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A PIERCE REPRESENTATION THEOREM FOR VARIETIES WITH BFC

WILLIAM ZULUAGA

Abstract. We generalize the Pierce representation theorem for (commutative) rings with unit to other algebraic categories with Definable Factor Congruences by using tools from topos theory. Of independent interest, we prove that an algebraic category with right existential definable factor congruences is coextensive if and only if has center stable by complements.

Contents

1. Introduction

By a variety with $\vec{0}$ and $\vec{1}$ we understand a variety $\mathbf{V}$ for which there are 0-ary terms 0_1 , ..., 0_n , 1_1 , ..., 1_n such that $\mathbf{V} \models \vec{0} \approx \vec{1} \to x \approx y$, where $\vec{0} = (0_1, ..., 0_n)$ and $\vec{1} = (1_1, ..., 1_n)$. If $\vec{a} \in A^n$ and $\vec{b} \in B^n$, we write $[\vec{a}, \vec{b}]$ for the n-uple $((a_1, b_1), ..., (a_n, b_n)) \in (A \times B)^n$. If $A \in \mathbf{V}$ then we say that $\vec{e} = (e_1, ..., e_n) \in A^n$ is a *central element* of A if there exists an isomorphism $\tau : A \to A_1 \times A_2$, such that $\tau(\vec{e}) = [\vec{0}, \vec{1}]$. Also, we say that $\vec{e}$ and $\vec{f}$ are a *pair of complementary central elements* of A if there exists an isomorphism $\tau : A \to A_1 \times A_2$ such that $\tau(\vec{e}) = [\vec{0}, \vec{1}]$ and $\tau(\vec{f}) = [\vec{1}, \vec{0}]$. As it is well known, the direct product representations $A \to A_1 \times A_2$ of an algebra A are closely related to the concept of factor congruence. A pair of congruences (θ, δ) of an algebra A is a pair of

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complementary factor congruences of A if $\theta \cap \delta = \Delta$ and $\theta \circ \delta = \nabla$. In such case θ and δ are called *factor congruences*. In most cases, the direct decompositions of an algebra are not unique; moreover, in general the pair $(\vec{e}, \vec{f})$ of complementary central elements does not determine the pair of complementary factor congruences $(ker(\pi_1\tau), ker(\pi_2\tau))$ where the $\pi_i' s$ are the canonical projections and τ is the isomorphism between A and $A_1 \times A_2$. We call such property the *determining property* (DP).

(DP) For every pair $(\vec{e}, \vec{f})$ of complementary central elements, there is a unique pair (θ, δ) of complementary factor congruences such that, for every $i = 1, ..., n$

$$(e_i, 0_i) \in \theta \text{ and } (e_i, 1_i) \in \delta \quad \text{and} \quad (f_i, 0_i) \in \delta \text{ and } (f_i, 1_i) \in \theta$$

Observe that (DP) is in some sense the most general condition guaranteeing that central elements have all the information about direct product decompositions in the variety. In [SanchezVaggione2009] it was proved that (DP) is equivalent to each one of the following conditions:

(DFC) **V** has definable factor congruences; i.e, there is a first order formula $\psi(\vec{z}, x, y)$ such that for every $A, B \in \mathbf{V}$

$$A \times B \models \psi([\vec{0}, \vec{1}], (a, b), (a', b')) \text{ iff } a = a'$$

(BFC) **V** has Boolean factor congruences, i.e., the set of factor congruences of any algebra in $\mathcal{V}$ is a Boolean sublattice of its congruence lattice.

Let **V** a variety with BFC. If the formula ψ of (DFC) is existencial we will say that **V** is a variety with exDFC. The aim of this work is to exhibit a representation theorem for varieties with exDFC in terms of internal conected models in toposes of sheaves over a Bolean algebra. The present work is motivated by the Pierce's representation theorem for integral rigs [Zuluaga2016] and Lawvere's strategic ideas about the topos-theoretic analysis of coextensive algebraic categories [Lawvere2008].

2. Preliminaries

2.1. Notation and basic results. If A is an algebra, we denote the congruence lattice of A by $Con(A)$. As usual, the join operation of $Con(A)$ is denoted by $\vee$. If $f : A \to B$ is an homomorphism we write $Ker(f)$ for the congruence of A, defined by $\{(a, b) \in A \times A \mid f(a) = f(b)\}$. The universal congruence on A is denoted by ∇^A and Δ^A denotes the identity congruence on A (or simply ∇ and Δ when the context is clear). If $S \subseteq A$, we write $\theta^A(S)$ for the least congruence containing $S \times S$. If $\vec{a}, \vec{b} \in A^n$, then $\theta^A(\vec{a}, \vec{b})$ denotes the congruence generated by $C = \{(a_k, b_k) \mid 1 \leq k \leq n\}$. If $\vec{a}, \vec{b} \in A^n$ and $\theta \in Con(A)$, we write $\vec{a} \equiv \vec{b}(\theta)$ or $[\vec{a}, \vec{b}] \in \theta$ to express that $(a_i, b_i) \in \theta$, for $i = 1, ..., n$. We use $FC(A)$ to denote the set of factor congruences of A. A variety **V** has Boolean factor congruences if for every $A \in \mathbf{V}$, the set $FC(A)$ is a distributive sublattice of $Con(A)$. We write $\theta \diamond \delta$ in $Con(A)$ to denote that θ and δ are complementary factor congruences of A. If $\theta \in FC(A)$, we use $\theta^{\star}$ to denote the factor complement of θ. If $\theta, \delta \in Con(A)$ we say that θ and δ *permutes* if $\theta \circ \delta = \delta \circ \theta$.

A *system over* $Con(A)$ is a $2n$-ple $(\theta_1, ..., \theta_n, x_1, ..., x_n)$ such that $(x_i, x_j) \in \theta_i \vee \theta_j$, for every i, j. A *solution* of the system $(\theta_1, ..., \theta_n, x_1, ..., x_n)$ is an element $x \in A$ such that $(x, x_i) \in \theta_i$ for every $1 \leq i \leq n$. Observe that if $\theta_1 \cap ... \cap \theta_n = \Delta^A$, thus the system $(\theta_1, ..., \theta_n, x_1, ..., x_n)$ has at least one solution.

2.2. Lemma. *Let θ and δ be congruences of A. The following are equivalent:*

1. *θ and δ permutes.*
2. *$\theta \vee \delta = \theta \circ \delta$*
3. *For every $x, y \in A$, the system (θ, δ, x, y) has a solution.*

Given two sets A_1,A_2 and a relation δ in $A_1 \times A_2$, we say that δ *factorizes* if there exist sets $\delta_1 \subseteq A_1 \times A_1$ and $\delta_2 \subseteq A_2 \times A_2$ such that $\delta = \delta_1 \times \delta_2$, where

$$\delta_1 \times \delta_2 = \{((a, b), (c, d)) \mid (a, c) \in \delta_1, (b, d) \in \delta_2\}$$

So, if $\delta \in Con(A_1 \times A_2)$ factorizes in δ_1, δ_2 it follows that $\delta_i \in Con(A_i)$, for $i = 1, 2$.

2.3. Lemma. [BigelowBurris1990] *Let $\mathbf{V}$ be a variety. The following are equivalent:*

1. *$\mathbf{V}$ has BFC.*
2. *$\mathbf{V}$ has factorable factor congruences. I.e. If $A, B \in \mathbf{V}$ and $\theta \in FC(A \times B)$, then θ factorizes.*

We say that a variety has the *Fraser-Horn property* (see [FraserHorn1970]) (FHP) if every congruence on a (finite) direct product of algebras factorizes.

Given a variety $\mathbf{V}$ and a set of variables X, we use $\mathbf{F_V}(X)$ to denote the free algebra of $\mathbf{V}$ freely generated by X (or simply $\mathbf{F}(X)$, if the context is clear). If $X = \{x_1, ..., x_n\}$, then we use $\mathbf{F_V}(x_1, ..., x_n)$ instead of $\mathbf{F_V}(\{x_1, ..., x_n\})$.
As a final remark we should recall that all the algebras considered along this work will always have finite n-ary function symbols and its type (unless necessary) will be omitted.

2.4. Generalities about Varieties with DFC. Let $\mathbf{V}$ a variety with $\vec{0}$ and $\vec{1}$ and suppose that has DFC. For every $A \in \mathbf{V}$, we write $Z(A)$ to denote the set of central elements of A and $\vec{e} \diamond_A \vec{f}$ to denote that $\vec{e}$ and $\vec{f}$ are complementary central elements of A. If $\vec{e}$ is a central element of A we write $\theta^A_{\vec{0},\vec{e}}$ and $\theta^A_{\vec{1},\vec{e}}$ for the unique pair of complementary factor congruences satisfying $\vec{e} \equiv \vec{0}(\theta^A_{\vec{0},\vec{e}})$ and $\vec{e} \equiv \vec{1}(\theta^A_{\vec{1},\vec{e}})$. It follows that $\vec{0}$ and $\vec{1}$ are central elements in every algebra A and the factor congruences associated to them are $\theta^A_{\vec{0},\vec{0}} = \Delta^A$, $\theta^A_{\vec{1},\vec{0}} = \nabla^A$ and $\theta^A_{\vec{0},\vec{1}} = \nabla^A$, $\theta^A_{\vec{1},\vec{1}} = \Delta^A$, respectively. If there is no place to confusion, we write $\theta^A_{\vec{0},\vec{e}}$ and $\theta^A_{\vec{1},\vec{e}}$ simply as $\theta_{\vec{0},\vec{e}}$ and $\theta_{\vec{1},\vec{e}}$. Since **V** has BFC, factor complements are unique so we obtain the following fundamental result

2.5. Theorem. *Let $\boldsymbol{V}$ a variety with DFC. The map $g : Z(A) \to FC(A)$, defined by $g(e) = \theta^A_{\vec{0},\vec{e}}$ is a bijection and its inverse $h : FC(A) \to Z(A)$ is defined by $h(\theta) = \vec{e}$, where $\vec{e}$ is the only $\vec{e} \in A^n$ such that $\vec{e} \equiv \vec{0}(\theta)$ and $\vec{e} \equiv \vec{1}(\theta^*)$.*

As a consequence of Lemma 2.2, the we obtain the following result for varieties with BFC

2.6. Lemma. *In every algebra A of a variety $\boldsymbol{V}$ with BFC every pair of factor congruences permutes.*

Those facts, allows us to define some operations in $Z(A)$ as follows: Given $\vec{e} \in Z(A)$, the *complement* $\vec{e}^{c_A}$ of $\vec{e}$, is the only solution to the equations $\vec{z} \equiv \vec{1}(\theta_{\vec{0},\vec{e}})$ and $\vec{z} \equiv \vec{0}(\theta_{\vec{1},\vec{e}})$. Given $\vec{e}, \vec{f} \in Z(A)$, the *infimum* $\vec{e} \wedge_A \vec{f}$ is the only solution to the equations $\vec{z} \equiv \vec{0}(\theta_{\vec{0},\vec{e}} \cap \theta_{\vec{0},\vec{f}})$ and $\vec{z} \equiv \vec{1}(\theta_{\vec{1},\vec{e}} \vee \theta_{\vec{1},\vec{f}})$. Finally, the *supremum* $\vec{e} \vee_A \vec{f}$ is the only solution to the equations $\vec{z} \equiv \vec{0}(\theta_{\vec{0},\vec{e}} \vee \theta_{\vec{0},\vec{f}})$ and $\vec{z} \equiv \vec{1}(\theta_{\vec{1},\vec{e}} \cap \theta_{\vec{1},\vec{f}})$.

As result, we obtain that $\mathbf{Z}(A) = (Z(A), \wedge_A, \vee_A, {}^{c_A}, \vec{0}, \vec{1})$ is a Boolean algebra which is isomorphic to $(FC(A), \vee, \cap, {}^*, \Delta^A, \nabla^A)$. Also notice that $\vec{e} \leq_A \vec{f}$ iff $\theta^A_{\vec{0},\vec{e}} \subseteq \theta^A_{\vec{0},\vec{f}}$ iff $\theta^A_{\vec{1},\vec{f}} \subseteq \theta^A_{\vec{1},\vec{e}}$. If the context is clear enough, we will not use the subscripts in the operations of $\mathbf{Z}(A)$.

We conclude this section with a result which will be useful in Section 9.7.

2.7. Lemma. [Badano2012] *Let $\boldsymbol{V}$ be a variety with DFC, $A \in \boldsymbol{V}$. For every $\vec{e}, \vec{f} \in Z(A)$, the following holds:*

1. *$\vec{a} = \vec{e} \wedge_A \vec{f}$ if and only if $[\vec{0}, \vec{a}] \in \theta_{\vec{0},\vec{e}}$ and $[\vec{a}, \vec{f}] \in \theta_{\vec{1},\vec{e}}$.*
2. *$\vec{a} = \vec{e} \vee_A \vec{f}$ if and only if $[\vec{1}, \vec{a}] \in \theta_{\vec{1},\vec{e}}$ and $[\vec{a}, \vec{f}] \in \theta_{\vec{0},\vec{e}}$.*

3. The universal property

In the Introduction we saw that for every variety with $\vec{0}$ and $\vec{1}$, having BFC is equivalent to the variety having definable factor congruences. In this section we introduce several definitions concerning with the different sorts of definability that arise at the light of this context. In addition, we present some useful results that arise from the universal property of principal congruences in varieties with BFC.

3.1. Definition. *Let $\boldsymbol{V}$ a variety with BFC.*

1. *A formula $\rho(\vec{z},x,y)$ defines $\theta_{\vec{1},\vec{e}}$ in terms of $\vec{e}$ if for every $A,B \in \boldsymbol{V}$, $a,b \in A$ and $c,d \in B$*
$$A \times B \models \rho([\vec{0},\vec{1}],(a,c),(b,d)) \textit{ iff } c = d$$
2. *A formula $\lambda(\vec{z},x,y)$ defines $\theta_{\vec{0},\vec{e}}$ in terms of $\vec{e}$ if for every $A,B \in \boldsymbol{V}$, $a,b \in A$ and $c,d \in B$*
$$A \times B \models \lambda([\vec{0},\vec{1}],(a,c),(b,d)) \textit{ iff } a = b$$

In the last case, we also say that ρ *defines $\theta_{\vec{0},\vec{e}}$ in terms of $\vec{e}^{c}$.*

Notice that if a formula ρ defines $\theta_{\vec{1},\vec{e}}$ in terms of $\vec{e}$, for every algebra $A \in \mathbf{V}$ and $\vec{e} \in Z(A)$, it follows that $\theta^{A}_{\vec{1},\vec{e}} = \{(a,b) \mid A \models \rho(\vec{1},a,b)\}$. A similar statement is obtained when a formula λ defines $\theta_{\vec{0},\vec{e}}$ in terms of $\vec{e}$.

