

A NOTE ON THE PERMUTATION BEHAVIOUR OF THE POLYNOMIAL $g_{n,q}$

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ABSTRACT. Let $q = 4$ and k a positive integer. In this short note, we present a class of permutation polynomials over $\mathbb{F}_{q^{3k}}$. We also present a generalization.

1. INTRODUCTION

For a prime power q , let \mathbb{F}_q denote the finite field with q elements. Let e be a positive integer. A polynomial $f \in \mathbb{F}_q[x]$ is called a *permutation polynomial* (PP) of \mathbb{F}_q if it induces a permutation of \mathbb{F}_q . Define

$$S_{k,q} = x + x^q + \cdots + x^{q^{k-1}} \in \mathbb{F}_p[x],$$

where k is a positive integer and $p = \text{Char}\mathbb{F}_q$. When q is fixed, we write $S_{k,q} = S_k$. Note that $S_e = \text{Tr}_{q^e/q}$, where $\text{Tr}_{q^e/q}$ is the *trace* function from \mathbb{F}_{q^e} to \mathbb{F}_q .

PPs over $\mathbb{F}_{q^{3k}}$, where q is a power of 2 have been studied in [2, 5, 7]. In [5], Hou confirmed a class of PPs over $\mathbb{F}_{4^{3k}}$ that was conjectured in [2]. In [7], Yuan and Ding generalized Hou's result and obtained several more classes of PPs of the form $L(x) + S_{2k}^a + S_{2k}^b$ over $\mathbb{F}_{q^{3k}}$, where q is a power of 2. In this paper, we present a new class of PPs over $\mathbb{F}_{4^{3k}}$, where $k > 0$ is an even integer.

Table 1 in [1] contains PPs over \mathbb{F}_{4^e} defined by the polynomial $g_{n,q}$ which was first introduced in [3] by Hou. Permutation property of $g_{n,q}$ was studied in [4, 2]. We call a triple of integers $(n, e; q)$ *desirable* if $g_{n,q}$ is a PP of \mathbb{F}_{q^e} . We refer the reader to [4] for more background of the polynomial $g_{n,q}$.

Desirable triples obtained from computer searches were presented in [1, 2, 4]. In [1], Hou and the author of the present paper discovered several new classes of PPs which explained several entries of Table 3 in [2]. In this short note, we explain yet another entry of Table 1 in [1].

The note is organized as follows.

In Section 2, we present a new class of PPs over $\mathbb{F}_{4^{3k}}$. In section 3, we explain an entry of Table 1 in [1] using the results obtained in Section 2. In Section 4, we present a generalization.

2. A CLASS OF PERMUTATION POLYNOMIALS OVER $\mathbb{F}_{q^{3k}}$

We recall two facts:

Fact 2.1. ([6, Theorem 7.7]) *Let $f \in \mathbb{F}_q[x]$. Then f is a PP of \mathbb{F}_q if and only if for all $a \in \mathbb{F}_q^*$, $\sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}_{q/p}(af(x))} = 0$, where $p = \text{Char}\mathbb{F}_q$ and $\zeta_p = e^{2\pi i/p}$.*

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Fact 2.2. ([2, Lemma 6.13]) *Let p be a prime and $f : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ a function. If there exists a $y \in \mathbb{F}_p^n$ such that $f(x+y) - f(x)$ is a nonzero constant for all $x \in \mathbb{F}_p^n$, then $\sum_{x \in \mathbb{F}_p^n} \zeta_p^{f(x)} = 0$.*

The following theorem is the main theorem of this paper.

Theorem 2.3. *Let $q = 4$, $e = 3k$ where $k > 0$ is an even integer. Then*

$$g = S_{k+1}^2 + S_{2k}^{q^k+1}$$

is a PP of \mathbb{F}_{q^e} .

Proof. By Fact 2.1, it suffices to show that

$$\sum_{x \in \mathbb{F}_{q^e}} (-1)^{\text{Tr}_{q^e/2}(ag(x))} = 0$$

for all $0 \neq a \in \mathbb{F}_{q^e}$. We write $\text{Tr} = \text{Tr}_{q^e/2}$.

Case 1. Assume $\text{Tr}_{q^e/q^k}(a) \neq 0$. By Fact 2.2, it suffices to show that there exists $y \in \mathbb{F}_{q^e}$ such that $\text{Tr}(ag(x+y) - ag(x))$ is a nonzero constant for all $x \in \mathbb{F}_{q^e}$.

