SUBVARIETIES WITH PARTIALLY AMPLE NORMAL BUNDLE

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Abstract. We show that local complete intersection subvarieties of smooth projective varieties, which have partially ample normal bundle, possess the G2-property. This generalizes results of Hartshorne and Bădescu-Schneider.

Introduction

Hartshorne [13, 14] investigated the cohomological properties of pairs (X, Y), where X is a projective scheme which is regular in a neighbourhood of a local complete intersection—lci, for short—subscheme Y with ample normal bundle. He showed that, on one hand, Y is G2 that is, the formal completion $\hat{X}_Y$ determines an étale neighbourhood of Y. On the other hand, the cohomology groups of coherent sheaves on the complement $X \backslash Y$ are finite dimensional, above appropriate degrees.

The ampleness of the normal bundle can be weakened. On the complex-analytic side, it suffices either a Hermitian metric with partially positive curvature (cf. [11, 7]). On the algebraic side, Bădescu-Schneider [4] addressed the *globally generated*, partially ample case (in the sense of Sommese) by reducing the problem to [13]. Their results mainly apply—due to the global generation of the normal bundle—to subvarieties of homogeneous varieties. A comprehensive reference for the algebraic approach is Bădescu [3].

Subvarieties with q-ample normal bundle have not been investigated yet. Here we are referring to the cohomological partial ampleness [2, 18]. It is less restrictive than Sommese's [17] and also more flexible, being a numerical condition. There are numerous subvarieties with partially ample, but neither ample nor globally generated normal bundle. Their ubiquity is, in our opinion, a strong motivation to systematically study their properties.

The main result of this article is stated below. It generalizes Hartshorne [13, Theorem 6.7], Bădescu-Schneider [4, Theorem 1], and strengthens as well the formality principle—for Y lci rather than smooth—due to Griffiths, Commichau-Grauert, Chen [11, 7, 6].

Theorem (cf. 2.6, 2.7) *Let X be a smooth irreducible projective variety defined over an algebraically closed field of characteristic zero, and Y a connected, lci subscheme, with $(\dim Y - 1)$-ample normal bundle. Then Y is G2 in X and the formality principle holds for (X, Y).*

We conclude the article with applications. It is worth mentioning that Voisin's strongly movable subvarieties [19] have non-pseudo-effective co-normal bundle, hence they enjoy the G2-property (cf. 3.5).

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1. Background material

Notation 1.1 We work over an algebraically closed field $\Bbbk$ of characteristic zero. Throughout the article, $\mathfrak{X}$ is a connected, noetherian formal scheme, regular and projective over $\Bbbk$; X stands for an irreducible projective variety—that is, reduced and irreducible—over $\Bbbk$.

Let Y be either a subscheme of definition of $\mathfrak{X}$—it is projective—, or a closed subscheme of X; in the latter case, we suppose X is non-singular along Y. Let $\dim Y$ be the maximal dimension of its components—we assume that all are at least 1-dimensional—, $\mathrm{codim}_X(Y) := \dim X - \dim Y$ (if $Y \subset X$). Let $\mathcal{I}_Y \subset \mathcal{O}_{\mathfrak{X}}$ (resp. $\subset \mathcal{O}_X$) be the sheaf of ideals defining Y; for $a \geqslant 0$, Y_a is the subscheme defined by $\mathcal{I}_Y^{a+1}$. The formal completion of X along Y is $\hat{X}_Y := \varinjlim Y_a$; it is regular and projective.

If Y is lci in $\mathfrak{X}$, we denote its normal sheaf by $\mathcal{N} = \mathcal{N}_Y := (\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$; it is locally free of rank ν. The structure sheaves of the various thickenings Y_a fit into the exact sequences:

$$0 \to \mathrm{Sym}^a(\mathcal{N}^\vee) \to \mathcal{O}_{Y_a} \to \mathcal{O}_{Y_{a-1}} \to 0,\ \forall a \geqslant 1. \tag{1.1}$$

For a coherent sheaf $\mathcal{G}$, we denote $h^t(\mathcal{G}) := \dim_{\Bbbk} H^t(\mathcal{G})$; for a field extension $K \hookrightarrow K'$, $\mathrm{trdeg}_K K'$ is the transcendence degree; $\mathrm{ct}^{A,B,\dots}$ stands for a real constant depending on the quantities $A, B, \dots$. A *line (resp. vector) bundle* is an *invertible (resp. locally free) sheaf.*

We recall some terminology due to Hironaka-Matsumura [15]. Suppose Y is connected; let $K(\hat{X}_Y)$ be the field of formal rational functions on X along Y (cf. [15, Lemma 1.4]).

- Y is G1 in X, if $H^0(\hat{X}_Y, \mathcal{O}_{\hat{X}_Y}) = \Bbbk$;
- Y is G2 in X, if $K(X) \hookrightarrow K(\hat{X}_Y)$ is finite;
- Y is G3 in X, if $K(X) \hookrightarrow K(\hat{X}_Y)$ is an isomorphism.

1.1. Cohomological q-ampleness.

This notion was introduced by Arapura and Totaro.

Definition 1.2 Let Y be a projective scheme, $\mathcal{A} \in \mathrm{Pic}(Y)$ an ample line bundle.

(i) (cf. [18, Theorem 7.1]) An invertible sheaf $\mathcal{L}$ on Y is *q-ample* if, for any coherent sheaf $\mathcal{G}$ on X, holds:

$$\exists\ \mathrm{ct}^{\mathcal{G}}\ \ \forall\, a \geqslant \mathrm{ct}^{\mathcal{G}}\ \ \forall\, t > q,\ H^t(Y, \mathcal{G} \otimes \mathcal{L}^a) = 0.$$

It's enough to verify the property for $\mathcal{G} = \mathcal{A}^{-k}, k \geqslant 1$ (cf. [18, Theorem 6.3, 7.1]).

(ii) (cf. [2, Lemma 2.1, 2.3]) A locally free sheaf $\mathcal{E}$ on Y is *q-ample* if $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1)$ on $\mathbb{P}(\mathcal{E}^\vee) := \mathrm{Proj}(\mathrm{Sym}^\bullet_{\mathcal{O}_Y} \mathcal{E})$ is q-ample. It is equivalent saying that, for any coherent sheaf $\mathcal{G}$ on Y, there is $\mathrm{ct}^{\mathcal{G}} > 0$ such that:

$$H^t(Y, \mathcal{G} \otimes \mathrm{Sym}^a(\mathcal{E})) = 0,\ \forall t > q,\ \forall a \geqslant \mathrm{ct}^{\mathcal{G}}.$$

The *q-amplitude of* $\mathcal{E}$, denoted $q^{\mathcal{E}}$, is the smallest integer q with this property. Note that $\mathcal{E}$ is q-ample if and only if so is $\mathcal{E}_{Y_{\mathrm{red}}}$ (cf. [18, Corollary 7.2]). Also, any locally free quotient $\mathcal{F}$ of $\mathcal{E}$ is still q-ample; indeed, $\mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}$.

