#### Torus classical conformal blocks

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#### Abstract

After deriving the classical Ward identity for the variation of the action under a change of the modulus of the torus we map the problem of the sphere with four sources to the torus. We extend the method previously developed for computing the classical conformal blocks for the sphere topology to the tours topology. We give the explicit results for the classical blocks up to the third order in the nome included and compare them with the classical limit of the quantum conformal blocks. The extension to higher orders is straightforward.

#### 1 Introduction

After the seminal papers of Zamolodchikov and Zamolodchikov [1, 2, 3] a lot of work has been performed about the structure and computation of the conformal blocks both of the sphere and higher genus surfaces [4, 5, 6, 7, 8, 9, 10, 11, 12].

In a previous paper [13] a simple iterative method was developed to compute the classical conformal block for the sphere topology. Essentially the idea was to expand the Heun equation in the parameter x which represents the modulus and to match such an expansion with the one obtained by performing the classical transformation of the equation with x = 0, the hypergeometric equation, when one operates on it a proper transformation of the variable.

In this paper we want to show how the same method can be applied to the one point torus amplitude. The procedure is simpler due to the special value assumed by three of the four singularities. To reach order n in the expansion in the nome  $\tilde{q} = e^{2\pi i \tau}$  one has to work out the corresponding expansion on the sphere to order 2n in x.

To relate such conformal block expansion with the quantum problem we work out the classical Ward identity, which gives the change of the classical action under a change of the modulus  $\tau$ , in terms of the accessory parameter of the problem. The procedure is rigorous and the equation we obtain agrees with the one derived by Piatek [8] from the classical limit of the linear differential equation obeyed by a correlation function containing a null field which is a consequence of the quantum Ward identity proved Eguchi and Ooguri [14].

Then we proceed to the comparison with the classical limit of the known quantum conformal blocks finding full agreement.

#### 2 The classical Ward identity

In this section we give a direct proof of the Ward identity relating the change of the torus action under a change of the modulus in terms of the accessory parameter and the Weierstrass  $\eta_1$  function.

The Liouville action on the torus represented by a parallelogram in the z-plane with periodic boundary conditions, is given by

$$S_z = \frac{1}{2\pi} \int_T (\frac{1}{2}\partial\phi \wedge \bar{\partial}\phi + e^{\phi}dz \wedge d\bar{z}) \frac{i}{2} - \frac{\eta_K}{4\pi i} \oint_{\epsilon_K} \phi(\frac{dz}{z - z_K} - \frac{d\bar{z}}{\bar{z} - \bar{z}_K}) - \eta_K^2 \log \epsilon_K^2 \quad (1)$$

where  $z_K$  are the location of the sources of strength  $\eta_K < 1/2$  i.e. elliptic singularities. Extension to parabolic singularities  $\eta_K = \frac{1}{2}$  poses no problem [15, 16]. The normalization of the action  $S_z$  is the same as the one adopted in [1] and [15].

Working with periodic boundary conditions is rather cumbersome; so we go over the Weierstrass representation of the torus given by the equations

$$u = \wp(z), \quad w^2 = 4(u - e_1)(u - e_2)(u - e_3), \quad e_j = \wp(\omega_j), \quad e_1 + e_2 + e_3 = 0.$$
 (2)

Due to the invariance of the area i.e.  $e^{\varphi} du \wedge d\bar{u} = e^{\phi} dz \wedge d\bar{z}$ , in the *u*-representation the field is given by

$$\varphi(u) = \phi(z) + \log J\bar{J}, \qquad J = \frac{dz}{du}.$$
 (3)

Substituting in eq.(1) we obtain [15]

$$S_{z} = S_{u} - \frac{1}{2} \sum_{K} \eta_{K} (1 - \eta_{K}) \log |4(u_{K} - e_{1})(u_{K} - e_{2})(u_{K} - e_{3})|^{2} - \frac{1}{4} \log |(e_{1} - e_{2})(e_{2} - e_{3})(e_{3} - e_{1})|^{2}$$

$$(4)$$

where

$$S_{u} = \frac{1}{2\pi} \int_{D_{\varepsilon}} (\frac{1}{2} \partial \varphi \wedge \bar{\partial} \varphi + e^{\varphi} du \wedge d\bar{u}) \frac{i}{2} - \frac{\eta_{K}}{4\pi i} \oint_{\varepsilon_{K}} \varphi(\frac{du}{u - u_{K}} - \frac{d\bar{u}}{\bar{u} - \bar{u}_{K}}) - \eta_{K}^{2} \log \varepsilon_{K}^{2}$$
$$- \frac{1}{16\pi i} \oint_{\varepsilon_{l}}^{d} \varphi(\frac{du}{u - e_{l}} - \frac{d\bar{u}}{\bar{u} - \bar{e}_{l}}) - \frac{1}{8} \log \varepsilon_{l}^{2} + \frac{1}{8\pi i} \frac{3}{2} \oint_{R}^{d} \varphi(\frac{du}{u} - \frac{d\bar{u}}{\bar{u}}) + \frac{1}{2} \left(\frac{3}{2}\right)^{2} \log R^{2}$$
(5)

