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EXTENSIONS OF RAMANUJAN'S RECIPROCITY THEOREM AND THE ANDREWS–ASKEY INTEGRAL

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Abstract. Ramanujan's reciprocity theorem may be considered as a three-variable extension of Jacobi's triple product identity. Using the method of q-partial differential equations, we extend Ramanujan's reciprocity theorem to a seven-variable reciprocity formula. The Andrews–Askey integral is a q-integral having four parameters with base q. Using the same method we extend the Andrews–Askey integral formula to a q-integral formula which has seven parameters with base q.

1. Introduction

In this paper we assume, unless otherwise stated, that $|q| < 1$ and use the standard product notation

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

If n is an integer or ∞, the multiple q-shifted factorials are defined as

$$(a_1, a_2, ..., a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n.$$

The celebrated Jacobi triple product identity is stated in the following proposition (see, for example [9, p. 1] and [12, p. 15]).

Proposition 1.1. *For $x \neq 0$, we have the triple product identity*

$$(q, x, q/x; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n.$$

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This identity is among the most important identity in mathematics, which has many interesting applications in number theory, combinatorics, analysis, algebra and mathematical physics. Some amazing extensions of this identity have been made by various authors. Ramanujan's ${}_1\psi_1$ summation formula and Bailey ${}_6\psi_6$ summation formula both contain this identity as a special case, and may be considered as two important extensions of this identity, and these two extensions have wider applications than Jacobi's triple product identity.

Ramanujan's reciprocity theorem and the Andrews–Askey integral formula may also be regarded as two extensions of Jacobi's triple product identity.

In this paper we will use the method of q-partial differential equations to extend Ramanujan's reciprocity theorem and Andrews–Askey integral formula to two more general q-formulae.

For simplicity, in this paper we use $\Delta(u, v)$ to denote the theta function

$$v(q, u/v, qv/u; q)_\infty. \tag{1.1}$$

The q-binomial coefficients are the q-analogs of the binomial coefficients, which are defined by

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

As usual, the basic hypergeometric series or q-hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_r \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n}{(q, b_1, b_2, ..., b_s; q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} z^n.$$

Now we introduce the definition of the Thomae–Jackson q-integral in q-calculus, which was introduced by Thomae [26] and Jackson [15].

Definition 1.2. *Given a function $f(x)$, the Thomae–Jackson q-integral of $f(x)$ on $[a, b]$ is defined by*

$$\int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{\infty}[bf(bq^n) - af(aq^n)]q^n.$$

If the function $f(x)$ is continuous on $[a, b]$, then, one can deduce that

$$\lim_{q\to 1}\int_a^b f(x)d_qx = \int_a^b f(x)dx.$$

In 1981, Andrews and Askey [4] established the following interesting q-beta integral formula using Ramanujan ${}_1\psi_1$ summation, which has four parameters $a, b, u.v$ with base q, which is now known as the Andrews–Askey integral.

Proposition 1.3. *If* $\max\{|au|, |bu|, |av|, |bv|\} < 1$ *and* $uv \neq 0$*, then, we have*

$$\int_u^v \frac{(qx/u, qx/v; q)_\infty}{(ax, bx; q)_\infty} d_q x = \frac{(1-q)\Delta(u,v)(abuv; q)_\infty}{(au, bu, av, bv; q)_\infty}.$$

Subsequently, in 1982, Al–Salam and Verma [1] found that Sears' nonterminating extension of the q-Saalschütz summation can be rewritten in the following simple form, see also [12, page 52].

Proposition 1.4. *If* $\max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|\} < 1$ *and* $uv \neq 0$*, then, we have*

$$\int_u^v \frac{(qx/u, qx/v, abcuvx; q)_\infty}{(ax, bx, cx; q)_\infty} d_q x = \frac{(1-q)\Delta(u,v)(abuv, acuv, bcuv; q)_\infty}{(au, bu, cu, av, bv, cv; q)_\infty}.$$

We call this q-integral formula the Al–Salam–Verma integral formula. This q-integral formula has five parameters with base q. When $c = 0$, this q-integral formula reduces to the Andrews–Askey integral in Proposition 1.3.

We have extended the Andrews–Askey integral formula or the Al–Salam–Verma integral formula to the following q-integral formula [20, Proposition 13.8], which has six parameters with base q. Extensions of the Andrews–Askey integral involving the terminating q-series have been discussed by [27] and [10].

Proposition 1.5. *If* a, b, c, d, u, v, r *are complex numbers such that* $\max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|, |abr/c|\} < 1$ *and* $uv \neq 0$*, then, we have the following* q*-integral formula:*

$$\int_u^v \frac{(qx/u, qx/v, abrx; q)_\infty}{(ax, bx, cx; q)_\infty} d_q x = \frac{(1-q)\Delta(u,v)(acuv, bcuv, abr/c; q)_\infty}{(au, av, bu, bv, cu, cv; q)_\infty} \times {}_3\phi_2\left(\begin{matrix} cu, cv, cuv/r \\ acuv, bcuv \end{matrix}; q, \frac{abr}{c}\right).$$

One of the main results of this paper is to extend the Andrews–Askey integral formula or the Al–Salam–Verma integral formula to the following integral formula, which has seven parameters a, b, c, d, r, u, v with bases q.

Theorem 1.6. *If a, b, c, d, u, v, r are complex numbers such that $uv \neq 0$ and $\max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|, |abr/c|\} < 1$, then, we have*

$$\begin{aligned}
&\int_u^v \frac{(qx/u, qx/v, acduvx, abrx; q)_\infty}{(ax, bx, cx, dx; q)_\infty} \\
&\qquad \times {}_3\phi_2\left(\begin{matrix} ar, ax, cx \\ acduvx, abrx \end{matrix}; q, bduv\right) d_qx \\
&= \frac{(1-q)\Delta(u,v)(acuv, aduv, bcuv, cduv, abr/c; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv; q)_\infty} \\
&\qquad \times {}_3\phi_2\left(\begin{matrix} cu, cv, cuv/r \\ acuv, bcuv \end{matrix}; q, \frac{abr}{c}\right).
\end{aligned}$$

When $d = 0$, the ${}_3\phi_2$ series in the integrand reduces to 1, and in the same time Theorem 1.6 becomes Proposition 1.5. So Theorem 1.6 is really an extension of Proposition 1.5.

Setting $r = cuv$ in Theorem 1.6, the ${}_3\phi_2$ series on the right-hand side of the equation in Theorem 1.6 equals 1, and we obtain the following proposition.

