

ASCENDING CHAIN CONDITION FOR F -PURE THRESHOLDS WITH FIXED EMBEDDING DIMENSION

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ABSTRACT. In this paper, we prove that the set of all F -pure thresholds of ideals with fixed embedding dimension satisfies the ascending chain condition. As a corollary, given an integer d , we verify the ascending chain condition for the set of all F -pure thresholds on all d -dimensional normal l.c.i. varieties. In the process of proving these results, we also show the rationality of F -pure thresholds of ideals on non-strongly F -regular pairs.

1. INTRODUCTION

A ring R of characteristic $p > 0$ is said to be F -finite if the Frobenius morphism $F : R \rightarrow R$ is finite. Suppose that X is a normal variety over an F -finite field k of characteristic $p > 0$. We further assume that X is sharply F -pure, that is, the Frobenius homomorphism $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ locally splits. Then, for every coherent ideal sheaf $\mathfrak{a} \subsetneq \mathcal{O}_X$, we can define the F -pure threshold $\text{fpt}(X; \mathfrak{a}) \in \mathbb{R}_{\geq 0}$ in terms of Frobenius splittings (see Definition 2.7 below). Recent studies ([TW04], [Tak13], [HnBWZ16]) reveal that F -pure thresholds have a strong connection to log canonical thresholds in characteristic 0. Moreover, as seen in [TW04], [MTW05] and [BS15], the F -pure threshold itself is an interesting invariant in both commutative algebra and algebraic geometry in positive characteristic.

In [Sat17], motivated by the ascending chain condition for log canonical thresholds in characteristic 0 ([Sho92], [dFEM10], [dFEM11] and [HMX14]), the author studied the ascending chain condition for F -pure thresholds. In *loc. cit.*, it was proved that the set of all F -pure thresholds of ideals on a fixed germ of a strongly F -regular pair satisfies the ascending chain condition, where strong F -regularity is a stronger condition than sharp F -purity (see Definition 2.2).

In this paper, we extend the result of [Sat17] to the case of sharply F -pure pair under some conditions. The first result in this paper deals with the ascending chain condition for F -pure thresholds on l.c.i. varieties, which is an positive characteristic analogue of [dFEM10, Theorem 1.3].

Theorem 1.1 (Corollary 4.10). *Fix an integer $n \geq 1$ and an F -finite field k of characteristic $p > 0$. Let T be a set of all n -dimensional normal l.c.i. varieties over k which are sharply F -pure. Then the set*

$$\{\text{fpt}(X; \mathfrak{a}) \mid X \in T, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

2010 *Mathematics Subject Classification.* 14B05, 13A35, 14M10.

Key words and phrases. ascending chain condition, F -pure threshold, parameter test module, F -jumping number, locally complete intersection.

satisfies the ascending chain condition.

In the proof of [dFEM10, Theorem 1.3], they use Inversion of Adjunction, which is the reason why they consider l.c.i. varieties. On the other hand, the theory of F -adjunction, introduced by Schwede ([Sch09]), can be applied even if the variety is not a locally complete intersection. Therefore, we can employ the same strategy in a more general setting as that of [dFEM10].

Suppose that (R, \mathfrak{m}) is an F -finite Noetherian normal local ring of characteristic $p > 0$ and Δ is an effective \mathbb{Q} -Weil divisor on $\text{Spec } R$. We further assume that the pair (R, Δ) is sharply F -pure (see Definition 2.2). In this case, we can define the F -pure threshold $\text{fpt}(R, \Delta; \mathfrak{a}) \in \mathbb{R}_{\geq 0}$ for every proper ideal $\mathfrak{a} \subseteq R$. The following is the main theorem of this paper, which extends the main theorem of [Sat17].

Main Theorem (Theorem 4.7). *Fix positive integers e and N . Suppose that T is any set such that every element of T is an F -finite Noetherian normal local ring (R, \mathfrak{m}, k) with $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq N$. Let $\text{FPT}(T, e) \subseteq \mathbb{R}_{\geq 0}$ be the set of all F -pure thresholds $\text{fpt}(R, \Delta; \mathfrak{a})$ such that*

- R is an element of T ,
- \mathfrak{a} is a proper ideal of R , and
- Δ is an effective \mathbb{Q} -Weil divisor on $X = \text{Spec } R$ such that (R, Δ) is sharply F -pure and $(p^e - 1)(K_X + \Delta)$ is Cartier, where K_X is a canonical divisor on X .

Then the set $\text{FPT}(T, e)$ satisfies the ascending chain condition.

In the process of proving the main theorem, we treat the rationality problem for F -pure thresholds. In characteristic 0, since log canonical thresholds can be computed by a single log resolution, it is obvious that the log canonical threshold of any ideal on any log \mathbb{Q} -Gorenstein pair is a rational number. In [dFEM10], they use the rationality to reduce the ascending chain condition for log canonical thresholds on l.c.i. varieties to that on smooth varieties.

However, in positive characteristic, the rationality of F -pure thresholds is a more subtle problem. In [ST14], Schwede and Tucker proved that the F -pure threshold of any ideal on any log \mathbb{Q} -Gorenstein strongly F -regular pair is a rational number. In this paper, we generalize their result to the case where the pair is not necessarily strongly F -regular, under the assumption that the Gorenstein index is not divisible by the characteristic.

Theorem 1.2 (Corollary 4.2). *Suppose that (R, \mathfrak{m}) is an F -finite Noetherian normal local ring of characteristic $p > 0$ and Δ is an effective \mathbb{Q} -Weil divisor on $X = \text{Spec } R$ such that (R, Δ) is sharply F -pure and $K_X + \Delta$ is \mathbb{Q} -Cartier with index not divisible by p . Then the F -pure threshold $\text{fpt}(R, \Delta; \mathfrak{a})$ is a rational number for every proper ideal $\mathfrak{a} \subseteq R$.*

In the proof of Theorem 1.2, we introduce a new variant of parameter test modules. Assume that Δ is \mathbb{Q} -Cartier and $t \geq 0$ is a real number. Then, we define the submodule $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \subseteq \omega_X$ as an approximation of the parameter test module $\tau(\omega_X, \Delta, \mathfrak{a}^t) \subseteq$

ω_X by small perturbations of Δ (see Definition 3.3). A real number $t \geq 0$ is called an F -jumping number of $(\omega_X, \Delta_{-0}; \mathfrak{a})$ if one of the following hold:

- (1) for every $\varepsilon > 0$, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \subsetneq \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t-\varepsilon})$, or
- (2) for every $\varepsilon > 0$, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \supsetneq \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t+\varepsilon})$.

The key ingredient of the proof of Theorem 1.2 is the rationality of F -jumping numbers of $(\omega_X, \Delta_{-0}; \mathfrak{a})$ (Corollary 3.8). Theorem 1.2 follows from the rationality and F -adjunction because the F -pure threshold $\text{fpt}(R, \Delta; \mathfrak{a})$ is equal to the first F -jumping number if $X = \text{Spec } R$ is regular (Proposition 4.1). We use Theorem 1.2 to reduce the main theorem to the ascending chain condition for F -pure thresholds of ideals on a fixed F -finite regular local ring, which has already been proved in [Sat17].

