

# RATIONAL CURVES ON ELLIPTIC K3 SURFACES

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ABSTRACT. We prove that any non-isotrivial elliptic K3 surface over an algebraically closed field  $k$  of arbitrary characteristic contains infinitely many rational curves. In the case when  $\text{char}(k) \neq 2, 3$ , we prove this result for any elliptic K3 surface. When the characteristic of  $k$  is zero, this result is due to the work of Bogomolov-Tschinkel and Hassett.

## 1. INTRODUCTION

Let  $X$  be a K3 surface over an algebraically closed field  $k$ . In [BT00, Corollary 3.28], Bogomolov and Tschinkel prove that when the characteristic of  $k$  is zero and  $X$  admits a non-isotrivial elliptic fibration then  $X$  contains infinitely many rational curves. Later, Hassett in [Has03, Section 9] handled the general case of arbitrary elliptic complex K3 surfaces. In this note, we extend the above results to the case where  $k$  has positive characteristic.

**Theorem 1.1.** *Let  $X$  be an elliptic K3 surface over an algebraically closed field  $k$ . Then  $X$  contains infinitely many rational curves in the following cases:*

- (1)  $X$  admits a non-isotrivial elliptic fibration;
- (2)  $\text{char}(k) \neq 2, 3$ .

In characteristic zero, this is the content of [BT00, Corollary 3.28] and [Has03, Section 9]. When  $k$  has positive characteristic, the main ingredients in case (1) are a result on the image of  $\ell$ -adic monodromy representations associated to non-isotrivial 1-dimensional families of elliptic curves, see Proposition 2.5. The proof is inspired from [BT00], though we simplify some arguments presented there. The proof in case (2) follows the arguments of Hassett in [Has03, Section 9]. This note is split into two parts. In the first section, some background on elliptic K3 surfaces is recalled. The main result is proved in the second section.

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## 2. BACKGROUND ON ELLIPTIC K3 SURFACES

Let  $k$  be an algebraically closed field of positive characteristic and  $\mathbb{P}_k^1$  the projective line over  $k$ . We recall some facts about elliptic K3 surfaces. For a more comprehensive introduction, see [Huy16, Chapter 11].

An elliptic K3 surface is a K3 surface  $X$  which admits a surjective morphism  $X \xrightarrow{\pi} \mathbb{P}_k^1$  whose generic fiber is a smooth integral curve of genus 1. If moreover the morphism  $\pi$  admits a section, then  $X$  is said to be a Jacobian elliptic K3 surface. The fibration is said to be non-isotrivial if not all the smooth fibers are isomorphic. For Jacobian elliptic K3 surfaces, the latter condition is equivalent to the fact that the  $j$ -invariant of the generic fiber is not in  $k$ .

**2.1. Tate-Shafarevich group.** Let  $X \xrightarrow{\pi} \mathbb{P}_k^1$  be an elliptic K3 surface. For every integer  $d \geq 0$ , one can associate to  $X$  an elliptic K3 surface  $J^d(X)$  as follows. If  $\eta$  denotes the generic point of  $\mathbb{P}_k^1$ , then the generic fiber  $X_\eta$  over  $k(\eta)$  is a smooth integral curve of genus 1. Then one can associate to it a smooth curve of genus 1,  $Jac^d(X_\eta)$ , which coarsely represents the étale sheafification of the functor

$$\text{Pic}^d : (\text{Sch}/k(\eta))^\circ \rightarrow (\text{Sets}), S \mapsto \text{Pic}^d(X_\eta \times S) / \sim .$$