Proposition 1.7. *If a, b, c, d, u, v, r are complex numbers such that $\max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|\} < 1$ and $uv \neq 0$, then, we have*

$$\begin{aligned}
&\int_u^v \frac{(qx/u, qx/v, abcuvx, acduvx; q)_\infty}{(ax, bx, cx, dx; q)_\infty} \\
&\qquad \times {}_3\phi_2\left(\begin{matrix} acuv, ax, cx \\ abcuvx, acduvx \end{matrix}; q, bduv\right) d_qx \\
&= \frac{(1-q)\Delta(u,v)(abuv, acuv, aduv, bcuv, cduv; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv; q)_\infty}.
\end{aligned}$$

In his lost notebook [23, p 40], Ramanujan stated the following beautiful reciprocity theorem without proof. This formula may be considered as a three-variable extension of Jacobi's triple product identity. This result, now known as Ramanujan's reciprocity theorem, was first proved by Andrews in his paper [3] in 1981. For another proof, see [7].

Theorem 1.8. *If $uv \neq 0$ and $av \neq q^{-m}, au \neq q^{-m}$, m=0, 1, 2, . . . , then, we have*

$$\begin{aligned}
\frac{(q, v/u, u/v; q)_\infty}{(au, av; q)_\infty} &= (1 - v/u)\sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}(v/u)^n}{(av; q)_{n+1}} \\
&\quad + (1 - u/v)\sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}(u/v)^n}{(au; q)_{n+1}}.
\end{aligned}$$

Andrews [3, Theorem 1] also derived a four-variable reciprocity theorem by using many summation and transformation formulae for basic hypergeometric series. Inspired by the work of Andrews, in 2003, we [18, Theorem 6] proved the following five-variable reciprocity formula by using the q-exponential operator to Ramanujan's ${}_1\psi_1$ summation.

Theorem 1.9. *For* $\max\{|au|, |av|, |cu|, |cv|, |du|, |dv|\} < 1$ *and* $uv \neq 0$, *then, we have*

$$v\sum_{n=0}^{\infty}\frac{(q/du, acuv; q)_n (dv)^n}{(av, cv; q)_{n+1}} - u\sum_{n=0}^{\infty}\frac{(q/dv, acuv; q)_n (du)^n}{(au, cu; q)_{n+1}}$$
$$= \frac{\Delta(u, v)(aduv, acuv, cduv; q)_\infty}{(au, av, cu, cv, du, dv; q)_\infty}.$$

Ramanujan's reciprocity formula is the special case $c = d = 0$ of Theorem 1.9, and Ramanujan ${}_1\psi_1$ summation formula is the special case $c = 0$ of Theorem 1.9. Setting $a = c = d = 0$ in Theorem 1.9 we can immediately obtain the Jacobi triple product identity.

On taking $cduv = q$ and $a = 0$, we can obtain the following very interesting Lambert series identity.

Proposition 1.10. *If* $cu \neq q^{-m}$ *and* $cv \neq q^{-m}$, *m=0, 1, 2, ..., then, we have the Lambert series identity*

$$v\sum_{n=0}^{\infty}\frac{(q/cu)^n}{1 - cvq^n} - u\sum_{n=0}^{\infty}\frac{(q/cv)^n}{1 - cuq^n} = \frac{v(q, q, u/v, qv/u; q)_\infty}{(cu, cv, q/cu, q/cv; q)_\infty}.$$

Recently, some other generalizations of Ramanujan's reciprocity formula have been found by various authors by rearranging some of the well-known q-formulae, see, for example [11, 16, 22]. In this paper we will give a completely new extension of Ramanujan's reciprocity formula.

For simplicity, we now introduce the notation ρ in the following definition.

Definition 1.11. *We use the notation* $\rho(a, b, c, d, r, u, v)$ *to denote the double q-series*

$$v\sum_{n=0}^{\infty}\frac{(q/du, acuv, bcuv; q)_n (dv)^n}{(av, bv, cv; q)_{n+1}}$$
$$\times\, {}_3\phi_2\left(\begin{matrix} q^{n+1}, vq^{n+1}/r, q/cu \\ avq^{n+1}, bvq^{n+1} \end{matrix}; q, \frac{abcruv}{q}\right).$$

Our generalization of Ramanujan's reciprocity formula is the following reciprocity formula which has seven parameters with base q.

Theorem 1.12. *If ρ is defined as in Definition 1.11 with $uv \neq 0$ and*

$$\max\{|au|, |av|, |bu|, |bv|, |cu|, |cv|, |du|, |dv|, |abr/d|, |abcruv/q|\} < 1,$$

then, we have the following seven-variable reciprocity formula:

$$\begin{aligned}&\rho(a, b, c, d, r, u, v) - \rho(a, b, c, d, r, v, u)\\ &= \frac{\Delta(u, v)(acuv, aduv, bcuv, bduv, cduv, abr/d; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv, abcruv/q; q)_\infty}\\ &\quad \times {}_3\phi_2\left(\begin{matrix} du, dv, duv/r \\ aduv, bduv \end{matrix}; q, \frac{abr}{d}\right).\end{aligned}$$

Setting $b = 0$ in Theorem 1.12, we immediately obtain Theorem 1.9.

Noting that when $r = duv$, the ${}_3\phi_2$ series reduces to 1. Thus, on putting $r = duv$ in Theorem 1.12, we immediately obtain the following beautiful q-formula.

Proposition 1.13. *If ρ is defined as in Definition 1.11 with $uv \neq 0$ and*

$$\max\{|au|, |av|, |bu|, |bv|, |cu|, |cv|, |du|, |dv|, |abcdu^2v^2/q|\} < 1,$$

then, we have the following six-variable reciprocity formula:

$$\begin{aligned}&\rho(a, b, c, d, duv, u, v) - \rho(a, b, c, d, duv, v, u)\\ &= \frac{\Delta(u, v)(abuv, acuv, aduv, bcuv, bduv, cduv; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv, abcdu^2v^2/q; q)_\infty}.\end{aligned}$$

The remainder of this paper is organized as follows. Some inequalities for q-series are discussed in Section 2. In Section 3, we introduce some important facts in q-differential calculus. Sections 4 and 5 are devoted to the proofs of Theorems 1.6 and 1.12. In Section 6, we will use Theorems 1.6 to derive a beta integral formula which including the Askey–Wilson integral as a special case. Some limiting cases of Theorem 1.9 are discussed in Section 7, and one notable example is the following formula:

$$\frac{(q; q)_\infty^4}{(qa, q/a; q)_\infty^2} = 1 + (1 - a)^2 \sum_{n=1}^{\infty} \frac{n(q/a)^n}{1 - aq^n} + (1 - 1/a)^2 \sum_{n=1}^{\infty} \frac{n(qa)^n}{1 - q^n/a}.$$

2. SOME INEQUALITIES FOR q-SERIES

For convenience, in this section, Sections 3 and 4, we assume that $0 < q < 1$.