Acknowledgments. The author wishes to express his gratitude to his supervisor Professor Shunsuke Takagi for his encouragement, valuable advice and suggestions. He was supported by JSPS KAKENHI Grant Number 17J04317.

2. PRELIMINARIES

2.1. F -singularities. In this subsection, we recall the definitions and some basic properties of F -singularities.

A ring R of characteristic $p > 0$ is said to be F -finite if the Frobenius morphism $F : R \rightarrow R$ is a finite ring homomorphism. A scheme X is said to be F -finite if for every open affine subscheme $U \subseteq X$, \mathcal{O}_U is F -finite. If R is an F -finite Noetherian normal domain, then R is excellent ([Kun76]) and $X = \text{Spec } R$ has a dualizing complex ω_X^\bullet , a canonical module ω_X and a canonical divisor K_X (see for example [ST17, p.4]).

Through this paper, all rings will be assumed to be F -finite of characteristic $p > 0$.

Definition 2.1. A pair (R, Δ) consists of an F -finite Noetherian normal local ring (R, \mathfrak{m}) and an effective \mathbb{Q} -Weil divisor Δ on X . A triple $(R, \Delta, \mathfrak{a}^t)$ consists of a pair (R, Δ) and a symbol \mathfrak{a}^t , where $\mathfrak{a} \subseteq R$ is an ideal and $t \geq 0$ is a real number.

Definition 2.2. Let $(R, \Delta, \mathfrak{a}^t)$ be a triple.

- (1) $(R, \Delta, \mathfrak{a}^t)$ is said to be *sharply F -pure* if there exist an integer $e > 0$ and a morphism $\varphi \in \text{Hom}_R(F_*^e R([\!(p^e - 1)\Delta\!]), R)$ such that $\varphi(F_*^e \mathfrak{a}^{\lceil t(p^e - 1) \rceil}) = R$.
- (2) $(R, \Delta, \mathfrak{a}^t)$ is said to be *strongly F -regular* if for every non-zero element $c \in R$, there exist an integer $e > 0$ and a morphism $\varphi \in \text{Hom}_R(F_*^e R([\!(p^e - 1)\Delta\!]), R)$ such that $\varphi(F_*^e (c\mathfrak{a}^{\lceil t(p^e - 1) \rceil})) = R$.

Lemma 2.3. Let $(R, \Delta, \mathfrak{a}^t)$ be a triple. Then the following hold.

- (1) If $(R, \Delta, \mathfrak{a}^t)$ is strongly F -regular, then it is sharply F -pure.
- (2) Suppose that $0 \leq \Delta' \leq \Delta$ is an \mathbb{Q} -Weil divisor and $0 \leq t' \leq t$ is a real number. If $(R, \Delta, \mathfrak{a}^t)$ is strongly F -regular (resp. sharply F -pure), then so is $(R, \Delta', \mathfrak{a}^{t'})$.
- (3) If (R, Δ) is strongly F -regular and $(R, \Delta, \mathfrak{a}^t)$ is sharply F -pure, then $(R, \Delta, \mathfrak{a}^s)$ is strongly F -regular for every $0 \leq s < t$.
- (4) Let $f \in R$ be a non-zero element and $\mathfrak{b} := f \cdot \mathfrak{a} \subseteq R$. Then, $(R, \Delta + t \text{div}_R(f), \mathfrak{a}^t)$ is strongly F -regular if and only if $(R, \Delta, \mathfrak{b}^t)$ is strongly F -regular.

(5) Let \widehat{R} be the \mathfrak{m} -adic completion and $\widehat{\Delta}$ the flat pullback of Δ to $\text{Spec } \widehat{R}$. Then, $(R, \Delta, \mathfrak{a}^t)$ is sharply F -pure if and only if $(\widehat{R}, \widehat{\Delta}, (\mathfrak{a}\widehat{R})^t)$ is sharply F -pure.

Proof. (1) and (2) follow from definitions. The proof of (3) is similar to that of [TW04, Proposition 2.2 (5)]. (4) follows from [Sch11, Lemma 3.1]. For (5), we define $I := \sum_{e, \varphi} \varphi(F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil}) \subseteq R$ (resp. $I' := \sum_{e, \varphi} \varphi(F_*^e (\mathfrak{a}\widehat{R})^{\lceil t(p^e-1) \rceil}) \subseteq \widehat{R}$), where e runs through all positive integers and φ runs through all elements in $\text{Hom}(F_*^e R(\lceil (p^e-1)\Delta \rceil), R)$ (resp. in $\text{Hom}_{\widehat{R}}(F_*^e \widehat{R}(\lceil (p^e-1)\widehat{\Delta} \rceil), \widehat{R})$). Then the triple $(R, \Delta, \mathfrak{a}^t)$ (resp. the triple $(\widehat{R}, \widehat{\Delta}, (\mathfrak{a}\widehat{R})^t)$) is sharply F -pure if and only if $I = R$ (resp. $I' = \widehat{R}$). Since

$$\text{Hom}_{\widehat{R}}(F_*^e \widehat{R}(\lceil (p^e-1)\widehat{\Delta} \rceil), \widehat{R}) \cong \text{Hom}(F_*^e R(\lceil (p^e-1)\Delta \rceil), R) \otimes_R \widehat{R},$$

we have $I' = I\widehat{R}$, which completes the proof. \square

Suppose that R is a ring of characteristic $p > 0$, $e > 0$ is a positive integer and $\mathfrak{a} \subseteq R$ is an ideal. Then we denote by $\mathfrak{a}^{\lceil p^e \rceil}$ the ideal of R generated by $\{f^{p^e} \in R \mid f \in \mathfrak{a}\}$. The following lemma is a variant of Fedder-type criteria.

Lemma 2.4 (cf. [Fed83], [HW02, Proposition 2.6]). *Suppose that (A, \mathfrak{m}) is an F -finite regular local ring of characteristic $p > 0$, $\mathfrak{a} \subseteq A$ is an ideal and $\Delta = \text{div}_A(f)/(p^e - 1)$ is an effective \mathbb{Q} -divisor with $f \in A$ and $e > 0$. Then, the triple $(A, \Delta, \mathfrak{a}^t)$ is sharply F -pure if and only if there exists an integer $n > 0$ such that*

$$f^{\frac{p^{en}-1}{p^e-1}} \mathfrak{a}^{\lceil t(p^{en}-1) \rceil} \not\subseteq \mathfrak{m}^{\lceil p^{en} \rceil}.$$

Proof. By the proof of [Sch08, Proposition 3.3], the triple $(A, \Delta, \mathfrak{a}^t)$ is sharply F -pure if and only if there exists an integer $n > 0$ and $\varphi \in \text{Hom}_R(F_*^{en} A((p^{en}-1)\Delta), A)$ such that $\varphi(F_*^{en} \mathfrak{a}^{\lceil t(p^{en}-1) \rceil}) = A$. Since

$$(p^{en}-1)\Delta = \text{div}_A(f^{\frac{p^{en}-1}{p^e-1}}),$$

the assertion follows from [Fed83, Lemma 1.6]. \square

Suppose that X is an F -finite Noetherian normal connected scheme, Δ is an effective \mathbb{Q} -Weil divisor on X , $\mathfrak{a} \subseteq \mathcal{O}_X$ is a coherent ideal sheaf and $t \geq 0$ is a real number. For any point $x \in X$, we denote by Δ_x the flat pullback of Δ to $\text{Spec } \mathcal{O}_{X,x}$.