Then  $J^d(X) \rightarrow \mathbb{P}_k^1$  is defined as the unique relatively minimal smooth model of  $Jac^d(X_\eta)$ . For  $d = 0$ , we denote it simply  $J(X)$  and it is a Jacobian elliptic K3 surface, see [Huy16, Chap.11, Section 4.1] or [CD89, Thm. 5.3.1] for more details. For every smooth fiber  $X_t$ ,  $t \in \mathbb{P}_k^1$ , the fiber  $J(X)_t$  is isomorphic to the Jacobian elliptic curve associated to  $X_t$ . Let  $J(X)^{sm} \subset J(X)$  be the open set of  $\pi$ -smooth points, viewed as a smooth group scheme over  $\mathbb{P}_k^1$ . Then the open  $\pi$ -smooth locus  $X^{sm} \rightarrow \mathbb{P}_k^1$  is a  $J(X)^{sm}$ -torsor over  $\mathbb{P}_k^1$ . Hence for an arbitrary Jacobian elliptic K3 surface  $Y \rightarrow \mathbb{P}_k^1$ , define the *Tate-Shafarevich group*  $\text{III}(Y)$  as the set of isomorphism classes of  $Y^{sm}$ -torsors over  $\mathbb{P}_k^1$ . The group structure on  $\text{III}(Y)$  depends on the choice of the section, however the isomorphism class does not.

**Proposition 2.1** (Chap.11, Section 5.2, 5.5(i), 5.6 [Huy16]). *Let  $X \rightarrow \mathbb{P}_k^1$  be a Jacobian elliptic K3 surface. The Tate-Shafarevich group  $\text{III}(X)$  is isomorphic to the Brauer group  $\text{Br}(X)$  of  $X$  and we have an injective map*

$$\text{III}(X) \hookrightarrow WC(X_\eta),$$

where  $WC(X_\eta)$  is the Weil-Châtelet group of the generic fiber of  $X \rightarrow \mathbb{P}_k^1$ .

Recall that the Brauer group of  $X$  is defined as the étale cohomology group  $H^2(X, \mathbb{G}_m)$  and recall also that for an elliptic curve  $E$  over a field  $K$ , the Weil-Châtelet group, denoted  $WC(E)$ , is defined as the set of isomorphism classes of torsors under  $E$  over  $K$ , see [Huy16, Chapter 11, Section 5.1].

For every positive integer  $d$  and for every smooth fiber  $X_t$ ,  $t \in \mathbb{P}_k^1$ ,  $J^d(X)_t$  is isomorphic to  $\text{Pic}^d(X_t)$ . Moreover, one has an isomorphism

$$\begin{array}{ccc} X & \xrightarrow{\sim} & J^1(X) \\ & \searrow \pi & \swarrow \pi_1 \\ & & \mathbb{P}_k^1 \end{array}$$

and  $J(J^d(X)) \simeq J(X)$ . In addition, the class  $[J^d(X)]$  of  $J^d(X)$  in  $\text{Br}(J(X))$  is equal to  $d[X]$ .

For every integers  $d, d'$ , we have natural rational maps of algebraic varieties

$$\begin{array}{ccc} J^d(X) \times_{\mathbb{P}_k^1} J^{d'}(X) & \dashrightarrow & J^{d+d'}(X) \\ & \searrow & \swarrow \\ & & \mathbb{P}_k^1 \end{array}$$

For a positive integer  $\ell$ , the diagonal embedding

$$J^1(X) \rightarrow \underbrace{J^1(X) \times_{\mathbb{P}_k^1} \cdots \times_{\mathbb{P}_k^1} J^1(X)}_{\ell \text{ times}}$$

composed with the rational map above defines a rational map  $\eta_\ell$  which fits into the following commutative diagram

$$\begin{array}{ccc} J^1(X) & \dashrightarrow^{\eta_\ell} & J^\ell(X) \\ & \searrow \pi & \swarrow \pi_\ell \\ & & \mathbb{P}_k^1 \end{array}$$

The map  $\eta_\ell$  is defined over the smooth locus of  $\pi$ .

**2.2. Rational curves.** Let  $X$  be a K3 surface over  $k$ . A rational curve on  $X$  is an integral closed subscheme  $C$  of dimension 1 and of geometric genus 0. Recall the following existence result, attributed to Bogomolov and Mumford, with a refinement of Li and Liedtke ([LL12, Theorem 2.1]).