Proposition 2.1. *If k is a nonnegative integer or ∞, a and b are two nonnegative numbers such that $0 \le b \le 1$, then, we have*

$$(-ab;q)_k \le (-a;q)_\infty.$$

If we further assume that $0 \le a \le 1$, then, we have

$$(ab;q)_k \ge (a;q)_\infty.$$

Proof. Keeping the fact $0 < q < 1$ in mind, we find that for any $0 \le j \le k-1$,

$$1 + abq^j \le 1 + aq^j.$$

On multiplying these inequalities together, we deduce that

$$(-ab;q)_k \le (-a;q)_k.$$

Since $(-aq^k;q)_\infty \ge 1$, we multiply $(-aq^k;q)_\infty$ to the right-hand side of the above inequality to arrive at the first inequality in the proposition. In the same way we can prove the second inequality. This completes the proof of Proposition 2.1. □

Proposition 2.2. *If $\max\{|b_1|, |b_2|, \ldots, |b_r|, |x|\} < 1$ and n is a nonnegative integer, then, we have*

$$\left| {}_{r+1}\phi_r \left(\begin{matrix} a, a_1q^n, \ldots, a_rq^n \\ b_1q^n, \ldots b_rq^n \end{matrix} ; q, x \right) \right| \le \frac{(-|ax|, -|a_1|, \ldots, -|a_r|; q)_\infty}{(|x|, |b_1|, \ldots, |b_r|; q)_\infty}.$$

Proof. Keeping $0 < q < 1$ in mind, using the triangle inequality and Proposition 2.1, we find that for $j \in \{1, 2, \ldots, r\}$,

$$|(a_jq^n;q)_k| \le \prod_{l=0}^{k-1}(|1 + |a_j|q^l) \le \prod_{l=0}^{\infty}(|1 + |a_j|q^l) = (-|a_j|;q)_\infty,$$

and

$$|(b_jq^n;q)_k| \ge \prod_{l=0}^{k-1}(|1 - |b_j|q^l) \ge \prod_{l=0}^{\infty}(|1 - |b_j|q^l) = (|b_j|;q)_\infty.$$

It follows that

$$\left| \frac{(a, a_1q^n, \ldots, a_rq^n; q)_k x^k}{(q, b_1q^n, \ldots b_rq^n; q)_k} \right| \le \frac{(-|a_1|, \ldots, -|a_r|)_\infty (-|a|;q)_k |x|^k}{(|b_1|, \ldots, |b_r|)_\infty (q;q)_k}.$$

Using this inequality and the triangle inequality, we conclude that

$$\left| {}_{r+1}\phi_r \left(\begin{matrix} a, a_1q^n, \ldots, a_rq^n \\ b_1q^n, \ldots b_rq^n \end{matrix} ; q, x \right) \right|$$
$$\le \frac{(-|a_1|, \ldots, -|a_r|)_\infty}{(|b_1|, \ldots, |b_r|)_\infty} \sum_{k=0}^{\infty} \frac{(-|a|;q)_k |x|^k}{(q;q)_k}.$$

On applying the q-binomial theorem to the right-hand side of the above inequality, we complete the proof of Proposition 2.2. □

It should be pointed out that Wang [28, Theorem 1.1] has obtained a similar inequality.

3. Some facts in q-differential calculus

Next we introduce some basic concepts in q-differential calculus.

Definition 3.1. *For any function $f(x)$ of one variable, the q-derivative of $f(x)$ with respect to x, is defined as*

$$\mathcal{D}_{q,x}\{f(x)\} = \frac{f(x)-f(qx)}{x},$$

and we further define $\mathcal{D}_{q,x}^0\{f\} = f$ and $\mathcal{D}_{q,x}^n\{f\} = \mathcal{D}_{q,x}\{\mathcal{D}_{q,x}^{n-1}\{f\}\}$.

The q-derivative was first introduced by L. Schendel [?] in 1877 and then by F. H. Jackson [14] in 1908, which is a q-analog of the ordinary derivative. The definition of the q-partial derivative can be found in [20].

Definition 3.2. *A q-partial derivative of a function of several variables is its q-derivative with respect to one of those variables, regarding other variables as constants. The q-partial derivative of a function f with respect to the variable x is denoted by $\partial_{q,x}\{f\}$.*

Definition 3.3. *A q-partial differential equation is an equation that contains unknown multivariable functions and their q-partial derivatives.*

The homogeneous Rogers–Szegő polynomials play an important role in the theory of orthogonal polynomials, which are defined by [19, 20]

$$h_n(a,b|q) = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}. \tag{3.1}$$

By multiplying two copies of the q-binomial theorem (see, for example [12, p. 8, Eq. (1.3.2)]), one can find that [19, 20]

$$\sum_{n=0}^{\infty} h_n(a,b|q)\frac{t^n}{(q;q)_n} = \frac{1}{(at,bt;q)_\infty}, \qquad |at|<1, |bt|<1. \tag{3.2}$$

It turn out that the q-partial differential equations is an important subject of study, we started the study of this subject in [20] and [21]. The following very useful expansion theorem for q-series can be found in [20, Proposition 1.6].

Theorem 3.4. *If $f(x,y)$ is a two-variable analytic function at $(0,0) \in \mathbb{C}^2$, then, f can be expanded in terms of $h_n(x,y|q)$ if and only if f satisfies the q-partial differential equation $\partial_{q,x}\{f\} = \partial_{q,y}\{f\}$.*

One of the most important formulae for the Rogers–Szegő polynomials is the following q-Mehler formula, which can be derived easily from Theorem 3.4, see [20, pp.219–220] for details.

Proposition 3.5. *For* $\max\{|asz|, |atz|, |bsz|, |btz|\} < 1$, *we have*

$$\sum_{n=0}^{\infty} h_n(a,b|q)h_n(s,t|q)\frac{z^n}{(q;q)_n} = \frac{(abstz^2;q)_\infty}{(asz, atz, bsz, btz;q)_\infty}.$$

In order to prove Theorems 1.6 and 1.12, we need the following proposition.