(iii) For a coherent sheaf $\mathcal{G}$ on Y, let $\mathrm{reg}^{\mathcal{A}}(\mathcal{G})$ be its Castelnuovo-Mumford regularity with respect to $\mathcal{A}$ and $\mathrm{reg}^{\mathcal{A}}_+(\mathcal{G}) := \max\{1, \mathrm{reg}^{\mathcal{A}}(\mathcal{G})\}$.

The q-amplitude enjoys *uniformity* and *sub-additivity* properties.

Theorem 1.3 (i) (cf. [18, Theorem 6.4, 7.1]) *Let Y be a projective scheme, $\mathcal{A}, \mathcal{L} \in \mathrm{Pic}(Y)$. We assume that $\mathcal{A}$ is sufficiently ample—Koszul-ample, cf.* [18, p. 733]*—, and $\mathcal{L}$ is q-ample.*

Then there are $\mathrm{ct}_1^{\mathcal{A},\mathcal{L}}, \mathrm{ct}_2^{\mathcal{A},\mathcal{L}} > 0$, *such that for any coherent sheaf* $\mathcal{G}$ *on* Y *holds:*

$$H^t(Y, \mathcal{G} \otimes \mathcal{L}^a) = 0, \ \ \forall t > q, \, \forall a \geqslant \mathrm{ct}_1^{\mathcal{A},\mathcal{L}} \cdot \mathrm{reg}_+^{\mathcal{A}}(\mathcal{G}) + \mathrm{ct}_2^{\mathcal{A},\mathcal{L}} .$$

(ii) (cf. [18, Theorem 3.4]) *If* $H^0(\mathcal{O}_Y) = \Bbbk$ *then, for a locally free sheaf* $\mathcal{E}$ *and coherent sheaf* $\mathcal{G}$ *on* Y, *one has*

$$\mathrm{reg}^{\mathcal{A}}(\mathcal{E} \otimes \mathcal{G}) \leqslant \mathrm{reg}^{\mathcal{A}}(\mathcal{E}) + \mathrm{reg}^{\mathcal{A}}(\mathcal{G}).$$

Hence it holds: $\mathrm{reg}_+^{\mathcal{A}}(\mathcal{E} \otimes \mathcal{G}) \leqslant \mathrm{reg}_+^{\mathcal{A}}(\mathcal{E}) + \mathrm{reg}_+^{\mathcal{A}}(\mathcal{G})$.

Theorem 1.4 (cf. [2, Theorem 3.1]) *Let* $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ *be an exact sequence of locally free sheaves on* Y. *Then it holds:* $q^{\mathcal{E}} \leqslant q^{\mathcal{E}_1} + q^{\mathcal{E}_2}$.

For products there is a better estimate.

Lemma 1.5 *Let* X_1, X_2 *be irreducible projective varieties and* $\mathcal{E}_1, \mathcal{E}_2$ *locally free sheaves on them, respectively. Let* $\mathcal{E}_1 \boxplus \mathcal{E}_2$ *be the direct sum of their pull-backs to* $X_1 \times X_2$. *Then we have:* $q^{\mathcal{E}_1 \boxplus \mathcal{E}_2} \leqslant \max\{q^{\mathcal{E}_1} + \dim X_2, q^{\mathcal{E}_2} + \dim X_1\}$.

Proof. Let $\mathcal{A}_1, \mathcal{A}_2$ be ample line bundles on X_1, X_2, respectively, $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ the tensor product of their pull-backs. For $k \geqslant 1$, $t > \max\{q^{\mathcal{E}_1} + \dim X_2, q^{\mathcal{E}_2} + \dim X_1\}, a \gg 0$, it holds:

$$\begin{aligned} &H^t\big(X_1 \times X_2, (\mathcal{A}_1^{-k} \boxtimes \mathcal{A}_2^{-k}) \otimes \mathrm{Sym}^a(\mathcal{E}_1 \boxplus \mathcal{E}_2)\big) \\ &= \bigoplus_{\substack{t_1+t_2=t,\\ a_1+a_2=a}} H^{t_1}\big(X_1, \mathcal{A}_1^{-k} \otimes \mathrm{Sym}^{a_1}(\mathcal{E}_1)\big) \otimes H^{t_2}\big(X_2, \mathcal{A}_2^{-k} \otimes \mathrm{Sym}^{a_2}(\mathcal{E}_2)\big) = 0. \end{aligned}$$ □

Lemma 1.6 *One has the equivalence:*

$$\mathcal{L} \in \mathrm{Pic}(Y) \text{ is } q\text{-ample} \ \Leftrightarrow\ \mathcal{L} \otimes \mathcal{O}_{Y'} \text{ is } q\text{-ample}, \ \forall\, Y' \subset Y \text{ irreducible.}$$

Proof. If $Y = Y' \cup Y''$ is the union of distinct closed subschemes, one has:

$$\begin{gathered} 0 \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y''} \to \mathcal{O}_{Y' \cap Y''} \to 0, \\ 0 \to \mathcal{I}_{Y'} \oplus \mathcal{I}_{Y''} \to \mathcal{O}_Y \to \mathcal{O}_{Y' \cap Y''} \to 0. \end{gathered}$$

Now tensor the exact sequences by $\mathcal{L}^m \otimes \mathcal{O}_Y(-k)$ and take their cohomology. □

1.2. **($\dim \boldsymbol{Y}$−1)-ample vector bundles on** Y**.** Subvarieties $Y \subset X$ with $(\dim Y - 1)$-ample normal bundle will play an essential role. The following is analogous to Totaro's result for invertible sheaves.

Proposition 1.7 (cf. [18, Theorem 9.1]) *Let* $\mathcal{E}$ *be a locally free sheaf on an irreducible projective variety* Y *(reduced, irreducible). The statements are equivalent:*

(i) $\mathcal{E}$ *is* $(\dim Y - 1)$-*ample.*

(ii) $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ *is not pseudo-effective, where* $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{E}^\vee)$.
In this case, we say that $\mathcal{E}^\vee$ is not pseudo-effective.

