λ-ANALOGUES OF r-STIRLING NUMBERS OF THE FIRST KIND

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Abstract. In this paper, we study λ-analogues of the r-Stirling numbers of the first kind which have close connections with the r-Stirling numbers of the first kind and λ-Stirling numbers of the first kind. Specifically, we give the recurrence relations for these numbers and show their connections with the λ-Stirling numbers of the first kind and higher-order Daehee polynomials.

1. Introduction

It is known that the Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (\text{see } [1,2,6-9,14]), \tag{1.1}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

For $\lambda \in \mathbb{R}$, the λ-analogue of falling factorial sequence is defined by

$$\begin{aligned}&(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \ (n \geq 1),\\ &(\text{see } [2,10,14,15,17]).\end{aligned} \tag{1.2}$$

In view of (1.1), we define λ-analogues of the Stirling numbers of the first kind as

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k)x^k, \quad (\text{see } [2,11-13,16,17]). \tag{1.3}$$

It is not difficult to show that

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} \binom{x}{l}_{\lambda} t^l = \sum_{l=0}^{\infty} \frac{(x)_{l,\lambda}}{l!} t^l, \quad (\text{see } [4,7-17]), \tag{1.4}$$

where $\binom{x}{l}_{\lambda}$ are the λ-analogues of binomial coefficients $\binom{x}{n}$ given by $\binom{x}{l}_{\lambda} = \frac{(x)_{l,\lambda}}{l!}$.

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The r-Stirling numbers of the first kind are defined by the generating function

$$\frac{1}{k!}\big(\log(1+t)\big)^k(1+t)^r=\sum_{n=k}^{\infty}S_1^{(r)}(n,k)\frac{t^n}{n!},\quad (\text{see } [3,20-23]). \tag{1.5}$$

where $k\in\mathbb{N}\cup\{0\}$ and $r\in\mathbb{R}$.

The unsigned r-Stirling numbers of the first kind are defined as

$$(x+r)(x+r+1)\cdots(x+r+n-1)=\sum_{k=0}^{n}{\begin{bmatrix}n+r\\k+r\end{bmatrix}}_r x^k,\quad (\text{see } [1,17,22]). \tag{1.6}$$

Thus, by (1.5), we get

$$(x+r)_n=(x+r)(x+r-1)\cdots(x+r-n+1)=\sum_{k=0}^{n}S_1^{(r)}(n,k)x^k,\quad (\text{see } [1]). \tag{1.7}$$

From (1.5) and (1.7), we note that

$$S_1^{(-r)}(n,k)=(-1)^{n-k}{\begin{bmatrix}n+r\\k+r\end{bmatrix}}_r. \tag{1.8}$$

The higher-order Daehee polynomials are defined by

$$\left(\frac{\log(1+t)}{t}\right)^k(1+t)^x=\sum_{n=0}^{\infty}D_n^{(k)}(x)\frac{t^n}{n!},\quad (\text{see } [5,18,19,24]). \tag{1.9}$$

When $x=0$, $D_n^{(k)}=D_n^{(k)}(0)$ are called the higher-order Daehee numbers. In particular, for $k=1$, $D_n(x)=D_n^{(1)}(x)$, $(n\geq 0)$, are called the ordinary Daehee polynomials.

In this paper, we consider λ-analogues of r-Stirling numbers of the first kind which are derived from the λ-analogues of the falling factorial sequence and investigate some properties for these numbers. Specifically, we give some identities and recurrence relations for the λ-analogues of r-Stirling numbers of the first kind and show their connections with the λ-Stirling numbers of the first kind and higher-order Daehee polynomials.

2. λ-analogues of r-Stirling numbers of the first kind

From (1.3) and (1.4), we have

$$\begin{aligned}(1+\lambda t)^{\frac{x}{\lambda}}&=\sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!}=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}S_{1,\lambda}(k,n)x^n\right)\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\left(n!\sum_{k=n}^{\infty}S_{1,\lambda}(k,n)\frac{t^k}{k!}\right)\frac{x^n}{n!}.\end{aligned} \tag{2.1}$$

On the other hand, we also have

$$(1+\lambda t)^{\frac{x}{\lambda}} = e^{\frac{x}{\lambda}\log(1+\lambda t)} = \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^n \frac{x^n}{n!}. \tag{2.2}$$

Therefore, by (2.1) and (2.2), we get the generating function for $S_{1,\lambda}(n,k)$, $(n,k\geq 0)$, which is given by

$$\frac{1}{n!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^n = \sum_{k=n}^{\infty} S_{1,\lambda}(k,n)\frac{t^k}{k!}. \tag{2.3}$$

Now, we define λ-analogues of r-Stirling numbers of the first kind as

$$\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} = \sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!}, \tag{2.4}$$

where $k\in\mathbb{N}\cup\{0\}$, and $r\in\mathbb{R}$.