Proposition 3.6. *The function $L(a,b,u,v,s,t)$ satisfies the q-partial differential equation $\partial_{q,a}\{L\} = \partial_{q,b}\{L\}$, where $L(a,b,u,v,s,t)$ is defined by*

$$\frac{(av, bv, abstu/v;q)_\infty}{(as, at, au, bs, bt, bu;q)_\infty} {}_3\phi_2\left(\begin{matrix} v/s, v/t, v/u \\ av, bv \end{matrix}; q, \frac{abstu}{v}\right).$$

Proof. It is easily seen that using $L(a,b,u,v,s,t)$ we can rewrite the formula in Proposition 1.5 in the form

$$L(a,b,u,v,s,t) = \frac{(v/s, v/t;q)_\infty}{(1-q)\Delta(s,t)(au, bu;q)_\infty} \int_s^t \frac{(qx/s, qx/t, abux;q)_\infty}{(ax, bx, vx/st;q)_\infty} d_q x.$$

Noting the definition of the q-partial differential equations and using a direct computation, we easily find that

$$\begin{aligned}&\partial_{q,a}\{L\} = \partial_{q,b}\{L\} \\ &= \frac{(v/s, v/t;q)_\infty}{(1-q)\Delta(s,t)} \int_s^t \frac{(x+u-aux-bux)\,(qx/s, qx/t, abuxq;q)_\infty}{(au, bu, ax, bx, vx/st;q)_\infty} d_q x,\end{aligned}$$

which indicates that Proposition 3.6 holds. □

4. The proof of Theorem 1.6

Recall the Sears ${}_3\phi_2$ transformation formula (see, for example [18, Theorem 3])

$$\begin{aligned}&{}_3\phi_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; q, \frac{b_1 b_2}{a_1 a_2 a_3}\right) \\ &= \frac{(b_2/a_3, b_1 b_2/a_1 a_2;q)_\infty}{(b_2, b_1 b_2/a_1 a_2 a_3;q)_\infty} {}_3\phi_2\left(\begin{matrix} b_1/a_1, b_1/a_2, a_3 \\ b_1, b_1 b_2/a_1 a_2 \end{matrix}; q, \frac{b_2}{a_3}\right).\end{aligned}$$

Using the Sears ${}_3\phi_2$ transformation formula, we easily conclude that

$$\begin{aligned}&{}_3\phi_2\left(\begin{matrix}ax, ar, cx\\ acduvx, abrx\end{matrix}q, bduv\right)\\ &=\frac{(abr/c, bcduvx;q)_\infty}{(bduv, abrx;q)_\infty}{}_3\phi_2\left(\begin{matrix}cduv, cduvx/r, cx\\ acduvx, bcduvx\end{matrix};q,\frac{abr}{c}\right).\end{aligned}$$

This transformation formula shows that Theorem 1.6 is equivalent to the following formula:

$$\begin{aligned}&\int_u^v \frac{(qx/u, qx/v, acduvx, bcduvx;q)_\infty}{(ax, bx, cx, dx;q)_\infty}\\ &\qquad\times {}_3\phi_2\left(\begin{matrix}cduv, cduvx/r, cx\\ acduvx, bcduvx\end{matrix}q,\frac{abr}{c}\right)d_qx\\ &=\frac{(1-q)\Delta(u,v)(acuv, aduv, bcuv, bduv, cduv;q)_\infty}{(au, av, bu, bv, cu, cv, du, dv;q)_\infty}\\ &\qquad\times {}_3\phi_2\left(\begin{matrix}cu, cv, cuv/r\\ acuv, bcuv\end{matrix};q,\frac{abr}{c}\right).\end{aligned}\tag{4.1}$$

In order to prove the identity (4.1), we need to prove the following lemma.

Lemma 4.1. *The q-integral in (4.1) is a two-variable analytic function of a and b, which is analytic at $(0,0)\in\mathbb{C}^2$.*

Proof. For the sake of convenience, we define the compact notation A_n and B_n by

$$\begin{aligned}A_n(a,b,c,d,r,u,v) &:= {}_3\phi_2\left(\begin{matrix}cduv, cdvu^2q^n/r, cuq^n\\ acdvu^2q^n, bcdvu^2q^n\end{matrix};q,\frac{abr}{c}\right),\\ B_n(a,b,c,d,r,u,v) &:= u(1-q)\frac{(q^{n+1}, q^{n+1}u/v, acdvu^2q^n, bcdvu^2q^n;q)_\infty}{(auq^n, buq^n, cuq^n, duq^n;q)_\infty}.\end{aligned}$$

Using the definition of q-integral, we find that the left-hand side of the equation in (4.1) can be written as

$$\begin{aligned}&\sum_{n=0}^\infty A_n(a,b,c,d,r,v,u)B_n(a,b,c,d,r,v,u)q^n\\ &\quad-\sum_{n=0}^\infty A_n(a,b,c,d,r,u,v)B_n(a,b,c,d,r,u,v)q^n.\end{aligned}\tag{4.2}$$

Next we will prove that this series converges to a two-variable analytic function of a and b at $(0,0)\in\mathbb{C}^2$. It is obvious that the first summation can be obtained from the second one by interchanging u and v. Thus we only need consider the second summation, namely,

$$\sum_{n=0}^{\infty} A_n(a,b,c,d,r,u,v)B_n(a,b,c,d,r,u,v)q^n. \tag{4.3}$$

We divide our proof into two cases according $cr \neq 0$ and $cr = 0$. We only prove the $cr \neq 0$ case and the $cr = 0$ case can be proved in the same way. Without loss of generality, we can assume that

$$\max\{|a|,|b|,|d|\} < 1 \text{ and } 0 < |c|,|r|,|u|,|v| < 1.$$

Using Propositions 2.1 and 2.2 and doing some simple calculations, we find that

$$\begin{aligned}|A_n(a,b,c,d,r,u,v)| &\leq \frac{(-|abduvr|,-|cdvu^2/r|,-|cu|;q)_\infty}{(|abr/c|,|acdvu^2|,|bcdvu^2|;q)_\infty}\\ &\leq \frac{(-1,-|r|,-|1/r|;q)_\infty}{(|r/c|,|u|,|u|;q)_\infty}.\end{aligned}$$

On making use of Proposition 2.1 and some elementary calculations, we deduce that

$$|B_n(a,b,c,d,r,u,v)| \leq \frac{(-1;q)_\infty^3(-|qu/v|;q)_\infty}{(|u|;q)_\infty^4}.$$

Using the triangular inequality and the above two inequalities, we conclude that

$$\begin{aligned}&|\sum_{n=0}^{\infty} A_n(a,b,c,d,r,u,v)B_n(a,b,c,d,r,u,v)q^n|\\ &\leq \sum_{n=0}^{\infty} |A_n(a,b,c,d,r,u,v)||B_n(a,b,c,d,r,u,v)|q^n\\ &\leq \frac{(-1;q)_\infty^4(-|qu/v|,-|r|,-|1/r|;q)_\infty}{(1-q)(|u|;q)_\infty^6(|r/c|;q)_\infty}.\end{aligned}$$

This indicates that the series in (4.3) converges absolutely and uniformly for $|a| < 1$. It is easily to see that every term of this series is a analytic at $a = 0$. Thus, this series converges to an analytic function of a, which is analytic at $a = 0$.

By symmetry, this series also converges to an analytic function of b, which a analytic at $b = 0$. Hence the series in (4.2) converges to a two-variable analytic function of a and b, which is analytic at $(a,b) = (0,0) \in \mathbb{C}^2$.