(iii) *There is a dominant morphism* $\varphi : C_S \to Y$, *with* S *affine and* C_S *an integral curve over* S, *such that the following conditions are satisfied:*
  (1) $\varphi^*\mathcal{E}$ *admits a line sub-bundle* $\mathcal{M}$ *which is relatively ample for* $C_S \to S$;
  (2) *Let* $S_y \subset S$ *be the curves passing through the general point* $y \in Y$ *and* $\mathcal{M}_{S_y}$ *the restriction of* $\mathcal{M}$ *to* C_{S_y}.
  Then the points $\{[\mathcal{M}_{s,y}]\}_{s \in S_y}$, *corresponding to* $\mathcal{M}_{s,y}$, *cover an open subset of* $\mathbb{P}(\mathcal{E}_y)$.
  (For shorthand, we say that $\mathcal{M} \subset \varphi^*\mathcal{E}$ *is* movable.*)*

If Y *is reducible, the conditions* (ii), (iii) *must hold for all its irreducible components.*

Proof. The last statement follows from 1.6. Let $\mathcal{O}_Y(1)$ be an ample line bundle on Y. Its dualizing sheaf ω_Y is torsion free of rank one, and $\mathcal{O}_Y(-c) \subset \omega_Y \subset \mathcal{O}_Y(c)$ for some $c > 0$ (cf. [18, §9]); hence the $(\dim Y - 1)$-ampleness means: $H^0(Y, \omega_Y \otimes \mathcal{L} \otimes \mathrm{Sym}^a \mathcal{E}^\vee) = 0$, $\forall \mathcal{L} \in \mathrm{Pic}(Y)$, $a > \mathrm{ct}^{\mathcal{L}}$. It is equivalent to $H^0(Y, \mathcal{M} \otimes \mathrm{Sym}^a \mathcal{E}^\vee) = 0$, $\forall \mathcal{M} \in \mathrm{Pic}(Y)$, $a > \mathrm{ct}^{\mathcal{M}}$, and to:

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{M} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(a)) = 0, \forall \mathcal{M} \in \mathrm{Pic}(\mathbb{P}(\mathcal{E})), \forall a > \mathrm{ct}^{\mathcal{M}}.$$

The last condition is the $(\dim \mathbb{P}(\mathcal{E}) - 1)$-ampleness of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$; (i)⇔(ii) follows.

The equivalence (ii)⇔(iii) is the duality (cf. [5, Theorem 0.2]), for $X = \mathbb{P}(\mathcal{E})$. However, *loc. cit.* requires X to be smooth. Thus we must prove the following.

<u>*Claim*</u> Let $(X, \mathcal{O}_X(1))$ be a d-dimensional projective variety, $\mathcal{L} \in \mathrm{Pic}(X)$. It holds:

$$\mathcal{L} \text{ is } (d-1)\text{-ample} \iff \exists \text{ movable curve } C \to X \text{ such that } \mathcal{L} \cdot C > 0.$$

(⇒) Let $\tilde{X} \xrightarrow{\sigma} X$ be a desingularization of X with exceptional locus E. Then $\sigma^*\mathcal{L}$ is $(d-1)$-ample. Indeed, we may assume that $\tilde{\mathcal{A}} := \big(\sigma^*\mathcal{O}_X(1)\big)(-E)$ is ample on $\tilde{X}$, hence:

$$H^0(\tilde{X}, \sigma^*\mathcal{L}^{-m} \otimes \tilde{\mathcal{A}}^k) \subset H^0(X, \mathcal{L}^{-m} \otimes \mathcal{O}_X(k) \otimes \sigma_*\mathcal{O}_{\tilde{X}}) = 0,\ k > 0,\ m \gg \mathrm{ct}^k.$$

For the last step, $\sigma_*\mathcal{O}_{\tilde{X}}$ is torsion-free of rank one, so $\mathcal{O}_X(-c) \subset \sigma_*\mathcal{O}_{\tilde{X}} \subset \mathcal{O}_X(c)$ for an appropriate $c > 0$. Then $\sigma^*\mathcal{L}^{-1}$ is not pseudo-effective, so there is a movable curve $C \to \tilde{X}$ such that $\sigma^*\mathcal{L} \cdot C > 0$.

(⇐) Let $C \to X$ be a movable curve and suppose $\mathcal{L}$ is not $(d-1)$-ample. There is $k_0 > 0$ and a strictly increasing sequence $\{m_t\}_t \subset \mathbb{Z}$, such that $H^0(X, \mathcal{L}^{-m_t} \otimes \mathcal{A}^{k_0}) \neq 0$. It follows: $0 \leqslant -m_t(\mathcal{L} \cdot C) + k_0\mathcal{O}_X(1) \cdot C$, $\forall\, t$, so $\mathcal{L} \cdot C \leqslant 0$, a contradiction. □

Observe that the notion of pseudo-effective vector bundle used in [5, §7] is more restrictive: it also requires that the projection of the non-nef locus of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ does not cover Y.

2. Finite dimensionality results and the G2 property

Hartshorne [13] investigated the cohomological properties of lci subvarieties with ample normal bundle and of their complements. Bădescu and Schneider [4] extended his results to subvarieties with Sommese-q-ample (globally generated) normal bundle, hence their applications mainly concern homogeneous spaces.

2.1. **Finite dimensionality.**
The following generalizes results in [13, Section 5].

Theorem 2.1 *Assume Y is lci, let $q^{\mathcal{N}}$ be the amplitude of its normal bundle $\mathcal{N}$.*

- (i) *Consider $\mathcal{L} \in \mathrm{Pic}(\mathfrak{X})$, let $q^{\mathcal{L}}$ be the amplitude of its restriction to Y. Let $\mathcal{F}$ be a locally free sheaf on $\mathfrak{X}$, of finite rank. Then the following statements hold:*
  - (a) *For $t < \dim Y - q^{\mathcal{N}}$, $H^t(\mathfrak{X}, \mathcal{F})$ is finite dimensional. In particular, if $q^{\mathcal{N}} \leqslant \dim Y - 1$ and $\mathfrak{X}$ is connected, then $H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \mathbb{k}$.*
  - (b) *$H^t(\mathfrak{X}, \mathcal{F} \otimes \mathcal{L}^{-b}) = 0$, for $t < \dim Y - (q^{\mathcal{N}} + q^{\mathcal{L}})$, $b \gg 0$.*
- (ii) *Let X be a projective scheme, non-singular along Y. Let $\mathcal{G}$ be a coherent sheaf on $X \backslash Y$ and $\mathcal{L} \in \mathrm{Pic}(X)$. The following statements hold:*
  - (a) *$H^t(X \backslash Y, \mathcal{G})$ is finite dimensional, $t \geqslant \dim X - \dim Y + q^{\mathcal{N}}$,*
  - (b) *$H^t(X \backslash Y, \mathcal{G} \otimes \mathcal{L}^b) = 0$, $t \geqslant \dim X - \dim Y + q^{\mathcal{N}} + q^{\mathcal{L}}$, $b \gg 0$.*

Proof. (i)(a) Use (1.1) and proceed as in *loc. cit.*, Theorem 5.1, Corollary 5.4.
(b)(cf. *loc. cit.*, Corollary 5.3) For $\mathcal{F} := \mathcal{F} \otimes \mathcal{O}_Y$, $\mathcal{L} := \mathcal{L} \otimes \mathcal{O}_Y$, is enough to show:

$$H^t(Y, \omega_Y \otimes \mathcal{F}^\vee \otimes \mathrm{Sym}^a(\mathcal{N}) \otimes \mathcal{L}^b) = 0, \quad \forall t > q^{\mathcal{N}} + q^{\mathcal{L}}, \forall a \geqslant 0, b \gg 0.$$

But $\mathrm{Sym}^a(\mathcal{N})\otimes\mathcal{L}^b$ is direct summand in $\mathrm{Sym}^{a+b}(\mathcal{N}\oplus\mathcal{L})$, and $\mathcal{N}\oplus\mathcal{L}$ is $(q^{\mathcal{N}}+q^{\mathcal{L}})$-ample. The vanishing holds for $a+b\geqslant \mathrm{ct}^{\mathcal{F}}$, *e.g.* $a\geqslant 0$, $b\geqslant \mathrm{ct}^{\mathcal{F}}$.
(ii) Use the formal duality [14, Theorem III.3.3] and the previous point. □

In [12, Exposé XIII, Conjecture 1.3], Grothendieck discusses the finite dimensionality of the cohomology groups of coherent sheaves on the complement of lci subvarieties. Hartshorne addressed the issue for smooth subvarieties of projective spaces (cf. [13, Corollary 5.7]).

