(t, ℓ)-STABILITY AND COHERENT SYSTEMS

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ABSTRACT. Let X be a non-singular irreducible complex projective curve of genus $g \geq 2$. The concept of stability of coherent systems over X depends on a positive real parameter α, given then a (finite) family of moduli spaces of coherent systems. We use (t, ℓ)-stability to prove the existence of coherent systems over X that are α-stable for all allowed $\alpha > 0$.

1. INTRODUCTION

Let X be a non-singular irreducible complex projective curve of genus $g \geq 2$. A coherent system of type (n, d, k) on X is a pair (E, V) where E is a vector bundle on X of rank n and degree d and $V \subset H^0(X, E)$ is a linear subspace of dimension k. For any real number α there is a concept of α-stability and there exist moduli spaces $G(\alpha; n, d, k)$ of α-stable coherent systems of type (n, d, k) (see [17] and [24]). A necessary condition for the non-emptiness of $G(\alpha; n, d, k)$ is that $\alpha > 0$. Thus, there is a family of moduli spaces $G(\alpha; n, d, k)$ of α-stable coherent systems of type (n, d, k) (see [17] and [24]) parameterised by $\mathbb{R}^+$. Moreover, there are finitely many critical values $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L$ of α; as α varies, the concept of α-stability remains constant between two consecutive critical values. We denote by $G_0(n, d, k)$ (resp. $G_L(n, d, k)$) the moduli spaces corresponding to $0 < \alpha < \alpha_1$ (resp. $\alpha > \alpha_L$). The moduli space $G_0(n, d, k)$ is related to the Brill-Noether loci, i.e. the subspaces of the moduli space of stable bundles consisting of those bundles with a prescribed number of sections (see §2). The study of coherent systems has been applied to prove, in some cases, the non-emptiness, irreducibility and the dimension of the Brill-Noether loci (see e.g. [7]).

Precise conditions for non-emptiness of $G(\alpha; n, d, k)$ are known when $k \leq n$ (see [6, Theorem 3.3]). For general curves, the first author in [8] gives a necessary and sufficient condition for $G(\alpha; n, d, n+1)$ to be non-empty, and describes geometric properties of $G(\alpha; n, d, n+1)$ (see also [3], [13] and [4]). For $k > n+1$, much less is known. There are general results due to M. Teixidor i Bigas [26], and E. Ballico [2]; they give numerical conditions that are sufficient for the non-emptiness of $G(\alpha; n, d, k)$. Teixidor i Bigas conditions are for generic curves and Ballico conditions are for very large degree.

One of the main tools used in [6] and [8], i.e. when $k \leq n+1$, was the existence of coherent systems $(E, V) \in G_L(n, d, k)$ that are α-stable for all $\alpha > 0$ allowed; in particular, the interest was on the non-emptiness of the scheme

$$U(n, d, k) := \{(E, V) \in G_0(n, d, k) \,|\, (E, V) \text{ is } \alpha\text{-stable for all } \alpha > 0 \text{ and } E \text{ is stable }\}.$$

The significance of $U(n, d, k)$ is further strengthened by the fact that a necessary condition for Butler's conjecture (see [13]) to hold, is the existence a generated coherent system in

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$U(n,d,k)$ (see in [11]). It is possible that is also a sufficient condition but we will not develop this point here.

In this paper we introduce a new technique in the study of the non-emptiness of $U(n,d,k)$ when $k \geq n+2$ that allows to ensures the existence of coherent systems in $G_0(n,d,k)$ that are α-stables for all $\alpha > 0$. The technique make use of the concept of (t,ℓ)-stability (see Definition 3.1), introduced by M.S. Narasimhan and S. Ramanan in [21] (see also [22]). The aim of this paper is to relate (t,ℓ)-stability of the vector bundle E with α-stability of the coherent system (E,V).

Write

$$\varepsilon = \begin{cases} 1 & \text{if } d \equiv g-1 \bmod n \\ 0 & \text{otherwise,} \end{cases}$$

For any positive integers $0 \leq a \leq g-1-\varepsilon$ denote by $A_a(n,d,k)$ the subscheme

$$A_a(n,d,k) := \{(E,V) \in G_0(n,d,k) : E \ \text{ is } (0,a)-\text{stable}\},$$

The next theorems (see Theorem 3.5 and 3.7) provides a criterion for the non-emptiness of $U(n,d,k)$. Let $M(n,d)$ be the moduli space of stable vector bundles over X of degree d and rank n.

