Cohomological invariants mod 2 of Weyl groups

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Let G be the Weyl group of a root system, i.e., a crystallographic finite Coxeter group, cf. [B], chap.VI, §4.1. Let k_0 be a field of characteristic $\neq 2$, let $H^\bullet(k_0) = \oplus_{n\geqslant 0} H^n(k_0, \mathbf{F}_2)$ and let $I_G = \mathrm{Inv}_{k_0}(G)$ be the ring of cohomological invariants mod 2 of G, as defined in [S], §4; it is a graded $H^\bullet(k_0)$-algebra. When G is of type A, it is isomorphic to a symmetric group Sym_n, and I_G is $H^\bullet(k_0)$-free of rank $1+[n/2]$, with an explicit basis $w_0 = 1, w_1, ..., w_{[n/2]}$, cf. [S], chap.VII.

In order to extend this description of I_G to the general case, define S_G to be the set of elements $g \in G$ with $g^2 = 1$; an element of S_G shall be called an *involution* of G. Let Σ_G be the set of conjugation classes of elements of S_G.

Theorem A. *There exists a natural injection $e : \Sigma_G \to I_G$ whose image is an $H^\bullet(k_0)$-basis of I_G.*

[Equivalently : the $H^\bullet(k_0)$-module I_G is canonically isomorphic to the set of all maps $\Sigma_G \to H^\bullet(k_0)$.]

The map e is compatible with the grading of I_G : if $g \in S_G$, define the *degree* of g to be be the multiplicity of -1 as an eigenvalue of g in the standard linear representation of G as a Coxeter group; let $\Sigma_{G,n}$ be the set of involution classes of degree n. *If $\sigma \in \Sigma_{G,n}$, then $e(\sigma)$ belongs to the n-th component I^n_G of I_G.*

Examples. 1. When $G = \mathrm{Sym}_n$, the elements of Σ_G are the conjugation classes of the products of i disjoint transpositions, with $2i \leqslant n$, and we recover the fact that $H^\bullet(k_0)$-free of rank $1 + [n/2]$, with a basis made up of elements of degree $0, 1, ..., [n/2]$. In that case the canonical basis is made up of the w_i^{gal}, which are closely related to the w_i mentioned above, cf. [S], §25.

2. When $G = \mathrm{Weyl}(\mathsf{E}_8)$, we have $|\Sigma_{G,n}| = 1$ for $0 \leqslant n \leqslant 8$, with the only exception of $n = 4$ where $|\Sigma_{G,n}| = 2$; and, of course, $\Sigma_{G,n} = \varnothing$ for $n > 8$. Hence I_G is a free $H^\bullet(k_0)$-module of rank 10, with a basis made up of elements of degree $0, 1, 2, 3, 4, 4, 5, 6, 7, 8$.

3. For E_7 and E_6, the degrees are $0, 1, 2, 3, 3, 4, 4, 5, 6, 7$ and $0, 1, 2, 3, 4$.

Definition of the map $e : \Sigma_G \to I_G$.

Let a be an element of I_G and let g be an involution of G of degree n. We first define a ≪ scalar product ≫ $\langle a, g\rangle$, which is an element of $H^\bullet(k_0)$. To do so, choose a splitting $g = s_1 \cdots s_n$, where the s_i are commuting reflections (recall that a reflection is an involution of degree 1); such a splitting always exists. Let $C = \langle s_1, ..., s_n\rangle$ be the group generated by the s_i, and let $a_C \in I_C$ be the image of a by the restriction map $I_G \to I_C$. The algebra I_C has a natural basis (α_I) indexed by the subsets I of $[1, n]$, cf. [S], §16.4. Let $\lambda_C \in H^\bullet(k_0)$ be the coefficient of $\alpha_{[1,n]}$ in a_C (≪ top coefficient ≫). One can show that λ_C *is independent of the chosen splitting of g, i.e., that it only depends on a and g.* We then define the scalar product $\langle a, g\rangle$ as λ_C; we have $\langle a, g\rangle = \langle a, g'\rangle$ if g and

g' are conjugate in G ; this allows us to define $\langle a, \sigma \rangle$ for every $\sigma \in \Sigma_G$. For a given σ, the map $a \mapsto \langle a, \sigma \rangle$ is $H^\bullet(k_0)$-linear ; if a has degree m, then $\langle a, \sigma \rangle$ has degree $m - n$ (one may view $a \mapsto \langle a, \sigma \rangle$ as an n-th fold residue map).
Example. Choose for a a Stiefel-Whitney class $w_i^{\rm gal}(\mathrm{Cox})$ of the Coxeter representation of G. One has $\langle a, \sigma \rangle = 0$ if $i \neq \deg(\sigma)$ and $\langle a, g \rangle = 1$ if $i = \deg(\sigma)$.

Theorem B.

(i) *If $a \in I_G$ is such that $\langle a, \sigma \rangle = 0$ for every σ, then $a = 0$.*

(ii) *Let n be an integer. For every $\sigma \in \Sigma_G$ of degree n, there exists $e(\sigma) \in I_G^n$ such that $\langle e(\sigma), \sigma \rangle = 1$ and $\langle e(\sigma), \sigma' \rangle = 0$ for every $\sigma' \neq \sigma$.*
[Note that, by (i), such an $e(\sigma)$ is unique.]

It is clear that Theorem B implies Theorem A.