Let S be a smooth projective variety and E a principal G-bundle on it, with G a connected linear algebraic group; let $P\subset G$ be a parabolic subgroup. Then $X:=E/P\xrightarrow{\pi}S$ is a locally trivial G/P-fibration. The co-ampleness (*ca*, for short) of homogeneous varieties has been explicitly computed by Goldstein [10]. By definition, $q^{\mathcal{T}_{G/P}}=\dim(G/P)-ca(G/P)$, hence $\mathcal{T}_{X,\pi}:=\mathrm{Ker}(\mathrm{d}\pi)$ is q-ample, for $q:=\dim X-ca(G/P)$.

Corollary 2.2 *Suppose $Y\subset X$ is a smooth S-family of subvarieties of relative codimension δ, $\dim Y>\dim S$; that is, $\mathrm{d}\pi_Y:\mathcal{T}_Y\to\pi_Y^*\mathcal{T}_S$ is surjective, $\mathrm{codim}_X(Y)=\delta$. Then $H^t(X\backslash Y,\mathcal{G})$ is finite dimensional for $t\geqslant\delta+\dim X-ca(G/P)$, for all coherent sheaves $\mathcal{G}$ on $X\backslash Y$.*

Hartshorne's result corresponds to $S=\{\text{point}\}$, $G/P\cong\mathbb{P}^n$, $t\geqslant\delta$.

Proof. The exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{T}_{Y,\pi_Y} \arrow[r] \arrow[d] & \mathcal{T}_Y \arrow[r] \arrow[d] & \pi_Y^*\mathcal{T}_S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{T}_{X,\pi}\restriction_Y \arrow[r] \arrow[d] & \mathcal{T}_X\restriction_Y \arrow[r] \arrow[d] & \pi^*\mathcal{T}_S\restriction_Y \arrow[r] & 0 \\
 & \mathcal{T}_{X,\pi}\restriction_Y\big/\mathcal{T}_{Y,\pi_Y} \arrow[r, equal] \arrow[d] & \mathcal{N}_{Y/X} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

shows that $\mathcal{N}_{Y/X}$ is a quotient of $\mathcal{T}_{X,\pi}\restriction_Y$, so is q-ample (cf. 1.2(ii)); apply 2.1(ii). □

2.2. **The G2 property.** Here we generalize [13, Section 6]. The difficulty to overcome is that several statements in there are proved for *curves*, the general case being obtained by induction on the dimension.

Lemma 2.3 (cf. [13, Lemma 6.1]) *Let $(Y,\mathcal{O}_Y(1))$ be a projective scheme, $\mathcal{L}\in\mathrm{Pic}(Y)$ and $\mathcal{E},\mathcal{F}$ locally free sheaves on Y. Let $h_{\mathcal{F}}(a,b):=h^0(Y,\mathcal{F}\otimes\mathrm{Sym}^a(\mathcal{E}^\vee)\otimes\mathcal{L}^{-b})$, $a,b\geqslant 1$.*

(i) *If $\mathcal{L}$ is $(\dim Y-1)$-ample, then it holds:*

$$h_{\mathcal{F}}(a,b)=0,\ \textit{for}\ \ b\geqslant \mathrm{ct}_1^{\mathcal{O}_Y(1),\mathcal{L},\mathcal{E}}\cdot a+\mathrm{ct}_2^{\mathcal{O}_Y(1),\mathcal{L},\mathcal{F}}. \tag{2.1}$$

(ii) *If $\mathcal{E}$ is $(\dim Y-1)$-ample, then it holds:*

$$h_{\mathcal{F}}(a,b)=0,\ \textit{for}\ \ a\geqslant \mathrm{ct}_1^{\mathcal{O}_Y(1),\mathcal{E},\mathcal{L}}\cdot b+\mathrm{ct}_2^{\mathcal{O}_Y(1),\mathcal{E},\mathcal{F}}. \tag{2.2}$$

Proof. We fix $\mathcal{O}_Y(1)$ sufficiently ample (cf. 1.3) and consider the regularity with respect to it. Also, we may assume that Y is irreducible; let ω_Y be its dualizing sheaf.

(i) There is $c_0 = c_0(Y) \geqslant 1$ such that $\mathcal{O}_Y(-c_0) \subset \omega_Y$, so it holds:

$$\begin{aligned} h^0(Y, \mathcal{F} \otimes \mathrm{Sym}^a(\mathcal{E}^\vee) \otimes \mathcal{L}^{-b}) &\leqslant h^0(Y, \omega_Y \otimes \mathcal{F}(c_0) \otimes \mathrm{Sym}^a(\mathcal{E}^\vee) \otimes \mathcal{L}^{-b}) \\ &= h^{\dim Y}(Y, \mathcal{F}^\vee(-c_0) \otimes \mathrm{Sym}^a(\mathcal{E}) \otimes \mathcal{L}^b). \end{aligned}$$

<u>*Claim*</u> The right hand side vanishes for b as in (2.1). Indeed, we replace $\mathcal{F} \rightsquigarrow \mathcal{F}(-c_0)$ and verify the statement for $h^{\dim Y}(\mathcal{F}^\vee \otimes \mathrm{Sym}^a(\mathcal{E}) \otimes \mathcal{L}^b)$. The effect of the replacement is $\operatorname{reg} \mathcal{F}^\vee \rightsquigarrow \operatorname{reg} \mathcal{F}^\vee - c_0$, with c_0 depending on Y. Now observe that is enough to prove the claim for Y reduced—so $H^0(\mathcal{O}_Y) = \Bbbk$—and for coherent sheaves $\mathcal{G}$ on Y.

