) \quad \{\pi_{\mathfrak{g}}^*(x_a), \pi_{\mathfrak{g}}^*(x_b)\} = f_{ab}^c \pi_{\mathfrak{g}}^*(x_c).$$

Upon noticing that a classical Jordan – Schwinger map with  $\pi^*(x_a)$  ranging within  $\mathcal{P}_1 \oplus \mathcal{P}_2$  can be defined, a (noncommutative) version of it is introduced in [20] as a vector space inclusion  $s_{\mathfrak{g}} : \mathfrak{g}^* \hookrightarrow \mathcal{P}_1 \oplus \mathcal{P}_2$  such that

$$(1.20) \quad [s_{\mathfrak{g}}(x_a), s_{\mathfrak{g}}(x_b)]_\theta = i\theta f_{ab}^c s_{\mathfrak{g}}(x_c).$$

On fixing a three dimensional Lie algebra  $\mathfrak{g}$ , the Moyal product in  $\mathbb{R}^4$  of functions of the variables  $s_{\mathfrak{g}}(x_a)$  is proven to depend only on the  $s_{\mathfrak{g}}(x_a)$  variables, so there exists a unital complex  $*$ -algebra  $A_{\mathfrak{g}} \subset \mathcal{M}^\theta$  which is given as the quotient

$$(1.21) \quad A_{\mathfrak{g}} = [u_1, u_2, u_3]/I_{\mathfrak{g}},$$

where (upon denoting  $u_a = s_{\mathfrak{g}}(x_a)$ )  $I_{\mathfrak{g}}$  is the two-sided ideal generated by  $[u_a u_b - u_b u_a - i\theta f_{ab}^c u_c]$  (see also [22]: we are realizing the universal enveloping algebra  $A_{\mathfrak{g}}$  as a subalgebra of  $\mathcal{M}^\theta$ ). We list the maps  $s_{\mathfrak{g}}$  from [20], starting by those having a (quadratic, i.e. in  $\mathcal{P}_2$ ) Casimir function  $C_{\mathfrak{g}}$ . In this list we do not consider the Abelian Lie algebra.

(1) For  $\mathfrak{g} = \mathfrak{su}(2)$ , with

$$[x_a, x_b] = \varepsilon_{ab}^c x_c,$$

we have

$$(1.22) \quad u_1 = \frac{1}{2}(q_1 q_2 + p_1 p_2), \quad u_2 = \frac{1}{2}(q_1 p_2 - q_2 p_1), \quad u_3 = \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2).$$

The Casimir function is given by  $C_{\mathfrak{su}(2)} = u_1^2 + u_2^2 + u_3^2$ . The identity

$$(1.23) \quad u_4^2 = u_1^2 + u_2^2 + u_3^2, \quad \text{with} \quad u_4 = \frac{1}{4}(q_1^2 + p_1^2 + q_2^2 + p_2^2),$$

shows that  $u_4$  is the quadratic Casimir for  $A_{\mathfrak{su}(2)}$ .

(2) For  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$  with

$$\begin{aligned} [x_1, x_2] &= -x_3, \\ [x_2, x_3] &= x_1, \\ [x_3, x_1] &= -x_2, \end{aligned}$$

we have

$$(1.24) \quad u_1 = \frac{1}{4}(q_1^2 + p_1^2 + q_2^2 + p_2^2), \quad u_2 = \frac{1}{4}(q_1^2 + q_2^2 - p_1^2 - p_2^2), \quad u_3 = \frac{1}{2}(q_1 p_1 + q_2 p_2).$$

The Casimir function is  $C_{\mathfrak{sl}(2, \mathbb{R})} = u_1^2 - u_2^2 - u_3^2$ . Analogously to the previous case, the identity

$$(1.25) \quad u_4^2 = u_1^2 - u_2^2 - u_3^2, \quad \text{with } u_4 = \frac{1}{2}(q_1 p_2 - q_2 p_1),$$

shows that  $u_4$  is the quadratic Casimir for  $A_{\mathfrak{sl}(2, \mathbb{R})}$ .

(3) For  $\mathfrak{g} = \mathfrak{e}(2)$ , with

$$\begin{aligned} [x_1, x_2] &= x_3, \\ [x_2, x_3] &= 0, \\ [x_3, x_1] &= x_2, \end{aligned}$$

we have

$$(1.26) \quad u_1 = q_1 p_2 - q_2 p_1, \quad u_2 = q_1, \quad u_3 = q_2.$$

The quadratic Casimir function is  $C_{\mathfrak{e}(2)} = (u_2^2 + u_3^2)/2$ .

(4) For  $\mathfrak{g} = \mathfrak{iso}(1, 1)$  with

$$\begin{aligned} [x_1, x_2] &= x_3, \\ [x_2, x_3] &= 0, \\ [x_3, x_1] &= -x_2, \end{aligned}$$

we have

$$(1.27) \quad u_1 = \frac{1}{2}(p_1^2 + p_2^2 - q_1^2 - q_2^2), \quad u_2 = q_2 + p_1, \quad u_3 = -q_1 - p_2.$$

The quadratic Casimir function is  $C_{\mathfrak{iso}(1,1)} = (u_3^2 - u_2^2)/2$ .

(5) for  $\mathfrak{g} = \mathfrak{h}(1)$  (the Heseinberg-Weyl Lie algebra), with

$$\begin{aligned} [x_1, x_2] &= x_3, \\ [x_2, x_3] &= 0, \\ [x_3, x_1] &= 0, \end{aligned}$$

we have

$$(1.28) \quad u_1 = q_1, \quad u_2 = q_2 p_1, \quad u_3 = q_2.$$

The quadratic Casimir function is  $C_{\mathfrak{h}(1)} = u_3^2$ .

The three dimensional Lie algebras having no Casimir correspond to the case  $a = 0$  in (1.18). A Jordan-Schwinger map for them is given by

$$(1.29) \quad u_1 = -h(q_1 p_1 + q_2 p_2) - c q_2 p_1 + b q_1 p_2, \quad u_2 = q_1, \quad u_3 = q_2.$$

Among the algebras we listed, the one corresponding to  $\mathfrak{g} = \mathfrak{su}(2)$  (which is closely related to the fuzzy sphere [33], also see [31] for a review on fuzzy spaces) has been intensively studied (see [45, 18, 40, 17, 43, 26])<sup>3</sup> as an example for a space with a Lie algebra type noncommutativity, so to provide, corresponding to a suitable differential calculus, a gauge action giving a renormalizable field theory.