From (2.3) and (2.4), we note that $S_{1,\lambda}^{(0)}(n,k) = S_{1,\lambda}(n,k)$, $(n\geq k\geq 0)$. Also, it is easy to show that

$$(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{r}{\lambda}} = \sum_{n=0}^{\infty}(x+r)_{n,\lambda}\frac{t^n}{n!}. \tag{2.5}$$

By (2.5), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}(x+r)_{n,\lambda}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\binom{x+r}{n}_{\lambda} t^n = (1+\lambda t)^{\frac{r}{\lambda}} e^{\frac{x}{\lambda}\log(1+\lambda t)} \\
&= \sum_{k=0}^{\infty} x^k \frac{1}{k!}\left(\frac{\log(1+\lambda t}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} \\
&= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.6}$$

Therefore, by comparing the coefficients on both sides of (2.6), we obtain the following theorem.

Theorem 2.1. *For $n\geq 0$, we have*

$$(x+r)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k.$$

Now, we observe that

$$\begin{aligned}
&\sum_{k=0}^{\infty} x^k \frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} \\
&= \left(\sum_{k=0}^{\infty} x^k \sum_{m=k}^{\infty} S_{1,\lambda}(m,k)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^l}{l!}\right) \\
&= \left(\sum_{m=0}^{\infty}\sum_{k=0}^{m} S_{1,\lambda}(m,k)x^k\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^l}{l!}\right) \qquad (2.7)\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}x^k\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}x^k\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus, by (2.6) and (2.7), we get

$$\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k = \sum_{k=0}^{n}\left(\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}\right)x^k. \qquad (2.8)$$

Therefore, by comparing the coefficients on both sides of (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$S_{1,\lambda}^{(r)}(n,k) = \sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}.$$

Now, we define λ-analogues of the unsigned r-Stirling numbers of the first kind as follows:

$$(x+r)(x+r+\lambda)(x+r+2\lambda)+\cdots(x+r+(n-1)\lambda) = \sum_{k=0}^{n}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} x^k. \qquad (2.9)$$

Note that $\lim_{\lambda\to 1}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} = \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r}$, $(n \geq k \geq 0)$.

By Theorem 2.1 and (2.9), we get

$$(x-r)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}^{(-r)}(n,k)x^k, \qquad (2.10)$$

and

$$(x-r)_{n,\lambda} = \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} x^k. \qquad (2.11)$$

From (2.10) and (2.11), we can easily derive the following equation (2.12).

$$S_{1,\lambda}^{(-r)}(n,k) = (-1)^{n-k} {\textstyle\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_{r,\lambda}, \ (n \geq k \geq 0). \tag{2.12}$$

For $n \geq 1$, by Theorem 2.1, we get

$$(x+r)_{n+1,\lambda} = \sum_{k=0}^{n+1} S_{1,\lambda}^{(r)}(n+1,k)x^k = \sum_{k=1}^{n+1} S_{1,\lambda}^{(r)}(n+1,k)x^k + (r)_{n+1,\lambda}. \tag{2.13}$$

On the other hand, by (1.2), we get

$$\begin{aligned}
&(x+r)_{n+1,\lambda} = (x+r)_{n,\lambda}(x+r-n\lambda) \\
&= x\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k - (n\lambda - r)\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k \\
&= \sum_{k=1}^{n} S_{1,\lambda}^{(r)}(n,k-1)x^k - \sum_{k=1}^{n}(n\lambda - r)S_{1,\lambda}^{(r)}(n,k)x^k + (r-n\lambda)(r)_{n,\lambda} + x^{n+1} \\
&= \sum_{k=1}^{n} \left\{ S_{1,\lambda}^{(r)}(n,k-1) - (n\lambda - r)S_{1,\lambda}^{(r)}(n,k) \right\} x^k + (r)_{n+1,\lambda} + x^{n+1}.
\end{aligned} \tag{2.14}$$

Therefore, by Theorem 2.1 and (2.14), we obtain the following theorem.