where  $D_{\varepsilon}$  is the double sheeted plane and the index d on the contour integrals means that a double turn has to be taken around the kinematical singularities  $e_l$ , l = 1, 2, 3 and at  $\infty$  in order to come back to the starting point. For a single source of strength  $\eta_s$  placed at the origin z = 0 i.e.  $u = \infty$  we have [17]

$$S_{u} = \frac{1}{2\pi} \int_{D_{\varepsilon}} \left(\frac{1}{2}\partial\varphi \wedge \bar{\partial}\varphi + e^{\varphi}du \wedge d\bar{u}\right) \frac{i}{2} - \frac{1}{16\pi i} \oint_{\varepsilon_{l}}^{d} \varphi\left(\frac{du}{u - e_{l}} - \frac{d\bar{u}}{\bar{u} - \bar{e}_{l}}\right) - \frac{1}{8}\log\varepsilon_{l}^{2}$$
$$- \frac{1}{8\pi i}(\eta_{s} - \frac{3}{2}) \oint_{R}^{d} \varphi\left(\frac{du}{u} - \frac{d\bar{u}}{\bar{u}}\right) + \frac{1}{2}\left(\eta_{s} - \frac{3}{2}\right)^{2}\log R^{2}$$
(6)

and

$$S_z = S_u - \frac{1}{4} \log |(e_1 - e_2)(e_2 - e_3)(e_3 - e_1)|^2 - 2\eta_s \log 2 .$$
(7)

The Liouville equation for  $\phi$  is

$$-\partial_z \partial_{\bar{z}} \phi + e^{\phi} = 2\pi \eta_s \delta^2(z), \quad 0 < \eta_s < \frac{1}{2}.$$
(8)

The auxiliary equation in the z-representation is given by

$$f''(z) + \delta_s \big( \wp(z) + \beta \big) f(z) = 0 \tag{9}$$

where  $\wp$  is Weierstrass's  $\wp$  function and  $\beta$  the accessory parameter and  $\delta_s = \eta_s(1-\eta_s) > 0$ . The auxiliary equation in the *u* variable is given by  $f'' + Q_u f = 0$  where by standard procedure [18] we have

$$Q_{u}(u) = \frac{\delta_{s}}{4} \frac{u+\beta}{(u-e_{1})(u-e_{2})(u-e_{3})} + \frac{3}{16} \left( \frac{1}{(u-e_{1})^{2}} + \frac{1}{(u-e_{2})^{2}} + \frac{1}{(u-e_{3})^{2}} \right)$$
(10)  
$$- \frac{2e_{1}}{(e_{1}-e_{2})(e_{1}-e_{3})(u-e_{1})} - \frac{2e_{2}}{(e_{2}-e_{1})(e_{2}-e_{3})(u-e_{2})} - \frac{2e_{3}}{(e_{3}-e_{1})(e_{3}-e_{2})(u-e_{3})} \right).$$

We recall that the complex parameters  $e_j = \wp(\omega_j)$  are related by  $e_1 + e_2 + e_3 = 0$ . Both the half-period  $\omega_1$  and the modulus  $\tau$  are functions of the  $e_j$ . At the end we are interested in the change of the action (1) under a change of the modulus  $\tau$  i.e. under a change of the  $e_j$  keeping and the half period  $\omega_1$  fixed. The result for the change of the action under a change of the position of the sources is

$$\frac{\partial S_u}{\partial u_K} = -\frac{B_K}{2} \tag{11}$$

where  $B_K$  are the accessory parameters of  $Q_u$  at the positions  $u_K$ , i.e.  $B_K/2$  is the residue of the simple pole of  $Q_u$  at  $u_K$ . This is the well known Polyakov relation. For a change of  $e_j$  we have [15]

$$\frac{\partial S_u}{\partial e_j} = -B_j \ . \tag{12}$$

Notice the factor 2 of difference between eq.(11) and eq.(12). For completeness we give in Appendix A a short derivation of (12) which holds for general hyperelliptic surfaces. In particular for the case of the torus with a single source we have using (7)

$$\frac{\partial S_z}{\partial e_1} = -\frac{\delta_s}{2} \frac{\beta + e_1}{(e_1 - e_2)(e_1 - e_3)}$$
(13)

and similar results for the derivative w.r.t.  $e_2$  and  $e_3$ . As we mentioned before we are interested in the variation of  $S_z$  when  $\omega_1$  is kept fixed. Under this conditions the  $e_j$  become functions of the modulus  $\tau$  and the change of the action  $S_z$  is given by

$$\delta S_z = -\frac{\delta_s}{2} \left[ \frac{\beta + e_1}{(e_1 - e_2)(e_1 - e_3)} \delta e_1 + \frac{\beta + e_2}{(e_2 - e_1)(e_2 - e_3)} \delta e_2 + \frac{\beta + e_3}{(e_3 - e_1)(e_3 - e_2)} \delta e_3 \right]$$
(14)

under the constraints  $\omega_1 = \text{const}$  and  $\delta e_1 + \delta e_2 + \delta e_3 = 0$ . To compute (14) we exploit the relation [19]<sup>1</sup>

$$\Delta^{1/4} = \frac{\pi^3}{4\omega_1^3} (\theta_1'(0|\tau))^2 \tag{15}$$

<sup>&</sup>lt;sup>1</sup>We use here and in the following the definition of the theta functions  $\theta_j$  adopted in [20, 21] and not the one of [19]

being  $\theta_1$  the elliptic theta function and  $\Delta$  the discriminant

$$\Delta = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 .$$
(16)