Interchanging u and v in (4.3) we immediately find that the first series in (4.2) is also analytic at $(a,b) = (0,0) \in \mathbb{C}^2$. In summary, the left-hand side of the equation in (4.1) is a two-variable analytic function of a and b, which is analytic at $(a,b)=(0,0)\in\mathbb{C}^2$. $\square$

Now we begin to prove (4.1) by using Lemma 4.1, Proposition 3.6 and Theorem 3.4.

Proof. Using the definition of L in Proposition 3.6 and some simple computations, we find that

$$L(a,b,duv,cduvx,x,r)=\frac{(acduvx,bdcuvx,abr/c;q)_\infty}{(ax,bx,ar,br,aduv,bduv;q)_\infty} \times {}_3\phi_2\left(\begin{matrix}cduv,cduvx/r,cx\\ acduvx,bcduvx\end{matrix};q,\frac{abr}{c}\right),$$

$$L(a,b,r,cuv,u,v)=\frac{(acuv,bcuv,abr/c;q)_\infty}{(au,av,bu,bv,ar,br;q)_\infty} \times {}_3\phi_2\left(\begin{matrix}cu,cv,cuv/r\\ acuv,bcuv\end{matrix};q,\frac{abr}{c}\right).$$

Using these two equations we can rewrite (4.1) in the form

$$\int_u^v \frac{(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}L(a,b,duv,cduvx,x,r)d_qx = \frac{(1-q)\Delta(u,v)(cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}L(a,b,r,cuv,u,v). \tag{4.4}$$

If we use $f(a,b)$ to denote the left-hand side of (4.4), then, $f(a,b)$ is analytic at $(0,0)\in\mathbb{C}^2$, and satisfies the q-partial differential equation $\partial_{q,a}\{f\}=\partial_{q,b}\{f\}$. Thus, by Theorem 3.4, there exists a sequence $\{\alpha_n\}$ independent of a and b such that

$$f(a,b)=\sum_{n=0}^\infty \alpha_n h_n(a,b|q).$$

On putting $b=0$ in this equation, using $h_n(a,0|q)=a^n$, and noting the definition of $f(a,b)$, we find that

$$f(a,0)=\frac{1}{(ar,aduv;q)_\infty}\int_u^v \frac{(qx/u,qx/v,acduvx;q)_\infty}{(ax,cx,dx;q)_\infty}d_qx=\sum_{n=0}^\infty \alpha_n a^n.$$

Applying the Al–Salam–Verma integral to the q-integral in this equation, we deduce that

$$\sum_{n=0}^\infty \alpha_n a^n = \frac{(1-q)\Delta(u,v)(acuv,cduv;q)_\infty}{(au,av,ar,cu,cv,du,dv;q)_\infty}. \tag{4.5}$$

If we use $g(a,b)$ to denote the right-hand side of (4.4), then, $g(a,b)$ is analytic at $(0,0) \in \mathbb{C}^2$, and satisfies the q-partial differential equation $\partial_{q,a}\{g\} = \partial_{q,b}\{g\}$. Thus, by Theorem 3.4, there exists a sequence $\{\beta_n\}$ independent of a and b such that

$$g(a,b) = \sum_{n=0}^{\infty} \beta_n h_n(a,b|q).$$

On putting $b = 0$ in this equation, using $h_n(a,0|q) = a^n$, and noting the definition of $g(a,b)$, we find that

$$g(a,0) = \sum_{n=0}^{\infty} \beta_n a^n = \frac{(1-q)\Delta(u,v)(acuv, cduv; q)_\infty}{(au, av, ar, cu, cv, du, dv; q)_\infty}.$$

Comparing this equation with (4.5), we find that $\alpha_n = \beta_n$, which implies that $f(a,b) = g(a,b)$, which shows that the identity in (4.1) holds. Thus we have proved Theorem 1.6 for $|a|$ and $|b|$ sufficiently small. By analytic continuation, we complete the proof of Theorem 1.6. □

5. The proof of Theorem 1.12

In order to prove Theorem 1.12, we first set up the following lemma.

Lemma 5.1. *The series on the left-hand side of the equation in Theorem 1.12 represents a two-variable analytic function of a and b, which is analytic at* $(0,0) \in \mathbb{C}^2$.

Proof. The proof can be divided into two cases according to $rcd \neq 0$ and $rcd = 0$. We only prove the $rcd \neq 0$ case and the $rcd = 0$ case can be proved similarly.

For the sake of simplicity, we will use the C_n and D_n to denote

$$C_n(a,b,c,d,u,v) := v\frac{(q/du, acuv, bcuv; q)_n (dv)^n}{(av, bv, cv; q)_{n+1}},$$
$$D_n(a,b,c,r,u,v) := {}_3\phi_2\left(\begin{matrix} q^{n+1}, vq^{n+1}/r, q/cu \\ avq^{n+1}, bvq^{n+1} \end{matrix}; q; \frac{abcruv}{q}\right).$$

Using these notations we can write the left-hand side of the equation in Theorem 1.12 as

$$\sum_{n=0}^{\infty} C_n(a,b,c,d,u,v) D_n(a,b,c,r,u,v) - \sum_{n=0}^{\infty} C_n(a,b,c,d,v,u) D_n(a,b,c,r,v,u). \tag{5.1}$$

Now we will show that this series converges to a two-variable analytic function of a and b at $(0,0)\in\mathbb{C}^2$. It is obvious that the second summation in the above equation can be obtained from the first summation by interchanging u and v, so we only need consider the first summation

$$\sum_{n=0}^{\infty} C_n(a,b,c,d,u,v)D_n(a,b,c,r,u,v) \tag{5.2}$$

Without loss of generality, we may assume that $\max\{|a|,|b|\}<1$ and in order to simplify the discussion, we only consider the case $rcd\neq 0$, as the case $rcd=0$ is similar. Using Proposition 2.1 and some simple calculation, we conclude that

$$|C_n(a,b,c,d,u,v)|\le \frac{|v|(-|1/du|,-|v|,-|v|;q)_\infty |dv|^n}{(|v|,|v|,|cv|;q)_\infty}.$$

On making use of Proposition 2.2 and a direct computation, we easily deduce that

$$\begin{aligned}|D_n(a,b,c,r,u,v)| &\le \frac{(-|abrv|,-q,-|v/r|;q)_\infty}{(|v|,|v|,|q/cu|;q)_\infty}\\ &\le \frac{(-|rv|,-q,-|v/r|;q)_\infty}{(|v|,|v|,|q/cu|;q)_\infty}.\end{aligned}$$

Using the triangular inequality and these two inequalities, we have

$$\begin{aligned}&\left|\sum_{n=0}^{\infty} C_n(a,b,c,d,u,v)D_n(a,b,c,r,u,v)\right|\\ &\le \sum_{n=0}^{\infty} |C_n(a,b,c,d,u,v)D_n(a,b,c,r,u,v)|\\ &\le |v|\frac{(-q,-|rv|,-|v/r|,-|1/du|;q)_\infty}{(|v|;q)_\infty^4(|cv|,|q/cu|;q)_\infty}\sum_{n=0}^{\infty}|dv|^n\\ &= \frac{|v|(-q,-|rv|,-|v/r|,-|1/du|;q)_\infty}{(1-|dv|)(|v|;q)_\infty^4(|cv|,|q/cu|;q)_\infty}.\end{aligned}$$

This shows that the series in (5.2) converges absolutely and uniformly for $\max\{|a|,|b|\}<1$. It is easily seen that every term of this series is analytic at $(a,b)=(0,0)\in\mathbb{C}^2$, thus this series converges to a two-variable analytic function of a and b which is analytic at $(0,0)\in\mathbb{C}^2$. □

Now we begin to prove Theorem 1.12 by Theorem 3.4, Proposition 3.6 and Lemma 5.1.