**Proposition 2.2** (Bogomolov-Mumford). *Let  $L$  be a non-trivial effective line bundle on a K3 surface  $X$  over  $k$ . Then  $L$  is linearly equivalent to a sum of effective rational curves.*

**2.3. Relative effective Cartier divisors.**

**Definition 2.3.** *Let  $X \rightarrow \mathbb{P}_k^1$  be an elliptic K3 surface. A relative effective Cartier divisor on  $X/\mathbb{P}_k^1$  is a closed subscheme  $\mathcal{M}$  on  $X$  such that  $\mathcal{M} \rightarrow \mathbb{P}_k^1$  is finite flat. If moreover  $\mathcal{M}$  is irreducible, it is called a multisection.*

Given an elliptic K3 surface  $X$  and a multisection  $\mathcal{M}$  on  $X$ , the map  $\mathcal{M} \rightarrow \mathbb{P}_k^1$  is finite flat and its degree is by definition the degree of  $\mathcal{M}$ .

Let  $X_0$  be a smooth fiber of  $X \rightarrow \mathbb{P}_k^1$  over a point  $0 \in \mathbb{P}_k^1$ . Then we have a map given by the intersection product

$$\mathrm{Pic}(X) \xrightarrow{(X_0, \cdot)} \mathbb{Z}.$$

It sends any multisection to its degree. The image of the above map is a non-zero subgroup of  $\mathbb{Z}$ , of finite index. Denote by  $d_X$  its index. It is called the degree of the elliptic fibration  $X \rightarrow \mathbb{P}_k^1$ . Remark that an elliptic fibration is Jacobian if and only if its degree is equal to one.

**Lemma 2.4.** *Let  $X \rightarrow \mathbb{P}_k^1$  be an elliptic K3 surface.*

- (1) *The order of  $[X]$  in  $\mathrm{Br}(J(X))$  is equal to  $d_X$ .*
- (2) *There exists a multisection of degree  $d_{\mathcal{M}} = d_X$  which is a rational curve.*
- (3) *There exists at least one multisection  $\mathcal{M}$  such that  $d_{\mathcal{M}} = d_X$  and which is moreover generically étale over  $\mathbb{P}_k^1$ .*

*Proof.* For (2), let  $\mathcal{M}$  be a multisection of degree  $d_X$ . By Proposition 2.2,  $\mathcal{M}$  is linearly equivalent to a sum of rational curves  $\sum_i C_i$ . Then there exists a unique curve  $C_i$  which is horizontal and all the others are vertical. Then  $C_i$  satisfies the desired properties.

For (1), notice that  $X_\eta$  is a torsor under the elliptic curve  $J(X)_\eta$  and that  $d_X$  is the index of  $X_\eta$ , i.e. is the greatest common divisor of the degrees of residue fields of closed points of  $X_\eta$  (see [Lic68, 1]). Since the order of  $X_\eta$  in  $WC(J(X)_\eta)$  is equal to its index by [Lic68, Theorem 1], it implies that the order of  $[X]$  is exactly  $d_X$ . By [Lic68, Section 5, Theorem 4]<sup>1</sup>, it is also equal to the minimal degree of residue fields of separable closed points. Hence there exists a closed separable point

<sup>1</sup>More precisely, see the proof given there.

in  $X_\eta$  of degree  $d_X$ . Taking its closure yields a separable multisection. This proves (3).  $\square$

**2.4. Monodromy.** Let  $X \xrightarrow{\pi} \mathbb{P}_k^1$  be an elliptic K3 surface. Let  $U$  be the largest Zariski open subset of  $\mathbb{P}_k^1$  over which the map  $\pi$  is smooth. Thus  $X_U \rightarrow U$  is a torsor under the smooth group scheme  $J(X)_U \rightarrow U$ . For  $b \in U$  a closed point and  $m$  prime to  $p := \text{char}(k)$ , the étale fundamental group  $\pi_1^{\text{ét}}(U, b)$  of  $U$  acts on the group of  $m$ -torsion points in  $J(X)_b$  and defines a group morphism

$$\rho : \pi_1^{\text{ét}}(U, b) \rightarrow \text{Aut} \left( \varprojlim_{\gcd(m,p)=1} J(X)_b[m] \right) = \prod_{\gcd(\ell,p)=1} \text{Aut}(\text{T}_\ell J(X)_b).$$