Taking the logarithmic differential of (15) at constant  $\omega_1$  we have

$$\frac{3}{2} \left( \frac{e_1}{(e_1 - e_2)(e_1 - e_3)} de_1 + \frac{e_2}{(e_2 - e_1)(e_2 - e_3)} de_2 + \frac{e_3}{(e_3 - e_2)(e_3 - e_1)} de_3 \right) = 2 \frac{\dot{\theta}_1'(0|\tau)}{\theta_1'(0|\tau)} d\tau = -i \frac{\pi}{2} \frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} d\tau = i \frac{6\omega_1}{\pi} \eta_1 d\tau$$
(17)

where the dot denotes the derivative w.r.t.  $\tau$  and we used [20]

$$\dot{\theta}_j(z|\tau) = -i\frac{\pi}{4}\theta_j''(z|\tau) \tag{18}$$

and [19]

$$\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} = -\frac{12\omega_1}{\pi^2}\eta_1 \tag{19}$$

where  $\eta_1 = \zeta(\omega_1)$ . This gives the sum of the three terms in eq.(14) with the  $e_j$  at the numerator. For the three terms proportional to  $\beta$  we notice that from

$$e_1 - e_2 = \frac{\pi^2}{4\omega_1^2} \theta_4^4(0|\tau), \quad q = e^{i\pi\tau}$$
(20)

we have

$$\frac{de_1 - de_2}{e_1 - e_2} = 4 \frac{\dot{\theta}_4(0|\tau)}{\theta_4(0|\tau)} d\tau = -i\pi \frac{\theta_4''(0,q)}{\theta_4(0,q)} d\tau$$
(21)

and similarly

$$\frac{de_2 - de_3}{e_2 - e_3} = 4\frac{\dot{\theta}_2(0|\tau)}{\theta_2(0|\tau)}d\tau = -i\pi\frac{\theta_2''(0,q)}{\theta_2(0,q)}d\tau$$
(22)

where we used again eq.(18). We employ now the formula [21]

$$\eta_1 = -e_i \omega_1 - \frac{\pi^2}{4\omega_1} \frac{\theta_{i+1}'(0,q)}{\theta_{i+1}(0,q)}$$
(23)

which by subtraction gives

$$\omega_1(e_1 - e_3) = \frac{\pi^2}{4\omega_1} \left( \frac{\theta_4''(0,q)}{\theta_4(0,q)} - \frac{\theta_2''(0,q)}{\theta_2(0,q)} \right)$$
(24)

and so using eqs.(21,22) and eq.(24) we have

$$\frac{de_1 - de_2}{(e_1 - e_2)(e_1 - e_3)} - \frac{de_2 - de_3}{(e_2 - e_3)(e_1 - e_3)}$$

$$= \frac{de_1}{(e_1 - e_2)(e_1 - e_3)} + \frac{de_2}{(e_2 - e_1)(e_2 - e_3)} + \frac{de_3}{(e_3 - e_1)(e_3 - e_2)} = -4i\frac{\omega_1^2}{\pi}d\tau .$$
(25)

Combining with eq.(17) we obtain

$$\left. \frac{\partial S_z}{\partial \tau} \right|_{\omega_1} = \frac{2i\delta_s \omega_1^2}{\pi} (\beta - \frac{\eta_1}{\omega_1}) \ . \tag{26}$$

Such equation was derived by Piatek [8] exploiting the classical limit of a linear differential equation containing the null-field  $\chi_{-\frac{b}{2}}$ , consequence of the quantum Ward identity given by Eguchi and Ooguri [14]. Here it has been derived directly from the classical action.

#### 3 The torus blocks

In [13] a simple iterative method was developed to compute the classical conformal block for the sphere topology with four sources. The auxiliary differential equation of the 4-point Liouville problem is given by

$$y''(z) + Q(z)y(z) = 0$$
(27)

with

$$Q = \frac{\delta_0}{z^2} + \frac{\delta}{(z-x)^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta - \delta_1}{z(z-1)} - \frac{C(x)}{z(z-x)(z-1)}$$
(28)

where  $\delta_j = (1 - \lambda_j^2)/4$  and C(x) is the accessory parameter.

The main idea of [13, 22] is to expand the Q appearing in eq.(28) in powers of x and to match such an expansion with the one obtained from  $Q_0$ 

$$Q_0 = \frac{\delta_\nu}{z^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_1 - \delta_\nu}{z(z-1)}$$
(29)

after performing on  $Q_0$  the transformation

$$z(v,x) = \frac{v - C - B_1/v - B_2/v^2 + \dots}{1 - C - B_1 - B_2 + \dots}$$
(30)

where

The transformed Q according to the well known rules is

$$Q_{v}(v) = Q_{0}(z(v)) \left(\frac{dz}{dv}\right)^{2} - \{z, v\}$$
(32)

where  $\{z, v\}$  is the Schwarz derivative of z w.r.t. v. Such a method generates an iterative structure for the coefficients  $c_j$  and  $b_{jk}$  and the derivatives of C w.r.t. x,  $C^{(n)}$  [13]. The procedure apart the known problem of the convergence of the series, is rigorous. As an example the explicit form of the expansion of the accessory parameter C(x) up to the third order included was given [13].