Proof. Using the definition of L in Proposition 3.6, we deduce that

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix} du, dv, duv/r \\ aduv, bduv \end{matrix}; q, \frac{abr}{d}\right) \\
&= \frac{(ar, br, au, bu, av, bv; q)_\infty}{(aduv, bduv, abr/d; q)_\infty} L(a, b, u, duv, v, r),
\end{aligned}
$$

$$
\begin{aligned}
&{}_3\phi_2\left(\begin{matrix} q^{n+1}, vq^{n+1}/r, q/cu \\ avq^{n+1}, bvq^{n+1} \end{matrix}; q, \frac{abcruv}{q}\right) \\
&= \frac{(av, bv, ar, br, acuvq^n, bcuvq^n; q)_\infty}{(avq^{n+1}, bvq^{n+1}, abcruv/q; q)_\infty} L(a, b, r, vq^{n+1}, v, cuvq^n).
\end{aligned}
$$

Using these two equations we can rewrite Theorem 1.12 in the form

$$
\begin{aligned}
(5.3)\qquad & v\sum_{n=0}^{\infty} \frac{(q/du; q)_n (dv)^n}{(cv; q)_{n+1}} L(a, b, r, vq^{n+1}, v, cuvq^n) \\
& - u\sum_{n=0}^{\infty} \frac{(q/dv; q)_n (du)^n}{(cv; q)_{n+1}} L(a, b, r, uq^{n+1}, u, cuvq^n) \\
& = \frac{\Delta(u, v)(cduv; q)_\infty}{(cu, cv, du, dv; q)_\infty} L(a, b, u, duv, v, r).
\end{aligned}
$$

If we use $f(a, b)$ to denote the left-hand side of this equation, then, from Lemma 5.1 we know that $f(a, b)$ is analytic at $(0, 0) \in \mathbb{C}^2$. Using Proposition 3.6, we easily see that $f(a, b)$ satisfies the q-partial differential equation $\partial_{q,a}\{f\} = \partial_{q,b}\{f\}$. Thus by Theorem 3.4, there exists a sequence $\{\alpha_n\}$ independent of a and b such that

$$
f(a, b) = \sum_{n=0}^{\infty} \alpha_n h_n(a, b).
$$

Putting $b = 0$ in this equation and using the fact $h_n(a, 0) = a^n$, we obtain

$$
f(a, 0) = \sum_{n=0}^{\infty} \alpha_n a^n.
$$

Noting the definition of $f(a, b)$ and using Theorem 1.9, we find that

$$
\begin{aligned}
&(ar, acuv; q)_\infty f(a, 0) \\
&= v\sum_{n=0}^{\infty} \frac{(q/du, acuv; q)_n (dv)^n}{(av, cv; q)_{n+1}} - u\sum_{n=0}^{\infty} \frac{(q/dv, acuv; q)_n (du)^n}{(au, cu; q)_{n+1}} \\
&= \frac{\Delta(u, v)(aduv, acuv, cduv; q)_\infty}{(au, av, cu, cv, du, dv; q)_\infty}.
\end{aligned}
$$

If we use $g(a,b)$ to denote the right-hand side of (5.3), then, $g(a,b)$ is analytic at $(0,0)\in\mathbb{C}^2$, and satisfies the q-partial differential equation $\partial_{q,a}\{g\}=\partial_{q,b}\{g\}$. Thus, by Theorem 3.4, there exists a sequence $\{\beta_n\}$ independent of a and b such that

$$g(a,b)=\sum_{n=0}^{\infty}\beta_n h_n(a,b|q).$$

On putting $b=0$ in this equation, using $h_n(a,0|q)=a^n$, and noting the definition of $g(a,b)$, we find that

$$g(a,0)=\sum_{n=0}^{\infty}\beta_n a^n=\frac{\Delta(u,v)(aduv,cduv;q)_\infty}{(ar,au,av,cu,cv,du,dv;q)_\infty}.$$

It follows that

$$\sum_{n=0}^{\infty}\alpha_n a^n=\sum_{n=0}^{\infty}\beta_n a^n.$$

Thus we have $\alpha_n=\beta_n$, which implies that $f(a,b)=g(a,b)$. Hence we have proved Theorem 1.12 for $|a|$ and $|b|$ sufficiently small and $0<q<1$. Using analytic continuation, this completes the proof of Theorem 1.12. □

6. A beta integral formula

Definition 6.1. *For $x=\cos\theta$, we define $h(x;a)$ and $h(x;a_1,a_2,\ldots,a_m)$ as follows:*

$$h(x;a)=(ae^{i\theta},ae^{-i\theta};q)_\infty=\prod_{k=0}^{\infty}(1-2q^kax+q^{2k}a^2),$$
$$h(x;a_1,a_2,\ldots,a_m)=h(x;a_1)h(x;a_2)\cdots h(x;a_m).$$

In this section we will use Theorem 1.6 to prove the following beta integral formula, which including the Askey–Wilson integral as a special case.