Further elaborating on our preliminary results in [35], in this paper we equip the algebras  $A_{\mathfrak{g}} \subset \mathcal{M}^\theta$  which correspond to a Lie algebra  $\mathfrak{g}$  having a quadratic Casimir function, with an exterior algebra and a differential calculus. These are realized as a suitable reduction of the calculus on  $\mathcal{M}^\theta$  given by  $(C_\wedge(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta), d)$ , where the derivations are defined by (1.16) (see [5, 36, 46]). We leave the remaining cases to a further work.

It is well known that the Moyal algebra  $\mathcal{M}^\theta$  is a normal space of distributions, and all its derivations are inner. For any algebra  $A_{\mathfrak{g}} \subset \mathcal{M}^\theta$ , the union of its inner and outer derivations close a Lie algebra  $\tilde{\mathfrak{g}}$  with  $\mathfrak{g} \subseteq \tilde{\mathfrak{g}} \subset \mathfrak{isp}(4, \mathbb{R})$ . The Lie algebra  $\tilde{\mathfrak{g}}$  is seen to act via inner derivations upon  $\mathcal{M}^\theta$ , and such action can be projected onto  $A_{\mathfrak{g}}$ . The set  $C_\wedge(\tilde{\mathfrak{g}}, A_{\mathfrak{g}})$  can be then described as a graded subalgebra of  $C_\wedge(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta)$ , the corresponding calculus  $(C_\wedge(\tilde{\mathfrak{g}}, A_{\mathfrak{g}}), d)$  as a reduction of  $(C_\wedge(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta), d)$ . Moreover, the differential calculus that we define on  $A_{\mathfrak{g}}$  turns out to have a frame, i.e. the exterior algebra is a free  $A_{\mathfrak{g}}$ -bimodule: this gives a way to study its cohomology.

Since the structure of the space of derivations for  $A_{\mathfrak{g}}$  strongly depends on the Lie algebra  $\mathfrak{g}$  being semisimple or not, the paper has two sections, which cover the two cases. The subsection 2.3 moreover shows that, for the considered differential calculi on  $A_{\mathfrak{g}}$  with semisimple  $\mathfrak{g}$ , with respect to a natural Cartan-Killing symmetric tensor on it, one can explicitly write a vector potential solving the Yang-Mills equations. This result mimics a result proven in [25, 39] for suitable classical fibrations.

## 2. DIFFERENTIAL CALCULUS ON $A_{\mathfrak{g}}$ FOR SEMISIMPLE LIE ALGEBRAS

Among the three dimensional Lie algebras considered in the introduction, the only semisimple ones are  $\mathfrak{su}(2)$  and  $\mathfrak{sl}(2, \mathbb{R})$ , which are the two real forms of  $\mathfrak{sl}(2, \mathbb{C})$ . It follows that the corresponding algebras  $A_{\mathfrak{g}}$  have no outer derivations, and that the Cartan-Killing form for them is non degenerate. Since the semisimplicity of  $\mathfrak{g}$  strongly affects the structure of the derivation based calculi on  $A_{\mathfrak{g}}$ , we start by analysing such a case.

**2.1. The case  $\mathfrak{g} = \mathfrak{su}(2)$ .** We begin upon analysing the classical (i.e. commutative) setting for this algebra, so we consider the symplectic vector space  $(\mathbb{R}^4, \omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2)$  equipped with the Euclidean metric  $g = dq_a \otimes dq_a + dp_a \otimes dp_a$ . Given the elements  $u_1, u_2, u_3$  as in (1.22) with  $\{u_a, u_b\} = \varepsilon_{abc} u_c$  and the element (1.23)

$$(2.1) \quad u_4 = \frac{1}{4}(q_1^2 + p_1^2 + q_2^2 + p_2^2),$$

the corresponding Hamiltonian vector fields  $(X_1, X_2, X_3)$  coincide with the right invariant vector fields which are tangent to the  $S^3$  orbits of the standard action of  $SU(2)$  upon  $\mathbb{R}^4 \setminus \{0\}$ , while  $X_4$  is the infinitesimal generator for the  $U(1)$  action giving the well known Kustaanheimo-Stiefel fibration  $\pi_{KS} : \mathbb{R}^4 \setminus \{0\} \xrightarrow{U(1)} \mathbb{R}^3 \setminus \{0\}$ . The set

$$A = \{f \in \mathcal{F}(\mathbb{R}^4 \setminus \{0\}) : X_4(f) = \{u_4, f\} = 0\}$$

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<sup>3</sup>Notice that such algebra had been considered in [23].

is the basis algebra for the  $\pi_{KS}$  fibration: one can equivalently say [8] that the derivations  $(X_1, \dots, X_4)$  give the projectable vector fields for such a fibration. As in [20], we define an algebra  $\tilde{A}$  as

$$(2.2) \quad \tilde{A} = \{f \in \mathcal{M}^\theta : [u_4, f]_\theta = 0\}.$$

Such an algebra can be realised as a suitable completion of the polynomial algebra  $P_{\mathfrak{su}(2)} = [u_1, u_2, u_3, u_4]/\mathfrak{I}_{\mathfrak{su}(2)}$  with  $\mathfrak{I}_{\mathfrak{su}(2)}$  the two-sided ideal generated by  $[u_a u_b - u_b u_a - i\theta \varepsilon_{abc} u_c]$  and by  $[u_4^2 - (u_1^2 + u_2^2 + u_3^2)]$ . Notice that  $\tilde{A}$  extends the algebra  $A_{\mathfrak{su}(2)}$  defined in (1.21) for the  $\mathfrak{su}(2)$  case since it contains the odd powers of  $u_4$ . The lines above show indeed that the algebra  $\tilde{A}$  is a noncommutative deformation of the commutative algebra  $\mathcal{F}(\mathbb{R}^3 \setminus \{0\})$ .