Let $y \in \mathbb{F}_{q^k}^*$. Then for $x \in \mathbb{F}_{q^e}$, we have

$$\begin{aligned} & \text{Tr}(ag(x+y) - ag(x)) \\ &= \text{Tr}[a(S_{k+1}^2(x+y) + S_{2k}^{q^k+1}(x+y) - S_{k+1}^2(x) - S_{2k}^{q^k+1}(x))] \\ &= \text{Tr}(a S_{k+1}^2(y)) \quad (S_{2k}(y) = 0 \text{ since } y \in \mathbb{F}_{q^k}) \\ &= \text{Tr}(a S_{k-1}^{2q}(y)) \\ &= \text{Tr}(b S_{k-1}(y)) \quad (a = b^{2q}) \\ &= \text{Tr}(b(y + y^q + \dots + y^{q^{k-2}})) \\ &= \text{Tr}(y(b^{q^{3k}} + b^{q^{3k-1}} + \dots + b^{q^{2k+2}})) \\ &= \text{Tr}_{q^k/2}(y(c^{q^{2k+2}} + \dots + c^{q^{3k}})), \quad \text{where } c = \text{Tr}_{q^e/q^k}(b) \neq 0. \end{aligned}$$

Since

$$\begin{aligned} & \gcd(\mathbf{x}^{2k+2} + \mathbf{x}^{2k+3} + \dots + \mathbf{x}^{3k}, \mathbf{x}^k + 1) \\ &= \gcd(1 + \mathbf{x} + \dots + \mathbf{x}^{k-2}, \mathbf{x}^k + 1) \\ &= 1, \end{aligned}$$

$c^{q^{2k+2}} + \dots + c^{q^{3k}} \neq 0$, and hence there exists $y \in \mathbb{F}_{q^k}$ such that $\text{Tr}(ag(x+y) - ag(x)) = \text{Tr}(y(b^{q^{3k}} + b^{q^{3k-1}} + \dots + b^{q^{2k+2}}))$ is a nonzero constant.

Case 2. Assume $\text{Tr}_{q^e/q^k}(a) = 0$. Then $a = b + b^{q^k}$ for some $b \in \mathbb{F}_{q^e}$. Since $a \neq 0$, we have $b \notin \mathbb{F}_{q^k}$. For $x \in \mathbb{F}_{q^e}$, we have

$$\begin{aligned} \text{Tr}(ag(x)) &= \text{Tr}(bg(x) + b^{q^k}g(x)) \\ &= \text{Tr}(bg(x) + bg(x)^{q^{2k}}) \\ &= \text{Tr}(b(g(x) + g(x)^{q^{2k}})). \end{aligned}$$

Note that

$$S_{2k}^{q^k} + S_{2k}^{q^{2k}} \equiv S_{2k} \pmod{\mathbf{x}^{q^e} - \mathbf{x}}$$

and

$$S_{k+1}^2 + S_{k+1}^{2q^{2k}} \equiv x^2 + x^{2q^k} + S_{2k}^2 + S_{2k}^{2q^k} \pmod{\mathbf{x}^{q^e} - \mathbf{x}}$$

Then we have

$$\begin{aligned}
(2.1) \quad g(\mathbf{x}) + g(\mathbf{x})^{q^{2k}} &\equiv S_{k+1}^2 + S_{2k}^{q^{k+1}} + S_{k+1}^{2q^{2k}} + S_{2k}^{q^{2k+1}} \pmod{\mathbf{x}^{q^e} - \mathbf{x}} \\
&= S_{k+1}^2 + S_{k+1}^{2q^{2k}} + S_{2k} \cdot (S_{2k}^{q^k} + S_{2k}^{q^{2k}}) \\
&\equiv x^2 + x^{2q^k} + S_{2k}^2 + S_{2k}^{2q^k} + S_{2k}^2 \pmod{\mathbf{x}^{q^e} - \mathbf{x}} \\
&= (\mathbf{x} + \mathbf{x}^{q^k} + S_{2k}^q)^2 \\
&= (S_{3k} + S_k^q)^2 \\
&\equiv (S_{2k}^{q^{k+1}})^2 \pmod{\mathbf{x}^{q^e} - \mathbf{x}}.
\end{aligned}$$