Theorem 6.2. *For $\max\{|a|,|b|,|c,|d|,|abr/c|\}<1$, we have the integral formula*

$$\frac{2\pi a_0}{(q,ac,ad,bc,cd,abr/c;q)_\infty}=\frac{2\pi(abdr;q)_\infty}{(q,ac,ad,bc,bd,cd,abr/c;q)_\infty}$$
$$=\int_0^{\pi}\frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}{}_3\phi_2\left(\begin{matrix}ce^{i\theta},ce^{-i\theta},c/r\\ac,bc\end{matrix};q,\frac{abr}{c}\right)d\theta.$$

Proof. For the sake of brevity, we use $I(x)$ to denote the function

$$I(x) = \frac{(acdx, abrx; q)_\infty}{(ax, bx, cx, dx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} ar, ax, cx \\ acdx, abrx \end{matrix} q, bd \right). \tag{6.1}$$

It is easily seen that $I(x)$ is analytic near $x = 0$. Thus, there exists a sequence $\{a_k\}_{k=0}^\infty$ independent of x such that

$$I(x) = a_0 + \sum_{k=1}^\infty a_k x^k.$$

By setting $x = 0$ in the above equation and using the q-binomial theorem, we conclude that

$$a_0 = \sum_{n=0}^\infty \frac{(ar; q)_n (bd)^n}{(q; q)_n} = \frac{(abdr; q)_\infty}{(bd; q)_\infty}. \tag{6.2}$$

Noting the definition of $\Delta(u, v)$ in (1.1) and using a simple computation, we easily find that

$$(e^{i\theta} - e^{-i\theta})\Delta(e^{i\theta}, e^{-i\theta}) = (q; q)_\infty h(\cos 2\theta; 1). \tag{6.3}$$

On replacing (u, v) by $(e^{i\theta}, e^{-i\theta})$ in Theorem 1.6 and noting the above equation , we deduce that

$$\begin{aligned}(e^{i\theta} - e^{-i\theta}) &\int_{e^{i\theta}}^{e^{-i\theta}} (qxe^{i\theta}, qxe^{-i\theta}; q)_\infty I(x) d_q x \\ &= \frac{(1-q)(q, ac, ad, bc, cd, abr/c; q)_\infty h(\cos 2\theta; 1)}{h(\cos\theta; a, b, c, d)} \\ &\quad \times {}_3\phi_2 \left(\begin{matrix} ce^{i\theta}, ce^{-i\theta}, c/r \\ ac, bc \end{matrix}; q, \frac{abr}{c} \right).\end{aligned} \tag{6.4}$$

Using the definition of the q-integral, we find that the left-hand side of this equation equals

$$\begin{aligned}&(1-q)(1-e^{-2i\theta}) \sum_{n=0}^\infty (q^{n+1}; q)_\infty (q^n e^{-2i\theta}; q)_\infty I(q^n e^{-i\theta}) q^n \\ &+ (1-q)(1-e^{2i\theta}) \sum_{n=0}^\infty (q^{n+1}; q)_\infty (q^n e^{2i\theta}; q)_\infty I(q^n e^{i\theta}) q^n.\end{aligned}$$

Inspecting the first series in the equation, we see that this series can be expanded in terms of the negative powers of $\{e^{-ki\theta}\}_{k=0}^\infty$, and the constant term of the Fourier expansion of this series is $(1-q)a_0$. Thus, there exists a sequence $\{\alpha_k\}_{k=1}^{\infty}$ independent of θ such that the first series equals

$$(1-q)a_0+\sum_{k=1}^{\infty}\alpha_k e^{-ik\theta}.$$

Replacing θ by $-\theta$, we immediately find that the second series is equal to

$$(1-q)a_0+\sum_{k=1}^{\infty}\alpha_k e^{ik\theta}.$$

Combining the above two expressions together, we arrive at

$$\begin{aligned}(e^{i\theta}-e^{-i\theta})&\int_{e^{i\theta}}^{e^{-i\theta}}(qxe^{i\theta},qxe^{-i\theta};q)_\infty I(x)d_qx\\ &=2(1-q)a_0+2\sum_{k=1}^{\infty}\alpha_k\cos k\theta.\end{aligned}$$

Comparing this equation with (6.3), we are led to the Fourier series expansion

$$\begin{aligned}&2(1-q)a_0+2\sum_{k=1}^{\infty}\alpha_k\cos k\theta\\ &=\frac{(1-q)(q,ac,ad,bc,cd,abr/c;q)_\infty h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}\\ &\quad\times{}_3\phi_2\left({ce^{i\theta},ce^{-i\theta},c/r \atop ac,bc};q,\frac{abr}{c}\right).\end{aligned}$$

On integrating the above equation over $[-\pi,\pi]$ and using the fact

$$\int_{-\pi}^{\pi}(\cos k\theta)d\theta=2\pi\delta_{k,0},$$

and noting that the integrand is an even function of θ, we deduce that

$$\begin{aligned}&\int_0^{\pi}\frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}{}_3\phi_2\left({ce^{i\theta},ce^{-i\theta},c/r \atop ac,bc};q,\frac{abr}{c}\right)d\theta\\ &=\frac{2\pi a_0}{(q,ac,ad,bc,cd,abr/c;q)_\infty}.\end{aligned}$$

Substituting the value of a_0 in (6.2) into this equation, we complete the proof of Theorem 6.2. □

Letting $r=c$ in Theorem 6.2, we immediately obtain the Askey–Wilson integral formula [6].

Theorem 6.3. *If* $max\{|a|,|b|,|c|,|d|\}<1$*, then, we have*

$$\int_0^{\pi} \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}d\theta = \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}.$$

Remark 6.4. By setting $u=e^{i\theta}$ and $v=e^{-i\theta}$ in Proposition 1.13 and then using the same argument as in the proof of Theorem 6.2, we can give a derivation of the Askey–Wilson integral.

The continuous q-Hermite polynomials $H_n(\cos\theta|q)$ is defined as

$$H_n(\cos\theta|q)=\sum_{k=0}^{n}\binom{n}{k}_q e^{i(n-2k)\theta}. \tag{6.5}$$

Using the definition of the homogeneous Rogers–Szegő polynomials defined in (3.1), it is easily seen that

$$H_n(\cos\theta|q)=h_n(e^{-i\theta},e^{i\theta}|q). \tag{6.6}$$

Putting $a=e^{-i\theta}$ and $b=e^{i\theta}$ in (3.2), one can find the following proposition.

Proposition 6.5. *For* $|t|<1$*, we have*

$$\sum_{n=0}^{\infty} H_n(\cos\theta|q)\frac{t^n}{(q;q)_n}=\frac{1}{(te^{i\theta},te^{-i\theta};q)_\infty}. \tag{6.7}$$

Remark 6.6. One of the most important properties of the q-Hermite polynomials is that they satisfy the following orthogonality relation, which was first proved by Szegő [25]:

$$\int_0^{\pi} H_m(\cos\theta|q)H_n(\cos\theta|q)h(\cos 2\theta;1)d\theta=2\pi(q;q)_n\delta_{m,n}/(q;q)_\infty.$$

This orthogonality relation has been used by several authors to evaluate the Askey–Wilson integral and other related q-beta integrals (see, for example [2, 13, 17]). We have just evaluated the Askey–Wilson integral without using the orthogonality relation for the q-Hermite polynomials. We can use a special case of the Askey–Wilson integral formula to give a new proof of the the orthogonality relation for the q-Hermite polynomials. The proof is as follows:

Putting $c=d=0$ in the Askey–Wilson integral formula, we immediately deduce that

$$\int_0^{\pi} \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b)}d\theta=\frac{2\pi}{(q,ab;q)_\infty}. \tag{6.8}$$

On multiplying two copies of the identity in (6.7), we find that for $\max\{|a|,|b|\}<1$,

$$\sum_{m,n=0}^{\infty} H_m(\cos\theta|q)H_n(\cos\theta|q)\frac{a^m b^n}{(q;q)_m(q;q)_n}=\frac{1}{h(\cos\theta;a,b)}.$$

Using Proposition 6.5, we can easily show that the above double series converges uniformly for $\max\{|a|,|b|\}<1$ on $0\le\theta\le\pi$.