Theorem 2.3. *For $1 \leq k \leq n$, we have*

$$S_{1,\lambda}^{(r)}(n+1,k) = S_{1,\lambda}^{(r)}(n,k-1) - (n\lambda - r)S_{1,\lambda}^{(r)}(n,k).$$

From (2.4), we note that

$$\begin{aligned}
\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} &= \frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k \sum_{l=0}^{\infty} \frac{r^l}{l!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l \\
&= \sum_{l=0}^{\infty} \binom{k+l}{l} r^l \frac{1}{(k+l)!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{k+l} \\
&= \sum_{l=0}^{\infty} \binom{k+l}{l} r^l \sum_{n=k+l}^{\infty} S_{1,\lambda}(n,k+l)\frac{t^n}{n!} \\
&= \sum_{l=0}^{\infty} r^l \binom{k+l}{l} \sum_{n=l}^{\infty} S_{1,\lambda}(n+k,k+l)\frac{t^{n+k}}{(n+k)!} \\
&= \sum_{n=0}^{\infty} \left(\frac{n!t^k}{(n+k)!} \sum_{l=0}^{n} r^l \binom{k+l}{l} S_{1,\lambda}(n+k,k+l) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.15}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}}&=\frac{t^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k(1+\lambda t)^{\frac{r}{\lambda}}\\&=\left(\sum_{l=0}^{\infty}D_l^{(k)}\frac{\lambda^l t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(r)_{m,\lambda}\frac{t^m}{m!}\right)\frac{t^k}{k!}\\&=\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}D_l^{(k)}\lambda^l(r)_{n-l,\lambda}\frac{t^n}{n!}\right)\frac{t^k}{k!}.\end{aligned}\tag{2.16}$$

Thus, by (2.15) and (2.16), we get

$$\sum_{l=0}^{n}r^l\frac{\binom{k+l}{l}}{\binom{n+k}{n}}S_{1,\lambda}(n+k,k+l)=\sum_{l=0}^{n}\binom{n}{l}D_l^{(k)}\lambda^l(r)_{n-l,\lambda}.\tag{2.17}$$

Therefore, by (2.17), we obtain the following theorem.

Theorem 2.4. *For $n\geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}D_l^{(k)}\lambda^l(r)_{n-l,\lambda}=\sum_{l=0}^{n}\frac{\binom{k+l}{l}}{\binom{n+k}{n}}r^lS_{1,\lambda}(n+k,k+l).$$

Now, we observe that

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}}&=\left(\sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^l}{l!}\right)\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k\\&=\sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.18}$$

Therefore, by (2.4) and (2.18), we obtain the following theorem.

Theorem 2.5. *For $n,k\geq 0$, with $n\geq k$, we have*

$$S_{1,\lambda}^{(r)}(n,k)=\sum_{m=k}^{n}\binom{n}{m}(r)_{n-m,\lambda}S_{1,\lambda}(m,k).$$

From (2.4), we note that

$$\begin{aligned}&\frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^m\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}}\\&=\frac{(m+k)!}{m!k!}\frac{1}{(m+k)!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{m+k}(1+\lambda t)^{\frac{r}{\lambda}}\\&=\binom{m+k}{m}\sum_{n=m+k}^{\infty}S_{1,\lambda}^{(r)}(n,m+k)\frac{t^n}{n!}.\end{aligned}\tag{2.19}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{m}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{k}(1+\lambda t)^{\frac{r}{\lambda}}\\
&=\left(\sum_{l=m}^{\infty}S_{1,\lambda}(l,m)\frac{t^{l}}{l!}\right)\left(\sum_{j=k}^{\infty}S_{1,\lambda}^{(r)}(j,k)\frac{t^{j}}{j!}\right)\\
&=\sum_{n=m+k}^{\infty}\left(\sum_{l=k}^{n-m}\binom{n}{l}S_{1,\lambda}^{(r)}(l,k)S_{1,\lambda}(n-l,m)\right)\frac{t^{n}}{n!}.
\end{aligned}\tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following theorem.

Theorem 2.6. *For $m,n,k\geq 0$ with $n\geq m+k$, we have*

$$\binom{m+k}{m}S_{1,\lambda}^{(r)}(n,m+k)=\sum_{l=k}^{n-m}\binom{n}{l}S_{1,\lambda}(l,k)S_{1,\lambda}(n-l,m).$$

By (2.3), we get

$$\begin{aligned}
\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^{n}}{n!}&=\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{k}(1+\lambda t)^{\frac{r}{\lambda}}(1+\lambda t)^{-\frac{r}{\lambda}}\\
&=\left(\sum_{l=k}^{\infty}S_{1,\lambda}^{(r)}(l,k)\frac{t^{l}}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{-\frac{r}{\lambda}}{m}\lambda^{m}t^{m}\right)\\
&=\left(\sum_{l=k}^{\infty}S_{1,\lambda}^{(r)}(l,k)\frac{t^{l}}{l!}\right)\left(\sum_{m=0}^{\infty}(-1)^{m}(r+(m-1)\lambda)_{m,\lambda}\frac{t^{m}}{m!}\right)\\
&=\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}^{(r)}(l,k)(-1)^{n-l}(r+(n-l-1)\lambda)_{n-l,\lambda}\right)\frac{t^{n}}{n!}.
\end{aligned}\tag{2.21}$$

Comparing the coefficients on both sides of (2.21), we have the following theorem.