Definition 2.5. With the notation above, we say that $(X, \Delta, \mathfrak{a}^t)$ is sharply F -pure if $(\mathcal{O}_{X,x}, \Delta_x, \mathfrak{a}_x^t)$ is sharply F -pure for every point $x \in X$.

Remark 2.6. Suppose that $X = \text{Spec } R$ is an affine scheme. Then, the above definition differs from the one given in [Sch08]. See [Sch10b].

Definition 2.7. With the notation above, assume that (X, Δ) is sharply F -pure. We define the F -pure threshold of $(X, \Delta; \mathfrak{a})$ by

$$\text{fpt}(X, \Delta; \mathfrak{a}) := \inf \{t \geq 0 \mid (X, \Delta, \mathfrak{a}^t) \text{ is not sharply } F\text{-pure}\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

When $X = \text{Spec } R$ is an affine scheme, we denote it by $\text{fpt}(R, \Delta; \mathfrak{a})$.

Lemma 2.8. *With the notation above, we assume that (X, Δ) is sharply F -pure. Then, $\text{fpt}(X, \Delta; \mathfrak{a}) = \min \{\text{fpt}(\mathcal{O}_{X,x}, \Delta_x; \mathfrak{a}_x) \mid x \in X\}$.*

Proof. We may assume that $X = \text{Spec } R$. For every $t \geq 0$, we consider $I_t := \sum_{e,\varphi} \varphi(F_*^e \mathfrak{a}^{\lceil t(p^e-1) \rceil}) \subseteq R$ as in the proof of Lemma 2.3 (5). Then, the set

$$Z_t := \{x \in X \mid (\mathcal{O}_{X,x}, \Delta_x, \mathfrak{a}_x^t) \text{ is not sharply } F\text{-pure}\} \subseteq X$$

is a closed set defined by the ideal I_t . Since R is Noetherian, there exists a real number $\varepsilon > 0$ such that Z_t is constant for all $\text{fpt}(X, \Delta; \mathfrak{a}) < t < \text{fpt}(X, \Delta; \mathfrak{a}) + \varepsilon$. Take a point $x \in Z_t$ for such t . Then we have $\text{fpt}(X, \Delta; \mathfrak{a}) = \text{fpt}(\mathcal{O}_{X,x}, \Delta_x; \mathfrak{a}_x)$, which completes the proof. \square

Proposition 2.9 ([Sch09, Theorem 5.5]). *Suppose that A is an F -finite regular local ring, $R = A/I$ is a normal ring and Δ_R is an effective \mathbb{Q} -Weil divisor on $\text{Spec } R$. Assume that the pair (R, Δ_R) is sharply F -pure and there exists an integer $e > 0$ such that $(p^e - 1)(K_X + \Delta_R)$ is Cartier. Then, there exists an effective \mathbb{Q} -Weil divisor Δ_A on $\text{Spec } A$ with the following properties:*

- (1) $(p^e - 1)\Delta_A$ is Cartier, and
- (2) Suppose that $\mathfrak{a} \subseteq R$ is an ideal and $\tilde{\mathfrak{a}} \subseteq A$ is the lift of \mathfrak{a} . Then we have $\text{fpt}(R, \Delta_R; \mathfrak{a}) = \text{fpt}(A, \Delta_A; \tilde{\mathfrak{a}})$.

2.2. Test ideals and parameter test modules. In this subsection, we recall the definitions and basic properties of test ideals and parameter test modules.

Definition 2.10. Suppose that R is an F -finite Noetherian normal domain, Δ is an effective \mathbb{Q} -Weil divisor on $X = \text{Spec } R$, $\mathfrak{a}, \mathfrak{b} \subseteq R$ are non-zero ideals and $t, s \geq 0$ are real numbers. The *test ideal* $\tau(R, \Delta, \mathfrak{a}^t \mathfrak{b}^s)$ (resp. the *parameter test module* $\tau(\omega_X, \Delta, \mathfrak{a}^t \mathfrak{b}^s)$) is the unique smallest non-zero ideal $J \subseteq R$ (resp. non-zero submodule $J \subseteq \omega_X$) such that

$$\varphi(F_*^e(\mathfrak{a}^{\lceil t(p^e-1) \rceil} \mathfrak{b}^{\lceil s(p^e-1) \rceil} J)) \subseteq J$$

for every integer $e \geq 0$ and every morphism $\varphi \in \text{Hom}_R(F_*^e R(\lceil (p^e - 1)\Delta \rceil), R)$ (resp. $\varphi \in \text{Hom}_R(F_*^e \omega_X(\lceil (p^e - 1)\Delta \rceil), \omega_X)$).

The test ideal and the parameter test module always exist ([Sch10a, Theorem 6.3] and [ST14, Lemma 4.2]). If $\mathfrak{b} = R$, then we write $\tau(R, \Delta, \mathfrak{a}^t)$ (resp. $\tau(\omega_X, \Delta, \mathfrak{a}^t)$). If $\mathfrak{a} = \mathfrak{b} = R$, then we write $\tau(R, \Delta)$ (resp. $\tau(\omega_X, \Delta)$). If $\mathfrak{a} = 0$, then we define $\tau(\omega_X, \Delta, \mathfrak{a}^t) = \tau(R, \Delta, \mathfrak{a}^t) := (0)$.

Lemma 2.11. *With the notation above, we assume that Δ is \mathbb{Q} -Cartier. Then the following hold.*

- (1) If $t \leq t'$ and $\Delta \leq \Delta'$, then $\tau(\omega_X, \Delta', \mathfrak{a}^{t'}) \subseteq \tau(\omega_X, \Delta, \mathfrak{a}^t)$.
- (2) ([ST14, Lemma 6.1]) There exists a real number $\varepsilon > 0$ such that if $t \leq t' \leq t + \varepsilon$, then $\tau(\omega_X, \Delta, \mathfrak{a}^{t'}) = \tau(\omega_X, \Delta, \mathfrak{a}^t)$.
- (3) ([ST14, Lemma 6.2]) There exists a real number $\varepsilon > 0$ such that if $t - \varepsilon \leq t' < t$, then $\tau(\omega_X, \Delta, \mathfrak{a}^{t'}) = \tau(\omega_X, \Delta, \mathfrak{a}^{t-\varepsilon})$.
- (4) ([ST14, Lemma 4.4]) Suppose that $\text{Tr}_R : F_* \omega_X \rightarrow \omega_X$ is the Grothendieck trace map ([BST15, Proposition 2.18]). Then we have

$$\text{Tr}_R(F_* \tau(\omega_X, \Delta, \mathfrak{a}^t)) = \tau(\omega_X, \Delta/p, \mathfrak{a}^{t/p}).$$

- (5) ([HT04, Theorem 4.2], cf. [BSTZ10, Lemma 3.26]) If \mathfrak{a} is generated by l elements and $l \leq t$, then $\tau(\omega_X, \Delta, \mathfrak{a}^t \mathfrak{b}^s) = \mathfrak{a} \tau(\omega_X, \Delta, \mathfrak{a}^{t-1} \mathfrak{b}^s)$.
- (6) ([Sch11, Lemma 3.1]) If $\mathfrak{b} = (f)$ is a non-zero principal ideal, then we have $\tau(\omega_X, \Delta, \mathfrak{a}^t \mathfrak{b}^s) = \tau(\omega_X, \Delta + s \operatorname{div}(f), \mathfrak{a}^t)$.
- (7) For an integer $r \geq 1$, we have $\tau(\omega_X, \Delta, \mathfrak{a}^{rt}) = \tau(\omega_X, \Delta, (\mathfrak{a}^r)^t)$.