We want now to adapt such a structure to the computation of the classical conformal blocks for the torus with one source.

The same Q which describes the sphere can be used to describe the torus provided we perform the following substitutions

$$\delta_0 = \delta = \delta_1 = 3/16 \tag{33}$$

and  $\delta_{\infty} = \frac{1-\lambda_s^2}{16} + \frac{3}{16}$  where the source dimension  $\delta_s$  appearing in the previous section is given by  $\delta_s = (1 - \lambda_s^2)/4$ . The value of the  $\delta$ 's appearing in eq.(33) induce the two-sheet structure of the *u*-plane describing the torus.

In order to exploit the results of [13] we maneuver eq.(10) in the form (28) to obtain

$$Q_{v}(v) = \frac{\delta_{s}}{4} \frac{v + \frac{\beta + e_{2}}{e_{1} - e_{2}}}{v(v - 1)(v - x)} + \frac{3}{16} \left( \frac{1}{v^{2}} + \frac{1}{(v - 1)^{2}} + \frac{1}{(v - x)^{2}} + \frac{2e_{1}}{(v - 1)^{2}} - \frac{2e_{2}}{xv(e_{1} - e_{2})} - \frac{2e_{3}}{x(x - 1)(e_{1} - e_{2})(v - x)} \right)$$
(34)

where x is related the the  $e_j$  by

$$x = \frac{e_3 - e_2}{e_1 - e_2}$$
 and  $v = \frac{u - e_2}{e_1 - e_2}$ . (35)

Eq.(34) compared to eq.(28) gives

$$\beta = -e_3 + \frac{4}{\delta_s} \left( -C\left(\frac{e_3 - e_2}{e_1 - e_2}, \delta_\nu\right) + \frac{3}{8} \frac{e_3}{(e_1 - e_2)} \right) (e_1 - e_2) .$$
(36)

The procedure originally adopted by [1, 2] is to start from a generic monodromy along the loop M embracing the origin and the point x and whose trace is denoted by  $\text{tr}M = -2\cos \pi \lambda_{\nu}$ . In this way C, or in the case of the torus  $\beta$  through eq.(36) becomes function of x and  $\delta_{\nu} = (1 - \lambda_{\nu}^2)/4$ .

This can be considered as an "off-shell"  $\beta$ . The on-shell  $\beta$  is obtained by choosing  $\delta_{\nu}$  as to have the single valuedness of the conformal field  $\varphi$  under the circuit M. As an illustration we give in Appendix B the perturbative computation of such  $\delta_{\nu}$ . The non perturbative procedure of [1, 8] is to fix the value of  $\delta_{\nu}$  from the saddle point which develops in the semiclassical limit of the quantum expression where the integration in dP of the dimension  $\Delta = \frac{Q^2}{4} + P^2$  is present. In all the procedure we work at  $\omega_1$  fixed and the  $e_j$  in eq.(36) are given as functions of the nome q by the relations [19, 20]

$$e_{1} = \frac{\pi^{2}}{12\omega_{1}^{2}} (\theta_{2}^{4}(0,q) + 2 \ \theta_{4}^{4}(0,q))$$

$$e_{2} = \frac{\pi^{2}}{12\omega_{1}^{2}} (\theta_{2}^{4}(0,q) - \theta_{4}^{4}(0,q))$$

$$e_{3} = -\frac{\pi^{2}}{12\omega_{1}^{2}} (2\theta_{2}^{4}(0,q) + \theta_{4}^{4}(0,q)) .$$
(37)

Thus we have simply to replace in (36)

$$C(x,\delta_{\nu}) = C(0,\delta_{\nu}) + xC'(0,\delta_{\nu}) + \frac{x^2}{2}C''(0,\delta_{\nu}) + \dots$$
(38)

where the  $C^{(n)}(0, \delta_{\nu})$  are the derivatives of C w.r.t. x and are given by the sphere procedure. We find

$$C(0) = \delta_{\nu} - \frac{3}{8}$$
(39)

$$C'(0) = -\frac{\delta_{\nu}}{2} - \frac{\delta_s}{8} + \frac{3}{8}$$
(40)

$$C''(0) = \frac{12\delta_{\nu} - 48\delta_{\nu}^2 + 8\delta_{\nu}\delta_s + \delta_s^2}{256\delta_{\nu}}$$
(41)

and higher derivatives are reported in Appendix C. Then exploiting eq.(36) and

$$\eta_1 = -\frac{\pi^2}{12\omega_1} \frac{\theta_1'''(0,q)}{\theta_1'(0,q)} = -\frac{\pi^2}{12\omega_1} (-1 + 24q^2 + 72q^4 + 96q^6 + 168q^8 + \cdots)$$
(42)