Altough in [SanchezVaggione2009], it was proved that the items 1. and 2. of the Definition 3.1 are equivalent (which is not trivial, since in general $\vec{0}$ and $\vec{1}$ are not interchangeables), such equivalence does not preserve the complexity of the formulas (c.f. [BadanoVaggione2013]). This situation motivates the need of introducing several definitions in terms of the complexity of the formulas envolved.

We say that a variety $\mathbf{V}$ with $\vec{0}$ and $\vec{1}$ has *right existentially defined factor congruences* (RexDFC) if the formula that defines $\theta_{\vec{1},\vec{e}}$ in terms of $\vec{e}$ is existential. Analogously, if the formula that defines $\theta_{\vec{0},\vec{e}}$ in terms of $\vec{e}$ is existential, we say that $\mathbf{V}$ has *left existentially defined factor congruences* (LexDFC). If $\mathbf{V}$ has RexDFC and LexDFC, we say that $\mathbf{V}$ has *twice existentially defined factor congruences* (TexDFC). Similar definitions arise when the considered formula is positive or equational (a finite conjunction of equations). In the positive case, we use the acronyms RpDFC, LpDFC and TpDFC to mean that the variety has *right positively defined factor congruences*, *left positively defined factor congruences* and *twice positively defined factor congruences*, respectively. For a further reading about varieties with equationally definable factor congruences the reader can consult [BadanoVaggione2013] and [BadanoVaggione2017].

3.2. Lemma. [Sanchez2010] *For every variety $\boldsymbol{V}$ with BFC the following holds:*

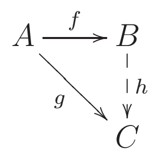
1. *RexDFC implies RpDFC.*
2. *LexDFC implies LpDFC.*
3. *TexDFC implies TpDFC.*

The following result expose the intimate relation between $\theta_{\vec{0},\vec{e}}$, and the complexity of the formula that defines it.

3.3. Lemma. [Badano2012] *Let* $\boldsymbol{V}$ *be variety with BFC.*

1. *If* $\boldsymbol{V}$ *has RpDFC, then for every* $A \in \boldsymbol{V}$ *and* $\vec{e}$ *central element of* A *we get that* $\theta^A_{\vec{1},\vec{e}} = \theta^A(\vec{1},\vec{e})$.
2. *If* $\boldsymbol{V}$ *has LpDFC, then for every* $A \in \boldsymbol{V}$ *and* $\vec{e}$ *central element of* A *we get that* $\theta^A_{\vec{0},\vec{e}} = \theta^A(\vec{0},\vec{e})$.
3. *If* $\boldsymbol{V}$ *has TpDFC, then for every* $A \in \boldsymbol{V}$ *and* $\vec{e}$ *central element of* A *we get that* $\theta^A_{\vec{1},\vec{e}} = \theta^A(\vec{1},\vec{e})$ *and* $\theta^A_{\vec{0},\vec{e}} = \theta^A(\vec{0},\vec{e})$.

We will say that an homomorphism $f : A \to P$ has the *universal property of identify the elements of* S, if for every homomorphism $g : A \to C$, such that $g(a) = g(b)$, for every $a, b \in S$; there exists a unique homomorphism $h : B \to C$, such that the diagram



commutes.

The following Lemma is an standard result in universal algebra. Nevertheless it is a key observation which will be useful for the rest of this paper.

3.4. Lemma. *Let* A *be an algebra with finite* n*-ary function symbols and* $S \subseteq A$*. Then, the canonic homomorphism* $\nu_S : A \to A/\theta(S)$ *has the universal property of identify all the elements of* S*.*

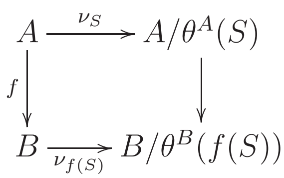
Recall that, as a consequence of Lemma 3.4, we get that, for every $\vec{a}, \vec{b} \in A^n$, the canonical homomorphism $A \to A/\theta(\vec{a}, \vec{b})$ has the universal property of identify the elements of the set $[\vec{a}, \vec{b}]$.

3.5. Corollary. *Let* $\boldsymbol{V}$ *a variety with BFC. Then:*

1. *If* $\boldsymbol{V}$ *has RexDFC, for every* $A \in \boldsymbol{V}$ *and* $\vec{e}$ *central element of* A*, the canonical morphism* $A \to A/\theta(\vec{1},\vec{e})$ *has the universal property of identify* $\vec{e}$ *with* $\vec{1}$.
2. *If* $\boldsymbol{V}$ *has LexDFC, for every* $A \in \boldsymbol{V}$ *and* $\vec{e}$ *central element of* A*, the canonical morphism* $A \to A/\theta(\vec{0},\vec{e})$ *has the universal property of identify* $\vec{e}$ *with* $\vec{0}$.

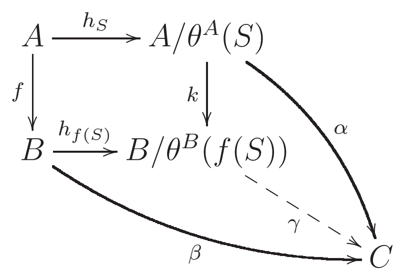
Proof. Apply Lemmas 3.2, 3.3 and 3.4. ■

3.6. Lemma. *Let A and B be algebras with finite n-ary function symbols and $f : A \to B$ an homomorphism. Then, for every $S \subseteq A$, the diagram*



is a pushout.

Proof. Let $a, b \in S$ and consider the following diagram:



Observe that,

$$h_{f(S)}(f(a)) = h_{f(S)}(f(b)).$$

Then, by Lemma 3.4, there exists a unique $k : A/\theta^A(S) \to B/\theta^B(f(S))$, such that the inner square commutes. Suppose now that $\alpha h_S = \beta f$. Thus, for $a, b \in S$ given, since $(a, b) \in \theta^A(S)$, we have that

$$\beta(f(a)) = \alpha(h_S(a))) = \alpha(h_S(b)) = \beta(f(b))$$

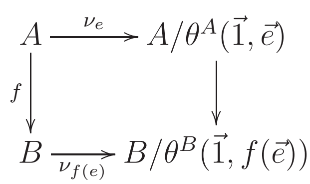
so again by Lemma 3.4, there exists a unique $\gamma : B \to C$, suh that the downward triangle commutes. Finally, to verify that the upper triangle commutes, notice that

$$(\gamma k)h_S = \gamma(kh_S) = \gamma(h_{f(S)}f) = \beta f = \alpha h_S$$

Since h_S is epi, we conclude that $\gamma k = \alpha$. ■

3.7. Corollary. *Let $\boldsymbol{V}$ be a variety with BFC, $A, B \in \boldsymbol{V}$, $f : A \to B$ be an homomorphism, $\vec{e}$ be a central element of A and $f(\vec{e}) = (f(e_1), ..., f(e_n))$, then:*

1. *If $\boldsymbol{V}$ has RexDFC, the diagram*



is a pushout.

2. *If* $\boldsymbol{V}$ *has LexDFC, the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\nu_e} & A/\theta^A(\vec{0},\vec{e}) \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \xrightarrow[\nu_{f(e)}]{} & B/\theta^B(\vec{0},f(\vec{e})) \end{array}$$

is a pushout.

Proof. Apply Lemmas 3.2, 3.3 and 3.6. ∎

4. Coextensivity and Center Stability

In the context of varieties with BFC one may be tempted to think that in general, homomorphisms preserves central elements and even complementary central elements. Unfortunately that is not case. Even in varieties with BFC having good properties like the Frasier Horn ones, the preservation of central elements is restricted to surjective homomorphisms (c.f. [Vaggione1996]). In this section we prove that the coextensivity of algebraic categories associated to varieties with RexDFC and center stable is equivalent to ask the variety having center stable by complements.

4.1. Definition. *A category with finite limits* $\mathcal{C}$ *is called extensive if has finite coproducts and the canonical functors* $1 \to \mathcal{C}/0$ *and* $\mathcal{C}/X \times \mathcal{C}/Y \to \mathcal{C}/(X+Y)$ *are equivalences.*

If the opposite $\mathcal{C}^{op}$ of a category $\mathcal{C}$ is extensive, we will say that $\mathcal{C}$ is coextensive. Classical examples of coextensive categories are the categories **Ring** and **dLat** of commutative rings with unit and bounded distributive lattices. In the following, we will use a characterization proved in [Carboni1993].

4.2. Proposition. *A category* $\mathcal{C}$ *with finite coproducts and pullbacks along its injections is extensive if and only if the following holds:*

1. *(Coproducts are disjoint.) For every* X *and* Y*, the square below is a pullback*

$$\begin{array}{ccc} 0 & \xrightarrow{!} & Y \\ \downarrow{\scriptstyle !} & & \downarrow{\scriptstyle in_1} \\ X & \xrightarrow[in_0]{} & X+Y \end{array}$$

2. *(Coproducts are universal.) For every* X, X_i, Y_i *with* $i = 0,1$ *and* $f : X \to Y_0 + Y_1$*, if the squares below are pullbacks*

$$\begin{array}{ccccc} X_0 & \xrightarrow{x_0} & X & \xleftarrow{x_1} & X_1 \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h_1} \\ Y_0 & \xrightarrow[y_0]{} & Y_0+Y_1 & \xleftarrow[y_1]{} & Y_1 \end{array}$$

then the cospan $X_0 \to X \leftarrow X_1$ *is a coproduct.*

Let $\mathbf{V}$ be a variety with BFC, $A, B \in \mathbf{V}$ and $f : A \to B$ an homomorphism. We will say that f *preserves pairs of complementary central elements* if preserves central elements; i.e, for all $e \in Z(A)$ it follows that $f(e) \in Z(B)$ and furthermore,

$$e_1 \diamond_A e_2 \Rightarrow f(e_1) \diamond_B f(e_2)$$

If every homomorphism between the algebras of $\mathbf{V}$ preserves central elements, we say that $\mathbf{V}$ has *stable center* (SC). If $\mathbf{V}$ has SC and every homomorphism between the algebras of $\mathbf{V}$ preserves central elements, we say that $\mathbf{V}$ has *center stable by complements* (CSC).

4.3. Remark. Observe that the definitions above are not trivial. For instance, let $\mathbf{L}$ be the variety of bounded lattices. It is known (see [Vaggione1999] and [FraserHorn1970]) that $\mathbf{L}$ is a variety with BFC. If $L = \mathbf{2} \times \mathbf{2}$ (with $\mathbf{2}$ the chain of two elements) and $M = \{0, 1, a, b, c\}$, with $\{a, b, c\}$ not comparables, it easily follows that L is subalgebra of M, but L is directly decomposable while M is not. So $\mathbf{L}$ is a variety which has not SC nor CSC.

Let $\mathbf{V}$ be a variety with BFC. We write $\mathcal{V}$ to denote the algebraic category associated to $\mathbf{V}$.

4.4. Lemma. *Let* $\mathbf{V}$ *be a variety with BFC. If* $\mathbf{V}$ *has RexDFC and CSC then, in* $\mathcal{V}$ *the products are stable by pushouts.*

Proof. Let $A, B \in \mathbf{B}$ and $f : A \to B$ be an homomorphism. If $A \cong A_1 \times A_2$, let us consider de diagram:

$$\begin{array}{ccccc} A_1 & \longleftarrow & A & \longrightarrow & A_2 \\ \downarrow & & \downarrow & & \downarrow \\ P_1 & \longleftarrow & B & \longrightarrow & P_2 \end{array}$$

Where P_1 and P_2 are the pushouts from the left and the right squares, respectively. If i denotes the isomorphism between A and $A_1 \times A_2$, then $A_j \cong A/Ker(\pi_j i)$ (with $j = 1, 2$). Since $Ker(\pi_1 i) \diamond Ker(\pi_2 i)$ in $Con(A)$, if $\vec{e}_j$ denotes the central element corresponding to $Ker(\pi_j i)$, thus from Lemmas 3.2 and 3.3 we have that $Ker(\pi_1 i) = \theta^A(\vec{1}, \vec{e}_2)$ and $Ker(\pi_2 i) = \theta^A(\vec{1}, \vec{e}_1)$. From, item 1. of Corollary 3.7, we get that $P_1 \cong B/\theta^B(\vec{1}, f(\vec{e}_2))$ and $P_2 \cong B/\theta^B(\vec{1}, f(\vec{e}_1))$. The universal property of pushouts implies that $B \to P_1$ coincides with $B \to \theta^B(\vec{1}, f(\vec{e}_2))$ and $B \to P_2$ with $B \to \theta^B(\vec{1}, f(\vec{e}_1))$. Since f pre-seserves pairs of complementary central elements by assumption, we can conclude that $B \cong B/\theta^B(\vec{1}, f(\vec{e}_2)) \times B/\theta^B(\vec{1}, f(\vec{e}_1)) \cong P_1 \times P_2$. ∎

4.5. Lemma. *Let* $\boldsymbol{V}$ *be a variety with BFC,* $A, B \in \boldsymbol{V}$ *and* $f : A \to B$ *an homomorphism that preserves central elements. If* $\boldsymbol{V}$ *has RexDFC and in* $\mathcal{V}$ *the binary products are stable by pushouts along* f*, thus* f *preserves pairs of complementary central elements.*