Within the classical setting, the rank of the space of derivations for  $A$  is 3, since we can write

$$(2.3) \quad X_4 = u_4^{-1} \sum_{j=1}^3 u_j X_j$$

on  $\mathcal{F}(\mathbb{R}^4 \setminus \{0\})$ , while the derivations on  $\mathcal{M}^\theta$  given by

$$(2.4) \quad D_\mu(f) = [u_\mu, f]_\theta, \quad \mu = 1, \dots, 4$$

are independent. The elements  $(u_1, \dots, u_4)$  from  $\tilde{A}$  give a (one dimensional) central extension  $\tilde{\mathfrak{g}}$  of the Lie algebra  $\mathfrak{g} = \mathfrak{su}(2)$  with respect to the  $*$ -product commutator in  $\mathcal{M}^\theta$ , with clearly  $\tilde{\mathfrak{g}} \subset \mathfrak{isp}(4, \mathbb{R})$ . The inclusion  $C_\wedge^1(\tilde{\mathfrak{g}}, \mathcal{M}^\theta) \subset C_\wedge^1(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta)$  has been studied in [35]. Using the definitions (1.2), the elements

$$(2.5) \quad \begin{aligned} \alpha_1 &= p_2 * dq_1 + p_1 * dq_2 - q_2 * dp_1 - q_1 * dp_2, \\ \alpha_2 &= -q_2 * dq_1 + q_1 * dq_2 - p_2 * dp_1 + p_1 * dp_2, \\ \alpha_3 &= p_1 * dq_1 - p_2 * dq_2 - q_1 * dp_1 + q_2 * dp_2, \\ \beta &= q_1 * dq_1 + q_2 * dq_2 + p_1 * dp_1 + p_2 * dp_2 \end{aligned}$$

are seen to satisfy the identities

$$(2.6) \quad \begin{aligned} \alpha_j(D_k) &= -2i\theta \delta_{jk} u_4, & \alpha_j(D_4) &= -2i\theta u_j \\ \beta(D_k) &= 0, & \beta(D_4) &= \theta^2 \end{aligned}$$

with  $j, k = 1, \dots, 3$ . Upon defining

$$(2.7) \quad \begin{aligned} \omega_j &= \frac{i}{2\theta} \alpha_j - \frac{1}{\theta^2} u_k \beta, \\ \omega_4 &= \frac{1}{\theta^2} u_4 \beta, \end{aligned}$$

we have

$$\omega_\mu(D_\sigma) = u_4 \delta_{\mu\sigma},$$

with  $\mu, \sigma = 1, \dots, 4$ . Since  $Z(\mathcal{M}^\theta) = \mathbb{C}$ , the elements  $\{\omega_\mu\}_{\mu=1, \dots, 4}$  are  $\mathbb{C}$ -linear maps from the Lie algebra  $\tilde{\mathfrak{g}}$  of derivations to  $\tilde{A}$ . Since  $u_4 \in Z(\tilde{A})$ , we extend  $\tilde{A}$  upon a localization, i.e. we define the element  $u_4^{-1}$  via the relations  $u_4^{-1} u_4 = u_4 u_4^{-1} = 1$  and  $u_4^{-1} u_k = u_k u_4^{-1}$  for  $k = 1, \dots, 3$ . Denoting the extended algebra by the same symbol, we consider the vector space  $\mathcal{D} \simeq \tilde{\mathfrak{g}}$  as the tangent space to the noncommutative space described by the algebra  $\tilde{A}$ , and clearly have that the elements

$$\varphi_\mu = u_4^{-1} \omega_\mu$$

provide a basis for  $\mathcal{D}^*$  which is  $\mathbb{C}$ -dual to  $\{D_\mu\}_{\mu=1,\dots,4}$ . We consider then the free  $\tilde{A}$ -bimodule  $C_\wedge^1(\tilde{\mathfrak{g}}, \tilde{A})$  generated by the elements  $\{\varphi_\mu\}$ . For  $f \in \tilde{A}$  the action of the exterior derivative upon  $\tilde{A}$  is given by

$$(2.8) \quad df = (D_\mu f)\varphi_\mu$$

where  $D_\mu f = [u_\mu, f]_\theta$ . The relations (1.1) and (1.2) can be used to define wedge products (so to have a graded algebra  $C_\wedge(\tilde{\mathfrak{g}}, \tilde{A}) = \bigoplus_{j=0}^4 C_\wedge^j(\tilde{\mathfrak{g}}, \tilde{A})$ ) and to extend the action of the exterior derivative. The same relations allow to prove also that, although  $\tilde{A}$  is not commutative, we have

$$(2.9) \quad \begin{aligned} f * \varphi_\mu &= \varphi_\mu * f, \\ \varphi_\mu \wedge \varphi_\sigma &= -\varphi_\sigma \wedge \varphi_\mu. \end{aligned}$$

Using such identities, it is also immediate to see that, for such a four dimensional differential calculus on  $\tilde{A}$ , it is

$$(2.10) \quad \begin{aligned} d\varphi_j &= -\frac{1}{2} \varepsilon_{jkl} \varphi_k \wedge \varphi_l & (j, k, l \in 1, \dots, 3) \\ d\varphi_4 &= 0. \end{aligned}$$

This means that the Maurer-Cartan equation for the differential calculus depends on the structure constants of the Lie algebra  $\tilde{\mathfrak{g}}$ , and its cohomology is related to the Eilenberg-Chevalley cohomology for  $\tilde{\mathfrak{g}}$ .

Notice that the elements  $\varphi_a$  cannot be realised as  $\sum_{a=1}^3 f_a du_a$ , so  $C_\wedge(\tilde{\mathfrak{g}}, \tilde{A})$  extends the differential calculus  $(\Omega_{\tilde{\mathfrak{g}}}, d)$  given as the smallest graded differential subalgebra of  $C_\wedge(\tilde{\mathfrak{g}}, \tilde{A})$  generated in degree 0 by  $\tilde{A}$  as described in the introduction.

**2.2. The case  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ .** The noncommutative algebra  $\tilde{A}$  can be defined, as in [20], upon setting

$$(2.11) \quad \tilde{A} = \{f \in \mathcal{M}^\theta : [u_4, f]_\theta = 0\},$$

where the elements  $\{u_1, u_2, u_3, u_4\} \in \mathcal{P}_2 \subset \mathcal{M}^\theta$  are given by (1.24) and (1.25).