Substituting this series into the left-hand side of (6.8) and then integrating term by term, and applying the q-binomial theorem to the right-hand side of (6.8), we find that

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{a^m b^n}{(q;q)_m(q;q)_n}\int_0^{\pi} H_m(\cos\theta|q)H_n(\cos\theta|q)h(\cos 2\theta;1)d\theta\\&=\frac{2\pi}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(ab)^n}{(q;q)_n}.\end{aligned}$$

Equating the coefficients of $a^m b^n$ on both sides of this equation, we arrive at the orthogonality relation for the q-Hermite polynomials.

7. Some limiting cases of Theorem 1.9

In this section we will discussed some limiting cases of Theorem 1.9.

Proposition 7.1. *For $a\ne q^{-m}, c\ne q^{-m}, d\ne q^{-m}, m=0,1,2,\ldots,$ we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(q/d,ac;q)_n d^n}{(a,c;q)_{n+1}}\\&\quad\times\left(n+1+\sum_{k=0}^{n}\frac{aq^k}{1-aq^k}+\sum_{k=0}^{n}\frac{cq^k}{1-cq^k}-\sum_{k=1}^{n}\frac{q^k}{d-q^k}\right)\\&=\frac{(q;q)_\infty^3(ac,ad,cd;q)_\infty}{(a,c,d;q)_\infty^2}.\end{aligned}$$

Proof. Keeping in mind that $\Delta(u,v)=(v-u)(q,qu/v,qv/u;q)_\infty$, dividing both sides of the equation in Theorem 1.9 by $v-u$, then letting $v\to u$ in the resulting equation, using L′Hôspital's rule and simplifying, we deduce that

$$\sum_{n=0}^{\infty}\frac{(q/du,acu^2;q)_n(du)^n}{(au,cu;q)_{n+1}} \times\left(n+1+\sum_{k=0}^{n}\frac{auq^k}{1-auq^k}+\sum_{k=0}^{n}\frac{cuq^k}{1-cuq^k}-\sum_{k=1}^{n}\frac{q^k}{du-q^k}\right) =\frac{(q;q)_\infty^3(acu^2,adu^2,cdu^2;q)_\infty}{(au,cu,du;q)_\infty^2}.$$

On replacing (au, bu, cu) by (a, b, c), we complete the proof of Proposition 7.1. □

On setting $a = c = 0$ in proposition 7.1, we immediately obtain the following proposition.

Proposition 7.2. *For $d \neq q^{-m}, m = 0, 1, 2, \ldots,$ we have*

$$\sum_{n=0}^{\infty}(q/d;q)_n d^n\left(n+1-\sum_{k=1}^{n}\frac{q^k}{d-q^k}\right)=\frac{(q;q)_\infty^3}{(d;q)_\infty^2}.$$

On putting $d = 0$ in Proposition 7.2, we immediately obtain the following identity of Jacobi [8, p. 14]:

$$(q;q)_\infty^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}.$$

On letting $d \to q$ Proposition 7.2, we obtain the Euler identity (see, for example [5, p. 280])

$$1-\sum_{n=1}^{\infty}(q;q)_{n-1}q^n=(q;q)_\infty.$$

On taking $d = -q$ in Proposition 7.2, we are led to the identity

$$1+\sum_{n=1}^{\infty}(-q)^n(-q;q)_{n-1}\left(2n+3+2\sum_{k=1}^{n-1}\frac{q^k}{1+q^k}\right)=\frac{(q;q)_\infty^3}{(-q;q)_\infty^2}.$$

Proposition 7.3. *For $a \neq q^m, m = \pm1, \pm2, \ldots,$ we have the Lambert series formula*

$$\frac{(q;q)_\infty^4}{(qa,q/a;q)_\infty^2}=1+(1-a)^2\sum_{n=1}^{\infty}\frac{n(q/a)^n}{1-aq^n}+(1-1/a)^2\sum_{n=1}^{\infty}\frac{n(qa)^n}{1-q^n/a}.$$

Proof. Setting $c = 0$ and $ad = q$ in proposition 7.1, we conclude that

$$\frac{(q;q)_\infty^4}{(a, q/a; q)_\infty^2} = \sum_{n=0}^{\infty} \frac{(q/a)^n}{1-aq^n}\left(n + \frac{1}{1-aq^n}\right) \tag{7.1}$$
$$= \frac{1}{(1-a)^2} + \sum_{n=1}^{\infty} \frac{n(q/a)^n}{1-aq^n} + \sum_{n=1}^{\infty} \frac{(q/a)^n}{(1-aq^n)^2}.$$

By a direct calculation, we can find the following elementary identity:

$$\sum_{n=1}^{\infty} \frac{(q/a)^n}{(1-aq^n)^2} = a^{-2} \sum_{n=1}^{\infty} \frac{n(qa)^n}{1-q^n/a}.$$

Substituting this equation into (7.1) and then multiplying both sides of the resulting equation by $(1-a)^2$, we complete the proof of Proposition 7.3. □

On taking $a = -1$ Proposition 7.3, we immediately conclude that

$$\frac{(q;q)_\infty^4}{(-q;q)_\infty^4} = 1 + 8\sum_{n=1}^{\infty} \frac{n(-q)^n}{1+q^n}.$$

On replacing q by $-q$ in this equation, we can obtain Jacobi's four-square identity (see, for example [8, p. 61])

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 32\sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}}.$$

On writing q by q^2 and then setting $a = q$ in the first identity in (7.1), we deduce that

$$\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4} = \sum_{n=0}^{\infty} \frac{nq^n}{1-q^{2n+1}} + \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{2n+1})^2}$$
$$= \sum_{n=0}^{\infty} \frac{nq^n}{1-q^{2n+1}} + \sum_{n=0}^{\infty} \frac{(n+1)q^n}{1-q^{2n+1}}$$
$$= \sum_{n=0}^{\infty} \frac{(2n+1)q^n}{1-q^{2n+1}},$$

which is equivalent to the Legendre four triangular numbers identity (see, for example [8, p. 72])

$$\left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right)^4 = \sum_{n=0}^{\infty} \frac{(2n+1)q^n}{1-q^{2n+1}}.$$

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