Proof. Take a canonical divisor K_X such that $-K_X$ is effective. Then we have $\tau(\omega_X, \Delta, \mathfrak{a}^t \mathfrak{b}^s) = \tau(R, \Delta - K_X, \mathfrak{a}^t \mathfrak{b}^s)$ ([ST14, Lemma 4.2]). Therefore, the assertions in (5) and (6) follow from the same assertions for test ideals. The proof of (7) is similar to the proof of (6). \square

Remark 2.12. Suppose that X is an F -finite Noetherian normal connected scheme which has a canonical module ω_X , $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ are coherent ideals, and $t, s \geq 0$ are real numbers. Since parameter test modules are compatible with localization ([HT04, Proposition 3.1]), we can define the parameter test module $\tau(\omega_X, \Delta, \mathfrak{a}^t \mathfrak{b}^s) \subseteq \omega_X$.

Lemma 2.13. *Let $(A, \Delta, \mathfrak{a}^t)$ be a triple such that A is a regular local ring.*

- (1) *If $(A, \Delta, \mathfrak{a}^t)$ is sharply F -pure, then for any rational numbers $0 < \varepsilon, \varepsilon' < 1$, the triple $(A, (1 - \varepsilon)\Delta, \mathfrak{a}^{t(1-\varepsilon')})$ is strongly F -regular.*
- (2) *If $(p^e - 1)\Delta$ is Cartier for an integer $e > 0$ and $(A, (1 - \varepsilon)\Delta, \mathfrak{a}^t)$ is strongly F -regular for every $0 < \varepsilon < 1$, then the triple $(A, \Delta, \mathfrak{a}^{t(1-\varepsilon')})$ is sharply F -pure for every $0 < \varepsilon' < 1$.*

Proof. For (1), we assume that the triple $(A, \Delta, \mathfrak{a}^t)$ is sharply F -pure. Since A is strongly F -regular ([HH89]), it follows from Lemma 2.3 (2) and (3) that $(A, (1 - \varepsilon)\Delta)$ is strongly F -regular for every $0 < \varepsilon < 1$. Then applying Lemma 2.3 (2) and (3) again, we see that $(A, (1 - \varepsilon)\Delta, \mathfrak{a}^{t(1-\varepsilon')})$ is strongly F -regular for every $0 < \varepsilon, \varepsilon' < 1$.

For (2), set $q := p^e$ and suppose that $\Delta = \operatorname{div}(f)/(q - 1)$ for some non-zero element $f \in A$. Take an integer $l > t$ such that \mathfrak{a} is generated by at most l elements and set $a_n := (l - t)/(q^n - 1)$ for every integer $n \geq 0$. Since for any triple, it is strongly F -regular if and only if the test ideal is trivial ([Tak04, Corollary 2.10], see also [Sch10a, Corollary 4.6]), we have $\tau(A, ((q^n - 1)/q^n)\Delta, \mathfrak{a}^t) = A$ for every integer $n \geq 0$.

Since A is regular local, we may identify test ideals on A with parameter test modules. Set $\varphi := \operatorname{Tr}_A^e(F_*^e(f \cdot -)) \in \operatorname{Hom}_A(F_*^e A((q - 1)\Delta, A))$. Then it follows from Lemma 2.11 (4), (5) and (6) that

$$\begin{aligned}
A &= \tau(A, ((q^n - 1)/q^n)\Delta, \mathfrak{a}^t) \\
&= \operatorname{Tr}_A^{en}(F_*^{en} \tau(A, f^{(q^n - 1)/(q - 1)} \mathfrak{a}^{tq^n})) \\
&= \varphi^n(F_*^{en} \tau(A, \mathfrak{a}^{tq^n})) \\
&\subseteq \varphi^n(F_*^{en} \mathfrak{a}^{\lceil tq^n - l \rceil}) \\
&\subseteq \varphi^n(F_*^{en} \mathfrak{a}^{\lceil (t - a_n)(q^n - 1) \rceil}),
\end{aligned}$$

which proves that $(A, \Delta, \mathfrak{a}^{t - a_n})$ is sharply F -pure for every integer $n \geq 0$. Since $\lim_{n \rightarrow \infty} a_n = 0$, the triple $(A, \Delta, \mathfrak{a}^{t(1-\varepsilon')})$ is sharply F -pure for every $0 < \varepsilon' < 1$. \square

2.3. Ultraproduct. In this subsection, we define the catapower of a Noetherian local ring and recall some properties.

Definition 2.14. Let \mathfrak{U} be a collection of subsets of \mathbb{N} . \mathfrak{U} is called an *ultrafilter* if the following properties hold:

- (1) $\emptyset \notin \mathfrak{U}$.
- (2) For every subsets $A, B \subseteq \mathbb{N}$, if $A \in \mathfrak{U}$ and $A \subseteq B$, then $B \in \mathfrak{U}$.
- (3) For every subsets $A, B \subseteq \mathbb{N}$, if $A, B \in \mathfrak{U}$, then $A \cap B \in \mathfrak{U}$.
- (4) For every subset $A \subseteq \mathbb{N}$, if $A \notin \mathfrak{U}$, then $\mathbb{N} \setminus A \in \mathfrak{U}$.

An ultrafilter \mathfrak{U} is called *non-principal* if the following holds:

- (5) If A is a finite subset of \mathbb{N} , then $A \notin \mathfrak{U}$.

By Zorn's Lemma, there exists a non-principal ultrafilter. From now on, we fix a non-principal ultrafilter \mathfrak{U} .

Definition 2.15. Let T be a set. We define the equivalence relation \sim on the set $T^{\mathbb{N}}$ by

$$(a_m)_m \sim (b_m)_m \text{ if and only if } \{m \in \mathbb{N} \mid a_m = b_m\} \in \mathfrak{U}.$$

We define the *ultrapower* of T as

$${}^*T := T^{\mathbb{N}} / \sim.$$

The class of $(a_m)_m \in T^{\mathbb{N}}$ is denoted by $\text{ulim}_m a_m$. If T is a ring (resp. local ring, field), then so is *T . Moreover, if T is an F -finite field of characteristic $p > 0$, then so is *T . (see [Sat17, Proposition 2.14]).