Indications on the proof of part (i) *of Theorem B.*

An induction argument shows that, if $\langle a, \sigma \rangle = 0$ for every σ, then the restriction of a to every « cube » (i.e., subgroup generated by commuting reflections) is 0. In that case, if the characteristic of k_0 is good for G, the arguments of [S], §25, show that $a = 0$. This already covers the case where the irreducible components of G are of classical type, since every characteristic $\neq 2$ is good. The exceptional types can be reduced to the classical ones, thanks to the fact that, if G is such a Weyl group, there exists a subgroup G' of G, generated by a subset of S_G (hence also a Weyl group), which is of classical type, and has *odd index* in G : for G of type $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2$, one takes G' of type $\mathsf{D}_5, \mathsf{A}_1 \times \mathsf{D}_6, \mathsf{D}_8, \mathsf{B}_4, \mathsf{A}_1 \times \mathsf{A}_1$, respectively ; one has $(G : G') = 27, 63, 135, 3, 3$. One then uses the fact that the restriction map $I_G \to I_{G'}$ is injective, cf. [S], prop.14.4, and that every cube of G is conjugate to a cube of G'.

Indications on the proof of part (ii) *of Theorem B.*

We need to construct enough cohomological invariants. For most Weyl groups, this is done by using Stiefel-Whitney classes. For instance, for Weyl(E_6), one takes the $w_i^{\rm gal}(\mathrm{Cox})$, $i = 0, 1, 2, 3, 4$. There are however three cases where we have to do otherwise. For each one, there are two distinct classes of involutions σ, σ' of the same degree n for which it is hard to find $a \in I_G^n$ with $\langle a, \sigma \rangle = 0,\ \langle a, \sigma' \rangle = 1$. These cases are : D_{2n}, $n \geqslant 3$; E_7, $n = 3$ and 4 ; E_8, $n = 4$.

For those, we use the relation given by Milnor's conjecture (now Voevodsky's theorem) between Witt invariants and cohomological invariants mod 2. The method applies to every linear group $\mathcal{G}$ over k_0. The ring $\mathrm{Inv}_{k_0}(\mathcal{G}, W)$ of Witt invariants of $\mathcal{G}$ (as defined in [S], §27.3) has a natural filtration : an invariant h has filtration $\geqslant n$ if, for every extension k/k_0 and every $\mathcal{G}$-torsor t of $\mathcal{G}$ over k, the element $h(t)$ of the Witt ring $W(k)$ belongs to the n-th power of the canonical ideal of $W(k)$; in that case, h defines (via the Milnor construction) an element a_h of $\mathrm{Inv}_{k_0}^n(\mathcal{G}, \mathbf{F}_2)$ which is 0 if and only if the filtration of h is $> n$. We thus get *an injective map* $\mathrm{gr}^n \mathrm{Inv}_{k_0}(\mathcal{G}, W) \ \to\ \mathrm{Inv}_{k_0}^n(\mathcal{G}, \mathbf{F}_2)$.

We apply this to $\mathcal{G} = G$, where G is as in the three cases above. One can find a linear orthogonal representation of G whose Brauer character χ is such that $\chi(\sigma) - \chi(\sigma') = 2^n$. This gives a G-quadratic form, hence an element of

$\mathrm{Inv}_{k_0}(G,W)$; one modifies slightly that element to make it of filtration $\geqslant n$, so that it gives a cohomological invariant a of G of degree n, and one checks that $\langle a,\sigma\rangle - \langle a,\sigma'\rangle = 1$; that information is enough to conclude the proof.

Dependence of $\mathrm{Inv}_{k_0}(G)$ *on* $H^\bullet(k_0)$ *- Universal objects.*

(i) *Additive structure*

For the additive structure, $\mathrm{Inv}_{\mathbf{C}}(G)$ is a universal object, i.e., there is natural isomorphism of $\mathbf{F}_2$-vector spaces : $\mathrm{Inv}_{k_0}(G) \simeq \mathrm{Inv}_{\mathbf{C}}(G) \ \otimes_{\mathbf{F_2}} H^\bullet(k_0)$.

(ii) *Ring structure*

For the ring structure, it is $\mathrm{Inv}_{\mathbf{R}}(G)$ which is a universal object : there is a natural graded-$\mathbf{F}_2$-algebra isomorphism : $\mathrm{Inv}_{k_0}(G) \simeq \mathrm{Inv}_{\mathbf{R}}(G) \ \otimes_{H^\bullet(\mathbf{R})} H^\bullet(k_0)$. [In this formula, $H^\bullet(k_0)$ is viewed as an $H^\bullet(\mathbf{R})$-algebra via the unique homomorphism $H^\bullet(\mathbf{R}) \to H^\bullet(k_0)$ which maps the class of -1 in $H^1(\mathbf{R}) \simeq \mathbf{R}^\times/(\mathbf{R}^\times)^{\mathbf{2}}$ onto the class of -1 in $H^1(k_0) \simeq k_0^\times/(k_0^\times)^2$.]

References

[B] N. Bourbaki, *Groupes et Algèbres de Lie*, chap.IV-VI, Hermann, Paris, 1968.
[S] J-P. Serre, *Cohomological invariants, Witt invariants, and trace forms*, ULS 28, 1-100, AMS, 2003.

Note. After my lecture, Stefan Gille has pointed out to me that, using a different method (based on a theorem of Totaro, but not involving involutions), Christian Hirsch had already computed in 2009 the structure of the cohomological invariants of all the finite Coxeter groups, under some mild hypotheses on the ground field ; his method also applies to other types of invariants. Reference :

Christian Hirsch, *Cohomological invariants of reflection groups*, Diplomarbeit (Betreuer : Prof. Dr. Fabien Morel), Univ. München, 2009 ; available on arXiv :1805.04670[math.AG].

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