Indeed, for $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{Y_{\mathrm{red}}})$, there is $r > 0$ such that $\mathcal{I}^r = 0$, so $\mathcal{O}_Y$ admits a filtration (similar to (1.1)) by the quotients $\mathcal{I}^{k-1}/\mathcal{I}^k$, $1 \leqslant k \leqslant r$, which are $\mathcal{O}_{Y_{\mathrm{red}}}$-modules; now we may use the estimates for $\mathcal{F}^\vee \otimes (\mathcal{I}^{k-1}/\mathcal{I}^k)$ on Y_{red}, which is coherent. Property 1.3 yields:

$$H^{\dim Y}(Y, \mathcal{G} \otimes \mathrm{Sym}^a \mathcal{E} \otimes \mathcal{L}^b) = 0,\ \forall b \geqslant \mathrm{ct}_1^{\mathcal{O}_Y(1),\mathcal{L}} \cdot \mathrm{reg}_+(\mathcal{G} \otimes \mathrm{Sym}^a \mathcal{E}) + \mathrm{ct}_2^{\mathcal{O}_Y(1),\mathcal{L}}.$$

But $\mathrm{Sym}^a \mathcal{E}$ is a summand of $\mathcal{E}^{\otimes a}$, so $\mathrm{reg}_+(\mathcal{G} \otimes \mathrm{Sym}^a \mathcal{E}) \leqslant a \cdot \mathrm{reg}_+(\mathcal{E}) + \mathrm{reg}_+(\mathcal{G})$, thus (2.1) holds for $b \geqslant \mathrm{ct}_1^{\mathcal{O}_Y(1),\mathcal{L}} \cdot (a \cdot \mathrm{reg}_+(\mathcal{E}) + \mathrm{reg}_+(\mathcal{G})) + \mathrm{ct}_2^{\mathcal{O}_Y(1),\mathcal{L}}$.

(ii) We may assume that Y is reduced. If $\mathcal{G}$ is coherent on Y, $h^{\dim Y}(\mathcal{G} \otimes \mathrm{Sym}^a(\mathcal{E}) \otimes \mathcal{L}^b)$ vanishes for $a \geqslant \mathrm{ct}_1^{\mathcal{O}_Y(1),\mathcal{E}} \cdot \mathrm{reg}_+(\mathcal{G} \otimes \mathcal{L}^b) + \mathrm{ct}_2^{\mathcal{O}_Y(1),\mathcal{E}}$, and $\mathrm{reg}_+(\mathcal{G} \otimes \mathcal{L}^b) \leqslant b \,\mathrm{reg}_+ \mathcal{L} + \mathrm{reg}_+ \mathcal{G}$. □

Proposition 2.4 (cf. [13, Theorem 6.2, Corollary 6.6]) *Let the situation be as in 1.1. Suppose Y is lci and its normal bundle $\mathcal{N}$ is* $(\dim Y - 1)$*-ample, of rank ν. For any locally free sheaf $\mathcal{F}$ and invertible sheaf $\mathcal{L}$ on $\mathfrak{X}$, there is a polynomial of degree* $\dim Y + \nu$ *such that:*

$$h^0(\mathfrak{X}, \mathcal{F} \otimes \mathcal{L}^b) \leqslant P^{Y,\mathcal{L},\mathcal{F}}_{\dim Y + \nu}(b), \quad \textit{for } b \gg 0.$$

Proof. Let $\mathcal{A} \in \mathrm{Pic}(Y)$ be sufficiently (Koszul) ample, such that $\mathcal{A}^{-1} \subset \omega_Y$; denote $\mathcal{F} := \mathcal{F} \otimes \mathcal{O}_Y, \mathcal{L} := \mathcal{L} \otimes \mathcal{O}_Y$. For $\gamma := \mathrm{ct}_1^{\mathcal{A},\mathcal{N},\mathcal{L}} + 1, b > \mathrm{ct}_2^{\mathcal{A},\mathcal{N},\mathcal{F}}$ (cf. (2.2)), it holds:

$$h^0(\mathfrak{X}, \mathcal{F} \otimes \mathcal{L}^b) \leqslant \sum_{a=0}^{\infty} h^0(Y, \mathcal{F} \otimes \mathrm{Sym}^a(\mathcal{N}^\vee) \otimes \mathcal{L}^b) = \sum_{a=0}^{\gamma b} h^0(Y, \mathcal{F} \otimes \mathrm{Sym}^a(\mathcal{N}^\vee) \otimes \mathcal{L}^b).$$

Since $\mathcal{F} \subset (\mathcal{A}^{c_0})^{\oplus \mathrm{rk} \mathcal{F}}, c_0 = \mathrm{reg}_+^{\mathcal{A}} \mathcal{F}^\vee$, it is enough to consider $\mathcal{F} = \mathcal{A}^{c_0}$.

Consider $S := \mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(-1) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)})$ and $\mathcal{O}_S(1)$ the relatively ample invertible sheaf on it. The right hand side above can be re-written:

$$\begin{aligned} \text{rhs} &= \sum_{a=0}^{\gamma b} h^0(Y, \mathcal{A}^{c_0} \otimes \mathrm{Sym}^a(\mathcal{N}^\vee) \otimes \mathcal{L}^b) \leqslant \sum_{a=0}^{\gamma b} h^0(Y, \omega_Y \otimes \mathcal{A}^{c_0+1} \otimes \mathrm{Sym}^a(\mathcal{N}^\vee) \otimes \mathcal{L}^b) \\ &= \sum_{a=0}^{\gamma b} h^{\dim Y}(Y, \mathcal{A}^{-c_0-1} \otimes \mathrm{Sym}^a(\mathcal{N}) \otimes \mathcal{L}^{-b}) \\ &= \sum_{a=0}^{\gamma b} h^{\dim Y}(\mathbb{P}(\mathcal{N}^\vee), \mathcal{A}^{-c_0-1} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(a) \otimes \mathcal{L}^{-b}) = h^{\dim Y}(S, \mathcal{A}^{-c_0-1} \otimes \mathcal{O}_S(\gamma b) \otimes \mathcal{L}^{-b}). \end{aligned}$$

But $h^{\dim Y}(S, \mathcal{O}_S(\gamma b) \otimes \mathcal{L}^{-b})$ is dominated by a polynomial in b, depending on $\mathcal{O}_S(\gamma) \otimes \mathcal{L}^{-1}$, of degree at most $\dim S = \dim Y + \nu$ (cf. [16, 1.2.33]). To include $\mathcal{A}^{-c_0-1}$, use

$$0 \to \mathcal{A}^{-c_0-1} \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \to 0,\ \ \dim Y_1 = \dim Y - 1,$$

which yields: $\text{rhs} \leqslant h^{\dim X}(\mathcal{O}_S(\gamma b) \otimes \mathcal{L}^{-b}) + h^{\dim X - 1}(\mathcal{O}_S(\gamma b) \otimes \mathcal{L}^{-b}\restriction_{Y_1})$. □

With these preparations, the proof of the following theorem is identical to *loc. cit.*

Theorem 2.5 (cf. [13, Theorem 6.7]) *Let the situation be as in 1.1. We assume:*
- *Y is connected, lci,* $\dim Y \geqslant 1$*;*
- *the normal bundle* $\mathcal{N}$ *of* Y *is* $(\dim Y - 1)$*-ample.*

Then the following statements hold:
(i) $\mathrm{trdeg}_{\Bbbk} K(\mathfrak{X}) \leqslant \dim Y + \mathrm{rk}\mathcal{N}$*;*
(ii) *If* $\mathrm{trdeg}_{\Bbbk} K(\mathfrak{X}) = \dim Y + \mathrm{rk}\mathcal{N}$*, then* $K(\mathfrak{X})$ *is a finitely generated extension of* $\Bbbk$*.*

Corollary 2.6 (cf. [13, Corollary 6.8]) *Let X be a projective scheme, non-singular in a neighbourhood of a closed, connected, lci subscheme Y with* $(\dim Y - 1)$*-ample normal bundle. Then Y is G2 in X.*

Proof. Indeed, $K(X)$ is a subfield of $K(\hat{X}_Y)$, so $\mathrm{trdeg}_{\Bbbk} K(\hat{X}_Y) \geqslant \dim X = \dim Y + \nu$. Hence we are in the case (ii) of the previous theorem. □

The result is optimal, one can not conclude that Y is G3 (cf. [14, Example p. 199].