Proof. Let $A \in \mathcal{V}$ and $\vec{e}$ a central element of A. If $\vec{g}$ denotes the complementary central element of $\vec{e}$ we get that $\theta^A_{\vec{0},\vec{e}} = \theta^A(\vec{1}, \vec{g})$, so by Lemmas 3.2 and 3.3, we get that $A \cong A/\theta^A(\vec{1}, \vec{g}) \times A/\theta^A(\vec{1}, \vec{e})$. Let us, consider the diagram

$$\begin{array}{ccccc} A/\theta^A(\vec{1}, \vec{g}) & \longleftarrow & A & \longrightarrow & A/\theta^A(\vec{1}, \vec{e}) \\ \downarrow & & \downarrow f & & \downarrow \\ B/\theta^B(\vec{1}, f(\vec{g})) & \longleftarrow & B & \longrightarrow & B/\theta^B(\vec{1}, f(\vec{e})) \end{array}$$

By Corollary 3.7, both squares are pushouts, so, since the binary products are stable by pushouts along f by assumption, the span $B/\theta^B(\vec{1}, f(\vec{g})) \leftarrow B \to B/\theta^B(\vec{1}, f(\vec{e}))$ is a product. This fact implies directly that $\theta^B(\vec{1}, f(\vec{g})) \diamond \theta^B(\vec{1}, f(\vec{e}))$ in $Con(B)$. Since f preserves central elements by hypothesis, both $f(\vec{e})$ and $f(\vec{g})$ are central elements of B so, we conclude that $f(\vec{e}) \diamond_B f(\vec{g})$. ∎

4.6. Lemma. *If* $\boldsymbol{V}$ *is a variety with* $\vec{0}$ *and* $\vec{1}$*, then, in* $\mathcal{V}$ *the pushout of the projections of binary products is the terminal object.*

Proof. For every pair of $A, B \in \mathcal{V}$ the pushout of the projections $A \leftarrow A \times B \to B$ belongs to $\mathcal{V}$. It is clear that the projections send $[\vec{0}, \vec{1}] \in A \times B$ to $\vec{0}$ in A and to $\vec{1}$ in B, so $\vec{0} = \vec{1}$ in the pushout. Since $\mathbf{V}$ is a variety with $\vec{0}$ and $\vec{1}$, it follows that the pushout must be the terminal object. ∎

4.7. Proposition. *Let* $\boldsymbol{V}$ *be a variety with BFC. If* $\boldsymbol{V}$ *has RexDFC with SC, the following are equivalent:*

1. $\boldsymbol{V}$ *has CSC.*
2. $\mathcal{V}$ *is coextensive.*

Proof. Since $\mathbf{V}$ has BFC, thus is a variety with $\vec{0}$ and $\vec{1}$. So, by Lemma 4.6, we get that the pushout of the projections of binary products is the terminal object. Let us assume that $\mathbf{V}$ has SCC. From the Lema 4.4, the products are stable by pushouts. Hence, by the dual of the Proposition 4.2, $\mathcal{V}$ is coextensive. The reciprocal follows from Lemma 4.5. ∎

5. An axiomatization for connected models

In this section we prove that a variety with BFC has RexDFC (LexDFC) if and only if the factor congruence $\theta_{\vec{1},\vec{e}}$ associated to a central element $\vec{e}$, coincides with the principal congruence that identifies $\vec{1}$ with $\vec{e}$ ($\vec{0}$ with $\vec{e}$). This fact will allows us to prove that the theory of connected models for varieties with RexDFC (LexDFC) is definable by a finite set of first order formulas.

We will use the following (Grätzer) version of Maltsev's key observation on principal congruences.

5.1. Lemma. *Let A be an algebra and $a, b \in \boldsymbol{A}$, $\vec{c}$, $\vec{d} \in A^n$. Then $(a,b) \in \theta^A(\vec{c},\vec{d})$ if and only if there exist $(n+m)$-ary terms $t_1(\vec{x},\vec{u})$,...,$t_k(\vec{x},\vec{u})$ with k odd and $\vec{\lambda} \in A^m$ such that:*

$$\begin{array}{cc} a = t_1(\vec{c},\vec{\lambda}) & b = t_k(\vec{d},\vec{\lambda}) \\ t_i(\vec{c},\vec{\lambda}) = t_{i+1}(\vec{c},\vec{\lambda}),\ i\ even, & t_i(\vec{d},\vec{\lambda}) = t_{i+1}(\vec{d},\vec{\lambda}),\ i\ odd. \end{array}$$

We recall that a *principal congruence formula* is a formula $\pi(x,y,\vec{u},\vec{v})$ of the form

$$\exists_{\vec{w}}(x \approx t_1(\vec{u},\vec{w}) \wedge \bigwedge_{i\in E_k} (t_i(\vec{u},\vec{w}) \approx t_{i+1}(\vec{u},\vec{w})) \wedge \bigwedge_{i\in O_k} (t_i(\vec{v},\vec{w}) \approx t_{i+1}(\vec{v},\vec{w})) \wedge t_k(\vec{v},\vec{w}) \approx y))$$

where k is odd and t_i are terms of type τ. This fact allows us to restate the latter Lemma as

5.2. Lemma. *Let A be an algebra, $a, b \in \boldsymbol{A}$, $\vec{c}$, $\vec{d} \in A^n$. Then $(a,b) \in \theta^A(\vec{c},\vec{d})$ if and only if there exists a principal congruence formula π, such that $A \models \pi(a,b,\vec{c},\vec{d})$.*

5.3. Lemma. *Let $\boldsymbol{V}$ be a variety with DFC, $A \in \boldsymbol{V}$ and $\vec{e} \in Z(A)$:*

1. *If $\theta^A_{\vec{1},\vec{e}} = \theta^A(\vec{1},\vec{e})$ then $\theta_{\vec{1},\vec{e}}$ is definible by a formula of the form $\exists \bigwedge p \approx q$.*
2. *If $\theta^A_{\vec{0},\vec{e}} = \theta^A(\vec{0},\vec{e})$ then $\theta_{\vec{0},\vec{e}}$ is definible by a formula of the form $\exists \bigwedge p \approx q$.*

Proof. We only prove 1. because the proof of 2. is essentially the same. Let us write $P = \mathbf{F}(x,y) \times \mathbf{F}(y)$, where $\mathbf{F}(x,y)$ and $\mathbf{F}(y)$ are the free algebras generated by $\{x,y\}$ and $\{y\}$, respectively. By hyphotesis, $Ker(\pi_2) = \theta^{\mathbf{P}}_{[\vec{1},\vec{1}],[\vec{0},\vec{1}]} = \theta^P([\vec{1},\vec{1}],[\vec{0},\vec{1}])$. Since the pair $((x,y),(y,y)) \in Ker(\pi_2)$, from Lema 5.1, there exist $(n+m)$-ary terms $t_1(\vec{x},\vec{u})$,...,$t_k(\vec{x},\vec{u})$ with k odd and $\vec{u} \in P^m$ such that:

$$\begin{array}{cc} (x,y) = t_1^P[[\vec{1},\vec{1}],\vec{u}] & (y,y) = t_k^P[[\vec{0},\vec{1}],\vec{u}] \\ t_i^P[[\vec{1},\vec{1}],\vec{u}] = t_{i+1}^P[[\vec{1},\vec{1}],\vec{u}],\ i \in E_k, & t_i^P[[\vec{0},\vec{1}],\vec{u}] = t_{i+1}^P[[\vec{0},\vec{1}],\vec{u}],\ i \in O_k. \end{array} \tag{1}$$

where E_k and O_k refer to the even and odd naturals less or equal to k, respectively.

Since $\vec{u} \in P$, there are $\vec{P}(x,y) \in F(x,y)$ and $\vec{Q}(y) \in F(x,y)$, such that $\vec{u} = [\vec{P},\vec{Q}]$. Recall that $t_i^P[[\vec{R},\vec{S}],[\vec{P},\vec{Q}]] = (t_i^{F(x,y)}[\vec{R},\vec{P}], t_i^{F(y)}[\vec{S},\vec{Q}])$, for $1 \le i \le k$ and $[\vec{R},\vec{S}] \in P$, thus, from equation (1), we obtain that there exist $(n+m)$-ary terms $t_1(\vec{x},\vec{u})$,...,$t_k(\vec{x},\vec{u})$ with k odd, $\vec{P}(x,y) \in F(x,y)$ and $\vec{Q}(y) \in F(y)$, such that:

$$y = t_i^{F(y)}[\vec{0},\vec{Q}(y)], \text{for every } 1 \le i \le k$$

and

$$\begin{array}{cc} x = t_1^{F(x,y)}[\vec{1}, \vec{P}(x,y)] & y = t_k^{F(x,y)}[\vec{0}, \vec{P}(x,y)] \\ t_i^{F(x,y)}[\vec{1}, \vec{P}(x,y)] = t_{i+1}^{F(x,y)}[\vec{1}, \vec{P}(x,y)],\ i \in E_k & t_i^{F(x,y)}[\vec{0}, \vec{P}(x,y)] = t_{i+1}^{F(x,y)}[\vec{0}, \vec{P}(x,y)],\ i \in O_k \end{array}$$

Let $\varphi(x, y, \vec{z}) = \pi(x, y, \vec{1}, \vec{z})$. In order to to check that φ defines $\theta^A_{\vec{1},\vec{e}}$ in terms of $\vec{e}$ let us assume $A, B \in \mathcal{V}$ and $(a, b), (c, d) \in A \times B$. Since the free algebra functor $\mathbf{F} : \mathbf{Set} \to \mathcal{V}$ is left adjoint to the forgetful functor, in the case of $b = d$, the assingments $\alpha_A : \{x, y\} \to A$ and $\alpha_B : \{y\} \to B$, defined by $\alpha_A(x) = a$, $\alpha_A(y) = c$ and $\alpha_B(y) = b$ generate a unique pair of homomorphisms $\beta_A : \mathbf{F}(x, y) \to A$ and $\beta_B : \mathbf{F}(y) \to B$ extending α_A and α_B, respectively. Therefore, since $P \models \varphi((x, y), (y, y), [\vec{0}, \vec{1}])$, by applying $g = \beta_A \times \beta_B$ in (1), we obtain as result that $A \times B \models \varphi((a, b), (c, b), [\vec{0}, \vec{1}])$. On the other hand, if $A \times B \models \varphi((a, b), (c, d), [\vec{0}, \vec{1}])$, then there exist $[\vec{\varepsilon}, \vec{\delta}] \in A \times B$, such that

$$\begin{array}{cc} (a,b) = t_1^{A\times B}[[\vec{1}, \vec{1}], [\vec{\varepsilon}, \vec{\delta}]] & (c,d) = t_k^{A\times B}[[\vec{0}, \vec{1}], [\vec{\varepsilon}, \vec{\delta}]] \\ t_i^{A\times B}[[\vec{1}, \vec{1}], [\vec{\varepsilon}, \vec{\delta}]] = t_{i+1}^{A\times B}[[\vec{1}, \vec{1}], [\vec{\varepsilon}, \vec{\delta}]],\ i \in E_k, & t_i^{A\times B}[[\vec{0}, \vec{1}], [\vec{\varepsilon}, \vec{\delta}]] = t_{i+1}^{A\times B}[[\vec{0}, \vec{1}], [\vec{\varepsilon}, \vec{\delta}]],\ i \in O_k. \end{array} \tag{2}$$

So, since $t_i^{A\times B}[[\vec{j}, \vec{r}], [\vec{\varepsilon}, \vec{\delta}]] = (t_i^A[\vec{j}, \vec{\varepsilon}], t_i^B[\vec{r}, \vec{\delta}])$, for every $[\vec{j}, \vec{r}] \in A \times B$ and $1 \leq i \leq k$, from (2), we conclude that

$$\begin{array}{cc} b = t_1^B[\vec{1}, \vec{\varepsilon}] & d = t_k^B[\vec{1}, \vec{\varepsilon}] \\ t_i^B[\vec{1}, \vec{\varepsilon}] = t_{i+1}^B[\vec{1}, \vec{\varepsilon}],\ i \in E_k & t_i^B[\vec{1}, \vec{\varepsilon}] = t_{i+1}^B[\vec{1}, \vec{\varepsilon}],\ i \in O_k. \end{array}$$

Which by Lemma 5.1 means that $(b, d) \in \theta^B(\vec{1}, \vec{1})$. Since $\theta^B(\vec{1}, \vec{1}) = \theta^B_{\vec{1},\vec{1}}$ by assumption and $\theta^B(\vec{1}, \vec{1}) = \Delta^B$, we get that $b = d$. This concludes the proof.

■

5.4. Corollary. *Let **V** be a variety with DFC. The following are equivalent:*

1. ***V** has RexDFC if and only if, for every $A \in$ **V** and $\vec{e} \in Z(A)$, $\theta^A_{\vec{1},\vec{e}} = \theta^A(\vec{1}, \vec{e})$.*
2. ***V** has LexDFC if and only if, for every $A \in$ **V** and $\vec{e} \in Z(A)$, $\theta^A_{\vec{0},\vec{e}} = \theta^A(\vec{0}, \vec{e})$.*

Proof. In each item, the first implication follows from Lemmas 3.2, 3.3 and the last one is a consequence of Lemma 5.3. ■

We say that a set of formulas $\Sigma(\vec{z}, \vec{u})$ *defines the property* $\vec{e} \diamond_A \vec{f}$ *in* **V** if for every $A \in$ **V** and $\vec{e}, \vec{f} \in A^n$ it follows that $\vec{e} \diamond_A \vec{f}$ if and only if $A \models \sigma[\vec{e}, \vec{f}]$, for every $\sigma \in \Sigma$.