Theorem 1.1. *Assume that $0 \leq a \leq g-1-\varepsilon$ and $A_a(n,d,k) \neq \emptyset$ where $d \geq 2ng+s$ and $k \geq d+n(1-g)-t$ with s,t,a integers such that $0 \leq t \leq a$ and $2t-s \leq a$. Then $A_a(n,d,k) \subset U(n,d,k)$ and $U(n,d,k) \neq \emptyset$. Moreover, if $k \leq d+n(1-g)$ then $\emptyset \neq A_a(n,d,k) \subset U(n,d,k)$ and $U(n,d,k)$ has a component of the expected dimension and birational to a Grassmannian bundle over an open set of $M(n,d)$.*

Clifford's Theorem for α-semistable coherent systems (see [19]) states that if $d \leq 2gn$, then $k \leq \frac{d}{2}+n$. Given (n,d,k) denote by λ the difference $\lambda := d-2(k-n)$.

Theorem 1.2. *Assume that $0 \leq a \leq g-1-\varepsilon$. If $d \leq 2gn$ and $\lambda \leq a$ then $A_a(n,d,k) \subset U(n,d,k)$. Moreover, if $A_a(n,d,k) \neq \emptyset$ then $U(n,d,k) \neq \emptyset$.*

For lower degrees the non-emptiness of $A_a(n,d,k)$ depends on the non emptiness of a Brill Noether locus, which, for many cases, is still an open problem. Nevertheless, for rank 2 and 3 we prove (see Theorem 3.10 and 3.12)

Theorem 1.3. *Assume $k = 2+r$ with $r \geq 1$. If there exists an integer $0 \leq a \leq g-1-\varepsilon$ such that*

$$\max\left\{d-2g-a, \frac{d-a}{2}\right\} \leq r < d-2g+\frac{g-a+\delta-3}{2+r}, \tag{1.1}$$

then $A_a(2,d,k) \subset U(2,d,k)$. Moreover, $\emptyset \neq A_a(2,d,k) \subset U(2,d,k)$.

With the notation

$$\vartheta = \begin{cases} 1 & \text{if } d-a \equiv 0 \mod 3, \\ -1 & \text{if } d-a \equiv 1 \mod 3 \\ 0 & \text{otherwise,} \end{cases}$$

we have the following theorem for rank 3.

Theorem 1.4. *Assume $k = 3 + r$ with $r \geq 1$. If there exists an integer $0 \leq a \leq g - 1 - \varepsilon$ such that*

$$\max\left\{d - 3g - a, \frac{d-a}{2}\right\} \leq r < d - 3g + \frac{2g - 2a - 1 - \vartheta}{3 + r}, \tag{1.2}$$

then $A_a(3, d, k) \subset U(3, d, k)$. Moreover, $\emptyset \neq A_a(3, d, k) \subset U(3, d, k)$.

Our numerical conditions are for any curve and for coherent systems with a general or special bundles. Also they include large and lowers degrees, so they cover part of those conditions in [26] and [2], but more importantly they extends beyond theirs conditions. Our methods give results for special curves, in particular for hyperelliptic curves, and also for coherent systems with a general or a special bundles with values outside Teixidor's parallelograms (see [10, §5] and Remark 2.1). Furthermore, since we do not use the results of [26] and [2], our results give another proof of non-emptiness for those parts cover by Teixidor i Bigas and E. Ballico which are included in our results.

In Section 2 we give some of the relevant results of the theories of Brill-Noether and coherent systems. In Section 3 we recall the main results on (t, ℓ)-stability that we will use; and we then prove our main results.

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2. Brill-Noether theory and Coherent systems

In this section we recall the main results that we will use on the Brill-Noether Theory and on coherent systems over a non-singular irreducible complex projective curve X of genus $g \geq 2$. For a more complete treatment of the subjects, see [5] and [23] and [15] and the bibliographies therein.

2.1. **Brill-Noether Theory.** Let $M(n, d)$ (resp. $\widetilde{M}(n, d)$) denote the moduli space of stable (resp. S-equivalence classes of semistable) bundles of rank n and degree d on X. The *Brill-Noether loci* are defined by

$$B(n, d, k) := \{E \in M(n, d) \,|\, h^0(E) \geq k\},$$

$$\widetilde{B}(n, d, k) := \{[E] \in \widetilde{M}(n, d) \,|\, h^0(grE) \geq k\},$$

where $[E]$ denotes the S-equivalence class of E and grE is the graded object associated with E through a Jordan-Hölder filtration. Since the Brill-Noether loci $B(n, d, k)$ are defined as determinantal varieties they are locally closed subschemes of expected dimension

$$\rho(n, d, k) := n^2(g - 1) + 1 - k(k - d + n(g - 1)).$$

The number $\rho(n, d, k)$ is often referred to as the Brill-Noether number for (g, n, d, k). We see at once that:

(1) if $n(g - 1) < d$ and $0 \leq k \leq d + n(1 - g)$ then $B(n, d, k) = M(n, d)$.
(2) If $n(g - 1) < d < 2n(g - 1)$ and $k > d + n(1 - g)$, $B(n, d, k) \subsetneqq M(n, d)$.