Theorem 2.7. *For $n,k\geq 0$, with $n\geq k$, we have*

$$S_{1,\lambda}(n,k)=\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}^{(r)}(l,k)(-1)^{n-l}(r+\lambda(n-l-1))_{n-l,\lambda}.$$

From (1.9), we have

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \frac{t^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k(1+\lambda t)^{\frac{r}{\lambda}}\\ &= \frac{t^k}{k!}\left(\sum_{m=0}^{\infty}D_m^{(k)}\lambda^m\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^l}{l!}\right)\\ &= \frac{t^k}{k!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}D_m^{(k)}\lambda^m(r)_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.22}$$

On the other hand, by (2.4), we get

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \sum_{n=k}^{\infty}S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!}\\ &= \frac{t^k}{k!}\sum_{n=0}^{\infty}S_{1,\lambda}^{(r)}(n+k,k)\frac{n!k!}{(n+k)!}\frac{t^n}{n!}.\end{aligned} \tag{2.23}$$

Thus, by comparing the coefficients on both sides of (2.22) and (2.23), we get

$$\sum_{m=0}^{n}\binom{n}{m}D_m^{(k)}\lambda^m(r)_{n-m,\lambda} = \frac{1}{\binom{n+k}{n}}S_{1,\lambda}^{(r)}(n+k,k). \tag{2.24}$$

Therefore, by (2.24), we obtain the following theorem.

Theorem 2.8. *For $n, k \geq 0$, we have*

$$S_{1,\lambda}^{(r)}(n+k,k) = \binom{n+k}{n}\sum_{m=0}^{n}\binom{n}{m}D_m^{(k)}\lambda^m(r)_{n-m,\lambda}.$$

From (1.9), we note that

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \frac{t^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k(1+\lambda t)^{\frac{r}{\lambda}}\\ &= \frac{t^k}{k!}\sum_{n=0}^{\infty}\lambda^n D_n^{(k)}(\tfrac{r}{\lambda})\frac{t^n}{n!}.\end{aligned} \tag{2.25}$$

By (2.23) and (2.25), we get

$$S_{1,\lambda}^{(r)}(n+k,k) = \lambda^n\frac{(n+k)!}{n!k!}D_n^{(k)}(\tfrac{r}{\lambda}) = \lambda^n\binom{n+k}{n}D_n^{(k)}(\tfrac{r}{\lambda}),\ (n \geq 0). \tag{2.26}$$

In particular, for $r = 0$, from (2.21) and (2.26) we have

$$\begin{aligned}\lambda^n \binom{n+k}{k} D_n^{(k)} &= S_{1,\lambda}(n+k,k) \\ &= \sum_{l=k}^{n+k} \binom{n+k}{l} S_{1,\lambda}^{(r)}(l,k)(-1)^{n+k-l}(r+(n+k-l-1)\lambda)_{n+k-l,\lambda},\end{aligned} \tag{2.27}$$

where $n,k \geq 0$.

Therefore, by (2.27), we obtain the following theorem.

Theorem 2.9. *For $n,k \geq 0$, we have*

$$\lambda^n \binom{n+k}{k} D_n^{(k)} = \sum_{l=k}^{n+k} \binom{n+k}{l} S_{1,\lambda}^{(r)}(l,k)(-1)^{n+k-l}(r+(n+k-l-1)\lambda)_{n+k-l,\lambda}.$$

In addition,

$$\begin{aligned}D_n^{(k)} = &\frac{1}{\binom{n+k}{k}} \sum_{l=k}^{n+k} \binom{n+k}{l} \binom{l}{k} \left(\frac{1}{\lambda}\right)^{n+k-l} \\ &\times (r+(n+k-l-1)\lambda)_{n+k-l,\lambda}(-1)^{n+k-l} D_{l-k}^{(k)}(\tfrac{r}{\lambda}).\end{aligned}$$