This action preserves the Weil pairing and factors as follows:

$$\rho : \pi_1^{\text{ét}}(U, b) \rightarrow \prod_{\ell \wedge p=1} \text{SL}(\text{T}_\ell J(X)_b).$$

For every prime  $\ell$ , we denote by  $\rho_{\ell^\infty}$  the representation of  $\pi_1^{\text{ét}}(U, b)$  on the Tate module  $\text{T}_\ell J(X)_b$  and denote by  $\rho_\ell$  its reduction modulo  $\ell$ . Then  $\rho_{\ell^\infty}$  is simply the projection on the  $\ell$ -factor in the previous map. The monodromy group  $\Gamma$  is the image of  $\pi_1^{\text{ét}}(U, b)$  under  $\rho$ . The next result on the image of the monodromy group will be crucial in the proof of Theorem 1.1.

**Proposition 2.5** ([CH05]). *If the elliptic fibration is not isotrivial, then there exists a constant  $c(k)$  depending only on  $k$ , such that for every  $\ell > c(k)$  the morphism  $\rho_\ell$  is surjective.*

This is the content of [CH05, Theorem 1.1] where the surjectivity is proven for the reduction modulo  $\ell$ , then one uses Lemma 2 in [Ser98, IV-23]. Notice that in [CH05, Theorem 1.1], the base field is supposed to be finite but one can check that the proof given there works for perfect fields, as mentioned in the discussion after Theorem 1.1 in *loc.cit.*

### 3. PROOF OF THEOREM 1.1

If  $X$  has Picard rank  $\rho(X)$  at least 20, then the automorphism group of  $X$  is infinite and hence  $X$  contains infinitely many rational curves, see [Huy16, Chap.13, Remark 1.6] and [BT00, Theorem 4.1]. Hence we assume that  $\rho(X) \leq 19$ .

The elliptic surface  $X$  defines a class in the Tate-Shafarevich group  $\text{III}(J(X))$  of  $J(X)$ , which is isomorphic to the Brauer group  $\text{Br}(J(X))$  by Proposition 2.1. This class is a sum of two elements  $\alpha_p + \alpha$ , where  $\alpha$  has torsion prime to  $p$  and  $\alpha_p$  is torsion of order  $p^a$ , for some integer  $a$ . Here  $p$  is the characteristic of  $k$ . We will construct infinitely many multisections on  $X$  which are rational curves and whose degrees tend to infinity. This will be enough to prove Theorem 1.1. Denote by  $d_X$  the degree of  $X$  and let  $\ell$  be a prime number with residue 1 (mod  $p^a$ ) and

such that  $\ell > \max(d_X, c(k))$ , where  $c(k)$  is given by Proposition 2.5. The prime to  $p$  torsion part of  $\text{Br}(J(X))$  is a divisible group by [Huy16, Chap. 18, Example 1.5]. The Kummer exact sequence and the assumption on the Picard rank ensures furthermore that it is not trivial (see formula (1.8) *loc. cit.*). We can thus find an elliptic K3 surface  $\pi_\ell : X_\ell \rightarrow \mathbb{P}^1$  such that  $J(X_\ell) \simeq J(X)$ ,  $\ell[X_\ell, \pi_\ell] = [X, \pi]$  in  $\text{Br}(J(X))$  and  $d_{X_\ell} = \ell d_X$ . Take for instance  $\alpha_p + \alpha_\ell$ , where  $\alpha_\ell$  is a non-trivial element in  $\text{Br}(J(X))$  which satisfies  $\ell \cdot \alpha_\ell = \alpha$ . Hence  $J^\ell(X_\ell) \simeq X$  and we have a rational map defined at the end of section 2.1:

$$\begin{array}{ccc} X_\ell & \overset{\eta_\ell}{\dashrightarrow} & X \\ & \searrow \pi_\ell & \swarrow \pi \\ & & \mathbb{P}_k^1 \end{array}$$