(3) If $0 < d \leq n(g-1)$ for any $k \geq 1$, $B(n,d,k) \subsetneqq M(n,d)$.

Recall that a semistable vector bundle $E \in \widetilde{M}(n,d)$ is called special if $h^0(E) \cdot h^1(E) \neq 0$.

Remark 2.1. (1) The special bundles are also called Brill-Noether bundles.
(2) The problem of the non emptiness of a Brill-Noether locus, for many cases, is still an open problem. In [10] it is represented in the Brill-Noether map the values of (n,d,k) for which $B(n,d,k) \neq M(n,d)$ is not empty (see also [15]).
(3) The numerical conditions in [27], which are the same as those in [26], define the so-called Teixidor's parallelograms in the Brill-Noether map. In particular, in [10, §5] one can see the existence of values (n,d,k) outside the Teixidor's parallelograms with $B(n,d,k) \neq \emptyset$ (see e.g. [10, Figure 6] for genus $g = 10$).

Clifford's Theorem for special bundles (see [9]) gives the bound $h^0(E) \leq \frac{d}{2} + n$. For a special bundle $E \in \widetilde{M}(n,d)$ with $d \geq n(g-1)$ it follows immediately that:

(1) $E^* \otimes K$ is special of degree $\leq n(g-1)$;
(2) $h^1(E) \leq ng - \frac{d}{2}$;
(3) $h^0(E) = k_0 + \imath$ for some $\imath = 1, \ldots, ng - \frac{d}{2}$ and $k_0 = d + n(1-g)$.

The Brill-Noether loci define a natural filtration

$$\ldots B(n,d,k) \subseteqq B(n,d,k-1) \subseteqq \cdots \subseteqq B(n,d,1) \subseteqq B(n,d,0) = M(n,d).$$

$$\ldots \widetilde{B}(n,d,k) \subseteqq \widetilde{B}(n,d,k-1) \subseteqq \cdots \subseteqq \widetilde{B}(n,d,1) \subseteqq \widetilde{B}(n,d,0) = \widetilde{M}(n,d),$$

called the Brill-Noether filtration or just the BN-filtration on $M(n,d)$ (resp. in $\widetilde{M}(n,d)$). Note that if $B(n,d,k) \subsetneqq M(n,d)$, $B(n,d,k+1) \subset \mathrm{Sing}B(n,d,k)$, and for many cases (see [15]) $B(n,d,k+1) = \mathrm{Sing}B(n,d,k)$ and $B(n,d,k)$ has a component of the expected dimension.

Denote by $Y_k^{n,d}$, or simply by Y_k when (n,d) are understood, the scheme given by

$$Y_k^{n,d} := B(n,d,k) - B(n,d,k+1).$$

Note that for any $E \in Y_k$, $h^0(E) = k$.

Such schemes $\{Y_k\}$ define a schematic stratification (see [16] or [1]) on $M(n,d)$. Let $\pi_2 : X \times M(n,d) \to M(n,d)$ be the projection in the second factor. Working locally in the étale topology if necessary, we can assume without loss of generality that there exists a universal family $\mathcal{U}$ over $X \times M(n,d)$. Let $\mathcal{U}_k$ be the restriction of $\mathcal{U}$ to $X \times Y_k$. The sheaf $\pi_{2*}(\mathcal{U}_k)$ is locally free of rank k. Moreover, the Grassmannian bundle $Grass(s, \pi_{2*}\mathcal{U}_k)$ of s-dimensional subspaces has dimension

$$\dim Grass(s, \pi_{2*}\mathcal{U}_k) = \dim Y_k + s(k-s).$$

Remark 2.2. Let $k_0 := d + n(1-g)$.

(1) If $d > 2n(g-1)$ then $\pi_{2*}\mathcal{U}$ is locally free sheaf of rank $d + n(1-g)$. Moreover,

$$\dim Grass(k, \pi_{2*}\mathcal{U}) = \rho(n,d,k).$$

(2) If $d \geq n(g-1)$ then $\emptyset \neq Y_{k_0}$ is an open set and for $k \leq k_0$,

$$\dim Grass(k, \pi_{2*}\mathcal{U}_{k_0}) = \rho(n,d,k).$$

(3) For $k_0+\imath$ with $\imath = 1,\dots, ng-\frac{d}{2}$,

$$\dim Grass(k,\pi_{2*}\mathcal{U}_{k_0+\imath}) = \dim Y_{k_0+\imath} + k(k_0-k)+k\imath.$$