Definition 2.16 ([Scho10]). Suppose that (R, \mathfrak{m}) is a Noetherian local ring and $({}^*R, {}^*\mathfrak{m})$ is the ultrapower. We define the *catapower* $R_{\#}$ as the quotient ring

$$R_{\#} := {}^*R / (\cap_n ({}^*\mathfrak{m})^n).$$

Proposition 2.17 ([Scho10, Theorem 8.1.19]). *Suppose that (R, \mathfrak{m}, k) is a Noetherian local ring of equicharacteristic and \widehat{R} is the \mathfrak{m} -adic completion of R . We fix a coefficient field $k \subseteq \widehat{R}$. Then we have*

$$R_{\#} \cong \widehat{R} \widehat{\otimes}_k ({}^*k).$$

In particular, if (R, \mathfrak{m}) is an F -finite regular local ring, then so is $R_{\#}$.

Suppose that (R, \mathfrak{m}) is a Noetherian local ring, $R_{\#}$ is the catapower and $a_m \in R$ for every m . We denote by $[a_m]_m \in R_{\#}$ the image of $(a_m)_m \in R^{\mathbb{N}}$ by the natural projection $R^{\mathbb{N}} \rightarrow R_{\#}$. Let $\mathfrak{a}_m \subseteq R$ be an ideal for every $m \in \mathbb{N}$. We denote by $[\mathfrak{a}_m]_m \subseteq R_{\#}$ the image of the ideal $\prod_m \mathfrak{a}_m \subseteq R^{\mathbb{N}}$ by the projection $R^{\mathbb{N}} \rightarrow R_{\#}$.

Proposition-Definition 2.18 ([Gol98, Theorem 5.6.1]). Let $\{a_m\}_{m \in \mathbb{N}}$ be a sequence of real numbers such that there exist real numbers M_1, M_2 which satisfies $M_1 < a_m < M_2$ for every $m \in \mathbb{N}$. Then there exists a unique real number $w \in \mathbb{R}$ such that for every real number $\varepsilon > 0$, we have

$$\{m \in \mathbb{N} \mid |w - a_m| < \varepsilon\} \in \mathfrak{U}.$$

We denote this number w by $\text{sh}(\text{ulim}_m a_m)$ and call it the *shadow* of $\text{ulim}_m a_m \in {}^*\mathbb{R}$.

3. A VARIANT OF PARAMETER TEST MODULES

In this section, we define a variant of parameter test modules and prove the rationality of F -jumping numbers.

Proposition 3.1. *Let $(X = \text{Spec } R, \Delta, \mathfrak{a}^t)$ be a triple such that $\Delta = sD$ for some Cartier divisor D and $t = s = 1/(p^e - 1)$ for some integer $e > 0$. Then $\tau(\omega_X, (s - \varepsilon)D, \mathfrak{a}^t)$ is constant for all sufficiently small rational numbers $0 < \varepsilon \ll 1$.*

Proof. The proof is essentially the same as that of [ST14, Lemma 6.2]. We may assume that $\mathfrak{a} \neq 0$. Set $q = p^e$. For every integer $l \geq 0$, we define the l -th truncation of s in the base q by

$$\langle s \rangle_l := \frac{q^l - 1}{q^l(q - 1)} \in \mathbb{Q}.$$

Since the sequence $\{\langle s \rangle_l\}_{l \in \mathbb{N}}$ is a strictly ascending chain which converges to s , it is enough to prove that $\tau(\omega_X, \langle s \rangle_l \cdot D, \mathfrak{a}^t)$ is constant for all sufficiently large l .

Take the normalized blowup $\pi : Y \rightarrow X$ along \mathfrak{a} . Let G be the Cartier divisor on Y such that $\mathcal{O}_Y(-G) = \mathfrak{a} \cdot \mathcal{O}_Y$. Take the Grothendieck trace maps $\text{Tr}_\pi : \pi_* \omega_Y \rightarrow \omega_X$, $\text{Tr}_X : F_* \omega_X \rightarrow \omega_X$ and $\text{Tr}_Y : F_* \omega_Y \rightarrow \omega_Y$ ([BST15, Proposition 2.18]). As in [BST15, p.4], we have $\text{Tr}_X \circ F_*(\text{Tr}_\pi) = \text{Tr}_\pi \circ \pi_*(\text{Tr}_Y)$ and Tr_π is injective. In particular, we may consider $\pi_* \omega_Y$ as a submodule of ω_X .

By [ST14, Theorem 5.1], for every integer $l \geq 0$, there exists an integer m_l such that

$$\tau(\omega_X, \langle s \rangle_l \cdot D, \mathfrak{a}^t) = \text{Tr}_X^{em} (F_*^{em} \pi_*(\tau(\omega_Y, q^m(\langle s \rangle_l \cdot \pi^* D + tG)))) \quad (1)$$

for all $m \geq m_l$.

By Lemma 2.11 (3) and (6), there exists l_0 such that $\tau(\omega_Y, \langle s \rangle_l \cdot \pi^* D + tG)$ is constant for all $l \geq l_0$. For every integer $l \geq 0$, it follows from Lemma 2.11 (4) that the morphism

$$\beta_l := \text{Tr}_Y^e : F_*^e(\tau(\omega_Y, q(\langle s \rangle_l \cdot \pi^* D + tG))) \rightarrow \tau(\omega_Y, \langle s \rangle_l \cdot \pi^* D + tG)$$

is surjective. We denote the kernel by \mathcal{N}_l . Since \mathcal{N}_l is constant for all $l \geq l_0$ and $-G$ is π -ample, there exists an integer m' such that

$$R^1 \pi_*(\mathcal{N}_l \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-MG)) = 0$$

for all integers $l \geq 0$ and $M \geq (q^{m'} - 1)/(q - 1)$.

Take integers $m, n \geq 1$ and consider the surjection

$$\gamma_{n,m} := \text{Tr}_Y^e : F_*^e(\tau(\omega_X, q^m(\langle s \rangle_n \pi^* D + tG))) \rightarrow \tau(\omega_X, q^{m-1}(\langle s \rangle_n \pi^* D + tG)).$$

By Lemma 2.11 (5) and (6), $\gamma_{n,m}$ coincides with $\beta_{n-m} \otimes \mathcal{O}_Y(-(q^m - 1)/(q - 1) \cdot (\pi^* D + G))$ if $m < n$ and with $\beta_0 \otimes \mathcal{O}_Y(-q^m \langle s \rangle_n \pi^* D - (q^m - 1)/(q - 1) \cdot G)$ if $m \geq n$. Therefore, $\pi_* \gamma_{n,m}$ is surjective if $m \geq m'$.

Combining with the equation (1), we have

$$\tau(\omega_X, \langle s \rangle_l \cdot D, \mathfrak{a}^t) = \text{Tr}_X^{em'} (F_*^{em'} \pi_*(\tau(\omega_Y, q^{m'}(\langle s \rangle_l \cdot \pi^* D + tG))))$$

for every l . By the definition of l_0 , the right hand side is constant for all $l \geq l_0 + m'$. \square

Corollary 3.2. *Let $(X = \text{Spec } R, \Delta, \mathfrak{a}^t)$ be a triple such that $t \in \mathbb{Q}$ and Δ is \mathbb{Q} -Cartier. Then $\tau(\omega_X, (1 - \varepsilon)\Delta, \mathfrak{a}^t)$ is constant for all $0 < \varepsilon \ll 1$.*

Proof. By Lemma 2.11 (4) and (7), we may assume that there exists an integer $e > 0$ such that $(p^e - 1)\Delta$ is Cartier and $t = 1/(p^e - 1)$. Then the assertion follows from Proposition 3.1. \square

We define the new variant of the parameter test module as the left limit of the map $s \mapsto \tau(\omega_X, s\Delta, \mathfrak{a}^t)$ at $s = 1$.