Now, we observe that

$$\begin{aligned}\sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} e^{-\frac{r}{\lambda}\log(1+\lambda t)} \\ &= \left(\sum_{l=k}^{\infty} S_{1,\lambda}^{(r)}(l,k)\frac{t^l}{l!}\right) \sum_{m=0}^{\infty} (-1)^m r^m \frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^m \\ &= \left(\sum_{l=k}^{\infty} S_{1,\lambda}^{(r)}(l,k)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-1)^m r^m \sum_{j=m}^{\infty} S_{1,\lambda}(j,m)\frac{t^j}{j!}\right) \\ &= \sum_{n=k}^{\infty}\left(\sum_{j=0}^{n-k}\sum_{m=0}^{j}\binom{n}{j}(-1)^m r^m S_{1,\lambda}(j,m)S_{1,\lambda}(n-j,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.28}$$

Therefore, by comparing the coefficients on both sides of (2.28), we obtain the following theorem

Theorem 2.10. *For $n,k \geq 0$, with $n \geq k$, we have*

$$S_{1,\lambda}(n,k) = \sum_{j=0}^{n-k}\sum_{m=0}^{j}\binom{n}{j}(-1)^m r^m S_{1,\lambda}(j,m)S_{1,\lambda}(n-j,k).$$

For $m, n \geq 0$, we define λ-analogues of the Whitney's type r-Stirling numbers of the first kind as

$$\begin{aligned}(mx+r)_{n,\lambda} &= (mx+r)(mx+r-\lambda)(mx+r-2\lambda)\cdots(mx+r-(n-1)\lambda)\\ &= \sum_{k=0}^{n} T_{1,\lambda}^{(r)}(n,k|m)x^k.\end{aligned} \tag{2.29}$$

By (2.29), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(mx+r)_{n,\lambda}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} T_{1,\lambda}^{(r)}(n,k|m)x^k\right)\frac{t^n}{n!}\\ &= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} T_{1,\lambda}^{(r)}(n,k|m)\frac{t^n}{n!}\right)x^k.\end{aligned} \tag{2.30}$$

On the other hand, by binomial expansion, we get

$$\begin{aligned}\sum_{n=0}^{\infty}(mx+r)_{n,\lambda}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\binom{mx+r}{n}_{\lambda}t^n\\ &= (1+\lambda t)^{\frac{mx+r}{\lambda}} = (1+\lambda t)^{\frac{r}{\lambda}}e^{mx\left(\frac{\log(1+\lambda t)}{\lambda}\right)}\\ &= \sum_{k=0}^{\infty}\frac{m^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}}x^k.\end{aligned} \tag{2.31}$$

Comparing the coefficients on both sides of (2.30) and (2.31), the generating function for $T_{1,\lambda}^{(r)}(n,k|m)$, $(n,k \geq 0)$, is given by

$$\frac{m^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} = \sum_{n=k}^{\infty} T_{1,\lambda}^{(r)}(n,k|m)\frac{t^n}{n!}. \tag{2.32}$$

From (2.4) and (2.32), we note that

$$S_{1,\lambda}^{(r)}(n,k) = \frac{1}{m^k}T_{1,\lambda}^{(r)}(n,k|m),\ (n \geq k \geq 0). \tag{2.33}$$

It is known that the r-Whitney numbers are defined as

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k, \quad \text{(see [3])}. \tag{2.34}$$

By (1.3), we get

$$\begin{aligned}(mx+r)_{n,\lambda} &= \sum_{l=0}^{n} S_{1,\lambda}(n,l)(mx+r)^{l} \\ &= \sum_{l=0}^{n} S_{1,\lambda}(n,l) \sum_{j=0}^{l} m^{j} W_{m,r}(l,j)(x)_{j} \\ &= \sum_{j=0}^{n} \sum_{l=j}^{n} S_{1,\lambda}(n,l) m^{j} W_{m,.r}(l,j)(x)_{j} \\ &= \sum_{j=0}^{n} \sum_{l=j}^{n} S_{1,\lambda}(n,l) m^{j} W_{m,r}(l,j) \sum_{k=0}^{j} S_{1}(j,k)x^{k} \\ &= \sum_{k=0}^{n} \left(\sum_{j=k}^{n} \sum_{l=j}^{n} S_{1,\lambda}(n,l) S_{1}(j,k) m^{j} W_{m,r}(l,j) \right) x^{k}.\end{aligned} \tag{2.35}$$

Therefore, by (2.29) and (2.35), we obtain the following theorem.

Theorem 2.11. *For $n, k \geq 0$, with $n \geq k$, we have*

$$T_{1,\lambda}^{(r)}(n,k|m) = \sum_{j=k}^{n} \sum_{l=j}^{n} S_{1,\lambda}(n,l) S_{1}(j,k) m^{j} W_{m,r}(l,j).$$

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