2.2. **Coherent systems.** Let (E,V) be a coherent system of type (n,d,k) on X. A subsystem of (E,V) is a coherent system (F,W) such that $F\subset E$ is a subbundle of E and $W\subset H^0(F)\cap V$. For a real number $\alpha>0$, the α-slope of a coherent system (E,V) of type (n,d,k), denoted by $\mu_\alpha(E,V)$, is the quotient

$$\mu_\alpha(E,V) := \frac{d+\alpha k}{n}.$$

A coherent system (E,V) is α-stable (resp. α-semistable) if, for all proper subsystems (F,W),

$$\mu_\alpha(F,W) < \mu_\alpha(E,V) \quad \text{(resp. } \leq).$$

We denote by $G_0(n,d,k)$ the moduli spaces of α-stable coherent systems corresponding to small $\alpha>0$ and by $U(n,d,k)$ the subscheme

$$U(n,d,k) := \{(E,V)\in G_0(n,d,k)\,|\,(E,V) \text{ is } \alpha\text{-stable for all } \alpha>0 \text{ and } E \text{ is stable }\}.$$

The Clifford's Theorem for α-semistable coherent systems (see [19]) states that, for any α-semistable coherent system (E,V) of type (n,d,k),

$$k\leq \begin{cases} d+n(1-g) & \text{if } d\geq 2gn\\ \frac{d}{2}+n & \text{if } d<2gn.\end{cases} \tag{2.1}$$

There is a forgetful morphism

$$\Phi : G_0(n,d,k)\longrightarrow \widetilde{B}(n,d,k) : (E,V)\mapsto [E].$$

Remark 2.3. An easy computation shows that:

(1) if $E\in M(n,d)$ is stable, then, for any linear subspace $V\subset H^0(E)$ of dimension k, $(E,V)\in G_0(n,d,k)$.
(2) If $E\in B(n,d,k)$, $\Phi^{-1}(E) = Grass(k,H^0(E))$.
(3) If $E\in B(n,d,k)$ then $(E,V)\in U(n,d,k)$ if for all subsystems of type (n',d',k'), $\frac{k'}{n'}\leq\frac{k}{n}$. Moreover, if $(E,V)\in G_0(n,d,k)$ but $(E,V)\notin U(n,d,k)$, then there exists an $\alpha_i>0$ and an α_i-semistable coherent subsystem (F,W) of type (n',d',k'), such that $\frac{k}{n}\leq\frac{k'}{n'}$.

It is well known that if $d\geq 2n(g-1)$ and $k\leq d+n(1-g)$ then $G_0(n,d,k)$ is birational to the Grassmannian bundle $Grass(k,\pi_{2*}\mathcal{U})$ and $\dim G_0(n,d,k)=\rho(n,d,k)$. Moreover, if $k_0 = d+n(1-g)$, from Remark 2.2,(2),

$$\dim\Phi^{-1}(Y_{k_0}) = \dim Grass(k,\pi_{2*}\mathcal{U}_{k_0+\imath}) = \rho(n,d,k).$$

The following proposition computes the dimension of $\Phi^{-1}(Y_{k_0+\imath})\subset G_0(n,d,k)$ for $d\geq n(g-1)$ and $\imath = 0,1,\dots,ng-\frac{d}{2}$.

Proposition 2.4. *Let $d\geq n(g-1)$ and $k_0=d+n(1-g)$. If $\imath = 0,1,\dots,ng-\frac{d}{2}$ and $c=\dim M(n,d)-\dim Y_{k_0+\imath}$ then for any $0\leq k\leq k_0+\imath$,*

$$\dim\Phi^{-1}(Y_{k_0+\imath}) = \rho(n,d,k)+k\imath-c.$$

Proof. We know that $\Phi^{-1}(Y_{k_0+\imath}) \cong Grass(k, \pi_{2*}\mathcal{U}_{k_0+\imath})$, where $Grass(k, \pi_{2*}\mathcal{U}_{k_0+\imath})$ is a Grassmannian bundle of rank $k(k_0+\imath-k)$ over $Y_{k_0+\imath}$.

If $c = \dim M(n,d) - \dim Y_{k_0+\imath}$ then

$$\begin{aligned}\dim \Phi^{-1}(Y_{k_0+\imath}) &= \dim Grass(k, \pi_{2*}\mathcal{U}_{k_0+\imath}) \\ &= \dim Y_{k_0+\imath} + k(k_0+\imath-k) \\ &= \dim M(n,d) - c - k(k-d+n(g-1)) + k\imath \\ &= \rho(n,d,k) + k\imath - c,\end{aligned}$$

and this is precisely the assertion of the proposition.

□

3. (t,ℓ)-stability and Main Results

In this section we summarize without proofs the relevant material on (t,ℓ)-stability. For a deeper discussion of (t,ℓ)-stable bundles we refer the reader to [22] and [20] (see also [21]).