As in the previous case for  $\mathfrak{su}(2)$ , this algebra is a suitable completion of the polynomial algebra  $P_{\mathfrak{sl}(2, \mathbb{R})} = [u_1, u_2, u_3, u_4] / \mathfrak{I}_{\mathfrak{sl}(2, \mathbb{R})}$  with  $\mathfrak{I}_{\mathfrak{sl}(2, \mathbb{R})}$  the two-sided ideal generated by  $[u_a u_b - u_b u_a - i\theta f_{ab}^c u_c]$  and by  $u_4^2 - (u_1^2 - u_2^2 - u_3^2)$  with respect to the Lie algebra structure constants of  $\mathfrak{sl}(2, \mathbb{R})$ . Even in this case we see that  $\tilde{A}$  extends the algebra  $A_{\mathfrak{sl}(2, \mathbb{R})}$  defined in (1.21) for the  $\mathfrak{sl}(2, \mathbb{R})$  since it contains the odd powers of  $u_4$ . The algebra  $\tilde{A}$  gives a noncommutative deformation of the commutative algebra  $\mathcal{F}(\mathbb{R}^3 \setminus \{\|u\| = 0\})$  where we have denoted the three dimensional hyperbolic norm as  $\|u\|^2 = u_1^2 - u_2^2 - u_3^2$ .

The analysis for a derivation based differential calculus for  $\tilde{A}$  proceeds as in the  $\mathfrak{su}(2)$  case, so we limit ourselves to report that the elements  $(u_1, \dots, u_4)$  give a one dimensional central extension  $\tilde{\mathfrak{g}}$  of  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$  with respect to the  $*$ -commutator in  $\mathcal{M}^\theta$ . Given the basis of the tangent space  $\mathcal{D} \simeq \tilde{\mathfrak{g}}$  with basis elements given by the derivations

$$(2.12) \quad D_\mu(f) = [u_\mu, f]_\theta,$$

it is immediate to prove that the elements

$$(2.13) \quad \begin{aligned} \alpha_1 &= -p_2 * dp_1 + p_1 * dp_2 + q_2 * dq_1 - q_1 * dq_2, \\ \alpha_2 &= q_2 * dp_1 - q_1 * dp_2 + p_2 * dq_1 - p_1 * dq_2, \\ \alpha_3 &= -p_2 * dp_1 + p_1 * dp_2 - q_2 * dq_1 + q_1 * dq_2, \\ \beta &= p_2 * dq_1 - p_1 * dq_2 - q_2 * dp_1 + q_1 * dp_2 \end{aligned}$$

in  $C_{\wedge}^1(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta)$  allow to define the forms

$$(2.14) \quad \begin{aligned} \omega_j &= \frac{i}{2\theta} \alpha_j - \frac{1}{\theta^2} u_k \beta, \\ \omega_4 &= \frac{1}{\theta^2} u_4 \beta, \end{aligned}$$

that give

$$\omega_\mu(D_\sigma) = u_4 \delta_{\mu\sigma},$$

with  $a, b = 1, \dots, 4$ . After the natural localization provided by adding  $\tilde{A}$  the element  $u_4^{-1}$  we see that, as in the previous case, the elements  $\varphi_a = u_4^{-1} \omega_a$  give a basis for  $\mathcal{D}^*$  which is dual to the basis (2.12) for  $\mathcal{D}$ . Such a basis generates, as already described, the four dimensional calculus  $(C_{\wedge}(\tilde{\mathfrak{g}}, \tilde{A}), d)$  whose Maurer-Cartan relations are clearly given by

$$(2.15) \quad \begin{aligned} d\varphi_1 &= -d\varphi_2 \wedge \varphi_3, \\ d\varphi_2 &= d\varphi_3 \wedge \varphi_1, \\ d\varphi_3 &= d\varphi_1 \wedge \varphi_2, \\ d\varphi_4 &= 0. \end{aligned}$$

**2.3. A digression: Yang-Mills equations.** We consider the matrix with 1-form entries

$$(2.16) \quad \mathfrak{A} = \alpha \sum_{a=1}^3 (i\tau_a \otimes \varphi_a) + \beta(\sigma_0 \otimes \varphi_4),$$

which is a  $\tilde{\mathfrak{g}}$ -valued 1-form on  $\tilde{A}$  with  $\sigma_0 = 1_2$  the two dimensional identity matrix and the two dimensional matrices  $\tau$  given by the Pauli matrices for the  $\mathfrak{su}(2)$  case, while  $(\tau_1 = \sigma_1, \tau_{2,3} = i\sigma_{2,3})$  for  $\mathfrak{sl}(2, \mathbb{R})$ .

As already mentioned above, since both  $\mathfrak{su}(2)$  and  $\mathfrak{sl}(2, \mathbb{R})$  are semisimple, the corresponding Cartan - Killing form  $g_{ab} = f_{ak}^s f_{bs}^k$  is non degenerate. Therefore the tensor (with  $\lambda \in \mathbb{R}$ )

$$(2.17) \quad \gamma = \gamma_{\mu\nu} \varphi_\mu \otimes \varphi_\nu = u_4^2 g_{ab} \varphi_a \otimes \varphi_b + \lambda^2 \varphi_4 \otimes \varphi_4$$

gives a symmetric non degenerate 2-form on the cotangent space for the calculus on  $\tilde{A}$ . The problem of defining a consistent Hodge duality on the exterior algebra for a non commutative algebra has not a general solution<sup>4</sup>. Given the four dimensional calculi we have developped on  $\tilde{A}$ , the identities (2.9) allow to set, with respect to the volume form  $\Omega = (\text{sign } g) \lambda u_4^3 \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4$ , the relations

$$(2.18) \quad \begin{aligned} *_H \varphi_\mu &= \tilde{\gamma}_{\mu\sigma} i_{D_\sigma} \Omega, \\ *_H (\varphi_{\mu_1} \wedge \dots \wedge \varphi_{\mu_k}) &= \tilde{\gamma}_{\mu_1 \sigma_{a_1}} \dots \tilde{\gamma}_{\mu_k \sigma_{a_k}} i_{D_{a_1}} \dots i_{D_{a_k}} \Omega. \end{aligned}$$

Such relations consistently define a Hodge duality on the frame for the calculus, with  $\tilde{\gamma}_{\mu\nu} \gamma_{\nu\rho} = \delta_{\mu\rho}$ . The Hodge duality is then extended to the whole exterior algebra by requiring it to be  $\tilde{A}$ -linear. Given such a Hodge structure, and recalling that  $du_4 = 0$ , it is straightforward to prove the following