Let $\vec{e}, \vec{f} \in A^n$ and φ be the formula used in the proof of Lemma 5.3. We consider the following formulas:

$\tau_r(\vec{z},\vec{u}) = (\forall_x)(\varphi(x,x,\vec{z}))$
$\tau_s(\vec{z},\vec{u}) = (\forall_{x,y})(\varphi(x,y,\vec{z}) \to \varphi(y,x,\vec{z}))$
$\tau_t(\vec{z},\vec{u}) = (\forall_{x,y,v})(\varphi(x,v,\vec{z}) \land \varphi(v,y,\vec{z}) \to \varphi(x,y,\vec{z}))$
$\tau_i(\vec{z},\vec{u}) = (\forall_{x,y})(\varphi(x,y,\vec{z}) \land \varphi(x,y,\vec{u}) \to x \approx y)$
$\tau_p(\vec{z},\vec{u}) = (\forall_{x,y})(\exists_v)(\varphi(x,v,\vec{z}) \land \varphi(v,y,\vec{u}))$
$\tau_k(\vec{z},\vec{u}) = \bigwedge_{1\leq j\leq n} \varphi(1_j,z_j,\vec{z}) \land \bigwedge_{1\leq j\leq n} \varphi(0_j,u_j,\vec{z})$

And for every function symbol f in the languange of A:

$$\tau_f(\vec{z},\vec{u}) = (\forall_{l_1,\ldots,l_m,v_1,\ldots,v_m})(\bigwedge_{1\leq\alpha\leq m} \varphi(l_\alpha,v_\alpha,\vec{z}) \to \varphi(f(l_1,...,l_m),f(v_1,...,v_m),\vec{z}))$$

If we call $E_0 = \{\tau_\beta(\vec{z},\vec{u}) \mid \beta \in \{r,s,t,i,p,k\}\}$, $E_1 = \{\tau_\beta(\vec{u},\vec{z}) \mid \beta \in \{r,s,t,i,p,k\}\}$, $C = \{\tau_f(\vec{z},\vec{u}) \mid f \in \tau\}$, where τ is the type of A, let $\Sigma(\vec{z},\vec{u}) = E_0 \cup E_1 \cup C$.

5.5. Lemma. *Let* $\boldsymbol{V}$ *a variety with BFC.*

1. *If* $\boldsymbol{V}$ *has RexDFC, there exists a set of formulas* $\Sigma(\vec{z},\vec{u})$ *defining the property* $\vec{e}\diamond_A \vec{f}$ *in* $\boldsymbol{V}$.
2. *If* $\boldsymbol{V}$ *has LexDFC, there exists a set of formulas* $\Sigma(\vec{z},\vec{u})$ *defining the property* $\vec{e}\diamond_A \vec{f}$ *in* $\boldsymbol{V}$.

Proof. We prove 1. Let $A \in \mathbf{V}$, $\vec{e},\vec{f} \in A^n$ and suppose that $\mathbf{V}$ has LexDFC. We define the following relations in A:

$$L_{\vec{e}} = \{(a,b) \in A\times A \mid A \models \varphi[\vec{e},a,b]\} \quad L_{\vec{f}} = \{(a,b) \in A\times A \mid A \models \varphi[\vec{f},a,b]\}$$

Let $\vec{e},\vec{f} \in A^n$. Observe that formulas τ_r, τ_s and τ_t say that $L_{\vec{e}}$ is an equivalence relation on A. The set $\{\tau_f \mid \text{f is a symbol of function in the lenguage of A}\}$ says is that $L_{\vec{e}}$ is a congruence. The formula τ_i says that $L_{\vec{e}} \cap L_{\vec{f}} = \Delta^A$ and the formula τ_i says that $L_{\vec{e}} \circ L_{\vec{f}} = \nabla^A$. Finally, the formula τ_k says that $[\vec{1},\vec{e}] \in L_{\vec{e}}$ and $[\vec{0},\vec{f}] \in L_{\vec{e}}$.

It is clear that if $A \models \sigma[\vec{e},\vec{f}]$ for every $\sigma \in \Sigma$, then $L_{\vec{e}}$ and $L_{\vec{f}}$ are factor congruences of A such that $\vec{e} \equiv \vec{0}(L_{\vec{e}})$, $\vec{f} \equiv \vec{1}(L_{\vec{e}})$, $\vec{f} \equiv \vec{0}(L_{\vec{f}})$, $\vec{e} \equiv \vec{1}(L_{\vec{f}})$. That is, $L_{\vec{e}} = \theta^A_{\vec{0},\vec{e}}$ and $L_{\vec{f}} = \theta^A_{\vec{0},\vec{f}}$. Hence, $\vec{e},\vec{f} \in Z(A)$ and $\vec{e}\diamond_A \vec{f}$. On the other hand, if $\vec{e}\diamond_A \vec{f}$, from Lemmas 3.2, 3.3 and 5.3, we get that $L_{\vec{e}} = \theta^A_{\vec{0},\vec{e}} = \theta^A(\vec{0},\vec{e})$ and $L_{\vec{f}} = \theta^A_{\vec{0},\vec{f}} = \theta^A(\vec{0},\vec{f})$, so $A \models \sigma[\vec{e},\vec{f}]$ for every $\sigma \in \Sigma$. The proof of 2. is similar. ■

Again, let $\mathbf{V}$ be a variety with BFC. We write $\mathbf{V}_C$ to denote the class of connected (directly idecomposable) algebras of $\mathbf{V}$. If $A \in \mathbf{V}_C$, then we also say that A is a $\mathbf{V}$-connected algebra.

5.6. Corollary. *If* $\boldsymbol{V}$ *has RexDFC (or LexDFC), the class* $\boldsymbol{V}_C$ *is axiomatizable by a set of first order formulas.*

Proof. Suppose that $\mathbf{V}$ has RexDFC. Consider the set Σ from Proposition 5.5. It is immidiate that $A \in \mathbf{V}_{DI}$ if and only if in A the following axioms hold

$$\vec{0} \neq \vec{1} \text{ and } \forall_{\vec{e},\vec{f}} \bigwedge \Sigma(\vec{e},\vec{f}) \rightarrow ((\vec{e} = \vec{0} \wedge \vec{f} = \vec{1}) \vee (\vec{e} = \vec{1} \wedge \vec{f} = \vec{0})).$$

■

6. Connected models in a Topos

Let $\mathbf{V}$ be a variety and $\Sigma(\vec{x},\vec{y})$ the set of formulas of item 1. in Lemma 5.5. We call $\mathbb{V}$ to the theory given by the equations holding in $\mathbf{V}$ and the axiom

$$\top \vdash \bigwedge \Sigma(\vec{1},\vec{0}) \tag{3}$$

For a given topos $\mathcal{E}$, let $\mathbb{V}(\mathcal{E})$ be the category of internal models in $\mathcal{E}$ respect to $\mathbb{V}$. Observe that in **Set**, axiom (3) is equivalent to say that $\mathbf{V}$ is a variety with $\vec{0}$ and $\vec{1}$.

With the aim of understand what is a variety with RexDFC in a topos $\mathcal{E}$, observe that the proof of Lemma 5.3 suggest that we can get a weaker condition to make a variety with $\vec{0}$ and $\vec{1}$ be variety with BFC. That is: *Let* $\boldsymbol{V}$ *be a variety with* $\vec{0}$ *and* $\vec{1}$*; and let* $A, B \in \boldsymbol{V}$*. Consider the projection* $\pi_B : A \times B \to B$*. If* $Ker(\pi_B) = \theta^{A\times B}([\vec{1},\vec{1}],[\vec{0},\vec{1}])$ *for every* $A, B \in \boldsymbol{V}$*, then* $\boldsymbol{V}$ *has BFC. Furthermore,* $\boldsymbol{V}$ *has RexDFC.* The proof of this fact uses the same arguments of the one given for Lemma 5.3 in order to obtain an existential formula defining $Ker(\pi_B)$. This entails that $\mathbf{V}$ has (DFC) and consecuently (see the Introduction) $\mathbf{V}$ has (BFC).

So, let A be in $\mathbb{V}(E)$ and for every $1 \leq i \leq n$ consider the following composites:

$$1 \xrightarrow{\vec{0}} A^n \xrightarrow{\pi_i} A, \quad 0_i = \pi_i \circ \vec{0} \qquad\qquad 1 \xrightarrow{\vec{1}} A^n \xrightarrow{\pi_i} A, \quad 1_i = \pi_i \circ \vec{1}$$

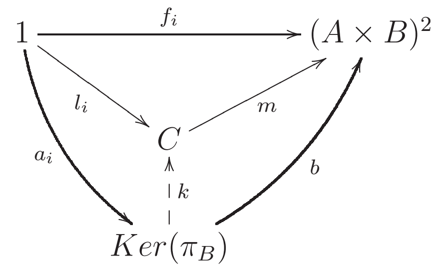
Thus, for every i and B in $\mathbb{V}(E)$ we obtain a morphism

$$1 \xrightarrow{f_i} (A \times B) \times (A \times B)$$

where $f_i = \langle\langle 1_i, 1_i\rangle, \langle 0_i, 1_i\rangle\rangle$. Consider the projection $\pi_B : A \times B \to B$ and let $b : Ker(\pi_B) \to (A\times B)^2$ be the morphism induced by the span $A\times B \leftarrow Ker(\pi_B) \to A\times B$. It easily follows that every f_i factors through b.

6.1. Definition. *A category of internal* $\mathbb{V}$*-models in* $\mathcal{E}$ *has the kernel determinig property (KDP) if for every pair of objects* A, B *in* $\mathbb{V}(E)$*,* $Ker(\pi_B)$ *is the least subobject of* $(A\times B)^2$

in $\mathbb{V}(\mathcal{E})$ *through which the collection* $\{f_i \mid 1 \leq i \leq n\}$ *factors. I,e:*



If $m : C \to (A \times B)^2$ *is a subobject such that* $ml_i = f_i$ *for every* $1 \leq i \leq n$*, then there exists a morphims* $k : Ker(\pi_B) \to C$ *(necessarily unique), such that* $mk = b$ *and* $ka_i = l_i$*.*

Inspired in Corollary 5.6 we introduce the following definition

6.2. Definition. *Let* $\mathbb{V}(E)$ *a category of internal* $\mathbb{V}$*-models in* $\mathcal{E}$ *with KDP. An internal* $\mathbb{V}$*-model* A *is connected if the following sequents hold*

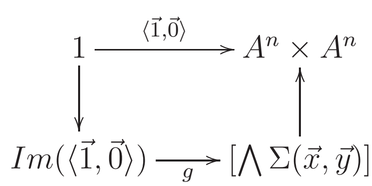
(C1) $\vec{0} = \vec{1} \vdash \bot$

(C2) $\bigwedge \Sigma(\vec{x}, \vec{y}) \vdash_{\vec{x},\vec{y}} (\vec{x} = \vec{0} \wedge \vec{y} = \vec{1}) \vee (\vec{x} = \vec{1} \wedge \vec{y} = \vec{0})$

in the internal logic of $\mathcal{E}$*.*

In the following, we write $\mathbb{C}_{\mathbb{V}}(\mathcal{E})$ for the theory of internal connected $\mathbb{V}$-models in $\mathcal{E}$.

Suppose A is in $\mathbb{C}_{\mathbb{V}}(\mathcal{E})$. Observe that, from axiom (3), there exists a morphism g such that the diagram below



commutes. Since $Im(\langle \vec{1}, \vec{0} \rangle) \cong [\vec{x} = \vec{1} \wedge \vec{y} = \vec{0}]$, we get that $[\vec{x} = \vec{1} \wedge \vec{y} = \vec{0}] \leq [\bigwedge \Sigma(\vec{x}, \vec{y})]$ in $Sub(A^n \times A^n)$. Thus, by completeness (c.f. D1.4.11 in [Johnstone2002]) we get that the sequent $(\vec{x} = \vec{1} \wedge \vec{y} = \vec{0}) \vdash_{\vec{x},\vec{y}} [\bigwedge \Sigma(\vec{x}, \vec{y})]$ holds in the internal logic of $\mathcal{E}$. Moreover, since

$$Im(\langle \vec{1}, \vec{0} \rangle) \cong Im(\langle \vec{0}, \vec{1} \rangle) \cong 1 \cong [\vec{x} = \vec{0} \wedge \vec{y} = \vec{1}]$$

from axiom (3), $Im(\langle \vec{0}, \vec{1} \rangle) \leq [\Sigma(\vec{x}, \vec{y})]$ in $Sub(A^n \times A^n)$, so the sequent $\top \vdash \bigwedge \Sigma(\vec{0}, \vec{1})$ holds in the internal logic of $\mathcal{E}$. By proceeding as before, we can deduce that the sequent $(\vec{x} = \vec{0} \wedge \vec{y} = \vec{1}) \vdash_{\vec{x},\vec{y}} [\bigwedge \Sigma(\vec{x}, \vec{y})]$ also holds in the internal logic of $\mathcal{E}$. We have proved the following

6.3. Lemma. *In $\mathbb{C}_{\mathbb{V}}(\mathcal{E})$ the sequent $(\vec{x} = \vec{1} \wedge \vec{y} = \vec{0}) \vee (\vec{x} = \vec{0} \wedge \vec{y} = \vec{1}) \vdash_{\vec{x},\vec{y}} [\bigwedge \Sigma(\vec{x},\vec{y})]$ holds.*

Let us consider the points $\langle \vec{0}, \vec{1} \rangle : 1 \to A^n \times A^n$ and $\langle \vec{1}, \vec{0} \rangle : 1 \to A^n \times A^n$. From axiom (3) it follows that, for every $\sigma \in \Sigma$, there exist a morphism $l_\sigma : 1 + 1 \to [\sigma(\vec{x},\vec{y})]$, such that the diagram below

$$\begin{array}{ccc} 1+1 & \xrightarrow{l_\sigma} & [\sigma(\vec{x},\vec{y})] \\ & {\scriptstyle [\langle\vec{0},\vec{1}\rangle,\langle\vec{1},\vec{0}\rangle]} \searrow & \downarrow \\ & & A^n \times A^n \end{array}$$

commutes, so $1{+}1 \leq [\sigma(\vec{x},\vec{y})]$ for every $\sigma \in \Sigma$. Hence, $1{+}1 \leq [\bigwedge \Sigma(\vec{x},\vec{y})]$ in $Sub(A^n \times A^n)$. Let us call $\alpha : 1 + 1 \to [\Sigma(\vec{x},\vec{y})]$ to the morphism that arise from the factorization of $1 + 1 \to A^n \times A^n$ along $[\Sigma(\vec{x},\vec{y})] \to A^n \times A^n$.

Since $[(\vec{x} = \vec{1} \wedge \vec{y} = \vec{0}) \vee (\vec{x} = \vec{0} \wedge \vec{y} = \vec{1})] \cong 1 + 1$, as result of the latter discution, we obtain a characterization for the internal connected $\mathbb{V}$-models in $\mathcal{E}$.