2.3. **A formality criterion.** One says that the *formal principle* holds for a pair (X, Y) consisting of a scheme X and a closed subscheme Y if the following condition is satisfied: for any other pair (Z, Y) such that $\hat{Z}_Y \cong \hat{X}_Y$, extending the identity of Y, there is an isomorphism between étale neighbourhoods of Y in X and in Z which induces the identity on Y.

Theorem 2.7 *In the situation 2.6, the formal principle holds for* (X, Y)*.*

This simplifies and strengthens [6, Theorem 3], since Y is only lci, rather than smooth.

Proof. Corollary 2.6 implies that Y is G2 in X. But, in this case, Gieseker proved (cf. [9, Theorem 4.2], [3, Corollary 9.20, 10.6]) that the formality holds for (X, Y). □

There are similar results in complex analytic setting. Griffiths [11] investigated the formality/rigidity of smooth subvarieties $Y \subset X$ whose normal bundle $\mathcal{N}_{Y/X}$ admits a Hermitian metric with curvature of signature (s, t), $s + t = \dim Y$, and proves in [*ibid.*, II. §2, 3] the rigidity of the embedding for $s \geqslant 2$. The main cohomological property of vector bundles admitting metrics of curvature with mixed signature (s, t) is that of being $(\dim Y - s)$-ample (cf. [1, Proposition 28, p. 257], [11, (7.28), p. 432]).

On the other hand, Commichau-Grauert [7, Satz 4] proved the formality for subvarieties with 1-positive normal bundle. Note that a 1-positive vector bundle on a smooth projective variety Y is $(\dim Y - 1)$-ample (cf. [7, Satz 2]).

We conclude that the cohomological approach adopted in this article yields under weaker assumptions the rigidity results obtained in [11, 7].

3. Examples of subvarieties with partially ample normal bundle

In this section we assume that X is a smooth projective variety.

3.1. **Elementary operations.**

Corollary 3.1 (i) *Let $Y_2 \subset Y_1 \subset X$ be connected lci,* $\dim Y_2 \geqslant 1$*. Suppose* $\mathcal{N}_{Y_2/Y_1}, \mathcal{N}_{Y_1/X}$ *are respectively q_2-, q_1-ample, with $q_1 + q_2 < \dim Y_2$. Then Y_2 is G2 in X.*
(ii) *Suppose Y_1, Y_2 are lci in X,* $\mathrm{codim}(Y_1 \cap Y_2) = \mathrm{codim}(Y_1) + \mathrm{codim}(Y_2)$*, and* $\mathcal{N}_{Y_j/X}$ *is q_j-ample, for $j = 1, 2$. Then* $\mathcal{N}_{Y_1 \cap Y_2/X}$ *is* $(q_1 + q_2)$*-ample.*
(iii) *Suppose $Y_j \subset X_j$ are connected lci and* $\mathcal{N}^{\vee}_{Y_j/X_j}$ *is not pseudo-effective (so $Y_j \subset X_j$ is G2), for $j = 1, 2$. Then $Y_1 \times Y_2$ is lci and G2 in $X_1 \times X_2$.*

(iv) *Let $f : X' \to X$ be a surjective, flat morphism. Suppose $Y \subset X$ is lci and $\mathcal{N}_{Y/X}$ is $(\dim Y - 1)$-ample. Then $Y' := f^{-1}(Y) \subset X$ is lci and $\mathcal{N}_{Y'/X'}$ is $(\dim Y' - 1)$-ample.*

Proof. (i)-(iii) are consequences of the sub-additivity 1.4 and 1.5, applied to appropriate normal bundle sequences. For (iv), note that f is equidimensional, $\mathcal{N}_{Y'/X'} = f^*\mathcal{N}_{Y/X}$. Apply Leray's spectral sequence to $Y' \to Y$. □

Corollary 3.2 *Suppose Y_1, Y_2 are lci in X, $\operatorname{codim}(Y_1 \cap Y_2) = \operatorname{codim}(Y_1) + \operatorname{codim}(Y_2)$, and $\mathcal{N}_{Y_j/X}$ is q_j-ample, for $j = 1, 2$. If $Y_1 \cap Y_2$ is connected and $q_2 < \dim(Y_1 \cap Y_2)$, e.g. $q_2 = 0$, then $Y_1 \cap Y_2$ is G2 in Y_1.*

Proof. Note that $\mathcal{N}_{Y_1\cap Y_2/Y_1} \cong \mathcal{N}_{Y_2/X}\restriction_{Y_1\cap Y_2}$. □

3.2. **Strongly movable subvarieties.** (cf. [19, Section 2]) A class of examples of subvarieties having the G2-property are the strongly movable subvarieties introduced by Voisin [19, Section 2], in the attempt to geometrically characterize big subvarieties.

Notation 3.3 Let $\mathcal{Y} \overset{(\pi,\rho)}{\subset} S \times X$ be a flat family of lci subschemes of X, with ρ dominant; then $\rho(\mathcal{Y})$ contains an open subset O of X. We may (and do) assume that $S, \mathcal{Y}$ are reduced, since so is X. The incidence variety Σ is the component of $(\pi_1, \pi_2)(\mathcal{Y} \times_X \mathcal{Y}) \subset S \times S$ containing the diagonal; π is a proper, so Σ is closed. One obtains the Cartesian diagram:

$$\begin{tikzcd}
\mathcal{Y}_{\Sigma_o} \arrow[r] \arrow[d, "\pi"] & \mathcal{Y}_{\Sigma} \arrow[r] \arrow[d] \arrow[rr, bend left, "\rho_\Sigma"] & \mathcal{Y} \times \mathcal{Y} \arrow[r, "(\pi_1,\rho_2)"'] \arrow[d, "(\pi_1,\pi_2)"'] & S \times X. \\
\Sigma_o \arrow[r] & \Sigma \arrow[r, "\iota"] & S \times S &
\end{tikzcd} \tag{3.1}$$

For $o \in S$, denote $\Sigma_o := \iota^{-1}(\{o\} \times S)$ and $\rho_o := \rho_\Sigma\restriction_{\Sigma_o}$.