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<sup>4</sup>When the commutative algebra of functions on two and three dimensional spheres is deformed within the quantum group setting, a Hodge duality on the corresponding exterior algebras has been studied in [11, 51, 49, 50, 30, 29]. Within the Drinfeld - Jimbo deformation formalism we refer the reader to [12, 1] and references therein for a consistent notion of a Hodge duality for a differential calculus suitably given by a twist quantization.

identities,

$$\begin{aligned}
\mathfrak{A} \wedge \mathfrak{A} &= -(2\alpha)d\mathfrak{A} \\
\mathfrak{F} &= d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A} = (1 - 2\alpha)d\mathfrak{A} \\
*_H \mathfrak{F} &= \lambda u_4^{-1}(2\alpha - 1)(\mathfrak{A} \wedge \varphi_4) \\
(2.19) \quad d(*_H \mathfrak{F}) + \mathfrak{A} \wedge (*_H \mathfrak{F}) - (*_H \mathfrak{F}) \wedge \mathfrak{A} &= \lambda u_4^{-1}(2\alpha - 1)\left(2 - \frac{1}{2\alpha}\right)(\mathfrak{A} \wedge \mathfrak{A} \wedge \varphi_4)
\end{aligned}$$

with  $*_H$  the Hodge star operator. The second line gives the curvature  $\mathfrak{F}$  corresponding to the vector potential  $\mathfrak{A}$ , the fourth line shows that the covariant derivative of  $*_H \mathfrak{F}$  is zero if and only if it is either  $\alpha = 1/2$  (a case that we discard since it corresponds to a zero curvature: notice that it comes from the pure gauge term given by the Maurer-Cartan form for  $\mathfrak{g}$ ) or  $\alpha = 1/4$ . This means that the vector potential  $\mathfrak{A}$  gives a non trivial solution of the homogeneous (i.e. sourceless) Yang-Mills equations for  $\alpha = 1/4$ . The above calculations show that the existence of this solution, which is valid for any value of the parameters  $\beta$  and  $\lambda \neq 0$  in the metric tensor  $g$ , only depends on the semisimplicity of the Lie algebra  $\mathfrak{su}(2)$ . This solution is the analogue on the noncommutative space  $\tilde{A}$  of the meron solution for the Yang-Mills equations which was introduced in [9], analysed within the principal and vector bundle formalism in [27], and extended to the quantum group setting in [51].

We close this digression upon noticing that, for the algebra  $\tilde{A}$  corresponding to the  $\mathfrak{su}(2)$  case, the Hodge - de Rham Laplacian operator given by

$$\square = *_H d *_H d = \delta_{ab}[u_a, [u_b, ]\theta]_\theta$$

coincides with the well studied Laplacian for the fuzzy sphere, whose spectral dimension is equal to 2. The problem of defining a Laplacian with spectral dimension equal to 3 can be studied within the fuzzy space approach, and an interesting example along this path is provided by [14, 15].

### 3. A DIFFERENTIAL CALCULUS ON $A_{\mathfrak{g}}$ FOR A NON SEMISIMPLE LIE ALGEBRA $\mathfrak{g}$

We begin this section upon recalling that, within the classical formalism of differential geometry, for a Lie algebra  $\mathfrak{g}$  given by  $[x_a, x_b] = f_{ab}^c x_c$  there exists a natural Poisson structure on  $\mathfrak{g}^*$  given by (notice that we are identifying the coordinates on  $\mathfrak{g}$  with those on  $\mathfrak{g}^*$ )

$$\Lambda_{\mathfrak{g}} = \frac{1}{2} f_{ab}^c x_c \frac{\partial}{\partial x_a} \wedge \frac{\partial}{\partial x_b}.$$

A vector field  $V = B_{ab} x_a \partial_b$  is a derivation for  $\mathfrak{g}$  if and only if it satisfies the relation

$$L_V \Lambda_{\mathfrak{g}} = 0.$$

For a non semisimple  $\mathfrak{g}$  there exist vector fields  $V$  whose action upon the corresponding universal enveloping algebra cannot be represented in terms of a commutator with an element in  $\mathfrak{g}$ . Such operators are naturally called *outer* derivations on  $\mathfrak{g}$ .

In this section we describe how the union of inner and outer derivations for a non semisimple Lie algebra  $\mathfrak{g}$  allows to define a tangent space for a differential calculus on  $A_{\mathfrak{g}}$  which turns out to be parallelisable.

**3.1. The case  $\mathfrak{g} = \mathfrak{e}(2)$ .** The Lie algebra  $\mathfrak{g} = \mathfrak{e}(2)$  is given in (1.26), so we have

$$\Lambda_{\mathfrak{e}(2)} = \frac{1}{2} (x_3 \partial_1 \wedge \partial_2 + x_2 \partial_3 \wedge \partial_1).$$

The vector fields

$$(3.1) \quad \begin{aligned} V_1 &= x_3 \partial_2 - x_2 \partial_3, \\ V_2 &= -x_3 \partial_1, \\ V_3 &= x_2 \partial_1 \end{aligned}$$

give the inner derivations for  $\mathfrak{e}(2)$ , with  $V_k(\cdot) = [x_k, \cdot]$ , while the vector field

$$(3.2) \quad V_E = x_2 \partial_2 + x_3 \partial_3$$

is an exterior derivation for the universal enveloping algebra corresponding to  $\mathfrak{e}(2)$ . From the relations

$$(3.3) \quad \begin{aligned} [V_1, V_E] &= 0, \\ [V_2, V_E] &= -V_2, \\ [V_3, V_E] &= -V_3 \end{aligned}$$

we see that the set  $\{V_\mu\}_{\mu=1,\dots,4} = \{V_1, V_2, V_3, V_E\}$  gives a basis for a Lie algebra  $\tilde{\mathfrak{e}}(2)$  which is a one dimensional extension of  $\mathfrak{e}(2)$ . We write the Lie algebra structure for  $\tilde{\mathfrak{e}}(2)$  as

$$(3.4) \quad [V_\mu, V_\nu] = \tilde{f}_{\mu\nu}{}^\rho V_\rho.$$

We define the noncommutative algebra  $A_{\mathfrak{e}(2)} \subset \mathcal{M}^\theta$  in terms of the elements  $(u_1, u_2, u_3)$  given in (1.26) as the quotient (1.21). Notice that this algebra coincides with  $\tilde{A}$  defined by