we have

$$\beta - \frac{\eta_{1}}{\omega_{1}}$$

$$= \frac{\pi^{2}}{\omega_{1}^{2}} \left( \frac{1}{\delta_{s}} \left( \frac{1}{4} - \delta_{\nu} \right) - q^{2} \frac{\delta_{s}}{2\delta_{\nu}} + q^{4} \frac{\delta_{s} \left( -96\delta_{\nu}^{3} + 3\delta_{s}^{2} - 5\delta_{\nu}\delta_{s}^{2} + 24\delta_{\nu}^{2} \left( -1 + 2\delta_{s} \right) \right)}{8\delta_{\nu}^{3} (3 + 4\delta_{\nu})}$$

$$+ q^{6} \left[ - \left( \delta_{s} \left( 384\delta_{\nu}^{6} + 6\delta_{s}^{4} - 19\delta_{\nu}\delta_{s}^{4} - 56\delta_{\nu}^{3}\delta_{s}^{2} \left( -1 + 2\delta_{s} \right) - 96\delta_{\nu}^{5} \left( -5 + 8\delta_{s} \right) \right) \right]$$

$$+ 3\delta_{\nu}^{2}\delta_{s}^{2} \left( -16 + 32\delta_{s} + 3\delta_{s}^{2} \right) + 48\delta_{\nu}^{4} \left( 3 - 12\delta_{s} + 10\delta_{s}^{2} \right) \right) / \left( 16\delta_{\nu}^{5} \left( 6 + 11\delta_{\nu} + 4\delta_{\nu}^{2} \right) \right) \right]$$

$$+ O(q^{8})$$

$$(43)$$

We can now compare the above results with the semiclassical limit of the quantum conformal blocks. The conformal block expansion for the torus 1-point function is given in general by [5, 23]

$$\langle \phi_{\alpha,\bar{\alpha}} \rangle_{\tau} = \operatorname{tr}(e^{-\tau_I H + i\tau_R P} \phi_{\alpha,\bar{\alpha}}(1,1)) \tag{44}$$

where  $\phi_{\alpha,\bar{\alpha}}(1,1)$  is the field on the sphere at point (1,1). Using

$$H = 2\pi (L_0 + \bar{L}_0) - \frac{\pi c}{6}, \quad P = 2\pi (L_0 - \bar{L}_0), \quad \tilde{q} = e^{2\pi i \tau}$$
(45)

we have

$$\langle \phi_{\alpha,\bar{\alpha}} \rangle_{\tau} = \left( \tilde{q}\tilde{\tilde{q}} \right)^{-\frac{c}{24}} \operatorname{tr}\left( \tilde{q}^{L_0} \tilde{\tilde{q}}^{\bar{L}_0} \phi_{\alpha,\bar{\alpha}}(1,1) \right) \,. \tag{46}$$

For Liouville theory we have the continuous spectrum [24]

$$\Delta = \frac{Q^2}{4} + P^2, \quad P \in R_+ ,$$
 (47)

and for the central charge we have

$$c = 1 + 6Q^2 = 1 + 6(b^{-1} + b)^2 .$$
(48)

Then

$$\langle \phi_{\alpha} \rangle_{\tau} = \int_{R_{+}} dP \quad \mathcal{F}^{\alpha}_{\Delta}(\tilde{q}) \quad \mathcal{F}^{\alpha}_{\Delta}(\bar{\tilde{q}}) \quad C^{\alpha}_{\Delta,\Delta}$$
(49)

and

$$\mathcal{F}^{\alpha}_{\Delta}(\tilde{q}) = \tilde{q}^{\Delta - \frac{c}{24}} \sum_{n=0} \tilde{q}^n F^{\alpha, n}_{\Delta}, \quad \tilde{q} = e^{2\pi i \tau} = q^2 .$$

$$\tag{50}$$

The conjecture [1, 8] is that for  $b \to 0$  and heavy charges i.e.  $\alpha = \eta_s/b$ ,  $\Delta = \frac{1}{b^2}(\frac{1}{4}+p^2) = \frac{\delta_{\nu}}{b^2}$  $\mathcal{F}^{\alpha}_{\Delta}(\tilde{q})$  exponentiates

$$\mathcal{F}^{\alpha}_{\Delta}(\tilde{q}) \to e^{\frac{1}{b^2}f(\eta_s, p, \tilde{q})} \tag{51}$$

with

$$f(\eta_s, p, \tilde{q}) = (\delta_{\nu} - \frac{1}{4})\log\tilde{q} + \lim_{b \to 0} (b^2 \log \sum_{n=0}^{\infty} F_{\Delta}^{\alpha, n} \tilde{q}^n) = (\delta_{\nu} - \frac{1}{4})\log\tilde{q} + \sum_{n=1}^{\infty} f_n(\eta_s, p)\tilde{q}^n$$
(52)

while at the same time we have

$$C^{\alpha}_{\Delta,\Delta} \to e^{-\frac{1}{b^2}S^{(3)}(\frac{1}{2}-ip,\eta_s,\frac{1}{2}+ip)}$$
 (53)

with  $S^{(3)}$  the classical three point action on the sphere. Then in the  $b \to 0$  limit the integral (49) can be computed by the saddle point method [1, 8] giving

$$e^{-\frac{1}{b^2}S^{cl}(\eta_s|\tau)} = e^{\frac{1}{b^2}(-S^{(3)}(\frac{1}{2} - ip_*, \eta_s, \frac{1}{2} + ip_*) + f(\eta_s, p_*, \tilde{q}) + f(\eta_s, p_*, \tilde{q}))}$$
(54)