Definition 3.3. Let $(X = \text{Spec } R, \Delta, \mathfrak{a}^t)$ be a triple such that $t \in \mathbb{Q}$ and Δ is \mathbb{Q} -Cartier. Then we define the submodule $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \subseteq \omega_X$ by $\tau(\omega_X, (1 - \varepsilon)\Delta, \mathfrak{a}^t)$ for sufficiently small $0 < \varepsilon \ll 1$.

Lemma 3.4. *Let $(X = \text{Spec } R, \Delta, \mathfrak{a}^t)$ be a triple such that $t \in \mathbb{Q}$ and Δ is \mathbb{Q} -Cartier. Then the following hold.*

- (1) *For any rational number $t < t'$, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t'}) \subseteq \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t)$.*
- (2) *For any real number $s \geq 0$, there exists $0 < \varepsilon$ such that $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{s'})$ is constant for every rational number $s < s' < s + \varepsilon$.*
- (3) *For any rational number $s > 0$, there exists $0 < \varepsilon$ such that $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{s'})$ is constant for every rational number $s - \varepsilon < s' < s$.*
- (4) *If \mathfrak{a} is generated by l elements and $t \geq l$, then we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) = \mathfrak{a}\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t-1})$.*
- (5) $\text{Tr}_X(F_*(\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t))) = \tau(\omega_X, (\Delta/p)_{-0}, \mathfrak{a}^{t/p})$.
- (6) *If $r\Delta$ is Cartier, then $\tau(\omega_X, (r+1)\Delta_{-0}, \mathfrak{a}^t) = \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \otimes \mathcal{O}_X(-r\Delta)$.*

Proof. (1), (4), (5) and (6) follow from Lemma 2.11. (2) follows from (1) and the ascending chain condition for the set of ideals in R .

For (3), we take a positive integer r such that rs is integer and $r\Delta$ is Cartier. By Lemma 2.11 (3), there exists $\delta > 0$ such that $\tau(\omega_X, (\mathfrak{a}^{rs}\mathcal{O}_X(-r\Delta))^{(1-\varepsilon)/r})$ is constant for all rational numbers $0 < \varepsilon < \delta$. We denote this module by M .

It follows from Lemma 2.11 (6) and (7) that for every rational number $0 < \varepsilon < \delta$, we have

$$\begin{aligned} \tau(\omega_X, (1 - \varepsilon)\Delta, \mathfrak{a}^{s(1-\varepsilon)}) &= \tau(\omega_X, \mathfrak{a}^{s(1-\varepsilon)}\mathcal{O}_X(-r\Delta)^{(1-\varepsilon)/r}) \\ &= \tau(\omega_X, (\mathfrak{a}^{rs}\mathcal{O}_X(-r\Delta))^{(1-\varepsilon)/r}) \\ &= M. \end{aligned}$$

By Lemma 2.11 (1), $\tau(\omega_X, (1 - \varepsilon)\Delta, \mathfrak{a}^{s(1-\varepsilon)}) = M$ for every $0 < \varepsilon, \varepsilon' < \delta$. Therefore, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{s(1-\varepsilon)}) = M$ for every rational number $0 < \varepsilon < \delta$. \square

Definition 3.5. Let $(X = \text{Spec } R, \Delta, \mathfrak{a}^t)$ be a triple such that t is not a rational number and Δ is \mathbb{Q} -Cartier. By Lemma 3.4 (2), there exists $\varepsilon > 0$ such that the submodule $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^s) \subseteq \omega_X$ is constant for every rational number $t < s < t + \varepsilon$. We denote this submodule of ω_X by $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t)$.

We note that even if t, t', s , and s' are not rational, the same assertions as in Lemma 3.4 (1), (2), (4), (5) and (6) hold.

Definition 3.6. Let $(X = \text{Spec } R, \Delta, \mathfrak{a})$ be a triple such that Δ is \mathbb{Q} -Cartier. A real number $t \geq 0$ is called an F -jumping number of $(\omega_X, \Delta_{-0}; \mathfrak{a})$ if one of the following hold:

- (1) for every $\varepsilon > 0$, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \subsetneq \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t-\varepsilon})$, or
- (2) for every $\varepsilon > 0$, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \supsetneq \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t+\varepsilon})$.

Lemma 3.7. *Let $q \geq 2$ and $l \geq 1$ be integers and $B \subseteq \mathbb{R}_{\geq 0}$ a subset. B is a discrete set of rational numbers if the following four properties hold:*

- (1) For any $x \in B$, $qx \in B$.
- (2) For any $x \in B$, if $x > l$, then $x - 1 \in B$.
- (3) For any real number $t \in \mathbb{R}_{\geq 0}$, there exists $\varepsilon > 0$ such that $B \cap (t, t + \varepsilon) = \emptyset$.
- (4) For any rational number $t \in \mathbb{Q}_{> 0}$, there exists $\varepsilon > 0$ such that $B \cap (t - \varepsilon, t) = \emptyset$.

Proof. Let D be the set of all accumulation points of B . By [BSTZ10, Proposition 5.5], we have $D = \emptyset$. This proves that B is a discrete set. If B contains a non-rational number, then by the assumptions (1) and (2), we have infinitely many elements in $B \cap [l - 1, l]$, which contradicts to the discreteness of B . \square

Corollary 3.8. *Let $(X = \text{Spec } R, \Delta, \mathfrak{a})$ is a triple such that Δ is \mathbb{Q} -Cartier. Then the set of all F -jumping numbers of $(\omega_X, \Delta_{-0}; \mathfrak{a})$ is a discrete set of rational numbers.*

Proof. It follows from Lemma 3.4 (5) that if t is an F -jumping number of $(\omega_X, \Delta_{-0}; \mathfrak{a})$, then pt is an F -jumping number of $(\omega_X, (p\Delta)_{-0}; \mathfrak{a})$. Therefore, we may assume that there exists an integer $e > 0$ such that $(p^e - 1)\Delta$ is Cartier.

Let l be the number of minimal generators of \mathfrak{a} and B be the set of all F -jumping numbers of $(\omega_X, \Delta_{-0}; \mathfrak{a})$. Then it follows from Lemma 3.4 that B , $q = p^e$ and l satisfy the assumptions in Lemma 3.7. \square

4. PROOF OF MAIN THEOREM

In this section, applying Corollary 3.8, we prove the rationality of F -pure thresholds (Corollary 4.2). We also prove that the shadow of F -pure thresholds coincides with the F -pure threshold on the catapower (Theorem 4.5). By combining them, we give the proof of the main theorem (Theorem 4.7).