Definition 3.4 Suppose the general member of $\mathcal{Y}$ is irreducible. We say that the family $\mathcal{Y}$ is *strongly movable*, if ρ_Σ is dominant; then $\mathcal{Y}_{\Sigma_o} \xrightarrow{\rho_o} X$ is dominant, for $o \in S$ general, and Y_o is strongly movable. An *arbitrary* family $\mathcal{Y}$ is *strongly movable* if so is its general member Y_o; that is, *all* the irreducible components of Y_o are strongly movable.

Proposition 3.5 *Let $\mathcal{Y}$ be as above, $o \in S$ a non-singular point such that Y_o is strongly movable. Then $\mathcal{N}_{Y_o/X}$ is $(\dim Y_o - 1)$-ample. Hence, if Y_o is connected, it is G2 in X.*

Proof. Let $\mathcal{T}_{S,o} \xrightarrow{\delta} H^0(Y_o, \mathcal{N}_{Y_o/X})$ be the infinitesimal deformation homomorphism. By 1.7, it is enough to prove that the restriction of $\mathcal{N}_{Y_o/X}$ to the irreducible components of Y_o are $(\dim Y_o - 1)$-ample. Recall that $\mathcal{N}_{Y_o/X}$ is $(\dim Y_o - 1)$-ample if and only if so is its restriction to $Y_{o,\mathrm{red}}$. For $\xi \in \mathcal{T}_{S,o}$, we denote $\hat{v}_\xi \in H^0(Y_{o,\mathrm{red}}, \mathcal{N}_{Y_o/X}\restriction_{Y_{o,\mathrm{red}}})$ the restriction of $\delta(\xi)$ to $Y_{o,\mathrm{red}}$. Henceforth, we replace Y_o by an irreducible component.

We must find a movable morphism $C \xrightarrow{\varphi} Y_{o,\mathrm{red}}$, an ample line bundle $\mathcal{L}_C \in \operatorname{Pic}(C)$, and a movable homomorphism $\mathcal{L}_C \to \varphi^*\mathcal{N}_{Y_o/X}$. We restrict ourselves to $\xi \in \mathcal{T}_{\Sigma_o,o} \subset \mathcal{T}_{S,o}$.
Claim 1 The vanishing locus of $\hat{v}_\xi$ is a non-empty, proper subset of $Y_{o,\mathrm{red}}$; for $\xi \in \mathcal{T}_{\Sigma_o,o}$ variable, the vanishing loci of $\hat{v}_\xi$ cover an open subset of $Y_{o,\mathrm{red}}$.

The vector ξ is determined by an arc $\operatorname{Spec}(\Bbbk[\![\epsilon]\!]) \xrightarrow{h} \Sigma_o$ through o. The defining property of Σ_o implies that $h(\epsilon) = y_\epsilon \in Y_o \cap Y_{h(\epsilon)}$, $h(0) = y \in Y_o$. Since Y_o is deformed at y in a tangential direction, we deduce $\delta(\xi)_y = 0$, so $\hat{v}_{\xi,y} = 0$.

We claim that $\hat{v}_\xi \neq 0$, for generic ξ, and their vanishing loci contain an open subset of $Y_{o,\mathrm{red}}$. Indeed, $\mathcal{Y}_{\Sigma_o} \xrightarrow{\rho_o} X$ is dominant, so $\mathcal{T}_{\Sigma_o,o} \xrightarrow{\mathrm{d}\rho_{o,y}} \mathcal{N}_{Y_{o,\mathrm{red}}/X,y}$ is surjective at a generic (smooth) point $y \in Y_{o,\mathrm{red}}$. Since Y_o is lci, a computation in local coordinates shows that there is a non-trivial homomorphism $\mathcal{N}_{Y_o/X,y} \to \mathrm{Sym}^k(\mathcal{N}_{Y_{o,\mathrm{red}}/X,y})$, for some $k > 0$ (*e.g.* $k = 1$, if Y_o is reduced at y), hence $\hat{v}_{\xi,y} \neq 0$. Second, Y_o is strongly movable, so the points $y \in Y_o$ where $\delta(\xi)_y = 0$, for some $\xi \in \mathcal{T}_{\Sigma_o,o}$, cover an open subset of Y_o.
Claim 2 Let $C \subset Y_o$ be a complete intersection curve which intersects the zero locus of $\hat{v}_\xi$ properly. By Claim 1, such curves are movable. Moreover, $\hat{v}_\xi$ extends to a pointwise injective homomorphism $\mathcal{L}_C \subset \mathcal{N}_{Y_o/X}\restriction_C$, where $\mathcal{L}_C$ is an ample line bundle. The latter is movable too, because $\mathrm{d}\rho_{o,y}$ is surjective at the generic point $y \in Y_o$. This is formalized as follows.

Let $\mathcal{C}_R \xrightarrow{f} X$ be a movable curve (R is a parameter variety). Consider the diagram

$$\begin{tikzcd}
g^*\operatorname{pr}^*\mathcal{N} \arrow[d] & \operatorname{pr}^*\mathcal{N} \arrow[d] & \mathcal{N} := \mathcal{N}_{Y_o/X}\restriction_{Y_{o,\mathrm{red}}} \arrow[d] \\
\mathcal{T}_{\Sigma_o,o} \times \mathcal{C}_R \arrow[u, bend left, "g^*v"] \arrow[r, "g=(\delta,f)"] & H^0(Y_{o,\mathrm{red}}, \mathcal{N}) \times Y_{o,\mathrm{red}} \arrow[u, bend left, "v"] \arrow[r, "\operatorname{pr}"] & Y_{o,\mathrm{red}}
\end{tikzcd}$$

where v is the evaluation map. We may suppose that $\mathcal{C}_R$ is such that its generic member intersects non-trivially and properly the zero locus of v. Note that g^*v yields a rational map $\mathcal{T}_{\Sigma_o,o} \times \mathcal{C}_R \dashrightarrow g^*\operatorname{pr}^*\mathbb{P}(\mathcal{N})$ which extends to a morphism outside a closed subscheme Z of codimension at least two. Its projection does not cover $\mathcal{T}_{\Sigma_o,o} \times R$, hence we obtain a movable, relatively ample $\mathcal{L} \subset g^*\operatorname{pr}^*\mathcal{N}$. $\square$

3.3. **Varieties whose cotangent bundle is not pseudo-effective.** Let $Y \subset X$ be a smooth subvariety, so $\mathcal{N}_{Y/X}$ is a quotient of $\mathcal{T}_X\restriction_Y$.

Notation 3.6 For shorthand, denote $\mathbb{P} := \mathbb{P}(\mathcal{T}_X)$ and $\mathbb{P}_Y := \mathbb{P}(\mathcal{T}_X\restriction_Y)$ its restriction to Y; let $\pi : \mathbb{P} \to X$ be the projection. Define $\mathrm{Mov}(\mathbb{P}_Y)_\mathbb{Q} \subset H_2(\mathbb{P}_Y;\mathbb{Q})$ to be the cone generated by the classes of movable curves on $\mathbb{P}_Y$ and $\mathrm{Mov}(\mathbb{P})_\mathbb{Q}$ similarly.