6.4. Lemma. *Let $\mathbb{V}(E)$ a category of internal $\mathbb{V}$-models in $\mathcal{E}$ with KDP. An internal $\mathbb{V}$-model A in $\mathcal{E}$ is connected if and only if the diagram below*

$$0 \xrightarrow{\;!\;} 1 \underset{\vec{0}}{\overset{\vec{1}}{\rightrightarrows}} A^n$$

is an equalizer in $\mathcal{E}$, and the morphism $\alpha : 1 + 1 \to [\Sigma(\vec{x},\vec{y})]$ is an iso.

Proof. Since $1 + 1 \leq [\Sigma(\vec{x},\vec{y})]$ in $Sub(A^n \times A^n)$, the result follows from apply Lemma 6.3 and the interpretations of the axioms (C1) and (C2) of Definition 6.2 in the internal logic of $\mathcal{E}$. ∎

7. Connected models in Coherent topoi

It is known that every distributive lattice D can be treaten as a *coherent category* (A1.4 in [Johnstone2002]). Its *coherent coverage* (A2.1.11(b) in [Johnstone2002]) is the function that sends each $d \in D$ to the set of finite families $\{d_i \leq d \mid i \in I\}$ such that $\bigvee_{i \in I} d_i = d$. As usual, the resulting topos of sheaves will be denoted by $\mathbf{Shv}(D)$. Binary covers $a \vee b = d$ of $d \in D$ will play an important role because in order to check that a presheaf $P : D^{op} \to \mathbf{Set}$ is a sheaf, it is enough to check the sheaf condition for binary covers.

Recall that every variety $\mathbf{V}$, is an algebrabic category over $\mathbf{Set}$. Thus a $\mathbf{V}$-model in $\mathbf{Shv}(D)$ is a functor $D^{op} \to \mathbf{V}$ such that the composite presheaf $D^{op} \to \mathbf{V} \to \mathbf{Set}$ is a sheaf.

The aim of this section is to characterize internal connected $\mathbb{V}$-models in $\mathbf{Shv}(D)$. Since the set of formulas Σ of Lemma 5.3 is not composed by equations, the theory $\mathbb{V}$ is not algebraic. So, we need to understand first what an internal $\mathbb{V}$-model in $\mathbf{Shv}(D)$

is. Observe that, from axiom (3), it follows that a **V**-model A in $\mathbf{Shv}(D)$ is an internal $\mathbb{V}$-model in $\mathbf{Shv}(D)$ if and only if axiom (3) holds in $A(d)$, for every $d \in D$. Hence an internal $\mathbb{V}$-model in $\mathbf{Shv}(D)$ is just a sheaf A such that $A(d)$ is an algebra of **V** with $\vec{0}$ and $\vec{1}$, for every $d \in D$.

7.1. Lemma. *Let* $\boldsymbol{V}$ *be a variety with* $\vec{0}$ *and* $\vec{1}$ *in* **Set**. *If* $\boldsymbol{V}$ *has RexDFC then* $\mathbb{V}(\widehat{D})$ *has KDP.*

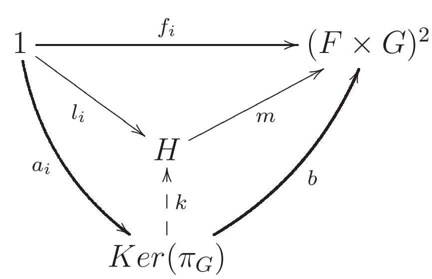
Proof. Since **V** is a variety with $\vec{0}$ and $\vec{1}$ then the axiom (3) holds. Let F, G in $\widehat{D}$. Since limits in $\widehat{D}$ are calculated pointwise, then if $\pi_G : F \times G \to G$, it follows that $Ker(\pi_G)(d) = Ker(\pi_{G(d)})$, for every $d \in D$. Thus, since **V** has RexDFC by hypothesis,

$$Ker(\pi_G)(d) = \theta^{F(d)\times G(d)}([\vec{1}_{F(d)}, \vec{1}_{G(d)}], [\vec{0}_{F(d)}, \vec{1}_{G(d)}]) = \theta^{F\times G}([\vec{1}, \vec{1}], [\vec{0}, \vec{1}])(d)$$

That is, $Ker(\pi_G)$ is the least subobject of $(F \times G)^2$ in $\mathbb{V}(\widehat{D})$ through which the collection $\{f_i \mid 1 \leq i \leq n\}$ factors. This concludes the proof. ∎

7.2. Corollary. *Let* $\boldsymbol{V}$ *be a variety with* $\vec{0}$ *and* $\vec{1}$ *in* **Set**. *If* $\boldsymbol{V}$ *has RexDFC then* $\mathbb{V}(\mathbf{Shv}(D))$ *has KDP.*

Proof. Let F, G in $\mathbf{Shv}(D)$, and let $H \to (F \times G)^2$ be a subobject in $\mathbf{Shv}(D)$ such that upper triangle in the diagram below



commutes in $\widehat{D}$, for every $1 \leq i \leq n$. From Lemma 7.1, $\mathbb{V}(\widehat{D})$ has KDP, so there exists a unique $k : Ker(\pi_G) \to H$, such that the left and the right triangles in the diagram above, commutes. Let $\mathbf{a} : \widehat{D} \to \mathbf{Shv}(D)$ be the sheafification respect to the coherent site. Since $\mathbf{a}$ preserves finite limits, then $\mathbf{a}(Ker(\pi_G))$ is the kernel of π_G in $\mathbf{Shv}(D)$. This concludes the proof. ∎

It is clear that $\mathbb{C}_{\mathbb{V}}$ neither is an algebraic theory; so to achieve our goal, we will require a little more effort. To do so, we will use specifically, a suitable description of binary coproducts in $\mathbf{Shv}(D)$ proved in ([CastiglioniMenniZuluaga2016]).

7.3. Lemma. [Binary coproducts in $\mathbf{Shv}(D)$] *For every* X, Y *in* $\mathbf{Shv}(D)$, *the coproduct* $X + Y$ *may be defined by*

$$(X + Y)(d) = \{(a, b, x, y) \mid a \vee b = d, a \wedge b = \perp, x \in X(a), y \in Y(b)\}$$

and, for any $(a, b, x, y) \in (X + Y)(d)$*,*

$$(X + Y)(c \leq d)(a, b, x, y) = (a, b, x, y) \cdot c = (a \wedge c, b \wedge c, x \cdot (a \wedge c), y \cdot (b \wedge c))$$

where $x \cdot c = X(c \leq d)(x) \in X(c)$ *and* $y \cdot c = Y(c \leq d)(y) \in Y(c)$*.*

In particular, from Lemma 7.3, $(1+1)(d) = \{(a, b) \mid a \vee b = d, a \wedge b = 0\}$. That is, $1 + 1$ is the "object of partitions" of D.

Recall that the variety of bounded distributive lattices is a variety with DFC, so for a bounded distributive lattice D the subobject $(1 + 1)(d)$ is isomorphic to the set $Z(\downarrow d)$; i.e, the center of $\downarrow d = \{a \in D \mid a \leq d\}$. On the other hand, if $\mathbf{V}$ is a variety with RexDFC, Corollary 7.2 tells us that $\mathbb{V}(\mathbf{Shv}(D))$ has KDP, so if $A : D^{op} \to \mathbf{Set}$ is an internal $\mathbb{V}$-model in $\mathbf{Shv}(D)$, then, $A(d)$ has BFC for every $d \in D$. This fact combined with Corollary 5.6 allows to say that, for every $d \in D$, the interpretation of $[\bigwedge \Sigma(\vec{x}, \vec{y})](d)$ brings an isomorphic description of $Z(A(d))$. Finally, from Lemma 6.4, we get that $\alpha : 1 + 1 \to [\bigwedge \Sigma(\vec{x}, \vec{y})]$ is a natural iso, so for every $d \in D$, the map $\alpha_d : Z(\downarrow d) \to Z(A(d))$ is bijective.

7.4. Proposition. *Let* $\boldsymbol{V}$ *be a variety with RexDFC. An internal* $\mathbb{V}$*-model* A *is connected in* $\mathbf{Shv}(D)$ *if and only if the following conditions hold:*

1. *If* $A(d) = 1$ *then* $d = 0$*.*
2. *For every* $c \leq d \in D$*, the diagram below*

$$\begin{array}{ccc} Z(A(d)) & \xrightarrow{\alpha_d} & Z(\downarrow d) \\ {\scriptstyle k_d}\downarrow & & \downarrow{\scriptstyle j_d} \\ Z(A(c)) & \xrightarrow[\alpha_c]{} & Z(\downarrow c) \end{array}$$

commutes, where, for every $c \in D$*,* $j_c(a) = a \wedge c$*,* $k_c(\vec{e}) = \vec{e} \cdot c$ *and* α_c *is an isomorphism.*

Proof. A restatement of Lemma 6.2 in the case of $\mathbf{Shv}(D)$. ■

8. The category of representations

8.1. Definition. *A representation (of a* $\boldsymbol{V}$*-model) is a pair* (D, X)*, consisting of a distributive lattice* D *and a* $\boldsymbol{V}$*-model in* $\mathbf{Shv}(D)$ *satisfying the equivalent conditions of Proposition 7.4.*

We now define a category $\mathfrak{R}$ whose objects are representations in the above sense. To describe the arrows in $\mathfrak{R}$ first recall that any morphism $f : D \to E$ between distributive lattices is in fact a morphism of sites (in the sense of Theorem VII.10.1 in

[MacLaneMoerdijk2012]) when D and E are considered as small categories equipped with the coherent topology. From Theorem VII.10.1 in *loc. cit.* there exists a geometric morphism $f : \mathbf{Shv}(E) \to \mathbf{Shv}(D)$, whose direct image f_* is defined as $f_*(X) = X \circ f$, for every $X \in \mathbf{Shv}(D)$; i.e. $f_*(X)(d) = X(f(d))$, for every $d \in D$.

We define now the maps in $\mathfrak{R}$. For representations (D, X) and (E, Y), an arrow $(D, X) \to (E, Y)$ in $\mathfrak{R}$ is a pair (f, φ) with $f : D \to E$ a morphism in $\mathbf{dLat}(\mathbf{Set})$ and $\varphi : X \to f_*Y$ a morphism in $\mathcal{V}(\mathbf{Shv}(D))$. If $(f, \varphi) : (D, X) \to (E, Y)$ and $(g, \gamma) : (C, W) \to (D, X)$ are maps in $\mathfrak{R}$ then we define the *composite* $(g, \gamma)(f, \varphi) : (C, W) \to (E, Y)$ as the pair $(fg, (g_*\varphi)\gamma)$. From the functoriality of f and the fact of $(fg)_* = g_*f_*$, it follows that composition in $\mathfrak{R}$ is well defined and is associative. Moreover, for every D in $\mathbf{dLat}(\mathbf{Set})$, the identity morphism id_D (as a morphism of sites) induces the identity morphism in $\mathbf{Shv}(D)$ so it easily follows that for every pair (D, X) in $\mathfrak{R}$, $id_{(D,X)} = (id_D, id_X)$.

For each morphism (D, X) in $\mathfrak{R}$ we define $\Gamma(D, X)$ as $X(1)$, and for every $(f, \varphi) : (D, X) \to (E, Y)$ in $\mathfrak{R}$, define $\Gamma(f, \varphi) = \varphi_1 : X(1) \to Y(f(1)) = Y(1)$. It easily follows that $\Gamma : \mathfrak{R} \to \mathcal{V}$ is a functor.

9. The representation of V-models

In this section we prove that every algebra with RexDFC and CSC in **Set** can be represented as an object of the category $\mathfrak{R}$.

Let $\mathbf{V}$ a variety with RexDFC and CSC. If A is in $\mathbf{V}$ and $\vec{e} \in Z(A)$, recall that from Lemmas 3.2 and 3.3 we get that $\theta^A_{\vec{1},\vec{e}} = \theta^A(\vec{1}, \vec{e})$.

9.1. Lemma. *Let A be in $\mathbf{V}$ and $\vec{e}, \vec{f} \in Z(A)$. The following holds:*

1. $\theta^A(\vec{1}, \vec{e} \wedge_A \vec{f}) = \theta^A(\vec{1}, \vec{e}) \vee \theta^A(\vec{1}, \vec{f})$.
2. $\theta^A(\vec{1}, \vec{e} \vee_A \vec{f}) = \theta^A(\vec{1}, \vec{e}) \cap \theta^A(\vec{1}, \vec{f})$.

Proof. We prove 1. By definition (Subsection 2.4), it is clear that $\theta^A(\vec{1}, \vec{e} \wedge_A \vec{f}) \subseteq \theta^A(\vec{1}, \vec{e}) \vee \theta^A(\vec{1}, \vec{f})$. On the other hand, since $\vec{e} \wedge_A \vec{f} \leq_A \vec{e}, \vec{f}$ thus $\theta^A(\vec{1}, \vec{e}), \theta^A(\vec{1}, \vec{f}) \subseteq \theta^A(\vec{1}, \vec{e} \wedge_A \vec{f})$, hence $\theta^A(\vec{1}, \vec{e}) \vee \theta^A(\vec{1}, \vec{f}) \subseteq \theta^A(\vec{1}, \vec{e} \wedge_A \vec{f})$. The proof of 2. is similar. ∎

9.2. Remark. Since $\theta^A(\vec{1}, \vec{1}) = \Delta^A$ and $\theta^A(\vec{1}, \vec{0}) = \nabla^A$, as a direct application of Lemma 9.1 it follows that the map $\phi : Z(A)^{op} \to FC(A)$ defined by $\phi(\vec{e}) = \theta^A(\vec{1}, \vec{e})$ is an iso of Boolean algebras.