Proposition 4.1. *Suppose that $(X = \text{Spec } A, \Delta)$ is a sharply F -pure pair such that A is regular and $(p^e - 1)\Delta$ is Cartier for some $e > 0$, and $\mathfrak{a} \subseteq A$ is a non-zero proper ideal. Then the F -pure threshold $\text{fpt}(A, \Delta; \mathfrak{a})$ coincides with the first jumping number of $(\omega_X, \Delta_{-0}; \mathfrak{a})$. In particular, it is a rational number.*

Proof. It is enough to show the equation

$$\text{fpt}(A, \Delta; \mathfrak{a}) = \sup \{s \geq 0 \mid \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^s) = \omega_X\}. \quad (2)$$

Set $t := \text{fpt}(A, \Delta; \mathfrak{a})$. Since A is regular local, we may identify ω_X with A . By Lemma 2.13 (1), we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t(1-\varepsilon)}) = \omega_X$ for every $0 < \varepsilon < 1$.

On the other hand, take any rational number s such that $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^s) = \omega_X$. It follows from Lemma 2.13 (2) that $(A, \Delta, \mathfrak{a}^{s(1-\varepsilon)})$ is sharply F -pure for every $0 < \varepsilon < 1$, which proves the equation (2). \square

Corollary 4.2 (Theorem 1.2). *Suppose that (R, Δ) is a sharply F -pure pair such that $(p^e - 1)(K_R + \Delta)$ is Cartier for some integer $e > 0$ and $\mathfrak{a} \subseteq R$ is an ideal. Then the F -pure threshold $\text{fpt}(R, \Delta; \mathfrak{a})$ is a rational number.*

Proof. By Lemma 2.3 (5), we may assume that R is a complete local ring. By Proposition 2.9, we may assume that R is a regular local ring. Hence, the assertion follows from Proposition 4.1. \square

Lemma 4.3. *Suppose that A is an F -finite regular local ring, $f \in A$ is a non-zero element, $\mathfrak{a} \subseteq A$ is an ideal, $e > 0$ is an integer and $t = u/v > 0$ is a rational number with integers $u, v > 0$. Set $\mathfrak{b} := f^v \cdot \mathfrak{a}^{(p^e-1)u} \subseteq A$ and $\Delta := \text{div}_A(f)/(p^e - 1)$. Assume that (A, Δ) is sharply F -pure. Then $t \leq \text{fpt}(A, \Delta; \mathfrak{a})$ if and only if $1/(v(p^e - 1)) \leq \text{fpt}(A; \mathfrak{b})$.*

Proof. We may assume that $\mathfrak{a} \neq (0)$. First, we assume that $t \leq \text{fpt}(A, \Delta; \mathfrak{a})$. By Lemma 2.13 (1), the triple $(A, (1 - \varepsilon)\Delta, \mathfrak{a}^{(1-\varepsilon)t})$ is strongly F -regular for every $0 < \varepsilon < 1$. It follows from Lemma 2.3 (4) that the triple $(A, \mathfrak{b}^{(1-\varepsilon)/(v(p^e-1))})$ is strongly F -regular, which proves the inequality $1/(v(p^e - 1)) \leq \text{fpt}(A; \mathfrak{b})$.

On the other hand, we assume that $1/(v(p^e - 1)) \leq \text{fpt}(A; \mathfrak{b})$. By Lemma 2.3 (3) and (4), the triple $(A, (1 - \varepsilon)\Delta, \mathfrak{a}^{(1-\varepsilon)t})$ is strongly F -regular for every $0 < \varepsilon < 1$. It follows from 2.3 (2) that the triple $(A, (1 - \varepsilon)\Delta, \mathfrak{a}^{(1-\varepsilon')t})$ is strongly F -regular for every $0 < \varepsilon, \varepsilon' < 1$. By Lemma 2.13 (2), we have $t \leq \text{fpt}(A, \Delta; \mathfrak{a})$. \square

Proposition 4.4. *Suppose that A is an F -finite regular local ring, $e > 0$ is an integer, $\Delta_m = \text{div}_A(f_m)/(p^e - 1)$ is an effective \mathbb{Q} -divisor on $\text{Spec } A$ for every $m \in \mathbb{N}$ and $\mathfrak{a}_m \subseteq A$ is a proper ideal for every $m \in \mathbb{N}$. Fix a non-principal ultrafilter \mathfrak{U} . Let $A_\#$ be the catapower of A and $\mathfrak{a}_\infty := [\mathfrak{a}_m]_m \subseteq A_\#$. Assume that (A, Δ_m) is sharply F -pure for every integer m . Then the following hold.*

- (1) $f_\infty := [f_m]_m \in A_\#$ is a non-zero element.
- (2) Set $\Delta_\infty := \text{div}_{A_\#}(f_\infty)/(p^e - 1)$. Then, $(A_\#, \Delta_\infty)$ is sharply F -pure.
- (3) For every rational number $t > 0$, we have $t \leq \text{fpt}(A_\#, \Delta_\infty; \mathfrak{a}_\infty)$ if and only if $\{m \in \mathbb{N} \mid t \leq \text{fpt}(A, \Delta_m; \mathfrak{a}_m)\} \in \mathfrak{U}$.

Proof. By Lemma 2.4, we have $f_m \notin \mathfrak{m}^{[p^e]}$ for every m . It follows from [Sat17, Lemma 2.19] that $f_\infty \notin \mathfrak{m}^{[p^e]}$, which proves (1) and (2). For (3), take integers $u, v > 0$ such that $t = u/v$ and set $\mathfrak{b}_m := f_m^v \cdot \mathfrak{a}_m^{u(p^e-1)}$ for every $m \in \mathbb{N} \cup \{\infty\}$. It follows from Lemma 4.3 that $\{m \in \mathbb{N} \mid t \leq \text{fpt}(A, \Delta_m; \mathfrak{a}_m)\} \in \mathfrak{U}$ if and only if $\{m \in \mathbb{N} \mid 1/(v(p^e - 1)) \leq \text{fpt}(A; \mathfrak{b}_m)\} \in \mathfrak{U}$. We first assume that $\{m \in \mathbb{N} \mid 1/(v(p^e - 1)) \leq \text{fpt}(A; \mathfrak{b}_m)\} \in \mathfrak{U}$. Since we have $\text{sh}(\text{ulim}_m \text{fpt}(A; \mathfrak{b}_m)) = \text{fpt}(A_\#; \mathfrak{b}_\infty)$ ([Sat17, Theorem 4.7]), we have $1/(v(p^e - 1)) \leq \text{fpt}(A_\#; \mathfrak{b}_\infty)$. Applying Lemma 4.3 again, we have $t \leq \text{fpt}(A_\#, \Delta_\infty; \mathfrak{a}_\infty)$.