 $p_*$  being the saddle point given by

$$\frac{\partial}{\partial p} \left( -S^{(3)}(\frac{1}{2} - ip, \eta_s, \frac{1}{2} + ip) + f(\eta_s, p, \tilde{q}) + f(\eta_s, p, \bar{\tilde{q}}) \right) = 0 .$$
 (55)

Then exploiting eq.(55) we have at the saddle point

$$\frac{\partial S^{cl}}{\partial \tilde{q}} = -\frac{\partial f}{\partial \tilde{q}} \ . \tag{56}$$

Using the classical Ward identity (26) we can then compare expansion (43) with the expansion of eq.(56). From eq.(52) we have

$$f_1 = \lim_{b \to 0} b^2 F^1, \quad f_2 = \lim_{b \to 0} b^2 \left( F^2 - \frac{1}{2} (F^1)^2 \right)$$
 (57)

$$f_3 = \lim_{b \to 0} b^2 \left( F^3 - F^1 F^2 + \frac{1}{3} (F^1)^3 \right) \quad \text{etc} .$$
(58)

We verified the agreement of this expansion for the order  $q^2$  and  $q^4$  with the classical limit of the quantum conformal blocks given in [8] and as for the  $q^6$  term we agree with the results of [11]. With the method described above one can easily go to higher orders.

### 4 Conclusions

In the present paper we extended the technique to compute the classical conformal blocks developed in [13] to the torus topology. We derived the classical Ward identity connecting the change of the action under a change of the modulus with the accessory parameter of the problem, directly from the action.

The classical limit for the quantum action allows to relate the quantum conformal blocks with the expansion obtained from the classical Ward identity. The comparison up to the sixth order in the nome q is performed with success and the calculation can be easily extended to higher orders.

## Appendix A

In this appendix we give a short derivation of the identities eq.(11), eq.(12) which where the main ingredients in proving the classical Ward identity (26). The result holds for all hyperelliptic surfaces with n sources both elliptic and parabolic [15]. Let  $u_K$  denote the position of the sources and  $u_l$  the position of the the kinematical singularities which define the hyperelliptic surface and generalize the  $e_l$  of the torus. In [15] we used the decomposition of the field  $\varphi$ 

$$\varphi = \varphi_M + \Omega \tag{59}$$

where  $\Omega$  is a real field which is equal to  $-2\eta_K \log(u-u_K)(\bar{u}-\bar{u}_K)$  in finite non overlapping disks around the sources and to  $-\frac{1}{2}\log(u-u_l)(\bar{u}-\bar{u}_l)$  around the kinematical singularities, and equal to  $-\frac{3}{2}\log u\bar{u}$  outside a disk of radius R which includes all singularities. Elsewhere  $\Omega$  is defined as a smooth field which connects smoothly with the field in the described regions. Notice that  $\Omega$  depends on the  $u_K, u_l$ . Substituting such decomposition into eq.(5) we obtain [15]

$$S_u = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int (\frac{1}{2} \partial \varphi_M \wedge \bar{\partial} \varphi_M - \varphi_M \partial \bar{\partial} \Omega - \frac{1}{2} \Omega \partial \bar{\partial} \Omega + e^{\varphi} du \wedge \bar{d} \bar{u}) \frac{i}{2} .$$
 (60)

Actually the integral is finite in the limit  $\varepsilon \to 0$  and so the limit symbol may be removed but then one has to remember that  $\partial \bar{\partial} \Omega$  is identically zero in the above described disks and not e.g. in  $D_K$  equal to  $-2\eta_K \partial_u (1/(\bar{u} - \bar{u}_K)) = -2\eta_K \pi \delta^2 (u - u_K)$ . We are interested in the derivative of eq.(60) w.r.t.  $u_j$ . In was proven in [17, 15] that for the torus with one source and also for the four point function on the sphere, the  $\beta$  and the parameter  $\kappa$ which appear in the solution  $\varphi_M$  are real-analytic functions of the  $u_j$  almost everywhere. A similar but weaker result holds for higher genus and higher point functions [25, 26]. This allows to take in eq.(60) the derivative operation under the integral sign; using the equations of motion for  $\varphi_M$  and simple integrations by parts one obtains [15]

$$\frac{\partial S_u}{\partial u_l} = -\frac{i}{4\pi} \oint_{u_l} \Omega_l \partial \varphi_M + \frac{i}{8\pi} \int \partial (\varphi_{Ml} \bar{\partial} \varphi_M) - \bar{\partial} (\varphi_{Ml} \partial \varphi_M) \tag{61}$$

where the subscript l means the derivative w.r.t.  $u_l$ . The local uniformizing variable around  $u_l$  is s with  $s^2 = u - u_l$ . For  $Q_s$  we have

$$Q_s = 2B_l + O(s) . ag{62}$$

The two independent solution of  $f'' + Q_s f = 0$  around s = 0 are given by

$$f_1 = 1 + a_1 s - B_l s^2 + a_3 s^3 + \dots$$
,  $f_2 = s + b_2 s^2 + b_3 s^3 + \dots$  (63)