$$\tilde{A} = \{f \in \mathcal{M}^\theta : [u_C, f]_\theta = 0\},$$

with  $u_C = \frac{1}{2}(u_2^2 + u_3^2) = \frac{1}{2}(q_1^2 + q_2^2)$ . It is clear that the operators

$$(3.5) \quad D_k(f) = [u_k, f]_\theta$$

define inner derivations on  $\tilde{A} = A_{\mathfrak{e}(2)}$ . The operator  $D_E$ , whose action upon  $\tilde{A}$  is defined by setting

$$(3.6) \quad \begin{aligned} D_E : u_1 &\mapsto 0, \\ &: u_2 \mapsto (i\theta)u_2, \\ &: u_3 \mapsto (i\theta)u_3 \end{aligned}$$

on the generators and extended in terms of the Leibniz rule, is an outer derivation on  $\tilde{A}$ . This action can also be written as

$$(3.7) \quad D_E(f) = [u_E, f]_\theta \quad \text{for } f \in \tilde{A}$$

with

$$(3.8) \quad \mathcal{M}^\theta \ni u_E = -(q_1 p_1 + q_2 p_2).$$

This means that the action of the outer derivation  $D_E$  for  $\tilde{A}$  can be represented as a commutator on  $\tilde{A} \subset \mathcal{M}^\theta$  in terms of a quadratic element  $u_E \in S \subset \mathcal{M}^\theta$ . Notice that the element  $u_E$  is defined up to an arbitrary function of the quadratic Casimir  $u_C$ , but this does not affect any of the results we shall describe.

The set  $\{u_\mu\}_{\mu=1,\dots,4} = \{u_1, u_2, u_3, u_E\}$  is a Poisson subalgebra in  $(\mathcal{M}^\theta, *)$  which is isomorphic (up to the factor  $i\theta$ ) to the Lie algebra  $\tilde{\mathfrak{e}}(2)$  written in (3.4). We denote by  $\mathcal{D} \simeq \tilde{\mathfrak{e}}(2)$  the Lie algebra of derivations spanned by  $\{D_\mu\}_{\mu=1,\dots,4} = \{D_k, D_E\}$  and describe the differential calculus on  $\tilde{A}$  based on it.

Since  $\tilde{\mathfrak{e}}(2) \subset \mathfrak{isp}(4, \mathbb{R})$ , we consider the elements  $\{\alpha_\mu\}_{\mu=1,\dots,4} \in (C_\wedge(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta), \mathfrak{d})$  whose action is given by

$$(3.9) \quad \alpha_\mu(D_\nu) = [u_\nu, u_\mu]_\theta$$

and clearly observe that, on  $\mathcal{M}^\theta$ , it is  $\alpha_\mu = du_\mu$ . The elements in  $C_\wedge^1(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta)$  given by

$$\begin{aligned} \omega_1 &= \frac{1}{2}(u_3\alpha_2 - u_2\alpha_3), \\ \omega_2 &= -\frac{1}{2}(u_3\alpha_1 + u_2\alpha_E), \\ \omega_3 &= \frac{1}{2}(u_2\alpha_1 - u_3\alpha_E), \\ \omega_E &= \frac{1}{2}(u_2\alpha_2 + u_3\alpha_3) \end{aligned} \quad (3.10)$$

verify (with  $\mu = 1, \dots, 4$ )

$$\omega_\mu(D_\sigma) = (i\theta)u_C\delta_{\mu\sigma}, \quad (3.11)$$

so the elements

$$\varphi_\mu = -\frac{i}{\theta}u_C^{-1}\omega_\mu \quad (3.12)$$

give a basis for  $C_\wedge^1(\tilde{\mathfrak{e}}(2), A_{\mathfrak{e}(2)})$  (where we have – analogously to the localisation considered in the previous sections – extended the algebra  $A_{\mathfrak{e}(2)}$  upon adding<sup>5</sup> the term  $u_C^{-1}$  with  $u_C^{-1}u_k = u_ku_C^{-1}$  and  $u_Cu_C^{-1} = u_C^{-1}u_C = 1$ ) which is dual to the basis  $\{D_\mu\}_{\mu=1,\dots,4}$ . Using the (1.1) for a wedge product, the definition (1.2) to introduce a d operator and the definition (1.3) for the contraction operator, we have that  $(C_\wedge(\tilde{\mathfrak{e}}(2), A_{\tilde{\mathfrak{e}}(2)}), d)$  provides a four dimensional differential calculus on  $A_{\tilde{\mathfrak{e}}(2)}$ . As for the calculi described in the previous section (see (2.9)), the identities  $f * \varphi_\mu = \varphi_\mu * f$  and  $\varphi_\mu \wedge \varphi_\sigma = -\varphi_\sigma \wedge \varphi_\mu$  are valid. It is therefore immediate to prove that the action of the exterior derivative  $d : A_{\tilde{\mathfrak{e}}(2)} \rightarrow C_\wedge^1(\tilde{\mathfrak{e}}(2), A_{\tilde{\mathfrak{e}}(2)})$  can be written as

$$df = (D_\mu f)\varphi_\mu = ([u_\mu, f]_\theta)\varphi_\mu \quad (3.13)$$

in terms of the commutator structure on  $\mathcal{M}^\theta$ . The Maurer - Cartan relation for such a calculus comes from the Lie algebra structure of  $\tilde{\mathfrak{e}}(2)$  as in (3.4),

$$d\varphi_\rho = -\frac{1}{2}i\theta\tilde{f}_{\mu\nu}{}^\rho\varphi_\mu \wedge \varphi_\nu$$

and reads explicitly as follows

$$\begin{aligned} (3.14) \quad d\varphi_1 &= 0, \\ d\varphi_2 &= i\theta(\varphi_3 \wedge \varphi_1 + \varphi_2 \wedge \varphi_4), \\ d\varphi_3 &= i\theta(\varphi_2 \wedge \varphi_1 + \varphi_3 \wedge \varphi_4), \\ (3.15) \quad d\varphi_4 &= 0. \end{aligned}$$

One can also immediately see that the presence of a 1-form which dualises the outer derivation for  $A_{\mathfrak{e}(2)}$  allows to prove that centre  $Z(A_{\mathfrak{e}(2)})$  of the algebra is not in the kernel of the exterior derivative operator  $d$  introduced in this section. This result comes immediately upon computing that

$$du_C = 2(i\theta)u_C\varphi_4. \quad (3.16)$$

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<sup>5</sup>Notice that the algebra  $A_{\mathfrak{e}(2)}$  given by such a localisation can be seen as a noncommutative deformation of the algebra  $\mathcal{F}(\mathbb{R}^3 \setminus \{u_2^2 + u_3^2 = 0\})$ , i.e. of the functions defined on  $\mathbb{R}^3$  without a straight line.