By Lemma 2.4,  $X_\ell$  contains a rational multisection  $\mathcal{M}_\ell$  of degree  $d_{\mathcal{M}_\ell} = d_{X_\ell} = \ell d_X$ . If the restriction of  $\eta_\ell$  to  $\mathcal{M}_\ell$  is isomorphic to its images above  $\mathbb{P}_k^1$  then  $\eta_\ell(\mathcal{M}_\ell)$  is a rational curve on  $X$  of degree divisible by  $\ell$  which is the desired result. Otherwise, since the multiplication by  $\ell$  map is étale (by [Gro62, Théorème 2.5]), there exists infinitely many closed points  $b$  in the maximal open subset  $U \subset \mathbb{P}_k^1$  where  $\pi$  is smooth,  $\mathcal{M}_{\ell,U} \rightarrow U$  is smooth and two distinct points  $P_1, P_2$  in  $X_{\ell,b} \cap \mathcal{M}_\ell$  such that  $\ell \cdot (P_1 - P_2) = 0$  in  $J(X)_b$ . Thus, the point  $P_1 - P_2$  is a  $\ell$ -primitive torsion point in  $J(X)_b$ . Let  $J(X)_U[\ell] \rightarrow U$  be the relative effective Cartier divisor of  $J(X)_U \rightarrow U$  of  $\ell$ -torsion points.

Let  $J(X)_{U,\text{prim}}[\ell]$  be the relative effective Cartier divisor of non-zero  $\ell$ -torsion points. Since  $X_{\ell,U}$  is a  $J(X)_U$ -torsor over  $U$ , there is an induced map:

$$(1) \quad J(X)_{U,\text{prim}}[\ell] \times \mathcal{M}_{\ell,U} \rightarrow X_{\ell,U}.$$

The closure of the image in  $X_\ell$  is a curve of  $X_\ell$  which intersects  $\mathcal{M}_\ell$  infinitely many times by the non-injectivity of  $\eta_\ell$ . Hence  $\mathcal{M}_\ell$  is isomorphic to an irreducible component of  $J(X)_{U,\text{prim}}[\ell] \times_U \mathcal{M}_{\ell,U}$ .

**3.1. Non-isotrivial case.** For  $\ell$  large enough,  $J(X)_{U,\text{prim}}[\ell]$  is irreducible by Proposition 2.5. Hence via its first projection, the above map is surjective over  $J(X)_{U,\text{prim}}[\ell]$ . Since there are  $\ell^2 - 1$  torsion points in each fiber of  $J(X)_{U,\text{prim}}[\ell]$  over  $U$ , this implies

$$d_{\mathcal{M}_\ell} = \ell d_X \geq \ell^2 - 1.$$

This is a contradiction by our assumption on  $\ell$ .

**3.2. Isotrivial case.** We assume now that the elliptic fibration  $X \rightarrow \mathbb{P}_k^1$  is isotrivial. Then the elliptic fibration  $J(X) \rightarrow \mathbb{P}_k^1$  is also isotrivial. If the characteristic of  $k$  is different from 2 and 3, which will be assumed

henceforth, then we can proceed following the lines of [Has03, Section 9]. The image of the étale fundamental group of  $U$  by  $\rho_\ell$  factors through the automorphism group of the geometric generic fiber of  $J(X) \rightarrow \mathbb{P}_k^1$  which is cyclic of order 2, 4 or 6, see [Sil86, III.10]. Assume that the fibration  $J(X) \rightarrow \mathbb{P}_k^1$  has  $n_0$  degenerate fibers of type  $I_0^*$ ,  $n'_1$  degenerate fibers of type  $I_a$ ,  $a > 0$ ,  $n''_1$  degenerate fibers of type  $I_a^*$ ,  $a > 0$ ,  $n_2$  fibers of type  $II$  or  $II^*$ ,  $n_3$  fibers of type  $III$  or  $III^*$ , and  $n_4$  fibers of type  $IV$  or  $IV^*$ . For the definition of the type of singularities of fibers, see [Huy16, Chapter 11, Section 1.3].