Definition 3.1. *Let $t,\ell \in \mathbb{Z}$. A vector bundle E of rank n and degree d is (t,ℓ)-stable if, for all proper subbundles $F \subset E$,*

$$\frac{d_F + t}{n_F} < \frac{d+t-\ell}{n}.$$

Denote by $A_{t,\ell}(n,d)$ the set of (t,ℓ)-stable bundles of rank n and degree d. It is known that (t,ℓ)-stability is an open condition [22, Proposition 5.3] and that $A_{t,\ell}(n,d) \neq \emptyset$ if and only if

$$t(n-r) + r\ell < r(n-r)(g-1) + \delta_r \tag{3.1}$$

for all integers r with $1 \leq r \leq n-1$, where δ_r is the unique integer such that $0 \leq \delta_r \leq n-1$ and $r(n-r)(g-1) + \delta_r \equiv rd \bmod n$ [20, Proposition 1.9].

We are interested in the relation between the $(0,a)$-stable bundles and α-stable coherent systems. Write

$$\varepsilon = \begin{cases} 1 & \text{if } d \equiv g-1 \bmod n \\ 0 & \text{otherwise,} \end{cases}$$

Proposition 3.2. *For any $0 \leq a \leq g-1-\varepsilon$, $A_{0,a}(n,d) \neq \emptyset$ and it is an open set of the moduli space $M(n,d)$. Moreover $A_{0,g-1}(n,d) \neq \emptyset$ if and only if $d \not\equiv g-1 \bmod n$.*

Proof. From the inequalities (3.1) we have that for any $0 \leq a \leq g-1-\varepsilon$, $A_{0,a}(n,d) \neq \emptyset$. The $(0,a)$-stability implies that

$$\mu(F) < \mu(E) - \frac{a}{n} \qquad i.e. \ \frac{a}{n} < \mu(E) - \mu(F) \tag{3.2}$$

for all subbundles of E. Therefore, $(0,a)$-stability implies stability. □

We have a filtration of open sets

$$\emptyset \neq A_{0,g-1-\varepsilon}(n,d) \subset \cdots \subset A_{0,1}(n,d) \subset A_{0,0}(n,d) = M(n,d).$$

Denote by $A_a(n,d,k)$ the open subscheme

$$A_a(n,d,k) := \{(E,V) \in G_0(n,d,k) : E \text{ is } (0,a)-\text{stable}\}.$$

If $\Phi : G_0(n,d,k) \longrightarrow \widetilde{B}(n,d,k)$ is the forgetful map then

$$\Phi(A_a(n,d,k)) = A_{(0,a)}(n,d) \bigcap B(n,d,k).$$

We see at once that $A_a(n,d,k) \neq \emptyset$ in the following cases.

Proposition 3.3. *If* $\dim A_{(0,a)}(n,d)^c < \dim B(n,d,k)$ *then* $A_a(n,d,k) \neq \emptyset$*. Moreover, if* $d \geq n(g-1)$ *and* $k \leq d + n(1-g)$ *then for any* $0 \leq a \leq g-1-\varepsilon$*,* $A_a(n,d,k) \neq \emptyset$*.*

Proof. We only need to make the following observation. If $A_{(0,a)}(n,d) \bigcap B(n,d,k) \neq \emptyset$ then $A_a(n,d,k) \neq \emptyset$. The hypotheses in the proposition give $A_{(0,a)}(n,d) \bigcap B(n,d,k) \neq \emptyset$. □

Remark 3.4. We have proved more, namely that if $\dim A_{(0,a)}(n,d)^c < \dim Y_r$ then $A_a(n,d,k) \neq \emptyset$ and for $r \geq k$,

$$Grass(k, \pi_{2*}\mathcal{U}_r)|_{Y_r \bigcap A_{(0,a)}(n,d)} \subset A_a(n,d,k) \subset G_0(n,d,k).$$

Moreover, $\dim Y_r + k(r-k) \leq \dim G_0(n,d,k)$.

The following theorems establish a relation between $(0,a)$-stable bundles and α-stable coherent systems with $\alpha > 0$. Moreover, they ensures, under certain conditions, the existence of coherent systems in $G_0(n,d,k)$ that are α-stables for all $\alpha > 0$.

From now on, a will be a positive integer such that $0 \leq a < g-1-\varepsilon$.

Theorem 3.5. *Assume* $A_a(n,d,k) \neq \emptyset$ *where* $d \geq 2ng + s$ *and* $k \geq d + n(1-g) - t$ *with* s,t *integers such that* $0 \leq t \leq a$ *and* $2t - s \leq a$*. Then* $A_a(n,d,k) \subset U(n,d,k)$ *and* $U(n,d,k) \neq \emptyset$*. Moreover, if* $k \leq d + n(1-g)$ *then* $\emptyset \neq A_a(n,d,k) \subset U(n,d,k)$ *and* $U(n,d,k)$ *has a component of the expected dimension and birational to a Grassmannian bundle over an open set of* $M(n,d)$*.*

Proof. Let $(E,V) \in A_a(n,d,k)$. We shall prove that under the hypothesis given (E,V) is α-stable for all $\alpha > 0$.