9.3. Lemma. *Let $\boldsymbol{V}$ be a variety with DFC and A an algebra of $\boldsymbol{V}$. If $\theta \diamond \delta$ in $Con(A)$, then for every $\vec{e} \in A$, the following are equivalent:*

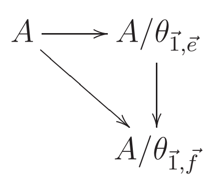
1. *$\vec{e} \in Z(A)$*
2. *$\vec{e}/\theta \in Z(A/\theta)$ and $\vec{e}/\delta \in Z(A/\delta)$.*

Proof. Let us assume $\theta \diamond \delta$ in $Con(A)$ and suppose $\vec{e} \in Z(A)$. Without loss of generality we can assume $\vec{e} = (\vec{0}, \vec{1})$ in $A = A_1 \times A_2$. Since DFC implies BFC (see the Introduction), from Lemma 2.3 there exist $\alpha_i \in Con(A_i)$, $(i = 1, 2)$ such that $\theta = \alpha_1 \times \alpha_2$. Thereby, via the canonical isomorphism between A/θ and $A/\alpha_1 \times A/\alpha_2$ we can conclude that $\vec{e}/\theta = (\vec{0}/\alpha_1, \vec{1}/\alpha_2)$, so $\vec{e}/\theta \in Z(A/\theta)$. The proof for $\vec{e}/\delta \in Z(A/\delta)$ is analogue. On the other hand, if $\vec{e} \in Z(A/\theta)$ and $\vec{e} \in Z(A/\delta)$, there exist $A_1, A_2, B_1, B_2 \in \mathbf{V}$ and isomorphisms $\tau_\theta : A/\theta \to A_1 \times A_2$, $\tau_\delta : A/\theta \to B_1 \times B_2$, such that $\tau_\theta(\vec{e}/\theta) = (\vec{0}_{A_1}, \vec{1}_{A_2})$ and $\tau_\delta(\vec{e}/\delta) = (\vec{0}_{B_1}, \vec{1}_{B_2})$. Since $\theta \diamond \delta$ by assumption, then $A \cong A/\theta \times A/\delta$, so, since $(A_1 \times A_2) \times (B_1 \times B_2) \cong (A_1 \times B_1) \times (A_2 \times B_2) = C$, if we write $C_1 = A_1 \times B_1$ and $C_2 = A_2 \times B_2$ there exists an isomorphism $\kappa : A \to C_1 \times C_2$ such that $\kappa(\vec{e}) = (\vec{0}, \vec{1})$. This concludes the proof. ∎

9.4. Lemma. *Let $\boldsymbol{V}$ be a variety with BFC and A and algebra of $\boldsymbol{V}$. If $\theta \in FC(A)$ and $\vec{z}/\theta \in Z(A/\theta)$, then there exists an $\vec{e} \in Z(A)$ such that $\vec{e}/\theta = \vec{z}/\theta$.*

Proof. Let δ be the factor congruence complementary to θ. Since $\nabla^A = \theta \circ \delta$ and $(\vec{z}, \vec{1}) \in \nabla^A$, then, there exists an $e \in A$ such that $(\vec{z}, \vec{e}) \in \theta$ and $(\vec{e}, \vec{1}) \in \delta$. It is clear that $\vec{z}/\theta = \vec{e}/\theta \in Z(A/\theta)$ and $\vec{e}/\delta = \vec{1}/\delta \in Z(A/\delta)$. Hence, by Lemma 9.3 we conclude that $\vec{e} \in Z(A)$. ∎

9.5. Lemma. *Let $\boldsymbol{V}$ be a variety with RexDFC and CSC. For every A algebra of $\boldsymbol{V}$ and every $\vec{e}, \vec{f} \in Z(A)$, if $\vec{f} \leq_A \vec{e}$ there exists a (necessarily unique map) $A/\theta_{\vec{1},\vec{e}} \to A/\theta_{\vec{1},\vec{f}}$ such that the diagram below*



commutes, where the horizontal and diagonal arrows are the respective canonical homomorphisms. Thereby, if $\vec{e} = \vec{f}$ then $A/\theta_{\vec{1},\vec{e}}$ is canonically iso to $A/\theta_{\vec{1},\vec{f}}$.

Proof. Again, from Lemmas 3.2 and 3.3, we obtain that for every $\vec{e} \in Z(A)$, $\theta_{\vec{1},\vec{e}} = \theta(\vec{1}, \vec{e})$. If $\vec{f} \leq_A \vec{e}$, then $\theta(\vec{1}, \vec{e}) \subseteq \theta(\vec{1}, \vec{f})$, so by Lemma 3.5 the result follows. ∎

As result, the assignment that sends $\vec{e} \in Z(A)$ to $A/\theta_{\vec{1},\vec{e}}$ is well defined so we obtain a functor $Z(A)^{op} \to \mathcal{V}$. In conclusion, we have obtained a $\mathbf{V}$-model $\overline{A}$ in $\widehat{Z(A)}$.

9.6. Lemma. *For every $\boldsymbol{V}$-model in* **Set** *the presheaf $\overline{A}$ in $\widehat{Z(A)}$ is a sheaf (respect to the coherent coverage on the lattice $Z(A)$).*

PROOF. Since $Z(A)$ is a Boolean algebra, to prove the statement it is enough to verify the sheaf condition for binary partitions, but this leads to a reformulation of item 3. in Lemma 2.2. ∎

9.7. LEMMA. *For every $\boldsymbol{V}$-model in* **Set** *the pair* $(Z(A), \overline{A})$ *is an object of* $\mathfrak{R}$.

PROOF. We use Lemma 7.4. Since A is an algebra with $\vec{0}$ and $\vec{1}$, it follows that for every $\vec{e} \in Z(A)$, $\overline{A}(\vec{e})$ such condition also holds. Observe that, in the case of $\mathbf{Shv}(D)$, the map $\alpha_{\vec{e}} : Z(\downarrow \vec{e}) \to Z(A/\theta(\vec{1}, \vec{e}))$, is canonically defined as $\alpha_{\vec{e}}(\vec{f}) = f/\theta(\vec{1}, \vec{e})$. We verify that $\alpha_{\vec{e}}$ is biyective. If $\alpha_{\vec{e}}(\vec{f}) = \alpha_{\vec{e}}(\vec{g})$, then $[\vec{f}, \vec{g}] \in \theta(\vec{1}, \vec{e})$. From Lemma 9.1, we have $\theta(\vec{1}, \vec{e}) \vee \theta(\vec{1}, \vec{f}) = \theta(\vec{1}, \vec{e} \wedge_A \vec{f})$ so $\theta(\vec{1}, \vec{e}) \subseteq \theta(\vec{1}, \vec{e} \wedge_A \vec{f})$. Since $\vec{e} \wedge_A \vec{g} = \vec{g}$ and $[\vec{1}, \vec{g}] \in \theta(\vec{1}, \vec{g})$, from the transitivity of $\theta(\vec{1}, \vec{g})$ we obtain that $[\vec{1}, \vec{f}] \in \theta(\vec{1}, \vec{g})$ so $\vec{g} \leq_A \vec{f}$. The verification of $\vec{f} \leq_A \vec{g}$ is similar. Thus α_e is injective. To check the surjectivity of $\alpha_{\vec{e}}$, let $\vec{f}/\theta(\vec{1}, \vec{e}) \in Z(A/\theta(\vec{1}, \vec{e}))$. From Lemma 9.4, there exists a $\vec{z} \in Z(A)$, such that $[\vec{f}, \vec{z}] \in \theta(\vec{1}, \vec{e})$. Since $[\vec{z}, \vec{1}] \in \theta(\vec{1}, \vec{z})$, then $[\vec{f}, \vec{1}] \in \theta(\vec{1}, \vec{e}) \vee \theta(\vec{1}, \vec{z}) = \theta(\vec{1}, \vec{e} \wedge_A \vec{f})$, again by Lemma 9.1. Thus we obtain that $\theta(\vec{f}, \vec{1}) \subseteq \theta(\vec{1}, \vec{e} \wedge_A \vec{f})$. From Lemma 2.7, $[\vec{e} \wedge_A \vec{z}, \vec{z}] \in \theta(\vec{1}, \vec{e})$, so, since $[\vec{f}, \vec{z}] \in \theta(\vec{1}, \vec{e})$, we get that $[\vec{f}, \vec{e} \wedge_A \vec{z}] \in \theta(\vec{1}, \vec{e})$. Hence, since $\vec{e} \wedge_A \vec{z} \leq_A \vec{f}$ and $\theta(\vec{f}, \vec{1}) \subseteq \theta(\vec{1}, \vec{e} \wedge_A \vec{f})$, we conclude that $\alpha_{\vec{e}}(\vec{e} \wedge_A \vec{z}) = f/\theta(\vec{1}, \vec{e})$. Finally, if $A/\theta(\vec{1}, \vec{e})$ is trivial, it follows that $[\vec{0}, \vec{1}] \in \theta(\vec{1}, \vec{e})$, thus $\theta(\vec{1}, \vec{0}) = \theta(\vec{1}, \vec{e})$. Hence, by Remark 9.2, $\vec{e}$ must be $\vec{0}$. This concludes the proof. ∎

10. RexDFC and CSC induce homomorphisms of Boolean algebras

As we saw in Section 4, not every variety with BFC has center stable. In this section we prove that a variety with RexDFC having center stable by complements is in fact a variety with the Fraser Horn Property. This result will allow us to prove that the every homomorphism f in the variety induces a Boolean algebra homomorphism between the centers of $dom(f)$ and $cod(f)$.

10.1. LEMMA. [Theorem 1 [FraserHorn1970]] *Let $\boldsymbol{K}$ be a variety and A, B be algebras of $\boldsymbol{K}$. The following are equivalent:*

1. *$\boldsymbol{K}$ has FHP.*
2. *For every $A, B \in \boldsymbol{K}$ and $\gamma \in Con(A \times B)$,*

$$\Pi_1 \cap (\Pi_2 \vee \gamma) \subseteq \gamma \text{ and } \Pi_2 \cap (\Pi_1 \vee \gamma) \subseteq \gamma$$

where Π_1 is the kernel of the projection on A and Π_2 is the kernel of the projection on B.

10.2. LEMMA. [Theorem 3 [FraserHorn1970]] *Let A and B be similar algebras. The following are equivalent:*

1. *$A \times B$ has FHP.*

2. For every $a, c \in A$ and $b, d \in B$,

$$\theta^{A\times B}((a,b),(c,d)) = \theta^A(a,c)\times\theta^B(b,d)$$

10.3. LEMMA. *Let A and B be algebras with finite n-ary function symbols and $f : A \to B$ an homomophism. If $(a,b) \in \theta^A(\vec{c},\vec{d})$, then $(f(a),f(b)) \in \theta^B(f(\vec{c}),f(\vec{d}))$. Thus, if $[\vec{a},\vec{b}] \in \theta^A(\vec{c},\vec{d})$ then $[f(\vec{a}),f(\vec{b})] \in \theta^A(f(\vec{c}),f(\vec{d}))$.*

PROOF. Apply Lemma 5.1. ■

Let $\mathbf{V}$ be a variety with DCF. As we have seen, for every algebra $A \in \mathbf{V}$ and $\vec{e} \in Z(A)$ the span $A/\theta_{\vec{0},\vec{e}} \leftarrow A \to A/\theta_{\vec{1},\vec{e}}$ is a product. Notice that in this case $\Pi_1 = \theta_{\vec{0},\vec{e}}$ and $\Pi_2 = \theta_{\vec{1},\vec{e}}$.

10.4. LEMMA. *Let $\boldsymbol{V}$ be a variety with RexDFC, $A \in \boldsymbol{V}$ and $\vec{e},\vec{f} \in Z(A)$ such that $\vec{e}\diamond_A\vec{f}$. If for every $\gamma \in Con(A)$, $\vec{e}/\gamma\diamond_{A/\gamma}\vec{f}/\gamma$, then $\boldsymbol{V}$ has the FHP.*

PROOF. Since $\mathbf{V}$ has RexDFC, from Lemmas 3.2 and 3.3, then, for every $\vec{e} \in Z(A)$, $\theta^A_{\vec{1},\vec{e}} = \theta^A(\vec{1},\vec{e})$. So, if $\vec{e}\diamond_A\vec{f}$ then $\theta^A_{\vec{0},\vec{e}} = \theta^A(\vec{1},\vec{f})$. We use Lemma 10.1. To do so, we prove $\theta^A(\vec{1},\vec{e})\cap(\theta^A(\vec{1},\vec{f})\vee\gamma) \subseteq \gamma$. Suppose $(x,y) \in \theta^A(\vec{1},\vec{e})\cap(\theta^A(\vec{1},\vec{f})\vee\gamma) \subseteq \gamma$, then, $(x,y) \in \theta^A(\vec{1},\vec{e})$ and there are $c_0,...,c_N \in A$, with $c_0 = x$ and $c_N = y$, such that $(c_{2i},c_{2i+1}) \in \theta^A(\vec{1},\vec{f})$ and $(c_{2i+1},c_{2(i+1)}) \in \gamma$. Since $A \to A/\gamma$ is clearly an homomorphism, from Lemma 10.3, we obtain that $(x/\gamma,y/\gamma) \in \theta^{A/\gamma}(\vec{1}/\gamma,\vec{e}/\gamma)$, $(c_{2i}/\gamma,c_{2i+1}/\gamma) \in \theta^{A/\gamma}(\vec{1}/\gamma,\vec{f}/\gamma)$ and $c_{2i+1}/\gamma = c_{2(i+1)}/\gamma$. From transitivity of $\theta^{A/\gamma}(\vec{1}/\gamma,\vec{f}/\gamma)$, we get that $(x/\gamma,y/\gamma) \in \theta^{A/\gamma}(\vec{1}/\gamma,\vec{f}/\gamma)$. Therefore, $(x/\gamma,y/\gamma) \in \theta^{A/\gamma}(\vec{1}/\gamma,\vec{e}/\gamma)\cap\theta^{A/\gamma}(\vec{1}/\gamma,\vec{f}/\gamma) = \Delta^{A/\gamma}$, since $\vec{e}/\gamma\diamond_{A/\gamma}\vec{f}/\gamma$ by assumption, so $(x,y) \in \gamma$. The proof of $\theta^A(\vec{1},\vec{f})\cap(\theta^A(\vec{1},\vec{e})\vee\gamma) \subseteq \gamma$ is similar. This concludes the proof. ■