3.2. **The case  $\mathfrak{g} = \mathfrak{iso}(1,1)$ .** The Lie algebra  $\mathfrak{iso}(1,1)$  comes as a real form of the complex Lie algebra  $\mathfrak{so}(2,1)$ . It follows that the elements

$$(3.17) \quad \begin{aligned} f_1 &= iu_1, \\ f_2 &= u_2, \\ f_3 &= iu_3, \end{aligned}$$

(where  $u_{k=1,\dots,3} \in \mathcal{S}$  are the generators of the Lie algebra  $\mathfrak{e}(2)$ , see (1.26)), give a realisation of  $A_{\mathfrak{iso}(1,1)}$ . Under the identification (3.17) one sees that the analysis performed in the previous subsection for the  $A_{\mathfrak{e}(2)}$  algebra can be – *mutatis mutandis* – carried through for the  $A_{\mathfrak{iso}(1)}$  case. Along this path we can then equip  $A_{\mathfrak{iso}(1,1)}$  with a derivation based four dimensional differential calculus.

3.3. **The case  $\mathfrak{g} = \mathfrak{h}(1)$ .** The Lie algebra  $\mathfrak{g} = \mathfrak{h}(1)$  corresponding to the Heisenberg-Weyl group is given in (1.28). The corresponding Poisson structure is given by

$$(3.18) \quad \Lambda = x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}.$$

For such a Lie algebra there are two inner derivations, whose action is given by the vector fields

$$(3.19) \quad \begin{aligned} V_1 &= x_3 \partial_2, \\ V_2 &= -x_3 \partial_1, \end{aligned}$$

and four exterior derivations, whose action is given by the vector fields

$$(3.20) \quad E_1 = x_1 \partial_1 + x_3 \partial_3,$$

$$(3.21) \quad E_2 = x_2 \partial_2 + x_3 \partial_3,$$

$$(3.22) \quad E_R = x_1 \partial_2 - x_2 \partial_1,$$

$$(3.23) \quad E_H = x_1 \partial_2 + x_2 \partial_1.$$

The commutator structure closed by these elements reads interesting Lie subalgebras. We have

$$(3.24) \quad \begin{aligned} [V_1, V_2] &= 0, \\ [E_1, E_2] &= 0, \\ [V_a, E_b] &= -\delta_{ab} V_a \end{aligned}$$

for  $a, b = 1, 2$ , with

$$(3.25) \quad \begin{aligned} [E_R, V_1] &= -V_2, & [E_R, V_2] &= V_1 \\ [E_H, V_1] &= V_2, & [E_H, V_2] &= V_1 \end{aligned}$$

and

$$(3.26) \quad \begin{aligned} [E_R, E_H] &= 2(E_1 - E_2), \\ [E_R, E_1] &= -E_H, & [E_R, E_2] &= E_H, \\ [E_H, E_1] &= -E_R, & [E_H, E_2] &= E_R. \end{aligned}$$

These relations show that  $\{V_a, E_b\}_{a,b=1,2}$  span a Lie algebra  $\tilde{\mathfrak{g}}$  that extends the abelian Lie algebra spanned by  $V_1, V_2$ . We see that also the outer derivations  $\{E_1, E_2, E_R, E_H\}$  alone span a Lie algebra.

Within the noncommutative setting, we consider  $A_{\mathfrak{h}(1)} \subset \mathcal{M}^\theta$  to be the algebra generated (and suitably completed) as in (1.21) by the elements  $u_1, u_2, u_3$  defined in (1.28). This algebra coincides with the algebra  $\tilde{A}$  defined by

$$\tilde{A} = \{f \in \mathcal{M}^\theta : [u_3, f]_\theta = 0\}.$$

The operators

$$(3.27) \quad D_k(f) = [u_k, f]_\theta$$

define inner derivations on  $A_{\mathfrak{h}(1)}$ . Analogously to what we described for the  $\mathfrak{g} = \mathfrak{e}(2)$  case, the operators  $D_{E_a}$  with  $a = 1, 2$  whose action upon  $A_{\mathfrak{h}(1)}$  is defined by setting

$$(3.28) \quad \begin{aligned} D_{E_1}(u_1) &= (i\theta)u_1, & D_{E_2}(u_1) &= 0, \\ D_{E_1}(u_2) &= 0, & D_{E_2}(u_2) &= (i\theta)u_2, \\ D_{E_1}(u_3) &= (i\theta)u_3, & D_{E_2}(u_3) &= (i\theta)u_3 \end{aligned}$$

on the generators and extended in terms of the Leibniz rule, are outer derivations for  $A_{\mathfrak{h}(1)}$ . This action can be written as

$$(3.29) \quad D_{E_a}(f) = [u_{E_a}, f]_\theta$$

with

$$(3.30) \quad \begin{aligned} u_{E_1} &= -(p_1q_1 + p_2q_2), \\ u_{E_2} &= -p_2q_2. \end{aligned}$$

The analogy with the analysis performed in the  $\mathfrak{e}(2)$  case is evident. The action of the outer derivation  $D_{E_a}$  for  $A_{\mathfrak{h}(1)}$  can be represented as a commutator on  $A_{\mathfrak{h}(1)} \subset \mathcal{M}^\theta$  in terms of the<sup>6</sup> quadratic element  $u_{E_a} \in S \subset \mathcal{M}^\theta$ .

It is nonetheless easy to prove that there are no elements  $u_R, u_H \in \mathcal{M}^\theta$  such that the action of the derivations  $D_R, D_H$  on  $A_{\mathfrak{h}(1)}$  associated to the vector fields  $E_R, E_H$ , given by

$$(3.31) \quad \begin{aligned} D_R(u_1) &= -(i\theta)u_2, & D_H(u_1) &= (i\theta)u_2, \\ D_R(u_2) &= (i\theta)u_1, & D_H(u_2) &= (i\theta)u_1, \\ D_R(u_3) &= 0, & D_H(u_3) &= 0 \end{aligned}$$

on the generators, can be written as a commutator with  $u_R, u_H$  as in (3.29).