Suppose for a contradiction that $(E,V) \notin U(n,d,k)$. From Remark 2.3(4) there exists an α_i-semistable coherent subsystem (F,W) of type (n',d',k'), such that $\frac{k}{n} \leq \frac{k'}{n'}$.

By hypothesis, one has

$$\frac{d + n(1-g) - t}{n} \leq \frac{k}{n} \leq \frac{k'}{n'}.$$

Assume $\mu(F) \geq 2g$. The Clifford bound (2.1) for coherent systems gives $\frac{k'}{n'} \leq \mu(F) + 1 - g$. Using this, together with the previous inequality, we obtain

$$\mu(E) + 1 - g - \frac{t}{n} \leq \frac{k'}{n'} \leq \mu(F) + 1 - g,$$

which implies

$$\mu(E) - \frac{a}{n} \leq \mu(E) - \frac{t}{n} \leq \mu(F),$$

since $0 \leq t \leq a$. This contradicts the $(0,a)$-stability of E (see (3.2)).

Assume now $\mu(F) < 2g$. The Clifford bound for (F,W) gives $\frac{k'}{n'} \leq \frac{\mu(F)}{2} + 1$. Hence

$$\mu(E) + 1 - g - \frac{t}{n} \leq \frac{k}{n} \leq \frac{k'}{n'} \leq \frac{\mu(F)}{2} + 1.$$

So, since E is $(0,a)$-stable,

$$\mu(E) - g - \frac{t}{n} \leq \frac{k}{n} \leq \frac{\mu(F)}{2} < \frac{\mu(E) - \frac{a}{n}}{2},$$

which implies

$$\mu(E) < 2g + 2\frac{t}{n} - \frac{a}{n} \leq 2g + \frac{s}{n}$$

since by hypothesis $2t - s \leq a$. This contradicts the assumption that $d \geq 2ng + s$. Hence, $(E,V) \in U(n,d,k)$ as required.

If $k \leq d + n(1-g)$, from Proposition 3.3, $A_a(n,d,k) \neq \emptyset$. Therefore the theorem follows from the observation that $\Phi(A_a(n,d,k))$ is an open set of $M(n,d)$.

□

Remark 3.6. (1) Note that the theorem does not involve any assumptions about $\Phi(G_0(n,d,k))$, it could be $M(n,d)$ or $\widetilde{B}(n,d,k) \neq M(n.d)$.

(2) In Theorem 3.5 the integer s could be negative, and is bounded by $2t - a \leq s$. In this case, if $k > d + n(1-g)$ then, $\Phi(G_0(n,d,k)) = \widetilde{B}(n,d,k) \neq M(n.d)$ and from Proposition 3.3, $A_a(n,d,k) \neq \emptyset$ if $\dim A_{(0,a)}(n,d)^c < \dim B(n,d,k)$.

(3) A slight change in the proof of Theorem 3.5 actually shows that if $(E,V) \in A_a(n,d,k)$ with E special and $h^0(E) = d + n(1-g) + \imath$ then $(E,V) \in U(n,d,k)$ if $d \geq 2ng + 2(t - \imath) - a$ and $d + n(1-g) - t \leq k$ when $0 \leq t - \imath \leq a$.

Clifford's Theorem for α-semistable coherent systems of type (n,d,k) and degree $0 < d \leq 2gn$ implies that $k \leq \frac{d}{2} + n$. We denote by λ the difference

$$\lambda := d - 2(k-n).$$

Theorem 3.7. *If $0 < d \leq 2gn$ and $\lambda \leq a$ then $A_a(n,d,k) \subset U(n,d,k)$. Moreover, if $A_a(n,d,k) \neq \emptyset$ then $U(n,d,k) \neq \emptyset$.*

Proof. Let $(E,V) \in A_a(n,d,k)$. Analysis similar to that in the proof of Theorem 3.5 shows that if $(E,V) \notin U(n,d,k)$ we get a contradiction. Indeed, suppose that there exists an α_i-semistable coherent subsystem $(F,W) \subset (E,V)$ of type (n',d',k'), such that $\frac{k}{n} \leq \frac{k'}{n'}$. Since E is $(0,a)$-stable, and hence stable,

$$\mu(F) < \mu(E) \leq 2g.$$

Thus, from Clifford's Theorem for coherent systems we have that $\frac{k'}{n'} \leq \frac{\mu(F)}{2} + 1$. Hence,

$$\frac{\mu(E)}{2} - \frac{\lambda}{2n} + 1 = \frac{k}{n} \leq \frac{k'}{n'} \leq \frac{\mu(F)}{2} + 1.$$

The assumption $\lambda \leq a$ implies that

$$\mu(E) \leq \mu(F) + \frac{\lambda}{n} \leq \mu(F) + \frac{a}{n}$$

which contradicts the $(0,a)$-stability of E. This gives $U(n,d,k) \neq \emptyset$, and the theorem follows.