For the converse implication, we assume that $\{m \in \mathbb{N} \mid 1/(v(p^e - 1)) \leq \text{fpt}(A; \mathfrak{b}_m)\} \notin \mathfrak{U}$. In this case, we have $\{m \in \mathbb{N} \mid 1/(v(p^e - 1)) > \text{fpt}(A; \mathfrak{b}_m)\} \in \mathfrak{U}$ and hence we have $1/(v(p^e - 1)) \geq \text{fpt}(A_\#; \mathfrak{b}_\infty)$. If $1/(v(p^e - 1)) = \text{fpt}(A_\#; \mathfrak{b}_\infty) = \text{sh}(\text{ulim}_m \text{fpt}(A; \mathfrak{b}_m))$, then by replacing by a subsequence, we may assume that the sequence $\{\text{fpt}(A; \mathfrak{b}_m)\}_m$ is a strictly ascending chain, which is contradiction to [Sat17, Main Theorem]. Therefore, we have $1/(v(p^e - 1)) > \text{fpt}(A_\#; \mathfrak{b}_\infty)$, which proves $t > \text{fpt}(A_\#, \Delta_\infty; \mathfrak{a}_\infty)$. \square

Theorem 4.5. *With the notation above, we have*

$$\text{sh}(\text{ulim}_m \text{fpt}(A, \Delta_m; \mathfrak{a}_m)) = \text{fpt}(A_\#, \Delta_\infty, \mathfrak{a}_\infty) \in \mathbb{Q}.$$

In particular, if the limit $\lim_{m \rightarrow \infty} \text{fpt}(A, \Delta_m; \mathfrak{a}_m)$ exists, then we have

$$\lim_{m \rightarrow \infty} \text{fpt}(A, \Delta_m; \mathfrak{a}_m) = \text{fpt}(A_{\#}, \Delta_{\infty}, \mathfrak{a}_{\infty}).$$

Proof. We first note that the shadow always exists because we have $\text{fpt}(A, \Delta_m; \mathfrak{a}_m) \leq \text{fpt}(A; \mathfrak{m}) = \dim A$ for all m . For any rational number $t > 0$, it follows from Proposition 4.4 that $t \leq \text{sh}(\text{ulim}_m \text{fpt}(A, \Delta_m; \mathfrak{a}_m))$ if and only if $t \leq \text{fpt}(A_{\#}, \Delta_{\infty}; \mathfrak{a}_{\infty})$, which completes the proof. \square

Corollary 4.6. *Suppose that $e > 0$ is an integer and (A, \mathfrak{m}) is an F -finite regular local ring of characteristic $p > 0$. Then the set*

$$\text{FPT}(A, e) := \{\text{fpt}(A, \Delta; \mathfrak{a}) \mid (A, \Delta) \text{ is sharply } F\text{-pure, } (p^e - 1)\Delta \text{ is Cartier, } \mathfrak{a} \subsetneq A\}$$

satisfies the ascending chain condition.

Proof. We assume the contrary. Then there exist sequences $\{\Delta_m\}_m$ and $\{\mathfrak{a}_m\}$ such that $\{\text{fpt}(A, \Delta_m; \mathfrak{a}_m)\}_{m \in \mathbb{N}}$ is a strictly ascending chain. Set $t := \lim_m \text{fpt}(A, \Delta_m; \mathfrak{a}_m)$. By Corollary 4.2 and Corollary 4.5, we have $t = \text{fpt}(A_{\#}, \Delta_{\infty}; \mathfrak{a}_{\infty}) \in \mathbb{Q}$.

Since t is rational and $\text{fpt}(A, \Delta_m; \mathfrak{a}_m) < t$ for all m , it follows from Proposition 4.4 that $\text{fpt}(A, \Delta_{\infty}; \mathfrak{a}_{\infty}) < t$, which is contradiction. \square

For a Noetherian local ring (R, \mathfrak{m}) , we denote by $\text{emb}(R)$ the embedding dimension of R .

Theorem 4.7 (Main Theorem). *Fix positive integers e and N . Suppose that T is any set such that every element of T is an F -finite Noetherian normal local ring (R, \mathfrak{m}) with $\text{emb}(R) \leq N$. Let $\text{FPT}(T, e)$ be the set of all F -pure thresholds $\text{fpt}(R, \Delta; \mathfrak{a})$ such that*

- R is an element of T ,
- \mathfrak{a} is a proper ideal of R , and
- Δ is an effective \mathbb{Q} -Weil divisor on $X = \text{Spec } R$ such that (R, Δ) is sharply F -pure and $(p^e - 1)(K_X + \Delta)$ is Cartier.

Then the set $\text{FPT}(T, e)$ satisfies the ascending chain condition.

Proof. Take an F -finite field k such that for every $(R, \mathfrak{m}) \in T$, there exists a field extension $R/\mathfrak{m} \subseteq k$. Set $A := k[[x_1, \dots, x_N]]$. Then it follows from Lemma 2.3 (6), Proposition 2.9 and Lemma 2.4 that we have the inclusion $\text{FPT}(T, e) \subseteq \text{FPT}(A, e)$, which proves that the set $\text{FPT}(T, e)$ satisfies the ascending chain condition. \square

Corollary 4.8. *Suppose that X is a normal variety over an F -finite field. Fix an integer $e > 0$. Let $\text{FPT}(X, e)$ be the set of all $\text{fpt}(X, \Delta; \mathfrak{a})$ such that*

- \mathfrak{a} is a proper coherent ideal sheaf on X and
- Δ is an effective \mathbb{Q} -Weil divisor on X such that (X, Δ) is sharply F -pure and $(p^e - 1)(K_X + \Delta)$ is Cartier.

The set $\text{FPT}(X, e)$ satisfies the ascending chain condition.

Proof. Set $T := \{\mathcal{O}_{X,x} \mid x \in X\}$. It follows from Lemma 2.8 that $\text{FPT}(X, e) \subseteq \text{FPT}(T, e)$, which completes the proof. \square

Lemma 4.9 (cf. [dFEM10, Proposition 6.3]). *Let (R, \mathfrak{m}) be an F -finite Noetherian normal local ring of dimension d . If R is a complete intersection and sharply F -pure, then $\text{emb}(R) \leq 2d$.*

Proof. Set $N := \text{emb}(R)$ and $c := N - d$. There exists an F -finite regular local ring A and a regular sequence $f_1, \dots, f_c \in A$ with $f_i \in \mathfrak{m}^2$ such that $R \cong A/(f_1, \dots, f_c)$. By [HW02, Proposition 2.6], we have $(f_1 \cdots f_c)^{p-1} \notin \mathfrak{m}^{[p]}$.

Since $f_i \in \mathfrak{m}^2$ for every i , we have $(f_1 \cdots f_c)^{p-1} \in \mathfrak{m}^{2c(p-1)}$. It follows from the inclusion $\mathfrak{m}^{N(p-1)+1} \subseteq \mathfrak{m}^{[p]}$ that we have $2c \leq N$, which proves $N \leq 2d$. \square

Corollary 4.10 (Theorem 1.1). *Let $n \geq 1$ be an integer. Suppose that T is any set such that every element of T is an n -dimensional Noetherian normal connected l.c.i. scheme which is sharply F -pure. Then, the set*

$$\{\text{fpt}(X; \mathfrak{a}) \mid X \in T, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

satisfies the ascending chain condition.

Proof. It follows from Lemma 4.9 that $\text{emb}(\mathcal{O}_{X,x}) \leq 2n$ for every $X \in T$ and every $x \in X$. Since every $X \in T$ is Gorenstein, we apply the main theorem. \square

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