Corollary 3.7 *Let $Y \subset X$ be a smooth subvariety such that $\mathcal{O}_{\mathbb{P}_Y}(1)$ is not pseudo-effective. Then $\mathcal{N}_{Y/X}$ is $(dimY - 1)$-ample, so Y is G2.*

Proof. Theorem 2.5 applies, since $\mathcal{N}_{Y/X}$ is $(\dim Y - 1)$-ample. $\square$

By using 1.7 we are going to show that, for $Y \subset X$ sufficiently general, the partial ampleness of $\mathcal{T}_X\restriction_Y$ implies the non-pseudo-effectiveness of the cotangent bundle of X. The latter is a numerical condition/restriction on the ambient variety. Examples include rationally connected varieties—see below—and, possibly, Calabi-Yau varieties (cf. [8, Corollary 6.12]).

Lemma 3.8 *Let $Y \xrightarrow{\iota} X$ be a subvariety, $\dim Y > 0$, and $H_2(\mathbb{P}_Y;\mathbb{Q}) \xrightarrow{\iota_*} H_2(\mathbb{P};\mathbb{Q})$ be the induced homomorphism. In the situations enumerated below, it holds:*

$$\iota_*\big(\mathrm{Mov}(\mathbb{P}_Y)\big)_\mathbb{Q} \subseteq \mathrm{Mov}(\mathbb{P})_\mathbb{Q}. \tag{3.2}$$

(i) *An algebraic group G acts on X with an open orbit O, such that the stabilizer of a point $x \in O$ acts with open orbit on $\mathcal{T}_{X,x}$, and $Y \cap O \neq \varnothing$.*
(ii) *$\Bbbk$ is uncountable, and Y is a very general member of a dominant family.*

Hence, if $\mathcal{O}_{\mathbb{P}_Y}(1)$ is not pseudo-effective, then $\mathcal{O}_\mathbb{P}(1)$ is the same.

Proof. (i) The G-translates of a movable curve on $\mathbb{P}_Y$ cover an open subset of $\mathbb{P}(\mathcal{T}_X)$.
(ii) Let $Y_S \subset S \times X$ be an S-flat family of subvarieties, with S affine, dominating X. Curves on X are parametrized by their Hilbert polynomial, of degree one, with integer coefficients. Let Π be the countable set of polynomials occurring for movable curves on $Y_s, s \in S$.

For $P \in \Pi$, denote $\mathrm{Hilb}^P_{Y_S/S} \xrightarrow{\pi_P} S$ the corresponding scheme. We are interested in the components corresponding to curves. For $s \in S$, let $\Pi_s \subset \Pi$ be the set of polynomials P_s such that π_{P_s} is not dominant; $\Pi_{\mathrm{rigid}} := \bigcup_{s\in S} \Pi_s$. The image of $\mathrm{Hilb}^{\Pi_{\mathrm{rigid}}}_{Y_S/S} \to S$ is a countable union of proper subvarieties. Take $s' \in S$ in the complement ($\Bbbk$ is uncountable); let $P_{s'}$ be the Hilbert polynomial of some movable curve $C_{s'} \subset Y_{s'}$. Then $P_{s'} \notin \Pi_{\mathrm{rigid}}$, so $\mathrm{Hilb}^{P_{s'}}_{Y_S/S} \xrightarrow{\pi_{P_{s'}}} S$ is surjective. Let $\Pi' := \Pi \backslash \Pi_{\mathrm{rigid}}$. The components of $\mathrm{Hilb}^{\Pi'}_{Y_S/S}$ (corresponding to movable curves) dominate S, so they are flat over the very general point $o \in S$.

We claim that movable curves on Y_o are movable on X. Indeed, for P_o as above, consider the universal curve $\mathcal{C}_S \subset \mathrm{Hilb}^{P_o}_{Y_S/S} \times_S Y_S$. The family $\mathcal{C}_o \subset \mathrm{Hilb}^{P_o}_{Y_o} \times Y_o$ dominates Y_o. By the continuity of $\mathrm{Hilb}^{P_o}_{Y_S/S} \times_S Y_S \to Y_S$, the same holds for $\mathcal{C}_s \subset \mathrm{Hilb}^{P_o}_{Y_s} \times Y_s$, for s near $o \in S$. Finally, $Y_S \to X$ is dominant, so $\mathcal{C}_S$ covers an open subset of X. □

Lemma 3.9 *Let X be a smooth rationally connected variety. Then $\mathcal{O}_{\mathbb{P}(\mathcal{T}_X)}(1)$ is not pseudo-effective, so $\mathcal{T}_X$ is $(\dim X - 1)$-ample.*

Proof. Consider a very free rational curve: a dominant morphism $\mathbb{P}^1 \times S \xrightarrow{\varphi} X$, where S is a variety, such that $\varphi^*\mathcal{T}_X\restriction_{\mathbb{P}^1\times\{s\}} \cong \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{G}$, $\forall s \in S$, with $\mathcal{G}$ globally generated. A nowhere vanishing section $g_s \in H^0(\mathbb{P}^1 \times \{s\}, \mathcal{G})$ yields the inclusion $j_{g_s} : \mathcal{O}_{\mathbb{P}^1}(1) \to \varphi^*\mathcal{T}_X\restriction_{\mathbb{P}^1\times\{s\}}$; we still denote by j_{g_s} the morphism $\mathbb{P}^1 \to \mathbb{P}(\mathcal{T}_X)$. Since $\mathcal{G}$ is globally generated, there is $\tilde{S} \subset S \times H^0(\mathcal{G})$ open, such that $\tilde{S} \to \mathrm{Morphisms}(\mathbb{P}^1, \mathbb{P}(\mathcal{T}_X))$, $\{s\} \times \{g_s\} \mapsto j_{g_s}$, yields a movable rational curve $\mathbb{P}^1 \times \tilde{S} \xrightarrow{\tilde{\varphi}} \mathbb{P}(\mathcal{T}_X)$ satisfying $\tilde{\varphi}^*\mathcal{O}_{\mathbb{P}(\mathcal{T}_X)}(-1)\restriction_{\mathbb{P}^1\times\{\tilde{s}\}} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, $\forall \tilde{s} \in \tilde{S}$. □

It is well-known—particularly for projective spaces—that the G3 property of the diagonal $\Delta_X := \{(x, x) \mid x \in X\} \subset X \times X$ imply important connectedness results for the intersections of subvarieties in X (cf. [3, Ch. 11]). Our results yield the G2-property of the diagonal; obviously, it is less than the G3-property, but it holds for a larger class of varieties.

Proposition 3.10 *Let X be a smooth projective variety, whose cotangent bundle is not pseudo-effective (e.g. rationally connected). Then the diagonal Δ_X is G2 in $X \times X$.*

Proof. The normal bundle of Δ_X is isomorphic to $\mathcal{T}_X$; we conclude by 2.6. □

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