10.5. COROLLARY. *Let $\boldsymbol{V}$ be a variety with RexDFC. If $\boldsymbol{V}$ has SCC then has FHP.*

PROOF. Inmmediate from Lemma 10.4. ■

10.6. LEMMA. *Every variety $\boldsymbol{V}$ with FHP is TexDFC.*

PROOF. We want to prove there exists an existential formula φ which defines $\theta_{\vec{0},\vec{e}}$ in terms of $\vec{e}$. To do so, let $C \in \mathbf{V}$ and $\vec{e} \in Z(C)$. Let us consider $\varphi(x,y,\vec{z}) = \pi(x,y,\vec{0},\vec{z})$, where $\pi(x,y,\vec{0},\vec{z})$ is the formula of Lemma 5.2. It is clear that φ is existential. Let $A, B \in \mathbf{V}$ and $(a,b),(c,d) \in A\times B$. If $A\times B \models \varphi((a,b),(c,d),(\vec{0},\vec{1}))$, then from Lemma 5.2, $((a,b),(c,d)) \in \theta^{A\times B}((\vec{0},\vec{0}),(\vec{0},\vec{1}))$. Since $\mathbf{V}$ has FHP by hypothesis, then from Lemma 10.2 $\theta^{A\times B}((\vec{0},\vec{0}),(\vec{0},\vec{1})) = \theta^A(\vec{0},\vec{0})\times\theta^B(\vec{0},\vec{1}) = \Delta^A\times\nabla^B$. Hence, $a = c$. On the other hand, suppose $a = c$. Let $P = \mathbf{F}(x)\times\mathbf{F}(x,y)$, and consider the pair $((x,x),(x,y)) \in P$. Since $\mathbf{V}$ has FHP by assumption, thus, again by Lemma 10.2, $\theta^P((\vec{0},\vec{0}),(\vec{0},\vec{1})) = \theta^{\mathbf{F}(x)}(\vec{0},\vec{0})\times\theta^{\mathbf{F}(x,y)}(\vec{0},\vec{1}) = \Delta^{\mathbf{F}(x)}\times\nabla^{\mathbf{F}(x,y)}$. Observe that $((x,x),(x,y)) \in \theta^P((\vec{0},\vec{0}),(\vec{0},\vec{1}))$ and consider the assignments $\alpha_A : \{x\} \to A$ and $\alpha_B : \{x,y\} \to B$, defined as $\alpha_A(x) = a$ and $\alpha_B(x) = b$, $\alpha_B(y) = d$, respectively. From the left adjointness of the free functor $\mathbf{F} : \mathbf{Set} \to \mathcal{V}$ to the forgetful functor, there exist a unique pair of homomorphisms $\beta_A : \mathbf{F}(x) \to A$ and $\beta_B : \mathbf{F}(x,y) \to B$ extending α_A and α_B. Consider the morphism $g = \beta_A \times \beta_B : P \to A \times B$. From Lemma 10.3 we obtain that $(g(x,x), g(x,y)) = ((a,b),(a,c)) \in \theta^{A\times B}((\vec{0},\vec{0}),(\vec{0},\vec{1}))$. Hence, by Lemma 5.2, $A \times B \models \varphi((a,b),(c,d),(\vec{0},\vec{1}))$. Therefore, $\mathbf{V}$ has LexDFC. The proof for $\mathbf{V}$ has RexDFC is analogue. This concludes the proof. ■

As a straight consequence of Corollary 10.5 and Lemma 10.6 we obtain

10.7. Corollary. *Every variety **V** with RexDFC and CSC is TexDFC.*

10.8. Lemma. *Let **V** be a variety with RexDFC, $A, B \in$ **V** and $f : A \to B$ be an homomorphism. If f preserves central elements, then $f|_{Z(A)} : Z(A) \to Z(B)$ is a bounded lattice homomorphism.*

Proof. First of all, observe that from Corollary 10.7; and Lemmas 3.2 and 3.3, we get that for every $\vec{e} \in A$, $\theta^A_{\vec{0},\vec{e}} = \theta^A(\vec{0},\vec{e})$ and $\theta^A_{\vec{1},\vec{e}} = \theta^A(\vec{1},\vec{e})$. Now, since f is homomorphism, it is clear that preserves $\vec{0}$ and $\vec{1}$. So, if $\vec{e}_1, \vec{e}_2 \in Z(A)$, and $\vec{a} = \vec{e}_1 \wedge_A \vec{e}_2$, thus from Lemma 2.7, $[\vec{0},\vec{a}] \in \theta^A(\vec{0},\vec{e}_1)$ and $[\vec{a},\vec{e}_2] \in \theta^A(\vec{1},\vec{e}_1)$. Thus, since $f(\vec{e}) \in Z(A)$ for every $\vec{e} \in A$ by hypothesis; from Lemma 10.3 we get that $[\vec{0}, f(\vec{a})] \in \theta^A(\vec{0}, f(\vec{e}_1))$ and $[f(\vec{a}), f(\vec{e}_2)] \in \theta^B(\vec{1}, f(\vec{e}_1))$ so again by Lemma 2.7 we can conclude that $f(a) = f(\vec{e}_1) \wedge_B f(\vec{e}_2)$. The proof for the preservation of the join is similar. ■

A direct application of Lemma 10.8 gives as result

10.9. Corollary. *Let **V** be a variety with RexDFC and SCC. Then, for every $A, B \in$ **V** and every homomorphism $f : A \to B$, the map $f|_{Z(A)} : Z(A) \to Z(B)$ is an homomorphism of Boolean algebras.*

11. The representation theorem

For the rest of this section $\mathbf{V}$ will be a variety with RexDFC and CSC. Next we show that the functor $\Gamma : \mathfrak{R} \to \mathcal{V}$ has a fully faithful left adjoint.

Let A and B be $\mathbf{V}$-models in $\mathbf{Set}$ and let $f : A \longrightarrow B$ be a $\mathcal{V}$. Since $\mathbf{V}$ is CSC, from Corollary 10.9, the restriction of f to $Z(A)$ determines a morphism of boolean algebras $f : Z(A) \to Z(B)$. Such morphism is also a morphism of lattices so determines a geometric morphism $f : \mathbf{Shv}(Z(B))) \to \mathbf{Shv}(Z(B))$ whose direct image f_* is defined as $f_*(G)(\vec{e}) = G(f(\vec{e}))$, for every $G \in \mathbf{Shv}(Z(B))$.

11.1. Lemma. *Every morphism $f : A \to B$ in $\mathcal{V}$ determines a natural transformation $\overline{f} : \overline{A} \longrightarrow f_*(\overline{B})$ in $\mathbf{Shv}(Z(A))$.*

Proof. Let $\vec{e} \in Z(A)$. If $i_{\vec{e}} : A \to A/\theta^A(\vec{1},\vec{e})$ and $i_{f(\vec{e})} : B \to A/\theta^A(\vec{1}, f(\vec{e}))$ are the canonical homomorphisms, from Corollary 3.5

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle i_{\vec{e}}}\downarrow & & \downarrow{\scriptstyle i_{f(\vec{e})}} \\ A/\theta^A(\vec{1},\vec{e}) & \xrightarrow[f_e]{} & B/\theta^B(\vec{1},f(\vec{e})) \end{array}$$

it follows that there exists a unique morphism $f_{\vec{e}} : A/\theta^A(\vec{1},\vec{e}) \to B/\theta^B(\vec{1},f(\vec{e}))$ in $\mathcal{V}$, such that the diagram above commutes. Consider the assigment $\overline{f} : \overline{A} \longrightarrow f_*(\overline{B})$, defined as $\overline{f}_{\vec{e}} = f_{\vec{e}}$. We prove that $\overline{f}$ is natural in $\mathbf{Shv}(Z(A))$. Let $\vec{e}_1, \vec{e}_2 \in Z(A)$ with $\vec{e}_2 \leq_A \vec{e}_1$. From Lemma 10.8 it follows that $f(\vec{e}_2) \leq_B f(\vec{e}_1)$, so again by Corollary 3.5, the diagram below (where the rows of the right square are the canonical morphisms $A/\theta^A(\vec{1},\vec{e}_1) \to A/\theta^A(\vec{1},\vec{e}_2)$ and $B/\theta^B(\vec{1},f(\vec{e}_1)) \to B/\theta^B(\vec{1},f(\vec{e}_2))$, respectively),

$$\begin{array}{ccccc} A & \xrightarrow{i_{\vec{e}_1}} & A/\theta^A(\vec{1},\vec{e}_2) & \longrightarrow & A/\theta^A(\vec{1},\vec{e}_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{\vec{e}_1}} & & \downarrow{\scriptstyle f_{\vec{e}_2}} \\ B & \xrightarrow[i_{f(\vec{e}_1)}]{} & B/\theta^B(\vec{1},f(\vec{e}_1)) & \longrightarrow & B/\theta^B(\vec{1},f(\vec{e}_2)) \end{array}$$

commutes. Since $B/\theta^B(\vec{1},f(\vec{e}_1)) = f_*(\overline{B})(\vec{e})$, the result follows. ∎

Lemmas 9.7 and 11.1 allows us to define an assigment $\mathcal{F} : \mathcal{V} \to \mathfrak{R}$ as $\mathcal{F}(A) = (Z(A), \overline{A})$ and $\mathcal{F}(f : A \to B) = (f, \overline{f})$.

11.2. Lemma. *The assignment $\mathcal{F} : \mathcal{V} \to \mathfrak{R}$ is functorial.*

Proof. It is clear that $\mathcal{F}(id_A) = (id_A, id_{\overline{A}}) = id_{\mathcal{F}(A)}$. So, let $f : A \to B$ and $h : B \to C$ be morphisms in $\mathcal{V}$. Then, we get that $\mathcal{F}(hf) = (hf, \overline{hf})$ and $\mathcal{F}(h)\mathcal{F}(f) = (hf, f_*(\overline{h})\overline{f}))$.

$$\begin{array}{ccccc} A & \xrightarrow{\quad f \quad} & B & \xrightarrow{\quad h \quad} & C \\ {\scriptstyle i_{\vec{e}}}\downarrow & & {\scriptstyle i_{f(\vec{e})}}\downarrow & & \downarrow{\scriptstyle i_{hf(\vec{e})}} \\ \underbrace{A/\theta^A(\vec{1},\vec{e}) \xrightarrow[f_{\vec{e}}]{}} & & B/\theta^B(\vec{1},f(\vec{e})) & \xrightarrow[h_{f(\vec{e})}]{} & C/\theta^C(\vec{1},hf(\vec{e})) \\ & & hf_{\vec{e}} & & \end{array}$$

Since $i_{hf(\vec{e})} = i_{h(f(\vec{e}))}$ and $hf(\vec{e}) = h(f(\vec{e}))$, then from Corollary 3.5, the diagram above commutes for every $\vec{e} \in Z(A)$. Hence, since $\overline{h}_{f(\vec{e})}\overline{f}_{\vec{e}} = f_*(\overline{h})_{\vec{e}}\overline{f}_{\vec{e}}$, then $f_*(\overline{h})\overline{f} = \overline{hf}$. Thereby $\mathcal{F}(h)\mathcal{F}(f) = \mathcal{F}(hf)$. ∎

11.3. Lemma. *Let A be an algebra of $\boldsymbol{V}$ in* **Set** *and P be a $\boldsymbol{V}$-model in $\widehat{Z(A)}$. For every homomorphism $g : A \to P(\vec{1})$ in $\boldsymbol{V}$, the following are equivalent:*

1. *For every $\vec{e} \in Z(A)$, $g(\vec{e}) \cdot \vec{e} = \vec{1} \in P(\vec{1})$,*
2. *There exist a unique morphism of $\boldsymbol{V}$-models $\phi : \overline{A} \to P$ in $\widehat{Z(A)}$, such that $\phi_{\vec{1}} = g$.*

Proof. Let us assume $g(\vec{e})\cdot\vec{e}=\vec{1}\in P(\vec{1})$, for every $\vec{e}\in Z(A)$. Since $\vec{1}\cdot\vec{e}=\vec{1}$ for every $\vec{e}\in Z(A)$, from the universal property of $A\to A/\theta^A(\vec{1},\vec{e})$ (Corollary 3.5), for every $\vec{e}\in Z(A)$, there exists a unique homomorphism $A/\theta^A(\vec{1},\vec{e})\to P(\vec{e})$ in **V** such that the diagram below

$$\begin{array}{ccc} A\cong\overline{A}(\vec{1}) & \xrightarrow{\quad g \quad} & P(1) \\ \downarrow & & \downarrow \\ A/\theta^A(\vec{1},\vec{e})=\overline{A}(\vec{e}) & \xrightarrow[\phi_{\vec{e}}]{} & P(\vec{e}) \end{array}$$

commutes. Observe that Corollary 3.5 also grants that the collection $\{\phi_{\vec{e}}\mid\vec{e}\in Z(A)\}$ is natural. The proof of the last part follows from the naturality of ϕ. ■

11.4. Lemma. *Let A be an algebra of **V** in* **Set** *and (E,Y) in $\mathfrak{R}$. For every $g:A\to Y(1)$ in **V**, the following are equivalent:*

1. *There is a unique lattice morphism $f:Z(A)\to E$, such that, for every $\vec{e}\in Z(A)$, $g(\vec{e})\cdot f(\vec{e})=\vec{1}\in f_*(Y)(\vec{e})$.*
2. *There exists a unique $(f,\varphi):(Z(A),\overline{A})\to(E,Y)$ in $\mathfrak{R}$, such that $\phi_{\vec{1}}=g$.*

Proof. If we assume 2. then 1. is granted for the naturality of ϕ. On the other hand, by assuming 1., it follows that $f_*(Y)$ is in **Shv**$(Z(A))$, so from Lemma 11.3, for the map $g:A\to Y(f(\vec{1}))=f_*(Y)(\vec{1})$ there exists a unique morphism of **V**-models $\phi:\overline{A}\to f_*(Y)$ such that $\phi_{\vec{1}}=g$. Thereby, the uniqueness of $(f,\varphi):(Z(A),\overline{A})\to(E,Y)$ in $\mathfrak{R}$, easily follows. This concludes the proof. ■

Coming up next, we prove the main result of this paper.