So for $x \in \mathbb{F}_{q^e}$,

$$\begin{aligned}
\mathrm{Tr}(ag(x)) &= \mathrm{Tr}(bS_{2k}(x)^{2q^{k+1}}) \\
&= \mathrm{Tr}(cS_{2k}(x)) \quad (b = c^{2q^{k+1}}) \\
&= \mathrm{Tr}(c(x + x^q + \cdots + x^{q^{2k-1}})) \\
&= \mathrm{Tr}(x(c^{q^{3k}} + c^{q^{3k-1}} + \cdots + c^{q^{k+1}})).
\end{aligned}$$

Since

$$\begin{aligned}
&\mathrm{gcd}(\mathbf{x}^{k+1} + \mathbf{x}^{k+2} + \cdots + \mathbf{x}^{3k}, \mathbf{x}^{3k} + 1) \\
&= \mathrm{gcd}(1 + \mathbf{x} + \cdots + \mathbf{x}^{2k-1}, \mathbf{x}^{3k} + 1) \\
&= \mathbf{x}^k + 1,
\end{aligned}$$

we see that for $z \in \mathbb{F}_{q^{3k}}$,

$$z^{q^{3k}} + z^{q^{3k-1}} + \cdots + z^{q^{k+1}} = 0 \Leftrightarrow z \in \mathbb{F}_{q^k}.$$

Since $c \notin \mathbb{F}_{q^k}$, we have $c^{q^{3k}} + c^{q^{3k-1}} + \cdots + c^{q^{k+1}} \neq 0$. Therefore

$$\sum_{x \in \mathbb{F}_{q^e}} (-1)^{\mathrm{Tr}(ag(x))} = \sum_{x \in \mathbb{F}_{q^e}} (-1)^{\mathrm{Tr}(x(c^{q^{3k}} + c^{q^{3k-1}} + \cdots + c^{q^{k+1}}))} = 0.$$

□

3. POLYNOMIAL $g_{n,q}$

We first recall two facts:

Fact 3.1. ([4, Eq. 4.1].) *For $m, q \geq 0$, we have*

$$g_{m+q^a, a} = g_{m+1, q} + S_a \cdot g_{m, q}.$$

Fact 3.2. ([2, Theorem 6.1].) *Let $q \geq 4$ be even, and let*

$$n = 1 + q^{a_1} + q^{b_1} + \cdots + q^{a_{q/2}} + q^{b_{q/2}},$$

where $a_i, b_i \geq 0$ are integers. Then

$$g_{n,q} = \sum_i S_{a_i} S_{b_i} + \sum_{i < j} (S_{a_i} + S_{b_i})(S_{a_j} + S_{b_j}).$$

Corollary 3.3. *Let $q = 4$, $e = 6$, and $n = 65921 = (q - 3)q^0 + 2q^3 + q^4 + q^8$. Then*

$$g_{n,q} \equiv S_3^2 + S_4^{q^2+1} \pmod{\mathbf{x}^{q^e} - \mathbf{x}},$$

and $g_{n,q}$ is a PP of \mathbb{F}_{q^e} .

Proof. We write g_n for $g_{n,q}$. Then by Fact 3.2 we have

$$\begin{aligned} g_n &= S_3^2 + S_4 S_8 \\ &\equiv S_3^2 + S_4(S_6 + S_2) \pmod{\mathbf{x}^{q^e} - \mathbf{x}} \\ &= S_3^2 + S_4 S_4^{q^2} \\ &= S_3^2 + S_4^{q^2+1}. \end{aligned}$$

Let $k = 2$ in Theorem 2.3. Then It follows from Theorem 2.3 that g_n is a PP of \mathbb{F}_{q^e} . \square

4. A GENERALIZATION

The following theorem is a generalization of Theorem 2.3.

Theorem 4.1. *Let q be a power of 2. Let $L \in \mathbb{F}_{q^{3k}}[x]$ be a 2-linearized polynomial such that*

- (i) L permutes \mathbb{F}_{q^k} , and
- (ii) $L + L^{q^{2k}} \equiv S_{2k}^2 + S_{2k}^{2q^{k+1}} \pmod{\mathbf{x}^{q^{3k}} - \mathbf{x}}$.

Then $L + S_{2k}^{q^{k+1}}$ is a PP of $\mathbb{F}_{q^{3k}}$.

Proof. Let $g = L + S_{2k}^{q^{k+1}}$. In Case 1 in Theorem 2.3, we choose $y \in \mathbb{F}_{q^k}$ such that $\text{Tr}_{q^k/2}(L(y) \text{Tr}_{q^e/q^k}(a)) \neq 0$. In Case 2, (2.1) still holds because of condition 2. \square

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