□

For rank 2 and 3, we can prove that $U(n,d,k)\neq\emptyset$ for a wider range of values of d and k by computing the dimension of $A_{0,a}(n,d)^c := M(n,d)\setminus A_{0,a}(n,d)$. An estimate for this was given in [20, Theorem 1.10], but it is possible to compute it precisely using the Segre invariants. Recall (see [12]) that the m-Segre invariant $s_m(E)$ of a bundle of rank n and degree d is defined by

$$s_m(E) := \min_{F\subset E}\{md - nd_F \mid F \text{ a subbundle of rank } m \text{ of } E\},$$

that is,

$$\frac{s_m(E)}{mn} = min_{F\subset E}\{\mu(E)-\mu(F)\}. \tag{3.3}$$

Let $M(n,d,m,s)$ be the set of stable vector bundles of rank n and degree d such that the m-Segre invariant is s, that is

$$M(n,d,m,s) := \{E\in M(n,d) \mid s_m(E)=s\}.$$

In [25] (see also [12]) it was proved that for an integer $0 < s \leq m(n-m)(g-1)$ such that $s \equiv md \, mod \, n$, $M(n,d,m,s)$ is non empty and irreducible and

$$\dim\ M(n,d,m,s) = n^2(g-1)+1+s-m(n-m)(g-1).$$

In the following result we describe the $A_{0,a}(n,d)$ in terms of Segre invariants. First, we introduce the following notation

$$\tilde{s}_m := \max\{s \mid s\leq ma, \quad s\equiv\ md \, mod \, n\}, \tag{3.4}$$

and

$$s_\Delta := \min_m\{m(n-m)(g-1)-\tilde{s}_m\}. \tag{3.5}$$

Theorem 3.8. *For any* $0\leq a\leq g-1-\varepsilon$,

$$A_{0,a}(n,d) = \bigcap_{m=1}^{n-1}\left(\bigcup_{s>ma} M(n,d,m,s)\right).$$

Moreover, $dim\, A_{0,a}(n,d) = n^2(g-1)+1-s_\Delta$.

Proof. The first part follows immediately from (3.2) and (3.3).

The dimension of $(A_{0,a}(n,d))^c$ follows from the next equalities:

$$\begin{aligned}
\dim(A_{0,a}(n,d))^c &= \dim \bigcap_{m=1}^{n-1}\left[\left(\bigcup_{s>ma} M(n,d,m,s)\right)\right]^c \\
&= \dim \bigcup_{m=1}^{n-1}\left[\left(\bigcup_{s>ma} M(n,d,m,s)\right)^c\right] \\
&= \dim \bigcup_{m=1}^{n-1}\left[\left(\bigcup_{s\le ma} M(n,d,m,s)\right)\right] \\
&= \max_m \left\{\max_s\{\dim\left(M(n,d,m,s)\right)\}\right\} \\
&= \max_m \left\{\max_s\{n^2(g-1)+1+s-m(n-m)(g-1)\}\right\} \\
&= \max_s\{n^2(g-1)+\tilde{s}_m-m(n-m)(g-1)\} \\
&= n^2(g-1)+1-s_\Delta.
\end{aligned}$$

□

The following results are an application of Theorem 3.8 for vector bundles of rank 2 and 3.

Corollary 3.9. $\dim A_{0,a}(2,d)^c = 3g + a - \delta$, *where*

$$\delta = \begin{cases} 2 & \textit{if } a \equiv d \textit{ mod } 2 \\ 3 & \textit{otherwise,} \end{cases}$$

Proof. From Theorem 3.8

$$A_{0,a}(2,d)^c = \bigcup_{0\le s\le a} M(2,d,s)$$

and

$$\dim M(2,d,s) = 3g + s - 2$$

for $s \le g-1$ (see also [18, Proposition 3.1]). Since $s \equiv d \mod 2$, it follows that $\dim M(2,d,s)$ attains its maximum for $s \le a$ when $s = a$ if $a \equiv d \mod 2$ or when $s = a-1$ otherwise. The result follows. □

Theorem 3.10. *Assume $k = 2+r$ with $r \ge 1$. If there exists an integer $0 \le a \le g-1-\varepsilon$ such that*

$$\max\left\{d-2g-a, \frac{d-a}{2}\right\} \le r < d - 2g + \frac{g-a+\delta-3}{2+r}, \tag{3.6}$$

then $A_a(2,d,k) \subset U(2,d,k)$. Moreover, $\emptyset \ne A_a(2,d,k) \subset U(2,d,k)$.