11.5. Theorem. *The functor $\Gamma:\mathfrak{R}\to\mathcal{V}$ has a full and faithful left adjoint.*

Proof. Let A be an arbitrary algebra of **V**. From Lemma 9.7, $(Z(A),\overline{A})$ is an object of $\mathfrak{R}$. Let us to consider the (iso) map $A\to A/\theta^A(\vec{1},\vec{1})=\overline{A}(\vec{1})=\Gamma(Z(A),\overline{A})$. We prove this map is universal from A to Γ. To do so, let (C,X) be in $\mathfrak{R}$ and $g:A\to X(1)=\Gamma(C,X)$ be an arbitrary morphism of $\mathcal{V}$. From the center stability of **V**, for every $\vec{e}\in Z(A)$, $g(\vec{e})\in Z(X(1))$. Since (C,X) is in $\mathfrak{R}$, X is connected in **Shv**(C), so by Proposition 7.4, there are bijections $\alpha_1,\alpha_{g(\vec{e})}$ making the diagram below

$$\begin{array}{ccc} Z(X(1)) & \xrightarrow{\alpha_1} & Z(C) \\ {\scriptstyle k_{g(\vec{e})}}\downarrow & & \downarrow{\scriptstyle j_{g(\vec{e})}} \\ Z(X(g(\vec{e}))) & \xrightarrow[\alpha_{g(\vec{e})}]{} & Z(\downarrow g(\vec{e})) \end{array}$$

commutes, for every $\vec{e}\in Z(A)$ (with $k_{g(\vec{e})}(\vec{h})=\vec{h}\cdot g(\vec{e})$ and $j_{g(\vec{e})}(l)=l\wedge g(\vec{e})$). Let us define $f:Z(A)\to C$, as $f(\vec{e})=\alpha_{\vec{1}}(g(\vec{e}))$. By Lemma 10.8, f is a lattice morphism, thus $f_*(Y)$ is in $\mathbf{Shv}(Z(A))$. The commutativity of diagram above allows us to make the following calculation

$$\alpha_{f(\vec{e})}(g(\vec{e}) \cdot f(\vec{e})) = \alpha_{\vec{1}}(g(\vec{e})) \wedge f(\vec{e}) = f(\vec{e}) = \alpha_{f(\vec{e})}(\vec{1} \cdot f(\vec{e}))$$

Hence, $g(\vec{e}) \cdot f(\vec{e}) = \vec{1} \in f_*(Y)(\vec{e})$. Thereby, from Lemma 11.3, there exists a unique morphism of $\mathbf{V}$-models $\phi : \overline{A} \to f_*(Y)$ in $\mathbf{Shv}(Z(A))$, such that $\phi_{\vec{1}} = g$, so, by Lemma 11.4 there exist a unique $(f, \varphi) : (Z(A), \overline{A}) \to (E, Y)$ in $\mathfrak{R}$, such that $\phi_{\vec{1}} = g$.

12. Corollaries in terms of local homeos

It is a classical result that for any topological space X, the category $\mathbf{LH}/X$ of local homeomorphisms over X is equivalent to the topos $\mathbf{Shv}(X)$ of sheaves over the same space (see Section II.6 in [MacLaneMoerdijk2012]). The equivalence $\mathbf{Shv}(X) \to \mathbf{LH}/X$ sends a sheaf $P : \mathcal{O}(X) \to \mathbf{Set}$ to the *bundle of germs* of P defined as follows. For each $x \in X$, let $P_x = \varinjlim_{x \in U} P(U)$ where the colimit is taken over the poset of open neighborhoods of x (ordered by reverse inclusion). The family of P_x's determines a function $\pi : \sum_{x \in X} P_x \to X$. Also, each $s \in P(U)$ determines an obvious function $\dot{s} : U \to \sum_{x \in X} P_x$ such that $\pi\dot{s} : U \to X$ is the inclusion $U \to X$. The set $\sum_{x \in X} P_x$ is topologized by taking as a base of opens all the images of the functions $\dot{s}$. This topology makes π into a local homeo, the above mentioned bundle of germs.

Any basis B for the topology of X may be considered as a subposet $B \to \mathcal{O}(X)$. The usual Grothendieck topology on $\mathcal{O}(X)$ restricts along $B \to \mathcal{O}(X)$ and the resulting morphism of sites determines an equivalence $\mathbf{Shv}(B) \to \mathbf{Shv}(X)$; see Theorem II.1.3 in [MacLaneMoerdijk2012]. The composite equivalence $\mathbf{Shv}(B) \to \mathbf{Shv}(X) \to \mathbf{LH}/X$ is very similar to the previous one because, by finality (in the sense of Section IX.3 of [MacLane1971]), the colimit $P_x = \varinjlim_{x \in U} P(U)$ may be calculated using only basic open sets.

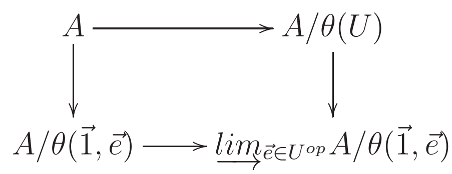
According to [Simmons1980], the *spectrum* of a distributive lattice D is the topological space σD whose points are the lattice morphisms $D \to \mathbf{2}$ and whose topology has, as a basis, the subsets $\sigma(a) \subseteq \sigma D$ (with $a \in D$) defined by $\sigma(a) = \{p \in \sigma D \mid p(a) = 1 \in \mathbf{2}\} \subseteq \sigma D$. In this way, we may identify D with the basis of its spectrum and obtain an equivalence $\mathbf{Shv}(D) \to \mathbf{LH}/\sigma D$. It assigns to each sheaf $P : D^{op} \to \mathbf{Set}$ the local homeomorphism whose fiber P_p over the point $p : D \to \mathbf{2}$ in σD is $P_p = \varinjlim_{p \in \sigma(a)} P(a)$.

Let $\mathbf{V}$ be a variety with RexDFC, A be an algebra of $\mathbf{V}$ and consider its center $Z(A)$. From the formulation of above, it can be proved that the points of $\sigma(Z(A))$ can be identified with the ultrafilters of $Z(A)$ and the basis $\{\sigma(\vec{e}) \mid \vec{e} \in Z(A)\}$ becomes a basis of clopens, making the space $\sigma(Z(A))$ a *Stone space* ([Johnstone1982]).

This facts, together with the ones considered before, allows us to obtain an equivalence $\mathbf{Shv}(Z(A)) \to \mathbf{LH}/\sigma Z(A)$ that sends a sheaf $P \in \mathbf{Shv}(Z(A))$ to a local homeomorphism over $\sigma Z(A)$, whose fiber P_U over an ultrafilter U in $\sigma Z(A)$ may be described as

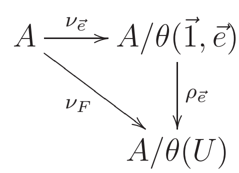
$$P_U = \varinjlim_{\vec{e}\in U} P(\vec{e})$$

12.1. Lemma. *Let* $\boldsymbol{V}$ *be a variety with RexDFC, A be an algebra of* $\boldsymbol{V}$. *For every ultrafilter U of Z(A), there exists a unique isomorphism* $A/\theta(U) \to \varinjlim_{\vec{e}\in U^{op}} A/\theta(\vec{1},\vec{e})$ *such that the following diagram*

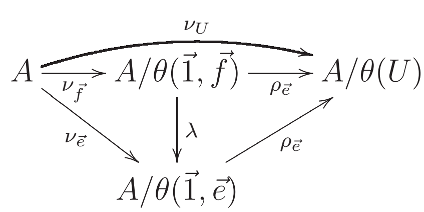


commutes for every $\vec{e} \in Z(A)$.

Proof. Let $\vec{e} \in Z(A)$ and U be an ultrafilter of $Z(A)$. If $\vec{e} \in U$, then $[\vec{1}, \vec{e}] \in \theta(U)$, so $\theta(\vec{1}, \vec{e}) \subseteq \theta(F)$. Thus, from Corollary 3.5, there exists a unique homomorphism $\rho_{\vec{e}}$: $A/\theta(\vec{1}, \vec{e}) \to A/\theta(U)$ such that the diagram



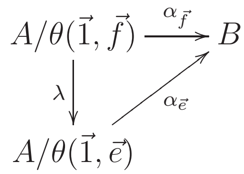
We prove that the sink $H = \{\rho_{\vec{e}} \mid \vec{e} \in U\}$ is a colimit. Let $\vec{e}, \vec{f} \in U$, such that $\vec{e} \leq_A \vec{f}$, then $\vec{e} \wedge_A \vec{f} = \vec{e}$. From Lemma 2.7, $[\vec{e}, \vec{f}] \in \theta(\vec{1}, \vec{e})$, then, since $[\vec{1}, \vec{e}] \in \theta(\vec{1}, \vec{e})$, we get that $[\vec{1}, \vec{f}] \in \theta(\vec{1}, \vec{e})$ and consequently $\theta(\vec{1}, \vec{f}) \subseteq \theta(\vec{1}, \vec{e})$. This proves that H is natural. In order to verfify that H is a cocone, let $\nu_{\vec{e}} : A \to A/\theta(\vec{1}, \vec{e})$, $\nu_{\vec{f}} : A \to A/\theta(\vec{1}, \vec{f})$ and $\nu_U : A \to A/\theta(U)$ be the canonical homomorphisms. From the diagram below



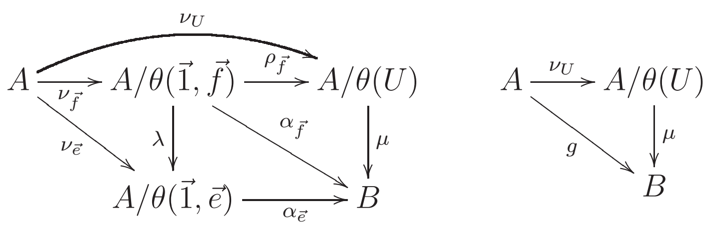
we obtain that $\nu_F = \rho_{\vec{f}}\nu_{\vec{f}} = \rho_{\vec{e}}\nu_{\vec{e}}$ and $\lambda\nu_{\vec{f}} = \nu_{\vec{e}}$. Then, from the following calculations

$$\rho_{\vec{f}}\nu_{\vec{f}} = \rho_{\vec{e}}\nu_{\vec{e}} = \rho_{\vec{e}}\lambda\nu_{\vec{f}}$$

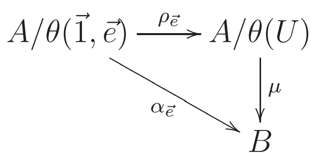
and the fact of $\nu_{\vec{f}}$ is epi, we conclude $\rho_{\vec{f}} = \rho_{\vec{e}}\lambda$. Finally, we check that H is universal. Let us to consider the commutative diagram



From the commutativity of the left diagram below,



we can deduce that the morphism $g = \alpha_{\vec{e}}\nu_{\vec{e}}$ identifies the elements of U, then, by Lemma 3.4, there exists a unique morphism $\mu : A/\theta(U) \to B$, such that the right upper diagram cummutes. From Corollary 3.5, for every $\vec{e} \in Z(A)$ we can conclude that the diagram



The result follows from the universal property of colimits. ■

Thus, if $\mathbf{V}$ is a variety with RexDFC and $A \in \mathbf{V}$, from Lemma 12.1 and the discussion along this section, in the particular case of the representing sheaf $\overline{A}$ in $\mathbf{Shv}(Z(A))$, the fiber over an ultrafilter U of $Z(A)$ is

$$\overline{A}_U = \underrightarrow{lim}_{\vec{e}\in U}\overline{A}(\vec{e}) = \underrightarrow{lim}_{\vec{e}\in U} A/\theta(\vec{1},\vec{e}) = A/\theta(U)$$

That is, if we consider the representing sheaf $\overline{A}$ in $\mathbf{Shv}(Z(A))$ as a local homemorphism over $\sigma Z(A)$, then the fiber over a point U in $\sigma Z(A)$ is the quotient of A by the principal congruence of A containing U.

Recall that a topos $\mathcal{E}$ with subobject classifier Ω is boolean, if the cospan $\top, \bot \colon 1 \to \Omega$ is a coproduct (or equivalently $\Omega \cong 1 + 1$). Thereby, if $\mathbf{V}$ is a variety with RexDFC, and $A \in V$, from Lemma 7.3, it turns that $\mathbf{Shv}(Z(A))$ is boolean a topos, and, in particular, a boolean coherent category. From Lemma 5.5 and Definition 6.2, it follows that the theory of internal connected $\mathbb{V}$-models $\mathbb{C}_{\mathbb{V}}$ is a first order theory, so if we call $\mathbb{C}'_{\mathbb{V}}$ to the Morleyzation of $\mathbb{C}_{\mathbb{V}}$, from D1.5.13 of [Johnstone2002], we obtain that $\mathbb{C}_{\mathbb{V}}(\mathbf{Shv}(Z(A))) \simeq \mathbb{C}'_{\mathbb{V}}(\mathbf{Shv}(Z(A)))$.

It is known that every point $x : 1 \to X$ of a topological space X determines a geometric morphism $\mathbf{Set} \to \mathbf{LH}/X$ whose inverse image $\mathbf{LH}/X \to \mathbf{Set}$ sends a local homeomorphism to the corresponding fiber over x. Since geometric morphisms preserve the interpretation of coherent sequents, they preserve internal connected $\mathbb{V}$-models. As as a consequence of the latter discussion, we obtain the following result.

12.2. Corollary. *Let* $\boldsymbol{V}$ *be a variety with RexDFC and CSC. Then, every algebra of* $\boldsymbol{V}$ *can be represented as the algebra of global sections of a local homeomorphism (over the Stone space* $\sigma Z(A)$*) whose fibers are* $\boldsymbol{V}$*-connected algebras.*

The Corollary 12.2 can be restricted even more, in order to obtain the last result of this paper. The proof is essentially the same of Corollary 14.2 in [CastiglioniMenniZuluaga2016].

12.3. Corollary. *Let* $\boldsymbol{V}$ *be a variety with RexDFC and CSC. Then, every algebra of* $\boldsymbol{V}$ *is a subdirect product of* $\boldsymbol{V}$*-connected algebras.*

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Departamento de Matemáticas, Universidad Nacional de Córdoba & CONICET
Email: `wijazubo@hotmail.com`