For the  $\varphi$  we have

$$\varphi = -\frac{1}{2}\log(u-u_l)(\bar{u}-\bar{u}_l) - 2\left[a_1s + \bar{a}_1\bar{s} - (B_l + \frac{a_1^2}{2})s^2 - (\bar{B}_l + \frac{\bar{a}_1^2}{2})\bar{s}^2 - \kappa^4 s\bar{s} + O(s^3)\right] + \text{const} = -\frac{1}{2}\log(u-u_l)(\bar{u}-\bar{u}_l) + \varphi_M .$$
(64)

Thus the contribution of the first integral in eq.(61) is

$$-\frac{i}{4\pi} \oint_{u_l}^d \Omega_l \partial \varphi_M = \frac{i}{4\pi} \oint \frac{1}{s^2} \left( a_1 ds - (2B_l + a_1^2) s ds - \kappa^4 \bar{s} ds + O(s^2) ds \right)$$
  
=  $-B_l - \frac{a_1^2}{2} = -B_l - \frac{1}{8} (\partial_s \varphi_M)^2 .$  (65)

Taking into account that around  $u_l$ ,  $\varphi_M = \varphi_M(s|\{u_j\})$  with  $s^2 = u - u_l$  the second integral in eq.(61) becomes

$$-\frac{i}{16}\oint \frac{1}{s}(\partial_s\varphi_M)^2 ds + \frac{i}{16}\oint \frac{1}{s}(\partial_s\varphi_M)(\partial_{\bar{s}}\varphi_M)d\bar{s} = \frac{1}{8}(\partial_s\varphi_M)^2 + 0$$
(66)

thus leaving the result

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$$\frac{\partial S_u}{\partial u_l} = -B_l \ . \tag{67}$$

The proof of eq.(11) goes along the same line but is simpler due to the fact that around  $u_K$ ,  $\varphi_M$  is a single valued function of u and thus the additional term in eq.(65) and the boundary contribution eq.(66) are absent. For the torus with a single source placed at  $z_s = 0$  due to symmetry under reflection  $\phi(z, \bar{z}) = \phi(-z, -\bar{z})$  we have  $\partial_s \varphi_M(e_j) = 0$ . Not so if the source is at  $z_s \neq 0$  with the u defined by eq.(2) or when we have more than one source but due to the cancellations discussed above the result is always (67).

### Appendix B

In this appendix as an illustration we perform the perturbative computation of the monodromy matrix along the loop M embracing the cut from  $e_3$  to  $e_2$ . This relates the value of  $\delta_{\nu}$  with the perturbative value of the accessory parameter  $\beta$ . As such  $\beta$  is already known we know, perturbatively, the on-shell value of  $\delta_{\nu}$ .

Two unperturbed, i.e.  $\delta_s = 0$ , solutions of the auxiliary equation

$$f''(u) + Q_u(u)f(u) = 0 (68)$$

are given by [18]

$$y_1(u) = [(u - e_1)(u - e_2)(u - e_3)]^{\frac{1}{4}} \equiv \Pi(u)$$
  
$$y_2(u) = \Pi(u)Z(u)$$
(69)

with

$$Z(u) = \int_{e_3}^{u} \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}$$
(70)

The cuts in eq.(69) and eq.(70) are chosen from  $e_1$  to  $+\infty$  and from  $e_2$  to  $e_3$ . If we go around the cut joining  $e_2$  to  $e_3$  the monodromies of the  $y_j(u)$  are

$$\tilde{y}_1(u) = -y_1(u)$$

$$\tilde{y}_2(u) = \tilde{y}_1(u)\tilde{Z}(u) = -y_1(u)(Z(u) + 2(\omega_2 - \omega_3)) = 2(\omega_3 - \omega_2)y_1(u) - y_2(u)$$
(71)

with trM = -2. To first order perturbation we have

$$\delta y_1(u) = \frac{\delta_s}{4w_{12}} \left( \Pi(u) \int_{e_3}^u \frac{(x+\beta)Z(x)}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} dx - \Pi(u)Z(u) \int_{e_3}^u \frac{x+\beta}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} dx \right)$$
(72)

and

$$\delta y_2(u) = \frac{\delta_s}{4w_{12}} \left( \Pi(u) \int_{e_3}^u \frac{(x+\beta)Z^2(x)}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} dx - \Pi(u)Z(u) \int_{e_3}^u \frac{(x+\beta)Z(x)}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} dx \right)$$
(73)

where  $w_{12}$  is the Wronskian  $w_{12} = y_1 y'_2 - y'_1 y_2 = 1/2$ . The continuation along the circuit M gives near  $e_3$ 

$$\delta \tilde{y}_1(u) = \frac{\delta_s}{4w_{12}} \left( (\Omega I_{11} - I_{12}) \Pi(u) + I_{11} \Pi(u) Z(u) \right) + O((u - e_3)^{\frac{5}{4}})$$
(74)

and

$$\delta \tilde{y}_2(u) = \frac{\delta_s}{4w_{12}} \left( (\Omega I_{12} - I_{22}) \Pi(u) + I_{12} \Pi(u) Z(u) \right) + O((u - e_3)^{\frac{5}{4}})$$
(75)

where

$$I_{11} = \oint \frac{(x+\beta)dx}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}}$$
  
=  $4 \int_{e_2}^{e_3} \frac{(x+\beta)dx}{\wp'(z)} = 4 \int_{\omega_2}^{\omega_3} (\wp(z)+\beta)dz = 2\Omega\beta + 4(\zeta(\omega_3) - \zeta(\omega_2)) ,$  (76)

with  $\omega_3 = \omega_1 + \omega_2$  and the contour integral is along the loop M embracing  $e_3, e_2$ .  $\zeta(z)$  is the elliptic zeta-function and  $\Omega = 2(\omega_2 - \omega_3)$ . The values of  $I_{12}$  and  $I_{22}$  do not intervene in the following calculation.