Proof. We begin by proving that $A_a(2,d,k) \ne \emptyset$. Since $\dim B(2,d,k) \ge \beta(2,d,k)$, it is sufficient by Proposition 3.3 to prove that $\dim A_{0,a}(2,d)^c < \beta(2,d,k)$. According to Corollary 3.9, this means we need to prove that

$$3g + a - \delta < 4(g-1) + 1 - (k)(r - d + 2g).$$

This follows from the second inequality in (3.6).

It remains to show that $A_a(2,d,k) \subset U(2,d,k)$. For this, we argue as in the proof of Theorem 3.5 and 3.7. Let $(E,V) \in A_a(2,d,k)$ and suppose $(E,V) \notin U(2,d,k)$. Let (F,W) be a subsystem of (E,V) of type $(1,d',k')$ such that $\frac{k}{2} \le k'$. From(3.6), we have $k \ge d+2-2g-a$. If $d' \ge 2g$,

$$\mu(E)+1-g-\frac{a}{n} \le k' \le \mu(F)+1-g,$$

which implies

$$\mu(E) \le \mu(F)+\frac{a}{n}.$$

This contradicts the $(0,a)$-stability of E. If $d' < 2g$,

$$\frac{k}{2} \le \frac{d'}{2}+1 \le \frac{d-a}{4}+1.$$

This contradicts the first inequality in (3.6). Hence, $\emptyset \ne A_a(2,d,k) \subset U(2,d,k)$ as claimed. □

For rank 3 Theorem 3.8 gives three different cases.

Corollary 3.11. *If* $0 \le a \le g-1-\varepsilon$ *then* $\dim A_{0,a}(3,d)^c = 7(g-1)+1+\tilde{s_2}$, *with* $\tilde{s_2} = \max\{s | s \le 2a, \quad s \equiv 2d \mod 3\}$. *Moreover,*

(1) *if* $d-a \equiv 0 \mod 3$ *then dim* $A_{0,a}(3,d)^c = 7(g-1)+2a+1$*;*

(2) *if* $d-a \equiv 1 \mod 3$ *then dim* $A_{0,a}(3,d)^c = 7(g-1)+2a-1$*;*

(3) *if* $d-a \equiv 2 \mod 3$ *then dim* $A_{0,a}(3,d)^c = 7(g-1)+2a$.

Proof. By hypothesis we have that $m = 1,2$. Now, using (3.4) and (3.5) we have $\tilde{s_1} \le a$ with $s_1 \equiv d \mod 3$ and $\tilde{s_2} \le 2a$ with $\tilde{s_2} \equiv 2d \mod 3$. Therefore $\tilde{s_1} \le \tilde{s_2}$ and $s_\Delta = 2(g-1)-\tilde{s}_2$. Now, the result follows from Theorem 3.8.

□

With the notation

$$\vartheta = \begin{cases} 1 & \text{if } d-a \equiv 0 \mod 3, \\ -1 & \text{if } d-a \equiv 1 \mod 3 \\ 0 & \text{otherwise,} \end{cases}$$

we have the following theorem for rank 3.

Theorem 3.12. *Assume* $k = 3+r$ *with* $r \ge 1$. *If there exists an integer* $0 \le a \le g-1-\varepsilon$ *such that*

$$\max\left\{d-3g-a, \frac{d-a}{2}\right\} \le r < d-3g+\frac{2g-2a-1-\vartheta}{3+r}, \tag{3.7}$$

then $A_a(3,d,k) \subset U(3,d,k)$. *Moreover,* $\emptyset \ne A_a(3,d,k) \subset U(3,d,k)$.

Proof. As in Theorem 3.10 we begin by proving that $A_a(3,d,k) \ne \emptyset$. If we prove that $\dim A_{0,a}(3,d)^c < \dim B(3,d,k)$, the assertion follows.

It is easily seen that we can conclude from the second inequality in (3.7) that

$$7(g-1)+2a+\vartheta<9(g-1)+1-k(r-d+3g),$$

hence that $\dim A_{0,a}(3,d)^c<\beta(3,d,k)\leq\dim B(3,d,k)$, and finally that $A_a(3,d,k)\neq\emptyset$.

To show that $A_a(3,d,k)\subset U(3,d,k)$ we argue as in the proof of Theorem 3.5, 3.7 and 3.8. We leave it to the reader to verify that if $(E,V)\in A_a(3,d,k)$ and $(E,V)\notin U(3,d,k)$ we get a contradiction using the first inequality in 3.7.

$\square$

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