By Equation (1),  $\mathcal{M}_{\ell,U}$  is an irreducible component of a principal homogeneous space under  $J(X)_{U,prim}[\ell]$ . Using Riemann-Hurwitz as in the proof of [Has03, Theorem 9.9] and noticing that the computations of the ramification contributions of degenerate fibers from [Has03, Table 1, page 259] hold for  $\ell$  large enough, see [N64, Chapitre III, 17], there exists  $C > 0$  such that  $g(\mathcal{M}_\ell) \geq C \cdot c(J)$  where  $g(\mathcal{M}_\ell)$  is the geometric genus of  $\mathcal{M}_\ell$  and

$$c(J) = \frac{1}{2}n_0 + n'_1 + n''_1 + \frac{5}{6}n_2 + \frac{3}{4}n_3 + \frac{2}{3}n_4 - 2.$$

Since  $\mathcal{M}_\ell$  is a rational curve, we infer that  $c(J) \leq 0$ . We use now the method of [Has03, Proposition 9.6] to classify K3 surfaces that satisfy the last condition. By Shioda-Tate formula [SS10, Theorem 6.3]), we have :

$$\rho(X) = 2 + \sum_{s \in \mathbb{P}^1(k)} (r_s - 1) + r(X)$$

where  $r_s$  denotes the number of irreducible components of a fiber  $X_s$  for  $s$  a closed point in  $\mathbb{P}_k^1$  and  $r(X)$  is the rank of the Mordell-Weil group of  $J(X)$ . On the other hand, the  $\ell$ -adic Euler formula ([Dol72, Theorem 1.1, Corollary 1.6]<sup>2</sup>) implies that:

$$(2) \quad 24 = \sum_{s \in \mathbb{P}^1(k)} [\chi(X_s)_\ell + \alpha_{s,\ell}]$$

where, for  $s \in \mathbb{P}_k^1(k)$ ,  $\chi(X_s)_\ell$  is the  $\ell$ -adic Euler characteristic of the fiber  $X_s$  and  $\alpha_{s,\ell}$  is its wild conductor defined in [Dol72, Section 1]. Recall that  $r_s = \chi(X_s)_\ell$  if the fiber  $X_s$  has reduction type  $I_a$  and otherwise  $r_s = \chi(X_s)_\ell - 1$ . Since the characteristic of  $k$  is different from 2 and 3, all the wild conductors above vanish.

Combining the two previous formulas, we get:

$$\begin{aligned} \rho(X) &= 2 + \sum_{\substack{s \in \mathbb{P}_k^1(k) \\ \text{of type } I_a}} (r_s - 1) + \sum_{\substack{s \in \mathbb{P}_k^1(k) \\ \text{not of type } I_a}} (r_s - 2) + r(X) \\ &= 26 - n'_1 - 2N + r(X) \end{aligned}$$

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<sup>2</sup>With the correct sign.

where  $N = n_0 + n''_1 + n_2 + n_3 + n_4$ . The assumption that  $c(J) \leq 0$  implies that

$$18 + r(X) + 3n'_1 + 2n''_1 + \frac{4}{3}n_2 + n_3 + \frac{2}{3}n_4 \leq \rho(X).$$

Hence either  $X$  has Picard rank equal to 22, or  $\rho(X) \leq 20$  and thus  $X$  is an element in the list given in [Has03, Proposition 9.6]. In all these cases,  $X$  is either a Kummer surface or its automorphism group is infinite. In both cases,  $X$  has infinitely many rational curves, see [BT05, Corollary 4.3] and [BT00, Lemma 4.9] for the second case.

**3.3. Situation in characteristic 2 and 3.** When the characteristic of  $k$  is equal to 2 or 3 and the elliptic fibration  $X \rightarrow \mathbb{P}_k^1$  is isotrivial then the classification above must be modified to take into account the wild ramification factors in Equation (2) which do not vanish in general, apart from special cases, see [SS10, Section 4.6, Table 2]. For example, we could have a K3 surface with a single cusp of conductor 24 for which  $c(J) = \frac{-7}{6}$  and  $\rho(X) \geq 2$ . It would be interesting to investigate these small rank situations.

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