Thus we have for the trace of M to first order

$$\operatorname{tr} M = -2\cos(\pi\lambda_{\nu}) = -2 + \frac{\delta_s}{4w_{12}}\Omega I_{11}$$
 (77)

Expanding we have to first order in  $\delta_s$ 

$$\delta_{\nu} = \frac{1 - \lambda_{\nu}^2}{4} = \frac{1}{4} \left( 1 - \frac{\delta_s}{4w_{12}\pi^2} \Omega I_{11} \right) \tag{78}$$

where  $I_{11}$  is given by eq.(76) and gives the relation between the accessory parameter  $\beta$ and the trace of the monodromy M. The direct perturbative computation of  $\beta$  [18] gives

$$\beta = \frac{\bar{\omega}_2 \zeta(\omega_1) - \bar{\omega}_1 \zeta(\omega_2)}{\bar{\omega}_2 \omega_1 - \bar{\omega}_1 \omega_2} = \frac{\eta_1}{\omega_1} - \frac{\pi}{4\omega_1^2 \tau_I} .$$
(79)

Substituting in (78) it gives

$$\delta_{\nu} = \frac{1 - \lambda_{\nu}^2}{4} = \frac{1}{4} \left( 1 + \frac{\delta_s}{\pi \tau_I} \right) > \frac{1}{4}$$
(80)

showing that the monodromy M is hyperbolic.

# Appendix C

We report here the values of the derivatives of the accessory parameter  $C(x, \delta_{\nu})$  which were used in the text to compute the torus conformal blocks

$$C^{(3)}(0,\delta_{\nu}) = 3(-48\delta_{\nu}^2 + \delta_s^2 + 4\delta_{\nu}(3+2\delta_s))/(512\delta_{\nu})$$
(81)

$$C^{(4)}(0, \delta_{\nu}) = \left(-9 \, \delta_{s}^{4} / (65536 \, \delta_{\nu}^{3}) + 15 \, \delta_{s}^{4} / (65536 \, \delta_{\nu}^{2}) - 9 \, \delta_{s}^{2} (-41 + 2 \, \delta_{s}) / (8192 \, \delta_{\nu}) + 3 \, (729 + 492 \, \delta_{s} + 86 \, \delta_{s}^{2}) / 4096 + 3 \delta_{\nu} (-243 + 82 \, \delta_{s}) / 512 - 729 \, \delta_{\nu}^{2} / 256 \right) / (3 + 4 \, \delta_{\nu})$$

$$(82)$$

$$C^{(5)}(0, \delta_{\nu}) = (-15(88320 \ \delta_{\nu}^{5} + 9 \ \delta_{s}^{4} - 15 \ \delta_{\nu} \ \delta_{s}^{4} + 24 \ \delta_{\nu}^{2} \ \delta_{s}^{2}(-59 + 6 \ \delta_{s})) - 128 \ \delta_{\nu}^{4}(-345 + 118 \ \delta_{s}) - 16 \ \delta_{\nu}^{3}(1035 + 708 \ \delta_{s} + 130 \ \delta_{s}^{2})))/$$

$$(131072 \ \delta_{\nu}^{3}(3 + 4 \ \delta_{\nu}))$$

$$(83)$$

$$C^{(6)}(0, \delta_{\nu}) = (-45(17190912 \ \delta_{\nu}^{8} - 6 \ \delta_{s}^{6} + 19 \ \delta_{\nu} \ \delta_{s}^{6} + 56 \ \delta_{\nu}^{3} \ \delta_{s}^{4}(-111 + 2 \ \delta_{s}) - 2048 \ \delta_{\nu}^{7}(-20985 + 1454 \ \delta_{s}) - 3 \ \delta_{\nu}^{2} \ \delta_{s}^{4}(-1776 + 32 \ \delta_{s} + 3 \ \delta_{s}^{2}) - 16 \ \delta_{\nu}^{4} \ \delta_{s}^{2}(34893 - 5316 \ \delta_{s} + 305 \ \delta_{s}^{2}) - 256 \ \delta_{\nu}^{6}(-54561 + 31988 \ \delta_{s} + 1675 \ \delta_{s}^{2}) + 256 \ \delta_{\nu}^{5}(-25182 - 17448 \ \delta_{s} - 4439 \ \delta_{s}^{2} + 168 \ \delta_{s}^{3})))/ (16777216 \ \delta_{\nu}^{5}(6 + 11 \ \delta_{\nu} + 4 \ \delta_{\nu}^{2}))$$

$$(84)$$

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