In order to introduce a derivations based calculus on  $A_{\mathfrak{h}(1)}$  we consider the elements  $\{u_\sigma\}_{\sigma=1,\dots,4} = \{u_1, u_2, u_3, u_4 = -\mu p_1q_1 - \nu p_2q_2\}$ , with  $\mu, \nu \in \mathbb{R}$ . These elements span, with respect to the non commutative product, a family (depending on the parameters  $\mu, \nu$ ) of Lie subalgebras in  $\mathcal{M}^\theta$  which are isomorphic (up to the factor  $(i\theta)$ ) to the four dimensional Lie algebras  $\tilde{\mathfrak{h}}(1)$  which extend  $\mathfrak{h}(1)$  as they are spanned by the elements

$$\tilde{\mathfrak{h}}(1) = \{\mathfrak{h}(1), \mu E_1 + (\nu - \mu)E_2\}.$$

The relevant Lie algebra structure is given by

$$(3.32) \quad \begin{aligned} [u_1, u_2]_\theta &= (i\theta)u_3, \\ [u_1, u_4]_\theta &= -(i\theta)\mu u_1, \\ [u_2, u_4]_\theta &= (i\theta)(\mu - \nu)u_2, \\ [u_3, u_4]_\theta &= -(i\theta)\nu u_3. \end{aligned}$$

In analogy to what we wrote in the previous subsection, we describe the differential calculus on  $A_{\mathfrak{h}(1)}$  based on the Lie algebra  $\tilde{\mathfrak{h}}(1)$  derivations  $D_\sigma$  given by

$$(3.33) \quad D_\sigma(f) = [u_\sigma, f]_\theta.$$

Since  $\tilde{\mathfrak{h}}(1) \subset \mathfrak{isp}(4, \mathbb{R})$ , we consider the elements  $\{\alpha_\rho\}_{\rho=1,\dots,4} \in (C_\wedge(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta), \mathfrak{d})$  whose action is given by

$$(3.34) \quad \alpha_\rho(D_\sigma) = [u_\sigma, u_\rho]_\theta$$

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<sup>6</sup>Notice further that the elements  $u_{E_a}$  are defined up to an arbitrary function of the element  $u_3$ , but this does not affect any of the results given in this section.

and observe that  $\alpha_\rho = du_\rho$ . It is a long but straightforward calculation to see that, if  $\nu \neq 0$ , then the elements

$$\begin{aligned}
\omega_1 &= \frac{1}{2} (u_3 \alpha_2 + (\frac{\mu}{\nu} - 1) u_2 \alpha_3), \\
\omega_2 &= \frac{1}{2} (-u_3 \alpha_1 + \frac{\mu}{\nu} u_1 \alpha_3), \\
\omega_3 &= \frac{1}{2} ((1 - \frac{\mu}{\nu}) u_2 \alpha_1 - \frac{\mu}{\nu} u_1 \alpha_2 + (i\theta)(\frac{\mu}{\nu} - \frac{\mu^2}{\nu^2}) \alpha_3 - \frac{1}{\nu} u_3 \alpha_4), \\
\omega_4 &= \frac{1}{2\nu} u_3 \alpha_3
\end{aligned}
\tag{3.35}$$

verify

$$\omega_\rho(D_\sigma) = (i\theta)u_3\delta_{\rho\sigma}.$$
\tag{3.36}

If we denote still by  $A_{\mathfrak{h}(1)}$  the algebra extended by the localization defined by adding the generator  $u_3^{-1}$  corresponding to the Casimir function, with  $u_3^{-1}u_3 = u_3u_3^{-1} = 1$  and  $u_3u_j = u_ju_3$  for  $j = 1, 2$ , then<sup>7</sup> the elements

$$\varphi_\rho = -\frac{i}{\theta} u_3^{-1} \omega_\rho$$
\tag{3.37}

give a basis for  $C_\lambda^1(\tilde{\mathfrak{h}}(1), A_{\mathfrak{h}(1)})$ . For the corresponding differential calculus one has that  $d : A_{\tilde{\mathfrak{e}}(2)} \rightarrow C_\lambda^1(\tilde{\mathfrak{e}}(2), A_{\tilde{\mathfrak{e}}(2)})$  can be written as

$$df = (D_\rho f)\varphi_\rho = ([u_\rho, f]_\theta)\varphi_\rho.$$
\tag{3.38}

Also for this calculus, as in (2.9), it is  $f * \varphi_\mu = \varphi_\mu * f$  and  $\varphi_\mu \wedge \varphi_\sigma = -\varphi_\sigma \wedge \varphi_\mu$ . The Maurer - Cartan relation comes from the Lie algebra structure (3.32) of  $\tilde{\mathfrak{h}}(1)$ ; as expected, the differential calculus for the Lie algebras  $\tilde{\mathfrak{h}}(1)$  depends on the ratio  $\mu/\nu$ . It is immediate to see that the identity  $\alpha_3 = du_3$  given by (3.34) can be cast as

$$du_3 = (i\theta)\nu u_3 \varphi_4,$$
\tag{3.39}

thus proving that the centre  $Z(A_{\mathfrak{h}(1)})$  of the algebra is not in the kernel of the exterior derivative  $d$  defined in this section.

#### 4. CONCLUSION

At the end of the paper it is clear that the three dimensional (classical) spaces we mention in the title are given as the foliations of the codimension one regular orbits for the action of the Lie algebra  $\mathfrak{g}$  upon  $\mathfrak{g}^* \simeq \mathbb{R}^3$ . For the given noncommutative deformation of such spaces we have described a derivation based differential calculus and exhibited a frame for it. The natural step forward along the path we started is to analyse the spinor structures on such exterior algebras. We aim to develop such analysis in a future work.

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<sup>7</sup>Notice that such an extended algebra  $A_{\mathfrak{h}(1)}$  gives a noncommutative deformation of the classical algebra  $\mathcal{F}(\mathbb{R}^3 \setminus \{x_3 = 0\})$ , or more generally of the functions defined on  $\mathbb{R}^3$  without a plane.

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